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A note on q-difference equations for Ramanujan's integrals

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Abstract This short paper derives the relationship between solutions of q-difference equations and generating functions for q-orthogonal polynomials. The key of the method is to obtain the expression of certain q-orthogonal polynomials as solutions of q-difference equations. In addition, we show how to generalize Ramanujan's integrals by the technique of q-difference equation. More over, we find two generalized q-Chu–Vandermonde formulas from the perspective of the method of q-difference equations.

Keywords Solutions of q-difference equation · Generating function · Al-Salam–Carlitz polynomial · Ramanujan's integral

Mathematics Subject Classification 05A30 · 11B65 · 33D15 · 33D45 · 39A13

1 Introduction

The objective of this paper is to extend the work of Liu [14,15] and Liu and Zeng [17]. These authors have found a q-difference equation related to Rogers–Szegö polynomials [21] which can be used to find interesting transformation formulas. We do the same analysis for the more general Al-Salam–Carlitz polynomials [8]. We apply this approach to provide a generating function for Al-Salam–Carlitz polynomials, generalize Ramanujan's q-beta integrals and the q-Chu–Vandermonde summation formula.

Dedicated to David Goss.

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For further information about basic hypergeometric series and q-orthogonal polynomials, see [2,11,12,23].

In this paper, we follow the notations and terminology in [9] and suppose that $0 < q < 1$. The basic hypergeometric series ${}_r\phi_s$

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{\left(a_1, a_2, \ldots, a_r; q\right)_n}{\left(q, b_1, b_2, \ldots, b_s; q\right)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \tag{1.1} $$

converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$ and for terminating. The compact factorials of ${}_r\phi_s$ are defined, respectively, by

$$ (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \tag{1.2} $$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The Rogers–Szegö polynomials were introduced by Szegö in 1926 but were already studied earlier by Rogers in 1894–1895. A good definition can be found in the book by Barry Simon [20, Ex. (1.6.5), pp. 77–87].

The homogeneous Rogers–Szegö polynomials [18, p. 3]

$$ h_n(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b^k c^{n-k} \quad \text{and} \quad g_n(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} b^k c^{n-k}. \tag{1.3} $$

The Al-Salam–Carlitz polynomials were introduced by Al-Salam and Carlitz in 1965 [1, Eqs. (1.11) and (1.15)]

$$ \phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k \quad \text{and} \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k (aq^{1-k}; q)_k. \tag{1.4} $$

They play important roles in the theory of q-orthogonal polynomials. In fact, there are two families of these polynomials: one with continuous orthogonality and another with discrete orthogonality. They are given explicitly in the book of Koekoek–Swarttouw–Lesky [13, Eqs. (14.24) and (14.25), pp. 534–540].

The generalized Al-Salam–Carlitz polynomials [7, Eq. (4.7)]

$$ \phi_n^{(a,b,c)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b; q)_k}{(c; q)_k} x^k y^{n-k} \quad \text{and} $$

$$ \psi_n^{(a,b,c)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b; q)_k}{(c; q)_k} (-1)^k q^{\binom{k+1}{2}-nk} x^k y^{n-k}, \tag{1.5} $$

whose generating functions are [7, Eqs. (4.10) and (4.11)]

$$\sum_{n=0}^{\infty} \phi_n^{(a,b,c)}(x, y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(yt;q)_\infty} {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, xt\right], \quad \max\{|yt|, |xt|\} < 1, \tag{1.6}$$

$$\sum_{n=0}^{\infty} \psi_n^{(a,b,c)}(x, y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (yt;q)_\infty {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, xt\right], \quad |xt| < 1. \tag{1.7}$$

Liu [14,15] obtained several important results by using the following q-difference equations. Liu and Zeng [17] provide further applications of these q-difference methods to q-orthogonal polynomials.

