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Congruences for partition functions related to mock theta functions

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Abstract Partitions associated with mock theta functions have received a great deal of attention in the literature. Recently, Choi and Kim derived several partition identities from the third- and sixth-order mock theta functions. In addition, three Ramanujan-type congruences were established by them. In this paper, we present some new congruences for these partition functions.

Keywords Partition · t-Core partition · Cubic partition · Mock theta function · Ramanujan-type congruence

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1 Introduction

A partition of a positive integer n is a finite nonincreasing sequence of positive integers whose sum equals n. Furthermore, a partition is called a t-core partition if there are no hook numbers being multiples of t. Let $a_t(n)$ be the number of t-core partitions of n. It is known [18] that

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$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

Here and in what follows, we make use of the standard q-series notation (cf. [19]).

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$
$$(a)_\infty = (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1,a_2,\cdots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

In addition, the cubic partition, which was introduced by Chan [11,12] and named by Kim [21] in connection with Ramanujan's cubic continued fractions, is a 2-color partition where the second color appears only in multiples of 2. Let $a(n)$ denote the number of cubic partitions of n, then its generating function is

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}.$$

In his last letter to Hardy [9, pp. 220–223], Ramanujan defined 17 functions, which he called mock theta functions. Since then, there has been an intensive study of partition interpretations for mock theta functions; see [2–6].

Recently, Choi and Kim [15] obtained the following identity related to the third-order mock theta function,

$$\upsilon(q) + \upsilon_3(q,q;q) = 2\frac{\left(q^4;q^4\right)_\infty^3}{\left(q^2;q^2\right)_\infty^2},$$

where $\upsilon(q)$ is the third mock theta function and $\upsilon_3(q,q;q)$ is defined by Choi [14],

$$\upsilon(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},\quad \upsilon_3(q,q;q) = \sum_{n=0}^{\infty} q^n(-q;q^2)_n.$$

We remark that $\upsilon_3(q,q;q)$ is, in fact, identical to $\upsilon(-q)$; see Fine's book [17, Eq. (26.85)].

Choi and Kim also gave the following identities related to the sixth-order mock theta functions:

$$\Psi(q) + 2\Psi_-(q) = 3\frac{q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty},$$
$$2\rho(q) + \lambda(q) = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty},$$

where $\Psi(q)$, $\Psi_{-}(q)$, $\rho(q)$, and $\lambda(q)$ are the sixth-order mock theta functions,

$$\Psi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}},\quad \Psi_{-}(q)=\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n},$$
$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{\binom{n+1}{2}}(-q;q)_n}{(q;q^2)_{n+1}},\quad \lambda(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^n(q;q^2)_n}{(-q;q)_n}.$$

Meanwhile, Choi and Kim studied three analogous partition functions defined by

$$\sum_{n=0}^{\infty}b(n)q^n=\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}, \tag{1}$$
$$\sum_{n=0}^{\infty}c(n)q^n=\frac{q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{2}$$
$$\sum_{n=0}^{\infty}d(n)q^n=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{3}$$

where $b(n)$ denotes the number of partition pairs (λ,σ); σ is a partition into distinct even parts; and λ is a partition into even parts of which 2-modular diagram is 2-core, and both $c(n)$ and $d(n)$ can be regarded as 3-core cubic partitions.

In this paper, we mainly study Ramanujan-type congruences for these partition functions. This paper is organized as follows: In Sect. 2, we introduce some preliminary results. In the next two sections, we will prove some Ramanujan-type congruences for $b(n)$ and $c(n)$, respectively. In Sect. 5, by employing p-dissection formulas of Ramanujan's theta functions $\psi(q)$ and $f(-q)$ established by Cui and Gu [16] as well as (p,k)-parameter representations due to Alaca and Williams [1], we show some congruences for $d(n)$. Finally, we end this paper with several open problems.

