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Topological strings, quiver varieties, and Rogers–Ramanujan identities

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Abstract Motivated by some recent works on BPS invariants of open strings/knot invariants, we guess there may be a general correspondence between the Ooguri–Vafa invariants of toric Calabi–Yau 3-folds and cohomologies of Nakajima quiver varieties. In this short note, we provide a toy model to explain this correspondence. More precisely, we study the topological open string model of $\mathbb{C}^3$ with one Aganagic–Vafa brane $\mathcal{D}_\tau$, and we show that, when $\tau \leq 0$, its Ooguri–Vafa invariants are given by the Betti numbers of certain quiver variety. Moreover, the existence of Ooguri–Vafa invariants implies an infinite product formula. In particular, we find that the $\tau = 1$ case of such infinite product formula is closely related to the celebrated Rogers–Ramanujan identities.

Keywords Topological strings · Ooguri–Vafa invariants · Quiver varieties · Rogers–Ramanujan identities

Mathematics Subject Classification 14N35 · 14N10 · 11P84 · 05E05

1 Introduction

Topological string theory is the topological sector of superstring theory [72]. In mathematics, we use Gromov–Witten theory to describe the topological string theory, see [27] for a review. Topological string amplitude is the generating function of Gromov–Witten invariants which are usually rational numbers according to their definitions [7,51]. In 1998, Gopakumar and Vafa [23] found that topological string amplitude

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is also the generating function of a series of integer-valued invariants related to BPS counting in M-theory. Later, Ooguri and Vafa [64] extended the above result to open string case, we name the corresponding integer-valued invariants as OV invariants. Furthermore, the OV invariants are further refined by Labasitida, Mariño, and Vafa in [48], the resulted invariants are called LMOV invariants [52], which have been studied by many literatures, see [54,59] for the recent approaches.

A central question in topological string theory is how to define the GV/OV/LMOV invariants directly. There have been many works, for examples [33,34,39,58,66], devoted to the definition of GV invariants. However, to the author's knowledge, no direct related works study the definition of OV/LMOV invariants. But there are some attempts to explain the integrality of OV invariants through different mathematical models. In [43], Kucharski and Sulkowski related the OV invariants to the combinatorics on words. In the joint work with Luo [54], we investigated the LMOV invariants for resolved conifold which is the large N duality of the framed unknot [57]. Moreover, we found that the (reduced) topological string partition function of $\mathbb{C}^3$ is equivalent to the Hilbert–Poincare polynomial of certain cohomological Hall algebra of quiver. Very recently, a series of works due to Diaconescu et al. [14,16,17] showed that the (refined) GV invariants can be expressed in terms of the Betti numbers of certain character varieties of algebraic curves based on the main conjectures in [15,32]. By the analogues of quiver varieties and character varieties showed in [30], it is natural to expect there will be an explanation of the integrality of GV/OV invariants by using quiver varieties. It is also expected that a general toric Calabi–Yau/quiver variety correspondence may exist in geometry.

1.1 Open string model on $(\mathbb{C}^3, \mathcal{D}_\tau)$

In this short note, we provide a toy model to state this correspondence through numerical calculations. More precisely, we focus on the open topological string on $(\mathbb{C}^3, \mathcal{D}_\tau)$, where $\mathcal{D}_\tau$ is the framing $\tau \in \mathbb{Z}$ Aganagic–Vafa A-brane [4,5]. Its (reduced) topological string partition is given by

$$Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(g_s, \mathbf{x} = (x, 0, 0, \ldots)) = \sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n. \tag{1}$$

We define

$$f_n^\tau(q) = (q^{1/2} - q^{-1/2})[x^n]\mathrm{Log}\left(\sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n\right), \tag{2}$$

where $[x^n]g(x)$ denotes the coefficient of x^n in the series $g(x) \in \mathbb{Z}[[x]]$ and Log is the plethystic logarithm introduced in Sect. 2.2. Applying the work of Ooguri and Vafa [64] to this open string model $(\mathbb{C}^3, \mathcal{D}_\tau)$, we formulate the following conjecture:

Conjecture 1.1 *For any $\tau \in \mathbb{Z}$, for a fixed integer $m \geq 1$, we have*

$$f_m^\tau(q) = \sum_{k\in\mathbb{Z}} N_{m,k}(\tau) q^{\frac{k}{2}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]. \tag{3}$$

In other words, for a fixed integer $m \geq 1$, there are only finitely many k, such that the integers $N_{m,k}(\tau)$ are nonzero.

The rest of this paper is devoted to study the Conjecture 1.1. We start with the $\tau = 0$ case for warming up. Recall the classical Cauchy identity for Schur functions [55],

$$\sum_{\lambda\in\mathcal{P}} s_\lambda(\mathbf{y}) s_\lambda(\mathbf{x}) = \prod_{i,j\geq 1} \frac{1}{(1 - x_i y_j)}, \tag{4}$$

where $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$, and $\mathcal{P}$ denotes the set of all the partitions. We consider the specialization $\mathbf{x} = (x, 0, 0, \ldots)$ and $\mathbf{y} = q^\rho = (q^{-1/2}, q^{-3/2}, q^{-5/2}, \ldots)$, the left- hand side of (4) becomes

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} s_\lambda(q^\rho) s_\lambda(\mathbf{x} = (x, 0, 0, \ldots)) &= \sum_{n\geq 0} s_n(q^\rho) x^n \\ &= \sum_{n\geq 0} \frac{(-1)^n q^{\frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n,\end{aligned} \tag{5}$$

and the right-hand side of (4) gives

$$\prod_{j\geq 1} (1 - xq^{-j+\frac{1}{2}})^{-1}. \tag{6}$$

Comparing to formulae (2) and (3) for when $\tau = 0$, by using the definition of plethystic logarithm Log, we obtain

$$N_{m,k}(0) = \begin{cases} 1, & \text{if } m = 1 \text{ and } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

However, for general $\tau \in \mathbb{Z}$, the Conjecture 1.1 is nontrivial. The first result of this paper is that, when $\tau \leq 0$, we find that the OV invariants $N_{n,k}(\tau)$ can be expressed in terms of the Betti number of certain quiver variety, which implies the Conjecture 1.1 for the case of $\tau \leq 0$.

1.2 Proof of the Conjecture 1.1 for the case of $\tau \leq 0$

We construct a quiver of one vertex with $1 - \tau$ infinite legs. Let $\mathcal{Q}_{\tilde{n}(1-\tau)}$ be the associated quiver variety of the representations in a dimension related to n and $1 - \tau$, we refer to [31] and Sect. 4 for this construction. Let $d_{\tilde{n}(1-\tau)} = \dim \mathcal{Q}_{\tilde{n}(1-\tau)}$.

There is a Weyl group S_n which acts on the compactly supported cohomology $H_c^{1-n+2d_{\tilde{n}(1-\tau)}-j}(\mathcal{Q}_{\tilde{n}(1-\tau)};\mathbb{C})$. Then, we have the following:

Theorem 1.2 *If n is odd*

$$N_{n,j}(\tau)=\begin{cases}0, & j \text{ is odd},\\ -(-1)^{(\tau-1)n}\dim(H_c^{1-n+2d_{\tilde{n}(1-\tau)}-j}(\mathcal{Q}_{\tilde{n}(1-\tau)};\mathbb{C})^{S_n}), & j \text{ is even}.\end{cases} \tag{7}$$

If n is even

$$N_{n,j}(\tau)=\begin{cases}0, & j \text{ is even},\\ -(-1)^{(\tau-1)n}\dim(H_c^{1-n+2d_{\tilde{n}(1-\tau)}-j}(\mathcal{Q}_{\tilde{n}(1-\tau)};\mathbb{C})^{S_n}), & j \text{ is odd}.\end{cases} \tag{8}$$

Therefore, as a direct corollary, we have shown

Corollary 1.3 *The Conjecture 1.1 holds for $\tau \leq 0$.*

Now the remain question is what about the case of $\tau \geq 1$? We do not know how to prove this case, but we find it is closely related to celebrated Rogers–Ramanujan identities (10) and (11).