Proposition 1 ([17, Eqs. (1.7) and (1.8)]) *Let $f(a, b)$ be a two-variable analytic function at $(0, 0) \in \mathbb{C}^2$. Then*

(A) *f can be expanded in terms of $h_n(a, b|q)$ if and only if f satisfies the functional equation*
$$bf(aq, b) - af(a, bq) = (b - a)f(a, b). \tag{1.8}$$

(B) *f can be expanded in terms of $g_n(a, b|q)$ if and only if f satisfies the functional equation*
$$af(aq, b) - bf(a, bq) = (a - b)f(aq, bq). \tag{1.9}$$

The method of q-difference equation is an effective way to obtain many results in q-series. For more information, please refer to [6,7,14,15].

Theorem 2 *Let $f(a, b, c, x, y)$ be a five-variable analytic function in a neighbourhood of $(a, b, c, x, y) = (0, 0, 0, 0, 0) \in \mathbb{C}^5$.*

(I) *If $f(a, b, c, x, y)$ can be expanded in terms of $\phi_n^{(a,b,c)}(x, y|q)$ if and only if*

$$\begin{aligned} &y\left[f(a, b, c, x, y) - \left(1 + q^{-1}c\right)f(a, b, c, qx, y) + q^{-1}cf\left(a, b, c, q^2x, y\right)\right] \\ &= x\Big\{\left[f(a, b, c, x, y) - f(a, b, c, x, qy)\right] \\ &\quad - (a + b)\left[f(a, b, c, qx, y) - f(a, b, c, qx, qy)\right] \\ &\quad + ab\left[f\left(a, b, c, q^2x, y\right) - f\left(a, b, c, q^2x, qy\right)\right]\Big\}. \end{aligned} \tag{1.10}$$

(II) *If $f(a, b, c, x, y)$ can be expanded in terms of $\psi_n^{(a,b,c)}(x, y|q)$ if and only if*

$$\begin{aligned} &q^{-1}y\left[f(a, b, c, x, y) - \left(1 + q^{-1}c\right)f\left(a, b, c, qx, y\right) + q^{-1}cf\left(a, b, c, q^2x, y\right)\right] \\ &= x\Big\{\left[f(a, b, c, x, y) - f\left(a, b, c, x, q^{-1}y\right)\right] \\ &\quad - (a + b)\left[f(a, b, c, qx, y) - f\left(a, b, c, qx, q^{-1}y\right)\right] \\ &\quad + ab\left[f\left(a, b, c, q^2x, y\right) - f\left(a, b, c, q^2x, q^{-1}y\right)\right]\Big\}. \end{aligned} \tag{1.11}$$

Remark 3 For $a = b = c = 0$ in Theorem 2, Eqs. (1.10) and (1.11) reduce to (1.8) and (1.9), respectively.

To determine if a given function is an analytic function in several complex variables, we often use the following Hartogs's theorem. For more information, please refer to Taylor [22, p. 28] and Liu [16, Theorem 1.8].

Proposition 4 (Hartogs's theorem [10, p. 15]) *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \subseteq \mathbb{C}^n$, then it is holomorphic (analytic) in D.*

In order to prove Theorem 2, we need the following fundamental property of several complex variables.

Proposition 5 ([19, p. 5, Proposition 1]) *If $f(x_1, x_2, \ldots, x_k)$ is analytic at the origin $(0, 0, \ldots, 0) \in \mathbb{C}^k$, then, f can be expanded in an absolutely convergent power series,*

$$f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k=0}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Proof of Theorem 2 From the Hartogs's theorem and the theory of several complex variables (see Propositions 4 and 5), we assume that