2 Preliminaries

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},\quad |ab|<1.$$

We now introduce the following Ramanujan's classical theta functions:

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \tag{4}$$
$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}=\frac{f_2^2}{f_1}, \tag{5}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = f_1. \tag{6}$$

One readily verifies

$$\varphi(-q) = \frac{f_1^2}{f_2}. \tag{7}$$

Here and in the sequel, we write $f_k := (q^k; q^k)_\infty$ for positive integers k for convenience.

We first require the following 2-dissections.

Lemma 1 *It holds that*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{8}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{9}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}. \tag{10}$$

Proof Here (8) comes from the 2-dissection of $\varphi(q)$ (cf. [8, p. 40, Entry 25]). For (9) and (10), see [26]. □

The following 3-dissections are also necessary.

Lemma 2 *It holds that*

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} \left(1 + 2qw(q^3) + 4q^2 w^2(q^3)\right), \tag{11}$$

$$\frac{1}{\psi(q)} = \frac{\psi^3(q^9)}{\psi^4(q^3)} \left(\frac{1}{w^2(q^3)} - \frac{q}{w(q^3)} + q^2\right), \tag{12}$$

where

$$w(q) = \frac{f_1 f_6^3}{f_2 f_3^3}. \tag{13}$$

Furthermore,

$$\frac{1}{f_1^3} = \frac{f_9^3}{f_3^{12}} \left(P^2(q^3) + 3qP(q^3) f_9^3 + 9q^2 f_9^6\right), \tag{14}$$

where

$$P(q) = f_1 \left(\frac{\varphi^3(-q^3)}{\varphi(-q)} + 4q \frac{\psi^3(q^3)}{\psi(q)}\right). \tag{15}$$

Proof For (11) and (12), see Baruah and Ojah [7]. For (14), see Wang [24]. Note that Wang [24] showed

$$P(q) = f_1 \left(1 + 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right).$$

We know from [23, Eqs. (3.2) and (3.5)] that

$$4q\frac{\psi^3(q^3)}{\psi(q)} = 4\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{6n+2}} - \frac{q^{3n+2}}{1-q^{6n+4}}\right),$$
$$\frac{\varphi^3(-q^3)}{\varphi(-q)} = 1 + 2\sum_{n\geq 0}\left(\frac{q^{6n+1}}{1-q^{6n+1}} + \frac{q^{6n+2}}{1-q^{6n+2}} - \frac{q^{6n+4}}{1-q^{6n+4}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right).$$

Hence, (15) follows immediately by the following trivial identity:

$$\frac{x}{1-x^2} = \frac{x}{1-x} - \frac{x^2}{1-x^2}.$$

□

Furthermore, we need

Lemma 3 ([16, Theorem 2.1]) *For any odd prime* p*,*

$$\psi(q) = q^{\frac{p^2-1}{8}}\psi\left(q^{p^2}\right) + \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right).$$

We further claim that for $0 \leq k \leq (p-3)/2$*,*

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Lemma 4 ([16, Theorem 2.2]) *For any prime* $p \geq 5$*,*

$$f(-q) = (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f\left(-q^{p^2}\right) + \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right).$$

We further claim that for $-(p-1)/2 \leq k \leq (p-1)/2$ *and* $k \neq (\pm p-1)/6$*,*

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Here for any prime $p \geq 5$,

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & p \equiv -1 \pmod 6. \end{cases}$$

At last, we require the following relations due to Alaca and Williams [1].

Lemma 5 *Let*

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)},$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Then