1.3 A Rogers–Ramanujan type identity

Combing(2), (3), and the definition of plethystic logarithm Log, Conjecture 1.1 can be rewritten in the form of infinite product (37).

Let us take a closer look at the case of $\tau = 1$. After some numerical computations by Maple 13 (see Sect. 5 for some of these numerical results), we observe the following rules for those integers $N_{m,k}(1)$:

- If m is even, $N_{m,k}(1) \geq 0$, and when m is odd, $N_{m,k}(1) \leq 0$.
- For a fix integer $m \geq 4$, we define the subset of $\mathbb{Z}$,

$$I_m = \{m+1, m+3, \ldots, m^2-2m-5, m^2-2m-3, (m-1)^2\} \subset \mathbb{Z}.$$

  then $N_{m,k}(1) = 0$, if $k \in \mathbb{Z} \setminus I_m$. Note that the last gap in I_m is $(m-1)^2 - (m^2 - 2m - 3) = 4$. Moreover, we let $I_1 = \{0\}$, $I_2 = \{1\}$, $I_3 = \{4\}$, according to the computations in Sect. 5.

Based on the above observations, let $n_{m,k} = (-1)^m N_{m,k}(1)$, we have the following refined form of the infinite product Formula (37) for $\tau = 1$.

Conjecture 1.4 *For a fixed $m \geq 1$, there are only finitely many positive integers $n_{m,k}$ for $k \in I_m$, such that*

$$\sum_{n\geq 0}\frac{a^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m\geq 1, l\geq 0}\prod_{k\in\mathbb{Z}}\frac{(1-a^{2m}q^{k+l+2m})^{n_{2m,2k+2m-1}}}{(1-a^{2m-1}q^{k+l+2m-1})^{n_{2m-1,2k+2m-2}}}. \tag{9}$$

Remark 1.5 After the email correspondence with Ole Warnaar [71], he suggested the author to rewrite the deformed Rogers–Ramanujan identity (57) into the form (9) which is related to the Rogers–Selberg identity [25].

Note that, (9) can be regarded as a Rogers–Ramanujan type identity. Recall the two classical Rogers–Ramanujan identities

$$\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{10}$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{11}$$

Formulae (10) and (11) were first discovered by Rogers [68], and then rediscovered by Ramanujan [26], Schur [69], and Baxter [6]. Now, there have been many different proofs and interpretations for them [1,8,22,49,70]. We refer to [25,71] for most modern understanding of the Rogers–Ramanujan identities.

These conjectural integers $n_{m,k}$ appearing in (9) are important. We expect an explicit formula for them. Let

$$g_m(q)=\sum_{k\in\mathbb{Z}}n_{m,k}q^k, \tag{12}$$

by using Maple 13, we have computed g_m for small m as showed in Sect. 5.

By our numerical computations, if we let $a=1$ and $a=q^{\frac{1}{2}}$, respectively, in Formula (9), then it recovers the Rogers–Ramanujan identities (10) and (11). Therefore, (9) can be regarded as an one-parameter- deformed Rogers–Ramanujan identity. From this point of view, integrality structures of topological string partitions provide a lot of infinite product formulas, which largely extend the explorations of Rogers–Ramanujan type formulae.

Finally, in order to give the reader some flavor of these numbers $n_{m,k}$, we compute the value $f^\tau(1)$ of (3) at $q=1$ from Mariño–Vafa formula [53,57] as follows:

$$f_m^\tau(1)=\frac{1}{m^2}\sum_{d|m}\mu(m/d)(-1)^{d\tau}\binom{d(\tau+1)-1}{d-1}, \tag{13}$$

where $\mu(n)$ denotes the Möbius function. We prove that

Theorem 1.6 *For any $m\geq 1$,*

$$f_m^\tau(1)=\frac{1}{m^2}\sum_{d|m}\mu(m/d)(-1)^{d\tau}\binom{d(\tau+1)-1}{d-1}\in\mathbb{Z}. \tag{14}$$

In particular, we obtain

Corollary 1.7 *For any $m \geq 1$,*

$$g_m(1) = \sum_k n_{m,k} = (-1)^m f_m^1(1) \in \mathbb{Z}. \tag{15}$$

The rest of this article is arranged as follows: in Sect. 2, we introduce the basic notations for partitions, symmetric functions, and plethystic operators. Then, we review the mathematical structures of topological strings in Sect. 3. We formulate the general Ooguri–Vafa conjecture by using plethystic operators and we present the explicit form of Ooguri–Vafa conjecture for the open string model $(\mathbb{C}^3, \mathcal{D}_\tau)$. In Sect. 4, we first review the main results of the work [31], as an application, we prove the Ooguri–Vafa conjecture for $(\mathbb{C}^3, \mathcal{D}_\tau)$ when $\tau \leq 0$. In Sect. 5, we focus on the Ooguri–Vafa conjecture for $(\mathbb{C}^3, \mathcal{D}_\tau)$ for the special case $\tau = 1$. We propose the deformed Rogers–Ramanujan type identity (9). Finally, we present a proof of the integrality of (14).

2 Symmetric functions and plethystic operators

2.1 Partitions and symmetric functions

A partition λ is a finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$. The length of λ is the total number of parts in λ and denoted by $l(\lambda)$. The weight of λ is defined by $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$. If $|\lambda| = d$, we say λ is a partition of d and denoted as $\lambda \vdash d$. The automorphism group of λ, denoted by $\mathrm{Aut}(\lambda)$, contains all the permutations that permute parts of λ by keeping it as a partition. Obviously, $\mathrm{Aut}(\lambda)$ has the order $|\mathrm{Aut}(\lambda)| = \prod_{i=1}^{l(\lambda)} m_i(\lambda)!$ where $m_i(\lambda)$ denotes the number of times that i occurs in λ. Define $\mathfrak{z}_\lambda = |\mathrm{Aut}(\lambda)| \prod_{i=1}^{\lambda} \lambda_i$.

Every partition is identified to a Young diagram. The Young diagram of λ is a graph with λ_i boxes on the ith row for $j = 1, 2, \ldots, l(\lambda)$, where we have enumerated the rows from top to bottom and the columns from left to right. Given a partition λ, we define the conjugate partition λ^t whose Young diagram is the transposed Young diagram of λ: the number of boxes on jth column of λ^t equals to the number of boxes on jth row of λ, for $1 \leq j \leq l(\lambda)$. For a box $x = (i, j) \in \lambda$, the hook length and content are defined to be $hl(x) = \lambda_i + \lambda_j^t - i - j + 1$ and $cn(x) = j - i$, respectively.

In the following, we will use the notation $\mathcal{P}_+$ to denote the set of all the partitions of positive integers. Let 0 be the partition of 0, i.e., the empty partition. Define $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$, and $\mathcal{P}^n$ the n tuple of $\mathcal{P}$.

The power sum symmetric function of infinite variables $\mathbf{x} = (x_1, \ldots, x_N, \ldots)$ is defined by $p_n(\mathbf{x}) = \sum_i x_i^n$. Given a partition λ, we define $p_\lambda(\mathbf{x}) = \prod_{j=1}^{l(\lambda)} p_{\lambda_j}(\mathbf{x})$. The Schur function $s_\lambda(\mathbf{x})$ is determined by the Frobenius formula

$$s_\lambda(\mathbf{x}) = \sum_\mu \frac{\chi_\lambda(\mu)}{\mathfrak{z}_\mu} p_\mu(\mathbf{x}), \tag{16}$$

where χ_λ is the character of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to λ, we have $\chi_\lambda(\mu) = 0$ if $|\mu| \neq |\lambda|$. The orthogonality of character formula gives

$$\sum_\lambda \frac{\chi_\lambda(\mu)\chi_\lambda(\nu)}{\mathfrak{z}_\mu} = \delta_{\mu\nu}. \tag{17}$$

