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On the polynomial Ramanujan sums over finite fields

Zhiyong Zheng[1]

Abstract The polynomial Ramanujan sum was first introduced by Carlitz (Duke Math J 14:1105–1120, 1947), and a generalized version by Cohen (Duke Math J 16:85–90, 1949). In this paper, we study the arithmetical and analytic properties of these sums, deriving various fundamental identities, such as Hölder formula, reciprocity formula, orthogonality relation, and Davenport–Hasse type formula. In particular, we show that the special Dirichlet series involving the polynomial Ramanujan sums are, indeed, the entire functions on the whole complex plane, and we also give a square mean values estimation. The main results of this paper are new appearance to us, which indicate the particularity of the polynomial Ramanujan sums.

Keywords Polynomial Ramanujan sums · Finite fields · Reciprocity formula · Orthogonality relation · Davenport–Hasse's type formula

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1 Introduction and results statement

One hundred years ago, Ramanujan was the first to appreciate the importance of the following exponential sum:

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$$c(m, n) = \sum_{\substack{d=1, \\ (d,n)=1}}^{n} e\left(\frac{md}{n}\right), \tag{1.1}$$

where $e(x) = e^{2\pi i x}$; n and m are positive integers. His interest in this sum originated in his desire to obtain expression for a variety of well-known arithmetical functions in the form of a series $\sum_n a_n c(m, n)$; in particular, he obtained some very famous identities, for example (see [29] and [41])

$$\sum_{n=1}^{+\infty} \frac{c(m, n)}{n} = 0, \quad \sum_{n=1}^{+\infty} \frac{c(m, n)}{m} = -\Lambda(n) \quad \text{and} \quad c(m, n) = \sum_{d|n, d|m} d\mu\left(\frac{n}{d}\right), \tag{1.2}$$

where $\mu(n)$ is the Möbius function; $\Lambda(n)$ is the Mangoldt function, and the last equality is usually said to be the Kluyver's equality. Following him, many other authors were also interested in this fascinating sum [6,7,36], especially it makes surprising appearances in singular series of the Hardy–Littlewood asymptotic formula for Waring problems and in the asymptotic formula of Vingradov on sums of three primes. For details, the reader is referred to [18].

Many mathematicians later tried to generalize this sum to find more and more applications. One of the most popular generalization was given by Cohen [10–12,14] (or see [46]) that

$$c_k(m, n) = \sum_{\substack{d=1, \\ (d,n^k)_k=1}}^{n} e\left(\frac{md}{n^k}\right), \tag{1.3}$$

where the g.c.d function $(a, b)_k$ is the greatest common kth power divisors of the integers a and b. In [10] and [12], Cohen presented an analogue of the Kluyver and Hölder formula for the above generalized Ramanujan sums

$$c_k(m, n) = \sum_{d|n, d^k|m} d^k \mu\left(\frac{n}{d}\right) = \phi_k(n)\mu(N)\phi_k^{-1}(N), \tag{1.4}$$

where $N^k = \frac{n^k}{(m,n^k)_k}$, and $\phi_k(n)$ is the Jordan totient function given by the following product expression (z is a complex number):

$$\phi_z(n) = n^z \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p^z}\right). \tag{1.5}$$

Various other generalizations were discussed in many papers including that of [3,9,16,32,35,42,50]. Here, we mention another interesting result, namely reciprocity formula. In [26,27], Johenson showed that

$$\frac{\mu(\bar{m})c(nm^*, m)}{m^*} = \frac{\mu(\bar{n})c(mn^*, n)}{n^*}, \tag{1.6}$$

where $\bar{n}$ denotes the largest square-free divisions of n, and $n^* = \frac{n}{\bar{n}}$.

In more recent years, people have more and more interests in this sum; it appeared in various other seemingly unrelated problems. In Algebra, Ramanujan sums as the super characters exhibit a new application of the theory of super characters [20], which recently developed by Diaconis–Issacs and Andre' [15], and Ramanujan sums in arithmetical semigroup [22]. In number theory, Ramanujan sums appeared in the study of Waring type formula [30], the distribution of rational numbers in short interval [28], equirepartition modulo odd integers [5], the distribution of average of Ramanujan sums [1,2,8,21,34,37,44,49], graph theory [17], symmetry classes of tensors [47], combinatorics [43], cyclotomic polynomials [19,33,48], and Mahler matrices [31]. In physics, Ramanujan sums have applications in the processing of low-frequency noise [39], and of long-period sequences [40], and in the study of quantum phase locking [38].