$$\begin{aligned}
f_1 &= 2^{-\frac{1}{6}} q^{-\frac{1}{24}} p^{\frac{1}{24}} (1-p)^{\frac{1}{2}} (1+p)^{\frac{1}{6}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{8}} k^{\frac{1}{2}}, \\
f_2 &= 2^{-\frac{1}{3}} q^{-\frac{1}{12}} p^{\frac{1}{12}} (1-p)^{\frac{1}{4}} (1+p)^{\frac{1}{12}} (1+2p)^{\frac{1}{4}} (2+p)^{\frac{1}{4}} k^{\frac{1}{2}}, \\
f_3 &= 2^{-\frac{1}{6}} q^{-\frac{1}{8}} p^{\frac{1}{8}} (1-p)^{\frac{1}{6}} (1+p)^{\frac{1}{2}} (1+2p)^{\frac{1}{24}} (2+p)^{\frac{1}{24}} k^{\frac{1}{2}}, \\
f_4 &= 2^{-\frac{2}{3}} q^{-\frac{1}{6}} p^{\frac{1}{6}} (1-p)^{\frac{1}{8}} (1+p)^{\frac{1}{24}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{2}} k^{\frac{1}{2}}, \\
f_6 &= 2^{-\frac{1}{3}} q^{-\frac{1}{4}} p^{\frac{1}{4}} (1-p)^{\frac{1}{12}} (1+p)^{\frac{1}{4}} (1+2p)^{\frac{1}{12}} (2+p)^{\frac{1}{12}} k^{\frac{1}{2}}, \\
f_{12} &= 2^{-\frac{2}{3}} q^{-\frac{1}{2}} p^{\frac{1}{2}} (1-p)^{\frac{1}{24}} (1+p)^{\frac{1}{8}} (1+2p)^{\frac{1}{24}} (2+p)^{\frac{1}{6}} k^{\frac{1}{2}}.
\end{aligned}$$

3 Congruences for $b(n)$

Theorem 1 *For* $n \geq 0$, $\alpha \geq 1$, *and prime* $p \geq 5$, *we have*

$$b\left(p^{2\alpha} n + \frac{(3j+p)p^{2\alpha-1} - 1}{3}\right) \equiv 0 \pmod 2, \tag{16}$$

where $j = 1, 2, \ldots, p-1$.

Proof In light of (1), we derive that

$$\sum_{n=0}^{\infty} b(n) q^n = \frac{f_4^3}{f_2^2} \equiv f_8 \pmod 2.$$

Applying Lemma 4, we deduce that, for any prime $p \geq 5$,

$$\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{3}\right) q^n \equiv (-1)^{\frac{\pm p-1}{6}} f(-q^{8p}) \pmod 2,$$

and

$$\sum_{n=0}^{\infty} b\left(p^2 n + \frac{p^2-1}{3}\right) q^n \equiv (-1)^{\frac{\pm p-1}{6}} f(-q^8) \pmod 2.$$

Moreover,

$$\sum_{n=0}^{\infty} b\left(p^3 n + \frac{p^4-1}{3}\right) q^n \equiv f(-q^{8p}) \pmod 2.$$

Hence, by induction on α, we derive that, for $\alpha \geq 1$,

$$\sum_{n=0}^{\infty} b\left(p^{2\alpha-1} n + \frac{p^{2\alpha}-1}{3}\right) q^n \equiv (-1)^{\alpha\left(\frac{\pm p-1}{6}\right)} f(-q^{8p}) \pmod 2.$$

This immediately leads to

$$b\left(p^{2\alpha-1}(pn+j) + \frac{p^{2\alpha}-1}{3}\right) \equiv 0 \pmod 2,$$

for $j = 1, 2, \ldots, p-1$. □

Remark 1 When studying the 1-shell totally symmetric plane partition function $f(n)$ (which is different to Ramanujan's theta function $f(-q)$ given in Sect. 2) introduced by Blecher [10], Hirschhorn and Sellers [20] proved that, for $n \geq 1$,

$$f(3n-2) = h(n),$$

with

$$\sum_{n=0}^{\infty} h(2n+1) q^n = \frac{f_2^3}{f_1^2}.$$

A couple of congruences modulo powers of 2 and 5 for $h(n)$ have been obtained subsequently; see [13,25,27]. We see from (1) that

$$b(2n) = h(2n+1).$$

One therefore may obtain some congruences for $b(n)$ as well. For example,

$$b(8n+6) \equiv 0 \pmod 4.$$

4 Congruences for $c(n)$

Theorem 2 *For $n \geq 0$, we have*

$$c(27n+24) \equiv 0 \pmod 9. \tag{17}$$

Proof We see from (2) and Lemma 2 that

$$\begin{aligned}\sum_{n=0}^{\infty} c(n)q^n &= \frac{qf_6^3}{\varphi(-q)\psi(q)}\\ &= qf_6^3\frac{\varphi^3(-q^9)\psi^3(q^9)}{\varphi^4(-q^3)\psi^4(q^3)}\big(1+2qw(q^3)+4q^2w^2(q^3)\big)\\ &\quad\times\left(\frac{1}{w^2(q^3)}-\frac{q}{w(q^3)}+q^2\right).\end{aligned}$$