We let $\Lambda(\mathbf{x})$ be the ring of symmetric functions of $\mathbf{x} = (x_1, x_2, \ldots)$ over the ring $\mathbb{Q}(q, t)$, and let $\langle \cdot, \cdot \rangle$ be the Hall pair on $\Lambda(\mathbf{x})$ determined by

$$\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda,\mu}. \tag{18}$$

For $\vec{\mathbf{x}} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$, denote by $\Lambda(\vec{\mathbf{x}}) := \Lambda(\mathbf{x}^1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}^n)$ the ring of functions separately symmetric in $\mathbf{x}^1, \ldots, \mathbf{x}^n$, where $\mathbf{x}^i = (x_1^i, x_2^i, \ldots)$. We will study functions in the ring $\Lambda(\vec{\mathbf{x}})$. For $\vec{\mu} = (\mu^1, \ldots, \mu^n) \in \mathcal{P}^n$, we let $a_{\vec{\mu}}(\vec{\mathbf{x}}) = a_{\mu^1}(\mathbf{x}^1) \cdots a_{\mu^n}(\mathbf{x}^n) \in \Lambda(\vec{\mathbf{x}})$ be homogeneous of degree $(|\mu^1|, \ldots, |\mu^n|)$. Moreover, the Hall pair on $\Lambda(\vec{\mathbf{x}})$ is given by $\langle a_1(\mathbf{x}^1) \cdots a_n(\mathbf{x}^n), b_1(\mathbf{x}^1) \cdots b_n(\mathbf{x}^n) \rangle = \langle a_1(\mathbf{x}^1), b_1(\mathbf{x}^1) \rangle \cdots \langle a_n(\mathbf{x}^n), b_n(\mathbf{x}^n) \rangle$ for $a_1(\mathbf{x}^1) \cdots a_n(\mathbf{x}^n), b_1(\mathbf{x}^1) \cdots b_n(\mathbf{x}^n) \in \Lambda(\vec{\mathbf{x}})$.

2.2 Plethystic operators

For $d \in \mathbb{Z}_+$, we define the dth Adams operator Ψ_d as the $\mathbb{Q}$-algebra map on $\Lambda(\vec{\mathbf{x}})$

$$\Psi_d(f(\vec{\mathbf{x}}; q, t)) = f(\vec{\mathbf{x}}^d; q^d, t^d). \tag{19}$$

Denote by $\Lambda(\vec{\mathbf{x}})^+$ the set of symmetric functions with degree ≥ 1. The plethystic exponential Exp and logarithm Log are inverse maps

$$\mathrm{Exp} : \Lambda(\vec{\mathbf{x}})^+ \to 1 + \Lambda(\vec{\mathbf{x}})^+, \ \mathrm{Log} : 1 + \Lambda(\vec{\mathbf{x}})^+ \to \Lambda(\vec{\mathbf{x}})^+, \tag{20}$$

respectively, defined by (see [30])

$$\mathrm{Exp}(f) = \exp\left(\sum_{d\geq 1} \frac{\Psi_d(f)}{d}\right), \ \mathrm{Log}(f) = \sum_{d\geq 1} \frac{\mu(d)}{d} \Psi_d(\log(f)), \tag{21}$$

where μ is the Möbius function. It is clear that

$$\mathrm{Exp}(f + g) = \mathrm{Exp}(f)\mathrm{Exp}(g), \ \mathrm{Log}(fg) = \mathrm{Log}(f) + \mathrm{Log}(g), \tag{22}$$

and $\mathrm{Exp}(x) = \frac{1}{1-x}$, if we use the expansion $\log(1-x) = -\sum_{d\geq 1} \frac{x^d}{d}$.

3 Integrality structures in topological strings

3.1 Closed strings and Gopakumar–Vafa conjecture

Let X be a Calabi–Yau 3-fold, the Gromov–Witten invariants $K_{g,Q}^X$ is the virtual counting of the number of holomorphic maps f from genus g Riemann surface C_g to X such that $f_*[C_g] = Q \in H_2(X, \mathbb{Z})$ [27]. Define

$$F^X(g_s, \omega) = \sum_{g\geq 0} g_s^{2g-2} F_g^X(\omega), \quad Z^X(g_s, \omega) = \exp(F^X(g_s, \omega)).$$

Usually, the Gromov–Witten invariants $K_{g,Q}^X$ are rational numbers. In 1998, Gopakumar and Vafa [23] conjectured that the generating function $F^X(g_s, \omega)$ of Gromov–Witten invariants can be expressed in terms of integer-valued invariants $N_{g,Q}^X$ as follows

$$\begin{aligned} F^X(g_s, \omega) &= \sum_{g\geq 0} g_s^{2g-2} \sum_{Q\neq 0} K_{g,Q}^X e^{-Q\cdot\omega} \\ &= \sum_{g\geq 0, d\geq 1} \sum_{Q\neq 0} \frac{1}{d} N_{g,Q}^X \left(2\sin\frac{dg_s}{2}\right)^{2g-2} e^{-dQ\cdot\omega}. \end{aligned} \tag{23}$$

The invariants $N_{g,Q}^X$ are called GV invariants in literatures. A central question in topological string is how to define the GV invariants directly. We refer to [33,34,39,58] for some progresses in this direction.

Obviously, genus 0 part of the Gopakumar–Vafa formula (23) yields the multiple covering formula [3]:

$$\sum_{Q\neq 0} K_{0,Q}^X e^{-Q\cdot\omega} = \sum_{Q\neq 0} N_{0,Q}^X \sum_{d\geq 1} \frac{1}{d^3} e^{-dQ\cdot\omega}. \tag{24}$$

By using the principle of mirror symmetry, around 1990, Candalas et al [13] calculated the numbers $N_{0,Q}^{X_5}$ from formula (24) for quintic Calabi–Yau 3-fold X_5, and found that $N_{0,Q}^{X_5}$ was equal to the number of rational curves of degree Q in X_5 which was hard to compute in enumerative geometry by classical method. This was the first important application of the topological string theory in mathematics.

When X is a toric Calabi–Yau 3-fold which is a toric variety with trivial canonical bundle [9]. Because of its toric symmetry, the geometric information of a toric Calabi–Yau 3-fold is encoded in a trivalent graph named "toric diagram" [2] which is the gluing of some trivalent vertices. The topological string partition function $Z^X(g_s, \omega) = \exp(F^X(g_s, \omega))$ of a toric Calabi–Yau 3-fold X can be computed by using the method of topological vertex [2,45]. The integrality of the invariants $N_{g,Q}^X$ for toric Calabi–Yau 3-fold X determined by Gopakumar–Vafa formula (23) was later proved by P. Peng [65] and Konishi [37].

3.2 Open strings and Ooguri–Vafa conjecture

Now we discuss the open topological strings. Let X be a Calabi–Yau 3-fold with a submanifold $\mathcal{D}$, we assume $\dim H_1(\mathcal{D}, \mathbb{Z}) = n$ with basis $\gamma_1, \ldots \gamma_n$. It is also expected that there are open Gromov–Witten invariants $K^{(X,\mathcal{D})}_{\vec{\mu},g,Q}$ determined by topological data $g, \vec{\mu}, Q$, such that $K^{(X,\mathcal{D})}_{\vec{\mu},g,Q}$ is the virtual counting of holomorphic maps f from genus g Riemann surface C_g with boundary ∂C_g to $(X, \mathcal{D})$, such that $f_*([C_g]) = Q \in H_2(X, \mathcal{D})$ and $f_*([\partial C_g]) = \sum_{i=1}^n \sum_{j\geq 1} \mu^i_j \gamma_i \in H_1(\mathcal{D}, \mathbb{Z})$. There are no general theory for open Gromov–Witten invariants, but see [38,50] for mathematical aspects of defining these invariants in special cases.