The main purpose of this paper is to generalize the Ramanujan sums to the polynomial case, and discuss various analogue properties of the classical case. The polynomial Ramanujan sum was first introduced by Carlitz [7], and a generalized version by Cohen [10]. To state their definition, let $\mathbb{F}_q$ be a finite field with $q = p^l$ elements, where p is a prime number; $\mathbb{F}_q[x]$ be the polynomial ring. Suppose H is a fixed polynomial in $\mathbb{F}_q[x]$, and $H \neq 0$, we first choose a complex-valued character of the additive group of the residue class ring $\mathbb{F}_q[x]/\langle H \rangle$; these characters are said to be the additive characters modulo H on $\mathbb{F}_q[x]$. If A is in $\mathbb{F}_q[x]$, let $A \equiv a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \pmod{H}$, where $m = \deg(H)$, we set an additive function modulo H on $\mathbb{F}_q[x]$ by $t(A) = a_{m-1}$. Then, for any A and B in $\mathbb{F}_q[x]$, we have $t(A + B) = t(A) + t(B)$, and $t(A) = t(B)$ if $A \equiv B \pmod{H}$, in particular, $t(A) = 0$ whenever $H|A$. To generalize this t-function modulo H, for any given polynomial G in $\mathbb{F}_q[x]$, we let $t_G(A) = t(GA)$. Clearly, t_G is also an additive function modulo H, that is

$$t_G(A + B) = t_G(A) + t_G(B) \quad \text{and} \quad t_G(A) = t_G(B) \text{ if } \quad A \equiv B \pmod{H}. \quad (1.7)$$

Next, let λ be a fixed non-principal character on $\mathbb{F}_q$, for example, one may choose $\lambda(a) = e\left(\frac{\text{tr}(a)}{p}\right)$ for $a \in \mathbb{F}_q$, where $\text{tr}(a)$ is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. We define a complex-valued function $E(G, H)$ on $F_q[x]$ by

$$E(G, H)(A) = \lambda(t_G(A)). \quad (1.8)$$

It is easy to see that $E(G, H)$ is an additive character modulo H on $\mathbb{F}_q[x]$, and

$$E(G, H)(A) = E(A, H)(G). \quad (1.9)$$

The polynomial Ramanujan sum modulo H on $\mathbb{F}_q[x]$ is given by (see Carlitz [7, 4.1])

$$\eta(G, H) = \sum_{\substack{D \bmod H, \\ (D,H)=1}} E(G, H)(D), \quad (1.10)$$

where the summation extends over a complete residue system modulo H in $\mathbb{F}_q[x]$. If $k \geq 1$ is a fixed integer, the generalized version of Cohen (see [10, 3.3]) is the following:

$$\eta_k(G, H) = \sum_{\substack{D \bmod H^k, \\ (D, H^k)_k = 1}} E(G, H^k)(D), \tag{1.11}$$

where the summation ranges over a complete residue system modulo H^k in $\mathbb{F}_q[x]$, and the g.c.d. function $(A, B)_k$ denotes the largest kth power common divisor (monic) of the polynomials A and B in $\mathbb{F}_q[x]$. We set $(A, B)_1 = (A, B)$ the usual g.c.d. function. If $D \bmod H^k$ with $(D, H^k)_k = 1$, which is said to be a k-reduced residue system modulo H according to Cohen [11,12]. Clearly, $\eta_1(G, H) = \eta(G, H)$.