$$f(a, b, c, x, y) = \sum_{k=0}^{\infty} A_k(a, b, c, y) x^k. \tag{1.12}$$

On one hand, substituting Eq. (1.12) into (1.10) yields

$$\begin{aligned} y\Bigg[&\sum_{k=0}^{\infty} A_k(a, b, c, y) x^k - \left(1 + q^{-1}c\right) \sum_{k=0}^{\infty} A_k(a, b, c, y)(qx)^k \\ &+ q^{-1}c \sum_{k=0}^{\infty} A_k(a, b, c, y)\left(q^2x\right)^k\Bigg] \\ &= x\Bigg\{\Bigg[\sum_{k=0}^{\infty} A_k(a, b, c, y) x^k - \sum_{k=0}^{\infty} A_k(a, b, c, qy) x^k\Bigg] \\ &- (a + b)\Bigg[\sum_{k=0}^{\infty} A_k(a, b, c, y)(qx)^k - \sum_{k=0}^{\infty} A_k(a, b, c, qy)(qx)^k\Bigg] \\ &+ ab\Bigg[\sum_{k=0}^{\infty} A_k(a, b, c, y)\left(q^2x\right)^k - \sum_{k=0}^{\infty} A_k(a, b, c, qy)\left(q^2x\right)^k\Bigg]\Bigg\}. \end{aligned} \tag{1.13}$$

By equating coefficients of x^k on both sides of Eq. (1.13), we have

$$A_k(a, b, c, y) = \frac{\left(1 - aq^{k-1}\right)\left(1 - bq^{k-1}\right)}{\left(1 - q^k\right)\left(1 - cq^{k-1}\right)} D_y A_{k-1}(a, b, c, y). \tag{1.14}$$

Iterating, we have

$$A_k(a,b,c,y)=\frac{(a,b;q)_k}{(q,c;q)_k}D_y^k A_0(a,b,c,y). \tag{1.15}$$

Letting $f(a,b,c,0,y)=A_0(a,b,c,y)=\sum_{n=0}^{\infty}\mu_n y^n$, we have

$$A_k(a,b,c,y)=\frac{(a,b;q)_k}{(q,c;q)_k}\sum_{n=0}^{\infty}\mu_n\frac{(q;q)_n}{(q;q)_{n-k}}y^{n-k}. \tag{1.16}$$

By Eq. (1.12), we have

$$\begin{aligned} f(a,b,c,x,y)&=\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}\sum_{n=0}^{\infty}\mu_n\frac{(q;q)_n}{(q;q)_{n-k}}y^{n-k}x^k\\ &=\sum_{n=0}^{\infty}\mu_n\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(a,b;q)_k}{(q,c;q)_k}x^k y^{n-k}=\sum_{n=0}^{\infty}\mu_n\phi_n^{(a,b,c)}(x,y|q). \end{aligned}$$

On the other hand, if $f(a,b,c,x,y)$ can be expanded in terms of $\phi_n^{(a,b,c)}(x,y|q)$, we can verify that $f(a,b,c,x,y)$ satisfies Eq. (1.10). The proof of Eq. (1.10) is complete. Similarly, we can deduce Eq. (1.11). The proof of Theorem 2 is complete. □

This paper is organized as follows. In Sect. 2, we generalize two generating functions for Andrews–Askey polynomials. In Sect. 3, we deduce generalizations of Ramanujan type q-beta integrals. In Sect. 4, we generalize q-Chu–Vandermonde formula.

2 Two generating functions for generalized Al-Salam–Carlitz polynomials

In this section, we generalize generating functions for Al-Salam–Carlitz polynomials.

Theorem 6 *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(a,b,c)}(x,y|q)\frac{(s/r;q)_n r^n}{(q;q)_n}=\frac{(sy;q)_\infty}{(ry;q)_\infty}{}_3\phi_2\left[\begin{matrix} a,b,s/r\\ c,sy\end{matrix};q,rx\right],$$
$$\max\{|rx|,|ry|\}<1, \tag{2.1}$$

$$\sum_{n=0}^{\infty}\psi_n^{(a,b,c)}(x,y|q)\frac{(r/s;q)_n s^n}{(q;q)_n}=\frac{(ry;q)_\infty}{(sy;q)_\infty}{}_3\phi_2\left[\begin{matrix} a,b,r/s\\ c,q/(sy)\end{matrix};q,\frac{qx}{y}\right],$$
$$\max\{|sy|,|qx/y|\}<1. \tag{2.2}$$