The total free energy and partition function of open topological string on X are defined by as follows

$$
\begin{aligned}
F^{(X,\mathcal{D})}(\mathbf{x}^1, \ldots, \mathbf{x}^n; g_s, \omega) &= -\sum_{g\geq 0} \sum_{\vec{\mu}\in\mathcal{P}^n\setminus\{0\}} \frac{\sqrt{-1}^{l(\mu)}}{|Aut(\vec{\mu})|} g_s^{2g-2+l(\vec{\mu})} \\
&\quad \times \sum_{Q\neq 0} K^{(X,\mathcal{D})}_{\vec{\mu},g,Q} e^{-Q\cdot\omega} \prod_{i=1}^n p_{\mu^i}(\mathbf{x}^i), \\
Z^{(X,\mathcal{D})}(\mathbf{x}^1, \ldots, \mathbf{x}^n; g_s, \omega) &= \exp(F^{(X,\mathcal{D})}(\mathbf{x}^1, \ldots, \mathbf{x}^n; g_s, \omega)).
\end{aligned}
\tag{25}
$$

We would like to calculate the partition function $Z^{(X,\mathcal{D})}(x^1, .., x^n; g_s, \omega)$ or the open Gromov–Witten invariants $K^{(X,\mathcal{D})}_{\vec{\mu},g,Q}$. For compact Calabi–Yau 3-folds, such as the quintic X_5, there are only a few works devoted to the study of its open Gromov–Witten invariants, for example, a complete calculation of the disk invariants of X_5 with boundary in a real Lagrangian was given in [67].

Suppose X is a toric Calabi–Yau 3-fold, and $\mathcal{D}$ is a special Lagrangian submanifold named as Aganagic–Vafa A-brane in the sense of [4,5]. The open string partition function $Z^{(X,\mathcal{D})}(x^1, .., x^n; g_s, \omega)$ can be computed by the method of topological vertex [2,45] or topological recursion developed by Eynard and Orantin [18]. The second approach was first proposed by Mariño [56], and studied further by Bouchard et al. [10]; the equivalence of these two methods was proved in [19,20].

The open Gromov–Witten invariants $K^{(X,\mathcal{D})}_{\vec{\mu},g,Q}$ are rational numbers in general. Just as in the closed string case [23], the open topological strings compute the partition function of BPS domain walls in a related superstring theory [64]. Ooguri and Vafa made the prediction that there are integers $N_{\vec{\mu};i,j}$ (OV invariants) such that

$$
F^{(X,\mathcal{D})}(\mathbf{x}^1, \ldots, \mathbf{x}^n; g_s, \omega) = \sum_{d\geq 1} \sum_{\vec{\mu}\in\mathcal{P}^n\setminus\{0\}} \frac{1}{d} \sum_{i,j} \frac{N_{\vec{\mu},i,j} a^{\frac{di}{2}} q^{\frac{dj}{2}}}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} s_{\vec{\mu}}(\vec{\mathbf{x}}), \tag{26}
$$

where $q = e^{\sqrt{-1}g_s}$ and $a = e^{-\omega}$.

Cleanly, one can formulate Ooguri–Vafa conjecture by using the Plethystic logarithm Log

Conjecture 3.1 *Let*

$$f_{\vec{\mu}}(q,a) = (q^{1/2} - q^{-1/2})\langle Log(Z^{(X,\mathcal{D})}(\mathbf{x}^1,\ldots,\mathbf{x}^n;q,a)), s_{\vec{\mu}}(\vec{\mathbf{x}})\rangle, \tag{27}$$

then we have

$$f_{\vec{\mu}}(q,a) = \sum_{i,j} N_{\vec{\mu},i,j} a^{\frac{i}{2}} q^{\frac{j}{2}} \in \mathbb{Z}\left[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}\right]. \tag{28}$$

Remark 3.2 These OV invariants $N_{\vec{\mu},i,j}$ were further refined to be the invariants $n_{\vec{\mu},g,Q}$ in [46–48]. See [54] for a more recent discussion about the LMOV invariants $n_{\vec{\mu},g,Q}$.

3.3 Open string model on $\mathbb{C}^3$

In this subsection, we focus on the open string model on $\mathbb{C}^3$ with Aganagic–Vafa A-brane $\mathcal{D}_\tau$, where $\tau \in \mathbb{Z}$ denotes the framing [4,5]. The topological (open) string partition function of $(\mathbb{C}^3, \mathcal{D}_\tau)$ is given by the Mariño–Vafa formula [57] which was proved by [53] and [63], respectively:

$$Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x};q) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(q;\tau)s_\lambda(\mathbf{x}), \tag{29}$$

and where

$$\mathcal{H}_\lambda(q;\tau) = (-1)^{|\lambda|\tau} q^{\frac{\kappa_\lambda \tau}{2}} \prod_{x\in\lambda} \frac{q^{cn(x)/2}}{q^{hl(x)/2} - q^{-hl(x)/2}}, \tag{30}$$

where $\kappa_\lambda = \sum_{i=1}^{l(\lambda)} \lambda_i(\lambda_i - 2i + 1)$.

The partition function $Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x};q)$ is in fact a certain generating function of terms which are the coefficients of highest order of a in the corresponding terms appearing in the open string partition function of the resolved conifold. That is why the parameter a does not appear in the expression $Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x};q)$. We refer to [54] for more details.

Applying the Ooguri–Vafa Conjecture 3.1 to $Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(g_s,\mathbf{x})$, it follows that for any $\tau \in \mathbb{Z}$ and $\mu \in \mathcal{P}_+$, we have

$$f_\mu^\tau(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\langle \mathrm{Log}(Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x};q)), s_\mu(\mathbf{x})\rangle \in \mathbb{Z}[q^{\pm\frac{1}{2}}]. \tag{31}$$

In particular, if we let $\mathbf{x} = (x,0,0,\ldots)$, then

$$\begin{aligned} Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(g_s,\mathbf{x}=(x,0,0,\ldots)) &= \sum_{n\geq 0} \mathcal{H}_{(n)}(q;\tau)x^n \\ &= \sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n, \end{aligned} \tag{32}$$

and

$$f_n^\tau(q) := f_{(n)}^\tau(q) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)[x^n]\mathrm{Log}\left(\sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n\right). \tag{33}$$

Therefore, formula (31) implies that for any $\tau \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$,

$$f_n^\tau(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]. \tag{34}$$

Therefore, we formulate the Ooguri–Vafa conjecture for $(\mathbb{C}^3, \mathcal{D}_\tau)$ as follows

Conjecture 3.3 *For any $\tau \in \mathbb{Z}$, for a fixed integer $n \geq 1$, we have*

$$f_n^\tau(q) = \sum_{k\in\mathbb{Z}} N_{n,k}(\tau) q^{\frac{k}{2}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]. \tag{35}$$

In other words, for a fixed integer $n \geq 1$, there are only finitely many k, such that the integers $N_{n,k}(\tau)$ are nonzero.

Now identity (33) is equivalent to

$$\mathrm{Log}\left(\sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n\right) = \sum_{n\geq 0}\sum_{k\in\mathbb{Z}} \frac{N_{n,k}(\tau) q^{\frac{k}{2}}}{\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)} x^n. \tag{36}$$

By using the properties of plethystic operators introduced in Sect. 2.2, we can write (32) in the form of infinite product as follows:

$$\begin{aligned}
\sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n &= \mathrm{Exp}\left(\sum_{n\geq 0}\sum_{k\in\mathbb{Z}} \frac{N_{n,k}(\tau) q^{\frac{k}{2}}}{\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)} x^n\right) \\
&= \mathrm{Exp}\left(-\sum_{n\geq 0}\sum_{k\in\mathbb{Z}}\sum_{l\geq 0} N_{n,k}(\tau) q^{\frac{k}{2}} q^{\frac{1}{2}+l} x^n\right) \\
&= \prod_{n\geq 0}\prod_{k\in\mathbb{Z}}\prod_{l\geq 0}\left(1 - q^{\frac{k}{2}} q^{\frac{1}{2}+l} x^n\right)^{N_{n,k}(\tau)},
\end{aligned} \tag{37}$$

where we have used the formal series

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots. \tag{38}$$

Remark 3.4 When we write the formula (32) into the form of infinite product, one can also use the formal series

$$\frac{1}{1-q^{-1}} = 1 + q^{-1} + q^{-2} + \cdots . \tag{39}$$

In order to make the connections to the Rogers–Ramanujan identities, here we need the infinite product form (37) by using the expansion (38).