By the above notations, it is easy to verify (see [10, (3.4) and (3.5)]) that

$$\eta_k(G_1, H) = \eta_k(G_2, H), \quad \text{if } G_1 \equiv G_2 \ (\bmod \ H^k), \tag{1.12}$$

$$\eta_k(G, H_1 H_2) = \eta_k(G, H_1)\eta_k(G, H_2), \quad \text{if } (H_1, H_2) = 1, \tag{1.13}$$

and

$$\eta_k(G, H) = \sum_{D|H, D^k|G} |D|^k \mu\left(\frac{H}{D}\right), \tag{1.14}$$

where and later, $\mu(H)$ is the Möbius function on $\mathbb{F}_q[x]$, $|D| = q^{\deg(D)}$ is the absolute value function on $\mathbb{F}_q[x]$, and $D|H$ means D is a monic divisor of H, and $\sum_{D|H}$ means D extending over all of monic divisors of H.

We note that the additive characters $E(G, H)$ given by (1.8) are, indeed, all of the additive characters ψ modulo H; in other words, for any additive character ψ modulo H, there exists a unique polynomial G in $\mathbb{F}_q[x]$, such that $\psi = E(G, H)$, and $\deg(G) < \deg(H)$ (See Lemma 2.1). Therefore, the polynomial Ramanujan sums $\eta(G, H)$ and $\eta_k(G, H)$ coincide with the classical sums $c(m, n)$ and $c_k(m, n)$ respectively.

The first result of this paper is to derive an analogue of Hölder formula for the polynomial sums. There is no essential difficulty to do this, but we make use of a simpler method to show (see Theorem 2.1) that

$$\eta(G, H) = \phi(H)\mu\left(\frac{H}{(G, H)}\right)\phi^{-1}\left(\frac{H}{(G, H)}\right),$$
$$\text{and} \quad \eta_k(G, H) = \phi_k(H)\mu(N)\phi_k^{-1}(N), \tag{1.15}$$

where $\phi(H)$ is the Euler totient function, and $\phi_k(H)$ is the Jordan totient function on $\mathbb{F}_q[x]$, and $N^k = \frac{H^k}{(G,H)_k}$. As we know, in the classical case, the proof of (1.4) by Cohen is more complicated (see Theorem 1 of [12]).

The second result is to present an analogue of the reciprocity formula for the polynomial Ramanujan sums (see Theorem 3.1). Let $\bar{H}$ be the largest square-free divisor of H, and $H^* = \frac{H}{\bar{H}}$, then we have

$$\frac{\mu(\bar{H})\eta(GH^*, H)}{|H^*|} = \frac{\mu(\bar{G})\eta(HG^*, G)}{|G^*|}. \tag{1.16}$$

The generalized sums $\eta_k(G, H)$ seemingly cannot share this kind of formula when $k > 1$; however, we derive the second reciprocity formula for $\eta_k(G, H)$ (see Theorem 3.3): if $D_1^k|H$ and $D_2^k|H$, then

$$\phi_k(D_2)\eta_k\left(\frac{H}{D_2^k}, D_1\right) = \phi_k(D_1)\eta_k\left(\frac{H}{D_1^k}, D_2\right). \tag{1.17}$$

The importance of (1.17) lies in the fact that it is equivalent to the Hölder formula (1.15), and plays a role in the proof of the orthogonal relation formula (4.13).

The main results of this paper are the following theorems on the special Dirichlet series involving in the polynomial Ramanujan sums. Let $\mathbb{A}$ be the set of monic polynomials of $\mathbb{F}_q[x]$. We define

$$\delta_k(s, G) = \sum_{H \in \mathbb{A}} \frac{\eta_k(G, H)}{|H|^s} = \sum_{n=0}^{+\infty} A(n)q^{-ns} \tag{1.18}$$

and

$$\tau_k(s, H) = \sum_{G \in \mathbb{A}} \frac{\eta_k(G, H)}{|G|^s} = \sum_{n=0}^{+\infty} B(n)q^{-ns}, \tag{1.18'}$$

where $s = c + it$ is a complex number, and

$$A(n) = \sum_{\substack{H \in \mathbb{A} \\ \deg(H)=n}} \eta_k(G, H), \quad B(n) = \sum_{\substack{G \in \mathbb{A} \\ \deg(G)=n}} \eta_k(G, H).$$

The last two infinite series in (1.18) and (1.18') are the definitions of $\delta_k(s, G)$ and $\tau_k(s, H)$, which tell us how to understand the special Dirichlet series in the middle of (1.18) and (1.18'). We show that