Employing Lemma 2, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} c(3n)q^n &= \frac{3q\varphi^3(-q^3)\psi^3(q^3)}{f_1^3\varphi(-q)\psi(q)}\\ &= \frac{3q\varphi^3(-q^9)\psi^3(q^9)f_9^3}{\varphi(-q^3)\psi(q^3)f_3^{12}}\left(P^2(q^3)+3qP(q^3)f_9^3+9q^2f_9^6\right)\\ &\quad\times\left(1+2qw(q^3)+4q^2w^2(q^3)\right)\left(\frac{1}{w^2(q^3)}-\frac{q}{w(q^3)}+q^2\right).\end{aligned} \tag{18}$$

Extracting terms involving q^{3n+2} and replacing q^3 by q in (18), it follows that

$$\sum_{n=0}^{\infty} c(9n+6)q^n = 12\frac{f_2^2f_3^{21}}{f_1^{16}f_6^6}+135q\frac{f_3^{12}f_6^3}{f_1^{13}f_2}+72q^2\frac{f_3^3f_6^{12}}{f_1^{10}f_2^4}+192q^3\frac{f_6^{21}}{f_1^7f_2^7f_3^6}.$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} c(9n+6)q^n &\equiv 3\frac{f_2^2f_3^{21}}{f_1^{16}f_6^6}+3q^3\frac{f_6^{21}}{f_1^7f_2^7f_3^6}\\ &\equiv 3\frac{f_2^2}{f_1}\left(\frac{f_3^{16}}{f_6^6}+q^3\frac{f_6^{18}}{f_3^8}\right) \pmod 9.\end{aligned}$$

Noting that f_2^2/f_1 contains no terms of the form q^{3n+2}, we have

$$\sum_{n=0}^{\infty} c(27n+24)q^n \equiv 0 \pmod 9.$$

□

Theorem 3 *For $n \geq 0$, we have*

$$c(45n+t) \equiv 0 \pmod{5}, \tag{19}$$

where $t = 9$ and 18.

Proof Referring to (18), we have

$$\sum_{n=0}^{\infty} c(9n)q^n = 45q\frac{f_2 f_3^{18}}{f_1^{15} f_6^3} + 90q^2\frac{f_3^9 f_6^6}{f_1^{12} f_2^2} + 288q^3\frac{f_6^{15}}{f_1^9 f_2^5}.$$

Hence,

$$\sum_{n=0}^{\infty} c(9n)q^n \equiv 3q^3 f_1 \frac{f_{30}^3}{f_5^2 f_{10}} \pmod{5}.$$

Since f_1 contains no terms of the form q^{5n+3} and q^{5n+4}, we have

$$c(9(5n+1)) = c(45n+9) \equiv 0 \pmod{5},$$

and

$$c(9(5n+2)) = c(45n+18) \equiv 0 \pmod{5}.$$

This yields that (19). □

Corollary 1 *For $n \geq 0$, we have*

$$c(45n+t) \equiv 0 \pmod{15}, \tag{20}$$

where $t = 9$ and 18.

Proof We know from [15, Theorem 4.2] that

$$c(3n) \equiv 0 \pmod{3},$$

which is indeed a direct consequence of (18). Hence, Corollary 1 follows by Theorem 3. □

5 Congruences for $d(n)$

Theorem 4 *For $n \geq 0$, $\alpha \geq 1$, and prime $p \geq 3$, we have*

$$d\left(2p^{2\alpha} + \frac{(8j+p)p^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod{2}, \tag{21}$$

where $j = 1, 2, \ldots, p-1$.