In the following two sections, we will show that when $\tau \in \mathbb{Z}$ and $\tau \leq 0$, these integers $N_{n,k}(\tau)$ can be interpreted as the Betti numbers of certain cohomologies of quiver varieties, which finishes the proof of Conjecture 3.3 for $\tau \leq 0$. As to the case of $\tau \geq 1$, we study carefully for the special case of $\tau = 1$, and we find that these integers $N_{n,k}(1)$ together with formula (37) give a deformed version of the famous Rogers–Ramanujan identities (10) and (11).

4 Cohomologies of quiver varieties

Motivated by the previous works in Gauge theory [41,42], Nakajima [61,62] introduced the quiver varieties and illustrated how to use them to construct the geometric representations of Kac–Moody algebras. From then on, quiver varieties became to be the central objects in mathematics, we refer to [40] for the introduction to quiver varieties. Quiver varieties have a lot of structures and applications, for example, they can be used to prove the famous Kac's conjectures [35].

4.1 Kac's conjecture

We follow the notations in [30,31]. Take a ground field $\mathbb{K}$, denote by $\Gamma = (I, \Omega)$ a quiver with $I = \{1, \dots, n\}$ the set of vertices, and Ω the set of edges of Γ. For $\gamma \in \Omega$, let $h(\gamma), t(\gamma) \in I$ denote the head and tail of γ. A representation of Γ of dimension $\mathbf{v} = \{v_i\}_{i \in I} \in (\mathbb{Z}_{\geq 0})^n$ over $\mathbb{K}$ is a collection of $\mathbb{K}$-linear maps $\phi_\gamma : \mathbb{K}^{v_{t(\gamma)}} \to \mathbb{K}^{v_{h(\gamma)}}$ for each $\gamma \in \Omega$ that can be identified with matrices by using the canonical base of $\mathbb{K}^m$. A representation is said to be absolutely indecomposable over $\mathbb{K}$, if it is nontrivial and not isomorphic to a direct sum of two nontrivial representations of Γ over $\mathbb{K}$. A indecomposable representation is said to be absolutely indecomposable over $\mathbb{K}$, if it is still indecomposable over any extension field of $\mathbb{K}$.

In order to study the representation theory of general quiver Γ, Kac [35] introduced $A_{\mathbf{v}}(q)$, the number of isomorphic classes of absolutely indecomposable representations of Γ with dimension $\mathbf{v} = (v_1, \dots, v_n)$ over finite field $\mathbb{F}_q$. Following the idea of [35], Hua firstly computed the Kac polynomial $A_{\mathbf{v}}(q)$ in the following form:

$$\sum_{\mathbf{v} \in \mathbb{Z}^n_{\geq 0} \ \{0\}} A_{\mathbf{v}}(q) \prod_{i=1}^{n} T_i^{v_i} = (q-1) \tag{40}$$
$$\cdot \mathrm{Log}\left(\sum_{(\pi^1,\dots,\pi^n) \in \mathcal{P}^n} \frac{\prod_{\gamma \in \Omega} q^{\langle \pi^{t(\gamma)}, \pi^{h(\gamma)} \rangle}}{\prod_{i \in I} q^{\langle \pi^i, \pi^i \rangle} \prod_{k \geq 1} \prod_{j=1}^{m_k(\pi^i)} (1-q^{-j})} \prod_{i=1}^{n} T_i^{|\pi^i|} \right),$$

where $\langle , \rangle$ is the pairing on partitions defined by

$$\langle \lambda, \mu \rangle = \sum_{i,j} \min(i, j) m_i(\lambda) m_j(\mu). \tag{41}$$

Kac [35] proved that $A_{\mathbf{v}}(q)$ has integer coefficients and made two remarkable conjectures:

(i) If Γ has no edge-loops, then the constant term of $A_{\mathbf{v}}(0)$ is equal to the multiplicity of the root $\mathbf{v}$ in the corresponding Kac–Moody algebra $\mathfrak{g}(\Gamma)$.

(ii) The Kac polynomial $A_{\mathbf{v}}(q)$ has nonnegative coefficients.

Conjecture (i) was proved by Hausel [29] and Conjecture (ii) was completely settled by Hausel et al. [31] by using the theory of Nakajima quiver varieties and computing via arithmetic Fourier transform. They introduced the following function which largely generalizes Hua's formula (40)

$$\begin{aligned} &\mathbb{H}(\mathbf{x}^1, \ldots, \mathbf{x}^n; q) := (q-1) \\ &\cdot \mathrm{Log}\left(\sum_{(\pi^1,\ldots,\pi^n)\in\mathcal{P}^n} \frac{\prod_{\gamma\in\Omega} q^{\langle \pi^{t(\gamma)}, \pi^{h(\gamma)} \rangle}}{\prod_{i\in I} q^{\langle \pi^i, \pi^i \rangle} \prod_{k\geq 1} \prod_{j=1}^{m_k(\pi^i)} (1-q^{-j})} \prod_{i=1}^{n} \tilde{H}_{\pi^i}(\mathbf{x}^i; q) \right), \end{aligned} \tag{42}$$

where $\tilde{H}_{\pi^i}(\mathbf{x}^i; q)$ is the (transformed) Hall–Littlewood polynomial introduced in [21].

For $s_{\vec{\mu}}(\vec{\mathbf{x}}) := s_{\mu^1}(\mathbf{x}^1) \cdots s_{\mu^n}(\mathbf{x}^n)$, we let

$$\mathbb{H}^s_{\vec{\mu}}(q) := \langle \mathbb{H}(\mathbf{x}^1, \ldots, \mathbf{x}^n; q), s_{\vec{\mu}}(\vec{\mathbf{x}}) \rangle. \tag{43}$$

4.2 The quiver varieties $\mathcal{Q}_{\tilde{\mathbf{v}}}$

In their remarkable work [31], Hausel et al. found the geometric interpretation of $\mathbb{H}^s_{\vec{\mu}}(q)$ by computing, via arithmetic Fourier transform, the dimension of certain cohomologies of Nakajima quiver varieties. Let us briefly recall the main results in [31].

We denote the space of all the representations of Γ over $\mathbb{K}$ with dimension $\mathbf{v}$ by

$$\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v}) := \bigoplus_{\gamma\in\Omega} \mathrm{Mat}_{v_{h(\gamma)}, v_{t(\gamma)}}(\mathbb{K}). \tag{44}$$

Let $\mathrm{GL}_{\mathbf{v}} = \prod_{i\in I} \mathrm{GL}_{v_i}(\mathbb{K})$ and $\mathfrak{gl}_{\mathbf{v}} = \prod_{i\in I} \mathfrak{gl}_{v_i}(\mathbb{K})$. The algebraic group $\mathrm{GL}_{\mathbf{v}}$ acts on $\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$ as

$$(g \cdot \phi)_\gamma = g_{v_{h(\gamma)}} \phi_\gamma g^{-1}_{v_{t(\gamma)}} \tag{45}$$

for any $g = (g_i)_{i\in I} \in \mathrm{GL}_{\mathbf{v}}$, $\phi = (\phi_\gamma)_{\gamma\in\Omega} \in \mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$. Since the diagonal center $(\lambda I_{v_i})_{i\in I} \in \mathrm{GL}_{\mathbf{v}}$ acts trivially on $\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$, the action reduces to an action of $\mathrm{G}_{\mathbf{v}} = \mathrm{GL}_{\mathbf{v}}/\mathbb{K}^\times$.