Theorem 1.1 *If $G \neq 0$, then $\delta_k(s, G)$ is an entire function on the whole complex plane, and we have*

$$\delta_k(s, G) = (1 - q^{1-s}) \sum_{D^k|G} |D|^{k-s}. \tag{1.19}$$

In particular, we have

$$\delta_k(1, G) = \sum_{H \in \mathbb{A}} \frac{\eta_k(G, H)}{|H|} = 0. \tag{1.20}$$

Furthermore, for any real numbers c and $T > 0$, we also have the following square mean value estimation:

$$\frac{1}{2T} \int_{-T}^{T} |\delta_k(c+it, G)|^2 \mathrm{d}t = \left(1 + q^{2(1-c)}\right) \sigma_0(2(k-c), G)$$

$$- 2q^{1-k} \sigma_1(2(k-c), G) + O\left(\frac{1}{T}\right), \qquad (1.21)$$

where (x is a real number)

$$\sigma_0(x, G) = \sum_{\substack{D_1^k | G, D_2^k | G \\ \deg(D_1) = \deg(D_2)}} |D_1|^x, \quad and \quad \sigma_1(x, G) = \sum_{\substack{D_1^k | G, D_2^k | G \\ \deg(D_1) = \deg(D_2) + 1}} |D_1|^x, \tag{1.22}$$

and the constant implied by "O" depends on q, G, and c only.

If $G = 0$, then $E(G, H^k)$ is the principal additive character modulo H^k, and $\eta_k(G, H) = \phi_k(H)$ by (1.11). It is easy to see that

$$\delta_k(s, G) = \zeta_{\mathbb{A}}^{-1}(s) \zeta_{\mathbb{A}}(s - k), \quad \text{if } \mathrm{Re}(s) > k + 1, \tag{1.23}$$

where $\zeta_{\mathbb{A}}$ is the zeta function on $\mathbb{F}_q[x]$ given by

$$\zeta_{\mathbb{A}}(s) = \sum_{H \in \mathbb{A}} \frac{1}{|H|^s}. \tag{1.24}$$

It is well known (see [45, Chap. 2]) that

$$\zeta_{\mathbb{A}}(s) = \left(1 - q^{1-s}\right)^{-1}, \quad \text{if } \mathrm{Re}(s) > 1. \tag{1.25}$$

Therefore, $\eta_k(s, G)$ has a simple pole $s = k + 1$ with residue $\frac{1}{\zeta_{\mathbb{A}}(k+1) \log q}$ at $s = k + 1$, when $G = 0$.

Theorem 1.2 *If H is a positive degree polynomial in $\mathbb{F}_q[x]$, then $\tau_k(s, H)$ is an entire function on the whole complex plane, and we have*

$$\tau_k(s, H) = (1 - q^{1-s})^{-1} \phi_{k(1-s)}(H), \tag{1.26}$$

where $\phi_{k(1-s)}(H)$ is the generalized Jordan totient function given by

$$\phi_z(H) = |H|^z \prod_{\substack{P | H \\ P \text{ irreducible}}} \left(1 - \frac{1}{|P|^z}\right). \tag{1.27}$$

Especially, if s = 1, we have

$$\tau_k(1, H) = \sum_{G \in \mathbb{A}} \frac{\eta_k(G, H)}{|G|} = -\frac{k \Lambda(H)}{\log q}, \tag{1.28}$$

where $\Lambda(H)$ is the Mangoldt function on $\mathbb{F}_q[x]$. Moreover, for any real numbers c and T with $c \neq 1$ and $T > 0$, we have

$$
\begin{aligned}
\frac{1}{2T} \int_{-T}^{T} |\tau_k(c+it, H)|^2 \mathrm{d}t = {} & \left|1 - q^{2c(1-c)}\right|^{-1} \\
& \times \sum_{\substack{D_1|H, D_2|H \\ \deg(D_1)=\deg(D_2)}} \mu\left(\frac{H}{D_1}\right) \mu\left(\frac{H}{D_2}\right) |D_1|^{2(1-c)} \\
& + 2|1 - q^{2(1-c)}|^{-1} \\
& \times \sum_{\substack{D_1|H, D_2|H \\ \deg(D_1)>\deg(D_2)}} \mu\left(\frac{H}{D_1}\right) \mu\left(\frac{H}{D_2}\right) \\
& \times |D_1|^{(k+2)(1-c)} |D_2|^{(2-k)(1-c)} + O\left(\frac{1}{T}\right), \quad (1.29)
\end{aligned}
$$

where the constant implied by "O" depends on q, H, and c only.