Proof From (3), one can see

$$\sum_{n=0}^{\infty} d(n)q^n = \frac{f_3^3}{f_1 f_2} \equiv f_6 \frac{f_3}{f_1^3} \pmod{2}.$$

With the help of (9), we have

$$\sum_{n=0}^{\infty} d(n)q^n \equiv \frac{f_4^6 f_6^4}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6^2 f_{12}^2}{f_2^7} \pmod{2}.$$

Hence,

$$\sum_{n=0}^{\infty} d(2n)q^n \equiv \frac{f_2^6 f_3^4}{f_1^9 f_6^2} \equiv \psi(q) \pmod{2}.$$

Invoking Lemma 3, for any odd prime p, we derive that

$$\sum_{n=0}^{\infty} d\left(2\left(pn + \frac{p^2-1}{8}\right)\right) q^n \equiv \psi(q^p) \pmod{2},$$

and

$$\sum_{n=0}^{\infty} d\left(2\left(p^2 n + \frac{p^2-1}{8}\right)\right) q^n \equiv \psi(q) \pmod{2}.$$

Furthermore,

$$\sum_{n=0}^{\infty} d\left(2p^3 n + \frac{p^4-1}{4}\right) q^n \equiv \psi(q^p) \pmod{2}.$$

It therefore follows by induction on α that for $\alpha \geq 1$,

$$\sum_{n=0}^{\infty} d\left(2p^{2\alpha-1} n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv \psi(q^p) \pmod{2}.$$

Thus, for $j = 1, 2, \ldots, p-1$,

$$d\left(2p^{2\alpha-1}(pn+j) + \frac{p^{2\alpha}-1}{4}\right) \equiv 0 \pmod{2},$$

which is the desired result. □

Theorem 5 *For $n \geq 0$, $\alpha \geq 1$, and prime $p \geq 5$, we have*

$$d\left(6p^{2\alpha}n + \frac{(24j+p)p^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod 3, \tag{22}$$

where $j = 1, 2, \ldots, p-1$.

Proof It follows by (11) and (12) that

$$\begin{aligned}\sum_{n=0}^{\infty} d(n) &= \frac{f_3^3}{\varphi(-q)\psi(q)}\\ &= f_3^3 \frac{\varphi^3(-q^9)\psi^3(q^9)}{\varphi^4(-q^3)\psi^4(q^3)}\left(1+2qw(q^3)+4q^2w^2(q^3)\right)\\ &\quad\times\left(\frac{1}{w^2(q^3)} - \frac{q}{w(q^3)} + q^2\right).\end{aligned} \tag{23}$$

So we get

$$\begin{aligned}\sum_{n=0}^{\infty} d(3n)q^n &= f_1^3 \frac{\varphi^3(-q^3)\psi^3(q^3)}{\varphi^4(-q)\psi^4(q)}\left(\frac{1}{w^2(q)} - 2qw(q)\right)\\ &= \frac{1}{f_2^2 f_6^3}\left(\frac{f_3^3}{f_1}\right)^3 - 2q\frac{f_6^6}{f_2^5}.\end{aligned}$$

Based on (10), we derive that

$$\sum_{n=0}^{\infty} d(6n)q^n = \frac{f_2^9 f_3^3}{f_1^8 f_6^3} + 3q\frac{f_2 f_6^5}{f_1^2 f_3} \equiv \frac{f_2^9 f_3^3}{f_1^8 f_6^3} \equiv f_1 \pmod 3.$$

Invoking Lemma 4, we arrive at that, for any prime $p \geq 5$,

$$\sum_{n=0}^{\infty} d\left(6\left(pn + \frac{p^2-1}{24}\right)\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}} f(-q^p) \pmod 3,$$

and

$$\sum_{n=0}^{\infty} d\left(6\left(p^2n + \frac{p^2-1}{24}\right)\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}} f(-q) \pmod 3.$$