Let $\overline{\Gamma}$ be the double quiver of Γ, namely, $\overline{\Gamma}$ has the same vertices as Γ, but the set of edges are given by $\overline{\Omega} := \{\gamma, \gamma^* | \gamma \in \Omega\}$, where $h(\gamma^*) = t(\gamma)$ and $t(\gamma^*) =$

$h(\gamma)$. By the trace pairing, we may identify $\mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v})$ with the cotangent bundle $\mathrm{T}^*\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$. We define the moment map

$$\begin{aligned} \mu_{\mathbf{v}} : &\mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v}) \to \mathfrak{gl}_{\mathbf{v}}^0 \\ &(x_\gamma)_{\gamma \in \overline{\Omega}} \mapsto \sum_{\gamma \in \Omega} [x_\gamma, x_{\gamma^*}], \end{aligned} \tag{46}$$

where $\mathfrak{gl}_{\mathbf{v}}^0 = \{(X_i)_{i\in I} \in \mathfrak{gl}_{\mathbf{v}} | \sum_{i\in I} \mathrm{Tr}(X_i) = 0\}$ is identified with the dual of the Lie algebra of $\mathrm{G}_{\mathbf{v}}$. It is a $\mathrm{G}_{\mathbf{v}}$-equivariant map. For $\xi = (\xi_i)_{i\in I} \in \mathbb{K}^I$ such that $\xi \cdot \mathbf{v} = \sum_i \xi_i v_i = 0$, then

$$(\xi_i I_{v_i})_{i\in I} \in \mathfrak{gl}_{\mathbf{v}}^0. \tag{47}$$

For such a $\xi \in \mathbb{K}^I$, the affine variety $\mu_{\mathbf{v}}^{-1}(\xi)$ inherits a $\mathrm{G}_{\mathbf{v}}$-action. The quiver variety $\mathcal{Q}_{\mathbf{v}}$ is the affine GIT quotient

$$\mu_{\mathbf{v}}^{-1}(\xi)//\mathrm{G}_{\mathbf{v}}. \tag{48}$$

The (related) quiver varieties were studied by many authors in past two decades, for example [11,44,61,62].

Let $\tilde{\Gamma}_{\mathbf{v}}$ on vertex set $\tilde{I}_{\mathbf{v}}$ be the quiver obtained from $(\Gamma, \mathbf{v})$ by adding at each vertex $i \in I$ a leg of length $v_i - 1$ with all the edges oriented towards the vertex i. Let $\tilde{\mathbf{v}} \in \mathbb{Z}_{\geq 0}^{\tilde{I}_{\mathbf{v}}}$ be the dimension vector with coordinate v_i at $i \in I \subset \tilde{I}_{\mathbf{v}}$ and with coordinates $(v_i - 1, v_i - 2, \ldots, 1)$ on the leg attached to the vertex $i \in I$. We let $\mathcal{Q}_{\tilde{\mathbf{v}}}$ be the quiver variety attached to the quiver $\tilde{\Gamma}_{\mathbf{v}}$ with parameter $\tilde{\xi}$ such that $\tilde{\xi} \cdot \tilde{\mathbf{v}} = 0$. Denote by $\tilde{C}_{\mathbf{v}}$ the Cartan matrix of the quiver $\tilde{\Gamma}_{\mathbf{v}}$, then

$$d_{\tilde{\mathbf{v}}} := 1 - \frac{1}{2}\tilde{\mathbf{v}}^t \tilde{C}_{\mathbf{v}} \tilde{\mathbf{v}} \tag{49}$$

equals $\frac{1}{2} \dim \mathcal{Q}_{\tilde{\mathbf{v}}}$ if $\mathcal{Q}_{\tilde{\mathbf{v}}}$ is nonempty.

Let $W_{\mathbf{v}} := S_{v_1} \times \cdots \times S_{v_n}$ be the Weyl group of the group $GL_{\mathbf{v}} := GL_{v_1} \times \cdots \times GL_{v_n}$, it acts on $H_c^*(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C})$ by the work of Nakajima [61,62]. We denote by $\chi^{\vec{\mu}} = \chi^{\mu^1} \cdots \chi^{\mu^n} : W_{\mathbf{v}} \to \mathbb{C}^\times$ the exterior product of the irreducible characters χ^{μ^i} of the symmetric group S_{v_i} in the notation of [55]. In particular, $\chi^{(v_i)}$ is the trivial character and $\chi^{(1^{v_i})}$ is the sign character $\epsilon_i : S_{v_i} \to \{\pm 1\}$.

The main result of [31] is

Theorem 4.1 *(Theorem 1.4 and Corollary 1.5 in* [31]*) We have*

$$\mathbb{H}^s_{\vec{\mu}}(q) = \sum_i \langle \rho^{2i}, \epsilon\chi^{\vec{\mu}} \rangle_{W_{\mathbf{v}}} q^{i - d_{\tilde{\mathbf{v}}}}, \tag{50}$$

where $\langle \rho^{2i}, \epsilon\chi^{\vec{\mu}} \rangle_{W_{\mathbf{v}}}$ is the multiplicity of $\epsilon\chi^{\vec{\mu}}$ in the representation ρ^{2i} of $W_{\mathbf{v}}$ in $H_c^{2i}(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C})$.

In particular, for $\vec{\mu} = (1^{\mathbf{v}}) = ((1^{v_1}), \ldots, 1^{v_n}) \in \mathcal{P}^n$, *we have*

$$\mathbb{H}^s_{1^{\mathbf{v}}}(q) = \sum_j \dim \left(H_c^{2j}(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C}\right)^{W_{\mathbf{v}}}) q^{j-d_{\tilde{\mathbf{v}}}}. \tag{51}$$

4.3 A special case

For the quiver $\Gamma = (I, \Omega)$, we attach $k_i \geq 1$ infinite legs to each vertex $i \in I$ of Γ. Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 1}$. We set all the arrows on the new legs point towards the vertex. Given a dimension vector $\mathbf{v} \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$, one can also construct quiver varieties similarly including the previous construction as the special case of all $k_i = 1$. More precisely, let $\tilde{\Gamma}(\mathbf{k})$ be the quiver obtained from $(\Gamma, \mathbf{v})$ by adding at each vertex $i \in I$ k_i infinite legs of the edges all oriented toward the vertex i. Denote by $\tilde{\mathbf{v}}(\mathbf{k})$ the dimension vector with coordinate v_i at $i \in I$ and with the same coordinates $(v_i - 1, v_i - 2, \ldots, 1, 0, 0, \ldots)$ on the k_i legs attached to the vertex $i \in I$. Now we let $\mathcal{Q}_{\tilde{\mathbf{v}}(\mathbf{k})}$ be the quiver variety associated to quiver $\tilde{\Gamma}(\mathbf{k})$.

Corollary 4.2 *([31], Proposition 3.4, by changing* $q \to q^{-1}$*) We have the identity*

$$(q^{-1} - 1) Log \left(\sum_{\mathbf{v} \in \mathbb{Z}^n_{\geq 0}} \frac{q^{\frac{1}{2}(\gamma(\mathbf{v}(\mathbf{k})) + \delta(\mathbf{v}(\mathbf{k})))}}{\prod_{i=1}^n (1-q)\cdots(1-q^{v_i})} (-1)^{\delta(\mathbf{v}(\mathbf{k}))} \prod_{i=1}^n T_i^{v_i} \right) \tag{52}$$
$$= \sum_{\mathbf{v} \in \mathbb{Z}^n_{\geq 0}} \mathbb{H}^s_{1^{\mathbf{v}(\mathbf{k})}}(q^{-1}) (-1)^{\delta(\mathbf{v}(\mathbf{k}))} \prod_{i=1}^n T_i^{v_i},$$

where

$$\gamma(\mathbf{v}(\mathbf{k})) = \sum_{i=1}^n (2 - k_i) v_i^2 - 2 \sum_{\gamma \in \Omega} v_{t(\gamma)} v_{h(\gamma)}, \quad \delta(\mathbf{v}(\mathbf{k})) = \sum_{i=1}^n k_i v_i, \tag{53}$$

and $(1^{\mathbf{v}(\mathbf{k})}) = ((1^{v_1})^{k_1}, \ldots, (1^{k_n})^{k_n})$, *where* $(1^{v_i})^{k_i}$ *denotes that* (1^{v_i}) *appears* k_i *times.*

By Theorem 4.1, we obtain

$$\mathbb{H}^s_{1^{\mathbf{v}(\mathbf{k})}}(q^{-1}) = \sum_j \dim \left(H_c^{2j}(\mathcal{Q}_{\tilde{\mathbf{v}}(\mathbf{k})}; \mathbb{C})^{W_{\mathbf{v}}} \right) q^{d_{\tilde{\mathbf{v}}(\mathbf{k})} - j}. \tag{54}$$

Now, we can finish the proof of Theorem 1.2.