Corollary 7 *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(a,b,c)}(x,y|q)(-1)^n q^{\binom{n}{2}}\frac{s^n}{(q;q)_n}=(sy;q)_\infty {}_2\phi_2\left[\begin{matrix}a,b\\ c,sy\end{matrix};q,sx\right], \tag{2.3}$$

$$\sum_{n=0}^{\infty}\psi_n^{(a,b,c)}(x,y|q)\frac{s^n}{(q;q)_n}=\frac{1}{(sy;q)_\infty}{}_3\phi_2\left[\begin{matrix}a,b,0\\ c,q/(sy)\end{matrix};q,\frac{qx}{y}\right],$$
$$\max\{|sy|,|qx/y|\}<1. \tag{2.4}$$

Remark 8 For $r=0$ in Theorem 6, Eqs. (2.1) and (2.2) reduce to Eqs. (2.3) and (2.4), respectively. For $s=0$ in Theorem 6, Eqs. (2.1) and (2.2) reduce to Eqs. (1.6) and (1.7), respectively.

Proof of Theorem 6 By the Weierstrass M-test, series $\sum_{n=0}^{\infty}M_n$ is convergent when $\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right|<1$. We check that both sides of Eq. (2.1) are convergent if $\max\{|rx|,|ry|\}<1$, that is,

$$\lim_{n\to\infty}\left|\frac{\phi_{n+1}^{(a,b,c)}(x,y|q)(s/r;q)_{n+1}r^{n+1}/(q;q)_{n+1}}{\phi_n^{(a,b,c)}(x,y|q)(s/r;q)_n r^n/(q;q)_n}\right|=|ry|<1,$$
$$\lim_{n\to\infty}\left|\frac{(a,b,s/r;q)_{n+1}(rx)^{n+1}/(q,c,sy;q)_{n+1}}{(a,b,s/r;q)_n(rx)^n/(q,c,sy;q)_n}\right|=|rx|<1.$$

We denote the right-hand side of Eq. (2.1) by $f(a,b,c,x,y)$, we can verify that $f(a,b,c,x,y)$ satisfies Eq. (1.10), so we have

$$f(a,b,c,x,y)=\sum_{n=0}^{\infty}\mu_n\phi_n^{(a,b,c)}(x,y|q) \tag{2.5}$$

and

$$f(a,b,c,0,y)=\sum_{n=0}^{\infty}\mu_n y^n=\frac{(sy;q)_\infty}{(ry;q)_\infty}=\sum_{n=0}^{\infty}\frac{(s/r;q)_n(yr)^n}{(q;q)_n}. \tag{2.6}$$

So $f(a,b,c,x,y)$ is equal to the left-hand side of (2.1). Similarly, we can obtain Eq. (2.2). The proof is complete. □

3 Generalizations of two of Ramanujan's integrals

The following two integrals of Ramanujan [3] are quite famous.

Proposition 9 ([3, Eqs. (2) and (3)]) *For* $0<q=\exp(-2k^2)<1$ *and* $m\in\mathbb{R}$. *Suppose that* $|abq|<1$, *we have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{\left(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q\right)_\infty} \mathrm{d}\, x = \sqrt{\pi} e^{m^2} \frac{\left(-aqe^{2mki}, -bqe^{-2mki}; q\right)_\infty}{(abq; q)_\infty}. \tag{3.1}$$

Suppose that $\max\left\{\left|aq^{1/2}e^{2mk}\right|, \left|bq^{1/2}e^{-2mk}\right|\right\} < 1$, *we have*

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2+2mx} \left(-aqe^{2kx}, -bqe^{-2kx}; q\right)_\infty \mathrm{d}\, x \\ = \sqrt{\pi} e^{m^2} \frac{(abq; q)_\infty}{\left(aq^{1/2}e^{2mk}, bq^{1/2}e^{-2mk}; q\right)_\infty}.\end{aligned} \tag{3.2}$$

Derivations of (3.1) and (3.2) for real values of the parameter m have been deduced by Askey [3]. Later on it became clear that these integrals are in fact valid for arbitrary complex values of the parameter m and they are thus instances of the standard Fourier transform with the exponential kernel by Atakishiyev and Feinsilver [5].