If $\deg(H) = 0$, then $\eta_k(G, H) = 1$ for any G in $\mathbb{F}_q[x]$, it follows that $\tau_k(s, H) = \zeta_\mathbb{A}(s)$, which has a simple pole $s = 1$ with residue $\frac{1}{\log q}$ at $s = 1$. If $c = 1$, the square mean value estimation is more complicated, and we will present it in another place.

The polynomial Ramanujan sum $\eta(G, H)$ in fact is a special type of Gauss sum on $\mathbb{F}_q[x]$. In [51], we presented an analogue of Davenport–Hasse's theorem for the polynomial Gauss sums (see [51, Theorem 1.3]). In the last section of this paper, we show that the generalized polynomial Ramanujan sums $\eta_k(G, H)$ also share this kind of Davenport–Hasse's formula (see Theorem 7.1).

Throughout this paper, P denotes an irreducible polynomial in $\mathbb{F}_q[x]$, $D|H$ means that D is a monic divisor of H; $\sum_{D|H}$ means D extending over all of monic divisors of H, and $|H| = q^{\deg(H)}$ is the absolute value function on $\mathbb{F}_q[x]$.

2 Preliminaries

We start this section by determining the construction of the additive character group modulo H on $\mathbb{F}_q[x]$ via $E(G, H)$.

Lemma 2.1 *For any ψ, an additive character modulo H on $\mathbb{F}_q[x]$, there exists a unique polynomial G in $\mathbb{F}_q[x]$ such that $\psi = E(G, H)$, and $\deg(G) < \deg(H)$.*

Proof For the sake of convenience, we write $\psi_G = E(G, H)$. By (1.8), we have $\psi_{G_1} = \psi_{G_2}$, if $G_1 \equiv G_2 \pmod{H}$, and hence we may set G in a complete residue system modulo H in $\mathbb{F}_q[x]$, so that $\deg(G) < \deg(H)$. Moreover, we have

$$
\psi_{G_1+G_2} = \psi_{G_1} \cdot \psi_{G_2} \quad \text{and} \quad \bar{\psi}_G = \psi_{-G}, \tag{2.1}
$$

where $\bar{\psi}_G$ is the usual conjugation of a complex number ψ_G. Since $\psi_G = \psi_0$, the principal additive character modulo H, if $G = 0$, or $H|G$. Conversely, we have

$\psi_G = \psi_0$ if and only if $H|G$. To show that statement, we see that λ is a non-principal additive character on $\mathbb{F}_q$ by assumption, then there is an element a in $\mathbb{F}_q$, so that $\lambda(a) \neq 1$. If $H \nmid G$, we let

$$R = (G, H) = x^k + a_{k-1}x^k + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x], \qquad (2.2)$$

where $0 \leq k \leq m - 1$, and $m = \deg(H)$. It follows that

$$ax^{m-1-k}R = ax^{m-1} + \cdots .$$

We note the following congruent equation in variable T that

$$GT \equiv ax^{m-1-k}R \pmod{H} \qquad (2.3)$$

is solvable in $\mathbb{F}_q[x]$; therefore, there exists a polynomial A in $\mathbb{F}_q[x]$ such that

$$GA \equiv ax^{m-1} + \cdots . \pmod{H}, \qquad (2.4)$$

and $t_G(A) = t(GA) = a$, so $\psi_G(A) = \lambda(t_G(A)) = \lambda(a) \neq 1$, and $\psi_G \neq \psi_0$. By (2.1), we have immediately

$$\psi_{G_1} \neq \psi_{G_2}, \quad \text{if } G_1 \not\equiv G_2 \pmod{H} \qquad (2.5)$$

because if $\psi_{G_1} = \psi_{G_2}$ then $\psi_{G_1-G_2} = \psi_0$, and $G_1 \equiv G_2 \pmod{H}$. This shows that ψ_G are different from each other when G running through a complete residue system of modulo H. Hence, there are exactly $|H| = q^m$ different characters ψ_G, but the number of additive characters modulo H on $\mathbb{F}_q[x]$ is exactly q^m, thus every character ψ is just of the form ψ_G. We complete the proof of Lemma 2.1. □

Next two lemmas are not new; one may find them in Carlitz [7] (see [7, (2.4), (2.5), and (2.6)]), but we give a more explicit expression here.