Proof For the framing $\tau \in \mathbb{Z}_{\leq 0}$, we take $k = 1 - \tau \in \mathbb{Z}_{\geq 1}$. Consider the one vertex quiver $\Gamma = \bullet$, we construct a new quiver $\Gamma(k)$ with the unique vertex attached with k infinite legs as shown above. Associate a dimension vector $\tilde{n}(k)$ to quiver $\Gamma(k)$, we have the quiver variety $\mathcal{Q}_{\tilde{n}(k)}$ by the construction showed previously. Now combining formulae (53) and (54) together in this special case, by the variable change $x = q^{1/2}T$, we obtain

$$(q^{1/2} - q^{-1/2})\mathrm{Log}\left(\sum_{n\geq 0} \frac{(-1)^{n(\tau-1)} q^{\frac{n(n-1)}{2}\tau + \frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n\right) \tag{55}$$
$$= -\sum_{n\geq 0} \mathbb{H}^s_{1^{n(k)}}(q)^{1/2-n/2}(-1)^{(\tau-1)n} x^n$$
$$= -\sum_{n\geq 0}\sum_j \dim\left(H_c^{2j}(\mathcal{Q}_{\tilde{n}(k)};\mathbb{C})^{S_n}\right) q^{\frac{1-n}{2} + d_{\tilde{n}(k)} - j}(-1)^{(\tau-1)n} x^n.$$

Therefore,

$$f_n^\tau(q) = -\sum_j \dim\left(H_c^{2j}(\mathcal{Q}_{\tilde{n}(1-\tau)};\mathbb{C})^{S_n}\right) q^{\frac{1-n}{2} + d_{\tilde{n}(1-\tau)} - j}(-1)^{(\tau-1)n}. \tag{56}$$

Comparing to Formula (35), we obtain the formulae (7) and (8) in Theorem 1.2. □

5 Deformed Rogers–Ramanujan identities

In the above section, we have interpreted and proved the integrality of Ooguri–Vafa invariants $N_{n,j}(\tau)$ for $\tau \in \mathbb{Z}$ and $\tau \leq 0$. It is natural to ask how about the case $\tau \geq 1$?

As shown in Sect. 3, Conjecture 3.3 leads to the conjectural infinite product formula (37).

In the following, we study carefully for Formula (37) in $\tau = 1$. We find that if we let $n_{m,k} := (-1)^m N_{m,k}$, then $n_{m,k}$ will be nonnegative. After some concrete computation by using Maple 13, we propose

Conjecture 5.1 *For a fixed integer $m \geq 1$, there exist finite many positive integers $n_{m,k}$, such that*

$$\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)}\left(q^{-\frac{1}{2}}x\right)^n = \prod_{m\geq 1}\prod_{k\in\mathbb{Z}}\prod_{l\geq 0}\left(1 - q^{\frac{k+1}{2}+l}x^m\right)^{(-1)^m n_{m,k}}. \tag{57}$$

In particular, when $x = q^{\frac{1}{2}}$ and $x = q^{\frac{3}{2}}$, these integers $n_{m,k}$ together with the formula (57) *yield the two Rogers–Ramanujan identities* (10) *and* (11).

Let us give some numerical checks for Conjecture 5.1. We introduce the polynomial

$$g_m(q) = \sum_{k\in\mathbb{Z}} n_{m,k} q^k. \tag{58}$$

By using Maple 13, we have computed the polynomial $g_m(q)$ for $1 \leq m \leq 18$. Here is a list for them when $m \leq 6$:

$$\begin{aligned}
g_1(q) &= 1 \\
g_2(q) &= q, \\
g_3(q) &= q^4, \\
g_4(q) &= q^5 + q^9, \\
g_5(q) &= q^6 + q^8 + q^{10} + q^{12} + q^{16}, \\
g_6(q) &= q^7 + 2q^9 + q^{11} + 3q^{13} + q^{15} + 2q^{17} + q^{19} + q^{21} + q^{25}.
\end{aligned}$$

If we let $x = q^{\frac{1}{2}}$, identity (57) becomes

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} &= \prod_{m\geq 1}\prod_{k\geq 0}\prod_{l\geq 0}\left(1-q^{\frac{m+k+1}{2}+l}\right)^{(-1)^m n_{m,k}} \\
&= \prod_i \prod_{l\geq 0}(1-q^{i+l})^{n_i},
\end{aligned} \tag{59}$$

where $n_i = \sum_{m+k+1=2i}(-1)^m n_{m,k}$, our computations imply that

$$n_i = \begin{cases} -1, & i = 5k+1 \text{ or } 5k+4, \ \text{for } k \geq 0, \\ 1, & i = 5k+2 \text{ or } 5k+5, \ \text{for } k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It turns out that formula (59) gives the first Rogers–Ramanujan identity (10).

Similarly, letting $x = q^{\frac{3}{2}}$, identity (57) becomes

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} &= \prod_{m\geq 1}\prod_{k\in\mathbb{Z}}\prod_{l\geq 0}\left(1-q^{\frac{3m+k+1}{2}+l}\right)^{(-1)^m n_{m,k}} \\
&= \prod_i \prod_{l\geq 0}(1-q^{i+l})^{r_i},
\end{aligned} \tag{60}$$

where $r_i = \sum_{3m+k+1=2i}(-1)^m n_{m,k}$, we find that

$$r_i = \begin{cases} -1, & i = 5k+2, \ \text{for } k \geq 0, \\ 1, & i = 5k+4, \ \text{for } k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence formula (60) gives the second Rogers–Ramanujan identity (11).

Formula (57) in Conjecture 5.1 is a formula of type "infinite sum = infinite product." We expect it could be interpreted by the denominator formula for some kinds of root system [36].

These conjectural integers $n_{m,k}$ appearing in Conjecture (5.1) are important. Although we have not obtained an explicit formula for them, we have an explicit formula for the value of $\sum_k n_{m,k}$ for any $m \geq 1$.

First, recall the definition of $f_\mu^\tau(q)$ in (31), we have

$$Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x};q)=\operatorname{Exp}\left(\frac{1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\sum_{\mu\in\mathcal{P}_+}f_\mu^\tau(q)s_\mu(\mathbf{x})\right). \tag{61}$$

By Formula (25) for the case of $(\mathbb{C}^3,\mathcal{D}_\tau)$, the open string free energy is given by

$$\begin{aligned}F^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x})&=\log Z^{(\mathbb{C}^3,\mathcal{D}_\tau)}(\mathbf{x};q) \\ &=-\sum_{\mu\in\mathcal{P}_+}\sum_{g\geq 0}\frac{\sqrt{-1}^{l(\mu)}}{|Aut(\mu)|}K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{\mu,g,\frac{|\mu|}{2}}g_s^{2g-2+l(\mu)}p_\mu(\mathbf{x}),\end{aligned} \tag{62}$$

which is the generating function the Gromov–Witten invariants $K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{\mu,g,\frac{|\mu|}{2}}$, and where $q=e^{ig_s}$.