In this section, we have the following generalization of Ramanujan's integrals.

Theorem 10 *For* $m \in \mathbb{R}$, $0 < q = \exp(-2k^2) < 1$. *Suppose that* $\max\{|abq|, |qc/a|\} < 1$, *we have*

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{\left(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q\right)_\infty} {}_3\phi_2\left[\begin{matrix} r, s, 0 \\ t, q^{1/2}e^{-2ikx}/a \end{matrix}; q, \frac{qc}{a}\right] \mathrm{d}\, x \\ = \sqrt{\pi} e^{m^2} \frac{\left(-aqe^{2mki}, -bqe^{-2mki}; q\right)_\infty}{(abq; q)_\infty} {}_3\phi_2\left[\begin{matrix} r, s, -e^{2mki}/b \\ t, 1/(ab) \end{matrix}; q, \frac{qc}{a}\right].\end{aligned} \tag{3.3}$$

Suppose that $\max\left\{\left|aq^{1/2}e^{2mk}\right|, \left|bq^{1/2}e^{-2mk}\right|, \left|cq^{1/2}e^{2mk}\right|\right\} < 1$, *we have*

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2+2mx} \left(-aqe^{2kx}, -bqe^{-2kx}; q\right)_\infty {}_2\phi_2\left[\begin{matrix} r, s \\ t, -aqe^{2kx} \end{matrix}; q, -cqe^{2kx}\right] \mathrm{d}\, x \\ = \sqrt{\pi} e^{m^2} \frac{(abq; q)_\infty}{\left(aq^{1/2}e^{2mk}, bq^{1/2}e^{-2mk}; q\right)_\infty} {}_3\phi_2\left[\begin{matrix} r, s, bq^{1/2}e^{-2mk} \\ t, abq \end{matrix}; q, cq^{1/2}e^{2mk}\right].\end{aligned} \tag{3.4}$$

Remark 11 For $c = 0$ in Theorem 10, Eqs. (3.3) and (3.4) reduce to Eqs. (3.1) and (3.2), respectively.

Proof of Theorem 10 It is easily seen that

$$\left(\left|q^{\frac{1}{2}}/a\right|; q\right)_n \le \left|\left(q^{\frac{1}{2}}e^{-2ikx}/a; q\right)_n\right| \le \left(-\left|q^{\frac{1}{2}}/a\right|; q\right)_n \tag{3.5}$$

and

$$\sum_{n=0}^{\infty}\frac{\left(|r|,|s|;q\right)_n}{\left(-|q|,-|t|,-\left|q^{1/2}/a\right|;q\right)_n}\left|\frac{qc}{a}\right|^n \leq \left|{}_3\phi_2\left[\begin{matrix} r,s,0 \\ t,q^{1/2}e^{-2ikx}/a \end{matrix};q,\frac{qc}{a}\right]\right|$$
$$\leq \sum_{n=0}^{\infty}\frac{\left(-|r|,-|s|;q\right)_n}{\left(|q|,|t|,\left|q^{1/2}/a\right|;q\right)_n}\left|\frac{qc}{a}\right|^n. \tag{3.6}$$