Furthermore, we have

$$\sum_{n=0}^{\infty} d\left(6\left(p^2\left(pn + \frac{p^2-1}{24}\right) + \frac{p^2-1}{24}\right)\right)q^n \equiv f(-q^p) \pmod 3.$$

Namely,

$$\sum_{n=0}^{\infty} d\left(6p^3 n + \frac{p^4-1}{4}\right) q^n \equiv f(-q^p) \pmod 3.$$

Thus, by induction on α, we derive that, for $\alpha \geq 1$,

$$\sum_{n=0}^{\infty} d\left(6p^{2\alpha-1} n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv (-1)^{\alpha\left(\frac{\pm p-1}{6}\right)} f(-q^p) \pmod 3.$$

This yields that, for $j = 1, 2, \ldots, p-1$,

$$d\left(6p^{2\alpha-1}(pn+j) + \frac{p^{2\alpha}-1}{4}\right) \equiv 0 \pmod 3,$$

which implies (22). □

Theorem 6 *For $n \geq 0$, $\alpha \geq 1$, and prime $p \geq 5$, we have*

$$d\left(6p^{2\alpha} n + \frac{(24j+9p)p^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod 9, \tag{24}$$

where $j = 1, 2, \ldots, p-1$.

Proof Extracting terms involving q^{3n+2} and replace q^3 by q in (23), then we derive that

$$\sum_{n=0}^{\infty} d(3n+2) q^n = \frac{3 f_3^3 f_6^3}{\varphi(-q)\psi(q) f_2^3} = \frac{3 f_3^3 f_6^3}{f_1 f_2^4}. \tag{25}$$

It follows by (10) that

$$\sum_{n=0}^{\infty} d(3n+2) q^n = 3\frac{f_3^3 f_6^3}{f_1 f_2^4} = 3\frac{f_4^3 f_6^5}{f_2^6 f_{12}} + 3q\frac{f_6^3 f_{12}^3}{f_2^4 f_4}.$$

Hence,

$$\sum_{n=0}^{\infty} d(6n+2) q^n = 3\frac{f_2^3 f_3^5}{f_1^6 f_6} \equiv 3 f_9 \pmod 9.$$

In view of Lemma 4, for any prime $p \geq 5$, we deduce that

$$\sum_{n=0}^{\infty} d\left(6\left(pn + \frac{3(p^2-1)}{8}\right) + 2\right) q^n \equiv 3(-1)^{\frac{\pm p-1}{6}} f(-q^{9p}) \pmod 9,$$

and

$$\sum_{n=0}^{\infty} d\left(6\left(p^2 n + \frac{3(p^2-1)}{8}\right)+2\right) q^n \equiv 3(-1)^{\frac{\pm p-1}{6}} f(-q^9) \pmod{9}.$$

Moreover,

$$\sum_{n=0}^{\infty} d\left(6\left(p^3 n + \frac{3(p^4-1)}{8}\right)+2\right) q^n \equiv 3 f(-q^{9p}) \pmod{9}.$$

Hence, by induction on $\alpha \geq 1$, we arrive at

$$\sum_{n=0}^{\infty} d\left(6\left(p^{2\alpha-1} n + \frac{3(p^{2\alpha}-1)}{8}\right)+2\right) q^n \equiv 3(-1)^{\alpha\left(\frac{\pm p-1}{6}\right)} f(-q^{9p}) \pmod{9},$$

which implies that for $j = 1, 2, \ldots, p-1$,

$$d\left(6\left(p^{2\alpha-1}(pn+j) + \frac{3(p^{2\alpha}-1)}{8}\right)+2\right) \equiv 0 \pmod{9}.$$

This leads to (24). □

Theorem 7 *For $n \geq 0$, we have*

$$d(45n+t) \equiv 0 \pmod{5}, \tag{26}$$

where $t = 17$ and 35.