An explicit expression for $K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{\mu,g,\frac{|\mu|}{2}}$ is obtained in [38] (cf. Formula (15) in [54]). In particular, for $m\geq 1$ and $g=0$, we have

$$K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{m,0,\frac{m}{2}}=\frac{(-1)^{m\tau}}{m^2}\binom{m(\tau+1)-1}{m-1}. \tag{63}$$

Then, we combine (61) and (62) together, and consider the specialization $\mathbf{x}=(x,0,0,\ldots)$, it follows that

$$\sum_{m\geq 1}\sum_{g\geq 0}K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{m,g,\frac{m}{2}}g_s^{2g}x^m=\sum_{d\geq 1}\frac{\sqrt{-1}g_s}{d(q^{\frac{d}{2}}-q^{-\frac{d}{2}})}\sum_{m\geq 1}\sum_k N_{m,k}(\tau)q^{\frac{dk}{2}}x^{dm}. \tag{64}$$

Taking the coefficients of x^m in (64), and considering the limit $g_s\to 0$, we obtain

$$K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{m,0,\frac{m}{2}}=\sum_{d|m}\frac{1}{d^2}\sum_k N_{m/d,k}(\tau), \tag{65}$$

where we have used

$$\lim_{g_s\to 0}\frac{\sqrt{-1}g_s}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}=\lim_{g_s\to 0}\frac{\sqrt{-1}g_s}{e^{\sqrt{-1}g_s\frac{d}{2}}-e^{-\sqrt{-1}g_s\frac{d}{2}}}=\frac{1}{d}. \tag{66}$$

Finally, by Möbius inversion formula, we have

$$f_m^\tau(1)=\sum_k N_{m,k}(\tau)=\sum_{d|m}\frac{\mu(\frac{m}{d})}{\left(\frac{m}{d}\right)^2}K^{(\mathbb{C}^3,\mathcal{D}_\tau)}_{d,0,\frac{d}{2}}, \tag{67}$$

where $\mu(d)$ is the Möbius function.

Therefore, by the expression (65) we obtain

Proposition 5.2 *For $m \geq 1$, the value $f_m^\tau(1)$ of (3) at $q = 1$ is given by*

$$f_m^\tau(1) = \frac{1}{m^2} \sum_{d|m} \mu(m/d)(-1)^{d\tau} \binom{d(\tau+1)-1}{d-1}. \tag{68}$$

In the following, we will prove that

Theorem 5.3 *For any $m \geq 1$ and $\tau \in \mathbb{Z}$,*

$$\frac{1}{m^2} \sum_{d|m} \mu(m/d)(-1)^{d\tau} \binom{d(\tau+1)-1}{d-1} \in \mathbb{Z}. \tag{69}$$

For $\tau \leq 0$, we have shown the integrality of Ooguri–Vafa invariant $N_{m,k}(\tau)$ in Theorem 1.2, so we only need to prove Theorem 5.3 for the case of $\tau > 0$ in the following.

In the author's joint work with Luo [54], we develop a systematic method to deal with the integrality of BPS numbers from string theory. We can apply our method directly to prove Theorem 5.3.

We define the following function, for nonnegative integer n and prime number p,

$$f_p(n) = \prod_{i=1,\, p\nmid i}^{n} i = \frac{n!}{p^{[n/p]}[n/p]!}. \tag{70}$$

Lemma 5.4 *(cf. Lemma 4.6 in [54]) For odd prime numbers p and $\alpha \geq 1$ or for $p = 2$, $\alpha \geq 2$, we have $p^{2\alpha} \mid f_p(p^\alpha n) - f_p(p^\alpha)^n$. For $p = 2, \alpha = 1$, $f_2(2n) \equiv (-1)^{[n/2]}$ (mod 4).*

Proof With $\alpha \geq 2$ or $p > 2$, $p^{\alpha-1}(p-1)$ is even, then

$$\begin{aligned}
&f_p(p^\alpha n) - f_p(p^\alpha(n-1)) f_p(p^\alpha) \\
&= f_p(p^\alpha(n-1)) \left(\prod_{i=1,\, p\nmid i}^{p^\alpha} (p^\alpha(n-1)+i) - f_p(p^\alpha) \right) \\
&\equiv p^\alpha(n-1) f_p(p^\alpha(n-1)) f_p(p^\alpha) \left(\sum_{i=1,\, p\nmid i}^{p^\alpha} \frac{1}{i} \right) \pmod{p^{2\alpha}} \\
&\equiv p^\alpha(n-1) f_p(p^\alpha(n-1)) f_p(p^\alpha) \left(\sum_{i=1,\, p\nmid i}^{[p^\alpha/2]} \left(\frac{1}{i} + \frac{1}{p^\alpha - i} \right) \right) \pmod{p^{2\alpha}} \\
&\equiv p^\alpha(n-1) f_p(p^\alpha(n-1)) f_p(p^\alpha) \left(\sum_{i=1,\, p\nmid i}^{[p^\alpha/2]} \frac{p^\alpha}{i(p^\alpha - i)} \right) \equiv 0 \pmod{p^{2\alpha}}.
\end{aligned}$$

Thus the first part of Lemma 5.4 is proved by induction. For $p = 2, \alpha = 1$, the remain argument is straightforward. □

Lemma 5.5 *For prime number p and $n = p^\alpha a$, $p \nmid a, \alpha \geq 1, \tau \geq 0$, $p^{2\alpha}$ divides*

$$(-1)^{\tau n}\binom{(\tau+1)n-1}{n-1} - (-1)^{\tau n/p}\binom{(\tau+1)n/p-1}{n/p-1}.$$

Proof By a straightforward computation, we have

$$\begin{aligned}
&(-1)^{\tau n}\binom{(\tau+1)n-1}{n-1} - (-1)^{\tau n/p}\binom{(\tau+1)n/p-1}{n/p-1}\\
&= (-1)^{\tau n}\binom{(\tau+1)n/p-1}{n/p-1}\left(\frac{f_p((\tau+1)n)}{f_p(\tau n)f_p(n)} - (-1)^{\tau(n-n/p)}\right).
\end{aligned} \tag{71}$$

For $p > 2$ or $p = 2, \alpha > 1$, then $n - n/p$ is even, thus (71) is divisible by $p^{2\alpha}$ by Lemma 5.4. For $p = 2, \alpha = 1$, (71) is divisible by 4 if

$$\left[\frac{(\tau+1)n}{4}\right] + \left[\frac{\tau n}{4}\right] + \left[\frac{n}{4}\right] - \tau\left(n - \frac{n}{2}\right) \equiv 0 \pmod 2$$

which depends only on $\tau \pmod 2$, verify for $\tau \in \{0, 1\}$ to get the results. □

Now, we can finish the proof of Theorem 5.3.

Proof For $m \geq 1$, we write $m = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where each $\alpha_i \geq 1$ and $p_1, \ldots, p_r$ are r distinct primes.

By the definition of Möbius function, only when $m/d = p_1^{\delta_1}p_2^{\delta_2}\cdots p_r^{\delta_r}$ for $\delta_i \in \{0, 1\}$, $\mu(m/d)$ is nonzero and

$$\mu(p_1^{\delta_1}p_2^{\delta_2}\cdots p_r^{\delta_r}) = (-1)^{\sum_{i=1}^r \delta_i}. \tag{72}$$

Therefore,

$$\begin{aligned}
&\sum_{d|m}\mu(m/d)(-1)^{d\tau}\binom{d(\tau+1)-1}{d-1}\\
&\quad= \sum_{\delta_i\in\{0,1\},1\leq i\leq r}(-1)^{\sum_{i=1}^r\delta_i}(-1)^{\tau n_\delta}\binom{(\tau+1)n_\delta-1}{n_\delta-1},
\end{aligned} \tag{73}$$

where $n_\delta = p_1^{\alpha_1-\delta_1}p_2^{\alpha_2-\delta_2}\cdots p_r^{\alpha_r-\delta_r}$. We need to show for any $1 \leq i \leq r$, (73) is divisible by $p_i^{2\alpha_i}$. Without loss of generality, we only show that (73) is divisible by $p_1^{2\alpha_1}$ in the following.

Indeed, (73) is equal to

$$\sum_{\delta_j\in\{0,1\},j\geq 2}(-1)^{\delta_2+\cdots+\delta_r}\left((-1)^{\tau n_{\delta'}}\binom{(\tau+1)n_{\delta'}-1}{n_{\delta'}-1}\right.$$
$$\left.-(-1)^{\tau n_{\delta'}/p_1}\binom{(\tau+1)n_{\delta'}/p_1-1}{n_{\delta'}/p_1-1}\right), \tag{74}$$

where $n_{\delta'}=p_1^{\alpha_1}p_2^{\alpha_2-\delta_2}\cdots p_r^{\alpha_r-\delta_r}=p^{\alpha_1}a$. Therefore, by Lemma 5.5, the formula (73) is divisible by $p_1^{2\alpha_1}$, hence we complete the proof of Theorem 5.3. ◻

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