Thus, we have

$$\left|\int_{-\infty}^{\infty}\frac{e^{-x^2+2mx}}{\left(aq^{1/2}e^{2ikx},bq^{1/2}e^{-2ikx};q\right)_\infty}\mathrm{d}x\right|\cdot\sum_{n=0}^{\infty}\frac{\left(|r|,|s|;q\right)_n}{\left(-|q|,-|t|,-\left|q^{1/2}/a\right|;q\right)_n}\left|\frac{qc}{a}\right|^n$$
$$\leq \left|\int_{-\infty}^{\infty}\frac{e^{-x^2+2mx}}{\left(aq^{1/2}e^{2ikx},bq^{1/2}e^{-2ikx};q\right)_\infty}{}_3\phi_2\left[\begin{matrix} r,s,0 \\ t,q^{1/2}e^{-2ikx}/a \end{matrix};q,\frac{qc}{a}\right]\mathrm{d}x\right|$$
$$\leq \left|\int_{-\infty}^{\infty}\frac{e^{-x^2+2mx}}{\left(aq^{1/2}e^{2ikx},bq^{1/2}e^{-2ikx};q\right)_\infty}\mathrm{d}x\right|\cdot\sum_{n=0}^{\infty}\frac{\left(-|r|,-|s|;q\right)_n}{\left(|q|,|t|,\left|q^{1/2}/a\right|;q\right)_n}\left|\frac{qc}{a}\right|^n. \tag{3.7}$$

Denoting the right-hand side of Eq. (3.3) by $f(r,s,t,c,a)$ and utilizing Eqs. (3.1) and (3.7), we have

$$|f(r,s,t,c,a)| \leq \left|\sqrt{\pi}e^{m^2}\frac{\left(-aqe^{2mki},-bqe^{-2mki};q\right)_\infty}{(abq;q)_\infty}\right|$$
$$\times\sum_{n=0}^{\infty}\frac{\left(-|r|,-|s|;q\right)_n}{\left(|q|,|t|,\left|q^{1/2}/a\right|;q\right)_n}\left|\frac{qc}{a}\right|^n$$
$$\leq \sqrt{\pi}e^{m^2}\frac{\left(-|aq|,-|bq|;q\right)_\infty}{(|abq|;q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{\left(-|r|,-|s|;q\right)_n}{\left(|q|,|t|,\left|q^{1/2}/a\right|;q\right)_n}\left|\frac{qc}{a}\right|^n. \tag{3.8}$$

From the Weierstrass M-test, we know that for $\max\{|abq|,|qc/a|\}<1$, the function $f(r,s,t,c,a)$ is uniformly absolutely convergent, so $f(r,s,t,c,a)$ is an analytic function of r,s,t,c and a for $\max\{|abq|,|qc/a|\}<1$ (see also [[17], p. 516]). Thus $f(r,s,t,c,a)$ is analytic near $(r,s,t,c,a)=(0,0,0,0,0)$ (see also [[4], p. 220] and

[[17], p. 511]). We can check that $f(r, s, t, c, a)$ satisfies Eq. (1.11), so the left-hand side of Eq. (3.3) equals

$$f(r, s, t, c, a) = \sum_{n=0}^{\infty} \mu_n \psi_n^{(r,s,t)}(c, a|q), \tag{3.9}$$

where

$$\begin{aligned} f(r, s, t, 0, a) &= \sum_{n=0}^{\infty} \mu_n a^n = \sqrt{\pi} e^{m^2} \frac{\left(-aqe^{2mki}, -bqe^{-2mki}; q\right)_\infty}{(abq; q)_\infty} \quad by\ (3.1) \\ &= \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{\left(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q\right)_\infty} \mathrm{d}\,x \\ &= \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{\left(bq^{1/2}e^{-2ikx}; q\right)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{\left(aq^{1/2}e^{2ikx}\right)^n}{(q; q)_n} \right\} \mathrm{d}\,x. \end{aligned}$$