Proof From (25), we have

$$\sum_{n=0}^{\infty} d(3n+2)q^n = \frac{3 f_3^3 f_6^3}{\varphi(-q)\psi(q) f_2^3}.$$

Again by (11), (12), and (14), we have

$$\sum_{n=0}^{\infty} d(9n+8)q^n = f_2 \cdot H,$$

where

$$H = \left(\frac{9 f_3^9 f_4 f_6^9}{f_1^3 f_2^{13} f_{12}^3} + \frac{9 f_3^3 f_4^2 f_6^{18}}{f_1 f_2^{16} f_{12}^6}\right) + q\left(\frac{27 f_3^6 f_6^9}{f_1^2 f_2^{13}} - \frac{18 f_4 f_6^{18}}{f_2^{16} f_{12}^3}\right)$$
$$+ q^2\left(\frac{36 f_3^9 f_{12}^6}{f_1^3 f_2^{10} f_4^2} + \frac{72 f_3^3 f_6^9 f_{12}^3}{f_1 f_2^{13} f_4} + \frac{108 f_1 f_6^{18}}{f_2^{16} f_3^3}\right)$$

$$-q^3\frac{72f_6^9f_{12}^6}{f_2^{13}f_4^2}+q^4\frac{144f_3^3f_{12}^{12}}{f_1f_2^{10}f_4^4}.$$

□

We next show a surprising congruence.

Lemma 6 *It holds that*

$$H\equiv 3\frac{f_{15}^3}{f_5f_{10}^2}\pmod 5. \tag{27}$$

Proof (Proof of Lemma 6) To prove (27), it suffices to show

$$H-3\frac{f_3^{15}}{f_1^5f_2^{10}}\equiv 0\pmod 5,$$

or equivalently,

$$\left(H-3\frac{f_3^{15}}{f_1^5f_2^{10}}\right)\frac{f_1^5f_3f_4^{10}f_6^{10}}{f_2^4f_{12}^6}\equiv 0\pmod 5,$$

since $\frac{f_1^5f_3f_4^{10}f_6^{10}}{f_2^4f_{12}^6}$ is invertible in the ring $\mathbb{Z}/5\mathbb{Z}[[q]]$. According to Lemma 5, it becomes

$$\frac{15p^2(1-p)(1+p)^5(2+p)^2(2+5p+12p^2+5p^3+2p^4)k^8}{32q^2(1+2p)}\equiv 0\pmod 5.$$

Lemma 6 follows obviously. □

We know from Lemma 6 that

$$\sum_{n=0}^{\infty}d(9n+8)q^n\equiv 3f_2\frac{f_{15}^3}{f_5f_{10}^2}\pmod 5.$$

Since $f_2=(q^2;q^2)_\infty$ contains no terms of the form q^{5n+1} and q^{5n+3}, we have

$$d(9(5n+1)+8)=d(45n+17)\equiv 0\pmod 5,$$

and

$$d(9(5n+3)+8)=d(45n+35)\equiv 0\pmod 5,$$

which leads to Theorem 7. □

Corollary 2 *For $n\geq 0$, we have*

$$d(45n+t)\equiv 0\pmod{15}, \tag{28}$$

where $t=17$ and 35.

Proof Again, we know from [15, Theorem 4.2] that

$$d(3n+2) \equiv 0 \pmod{3}.$$

It indeed follows directly from (25). We thus prove Corollary 2 by Theorem 7. □

6 Final remarks

We end this paper by raising the following congruences.

Question 1 We have

$$c(45n+21) \equiv 0 \pmod{5}, \tag{29}$$
$$c(63n+t) \equiv 0 \pmod{7}, \tag{30}$$

where $t = 30$, 48, and 57.

Question 2 We have

$$d(45n+41) \equiv 0 \pmod{5}, \tag{31}$$
$$d(63n+t) \equiv 0 \pmod{7}, \tag{32}$$

where $t = 32$, 50, and 59.

All these congruences have been verified by the authors using an algorithm due to Radu and Sellers [22]. However, since the modular form proofs are very routine and tedious, we here want to ask if there exist elementary proofs of these congruences.

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