So we have

$$f(r, s, t, c, a) = \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{\left(bq^{1/2}e^{-2ikx}; q\right)_\infty} \left\{ \sum_{n=0}^{\infty} \psi_n^{(r,s,t)}(c, a|q) \frac{\left(q^{1/2}e^{2ikx}\right)^n}{(q; q)_n} \right\} \mathrm{d}\,x, \tag{3.10}$$

which is equal to the left-hand side of Eq. (3.3) by Eq. (2.4). Similarly, we can gain Eq. (3.4). The proof of Theorem 10 is complete. □

4 Generalizations of q-Chu–Vandermonde formula

The q-Chu–Vandermonde formula is [9, Eq. (II.6)]

$${}_2\phi_1\left[\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right] = \frac{\left(c/a; q\right)_n}{(c; q)_n} a^n. \tag{4.1}$$

In this section, we now extend the q-Chu–Vandermonde formula.

Theorem 12 *For $n \in \mathbb{N}_0$, we have*

$$\begin{aligned} &\sum_{k=0}^{n} \frac{\left(q^{-n}, a; q\right)_k q^k}{(q, cd; q)_k} {}_3\phi_2\left[\begin{matrix} r, s, aq^k \\ t, qa/(cd) \end{matrix}; q, \frac{qg}{d}\right] \\ &= \frac{\left(cd/a; q\right)_n a^n}{(cd; q)_n} {}_3\phi_2\left[\begin{matrix} r, s, a \\ t, aq^{1-n}/(cd) \end{matrix}; q, \frac{qg}{d}\right], \quad |qg/d| < 1, \end{aligned} \tag{4.2}$$

$$\sum_{k=0}^{n}\frac{\left(q^{-n},a;q\right)_kq^k}{(q,cd;q)_k}{}_3s\phi_2\left[\begin{matrix}r,s,aq^k\\t,cdq^k\end{matrix};q,\frac{cg}{a}\right]$$
$$=\frac{\left(cd/a;q\right)_na^n}{(cd;q)_n}{}_3\phi_2\left[\begin{matrix}r,s,aq^n\\t,cdq^n\end{matrix};q,\frac{cgq^n}{a}\right],\quad |cg/a|<1. \tag{4.3}$$

Remark 13 For $g=0$ in Theorem 12, Eqs. (4.2) and (4.3) reduce to (4.1), respectively.

Proof of Theorem 12 First, we can rewrite Eq. (4.1) equivalently by

$$\sum_{k=0}^{n}\frac{\left(q^{-n},a;q\right)_kq^k}{(q;q)_k}\frac{\left(cdq^k;q\right)_\infty}{\left(cd/a;q\right)_\infty}=a^n\frac{\left(cdq^n;q\right)_\infty}{\left(cdq^n/a;q\right)_\infty}. \tag{4.4}$$

We denote the right-hand side of (4.2) by $F(r,s,t,g,d)$, we can check that $F(r,s,t,g,d)$ satisfies Eq. (1.11). By Eq. (4.4), we have

$$F(r,s,t,g,d)=\sum_{j=0}^{\infty}\mu_j\psi_j^{(r,s,t)}(g,d|q) \tag{4.5}$$

and

$$F(r,s,t,0,d)=\sum_{j=0}^{\infty}\mu_jd^j=a^n\frac{\left(cdq^n;q\right)_\infty}{\left(cdq^n/a;q\right)_\infty}$$
$$=\sum_{k=0}^{n}\frac{\left(q^{-n},a;q\right)_kq^k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{\left(aq^k;q\right)_j\left(cd/a\right)^j}{(q;q)_j}.$$

So we have

$$F(r,s,t,g,d)=\sum_{k=0}^{n}\frac{\left(q^{-n},a;q\right)_kq^k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{\left(aq^k;q\right)_j\left(c/a\right)^j}{(q;q)_j}\psi_j^{(r,s,t)}(g,d|q), \tag{4.6}$$

which is equal to the left-hand side of (4.2) by Eq. (2.2). Similarly, we can deduce Eq. (4.3). The proof is complete. □

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