Lemma 2.2 *If A is a monic polynomial in $\mathbb{F}_q[x]$, then we have*

$$E(GA, HA) = E(G, H). \qquad (2.6)$$

Proof For any $B \in \mathbb{F}_q[x]$, let

$$GB \equiv a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \pmod{H},$$

then

$$AGB \equiv A(a_{m-1}x^{m-1} + \cdots + a_0) \pmod{AH}.$$

Because A is a monic polynomial, we see that the function t_{GA} modulo HA just is the function t_G modulo H. It follows that

$$E(GA, HA)(B) = \lambda(t_G(B)) = E(G, H)(B), \qquad (2.7)$$

and the lemma follows immediately. □

Lemma 2.3 *Suppose* $A \in \mathbb{F}_q[x]$*, then we have*

$$\sum_{G \bmod H} E(G, H)(A) = \begin{cases} |H|, & \text{if } H|A, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

where the summation extends over a complete residue system modulo H*.*

Proof By Lemma 2.1, it just is the orthogonal relation formula; we have the lemma at once. □

The following lemma is a more complicated orthogonal relation, which will be used in the next section.

Lemma 2.4 *If* $D_1|H$*,* $D_2|H$*,* $(X, D_1^k)_k = 1$*, and* $(Y, D_2^k)_k = 1$*, where X and Y are two polynomials in* $\mathbb{F}_q[x]$ *with* $\deg(X) < k \deg(D_1)$ *and* $\deg(Y) < k \deg(D_2)$*, then we have*

$$\sum_{A+B \equiv G \pmod{H^k}} E(X, D_1^k)(A)E(Y, D_2^k)(B)$$
$$= \begin{cases} |H|^k E(X, D^k), & \text{if } X = Y, D_1 = D_2 = D, \\ 0, & \text{otherwise,} \end{cases} \tag{2.9}$$

where the summation ranges over a complete residue system modulo H^k*.*

Proof We first note that $E(G, H) = E(G, a^{-1}H)$, where a is the leading coefficient of H. Without loss of generality, we may suppose that H is a monic polynomial, and write $D_1 R_1 = H$, $D_2 R_2 = H$, and $B = G - A \pmod{H^k}$, then the left side of (2.9) is (see (1.9)) that

$$\sum_{A \bmod H^k} E\left(X, D_1^k\right)(A)E\left(Y, D_2^k\right)(G - A)$$
$$= E(Y, D_2^k)(G) \sum_{A \bmod H^k} E\left(X R_1^k, H^k\right)(A)E\left(Y R_2^k, H^k\right)(-A)$$
$$= E(Y, D_2^k)(G) \sum_{A \bmod H^k} E\left(A, H^k\right)\left(X R_1^k - Y R_2^k\right). \tag{2.10}$$

By Lemma 2.3, the inner sum in the above equality is zero, if $X R_1^k - Y R_2^k \not\equiv 0 \pmod{H^k}$, and $|H|^k$, if $X R_1^k \equiv Y R_2^k \pmod{H^k}$. Since $\deg(X) < k \deg(D_1)$ and $\deg(Y) < k \deg(D_2)$, we have $X R_1^k = Y R_2^k$, if $X R_1^k \equiv Y R_2^k \pmod{H^k}$. It follows that $X D_2^k = Y D_1^k$, and $D_1 = D_2$, $X = Y$, since $(X, D_1^k)_k = (Y, D_2^k)_k = 1$. We complete the proof of this lemma. □

Remark 2.1 According to Definition 3.3 of [24], we may give a modern way to approach the additive characters modulo H on $\mathbb{F}_q[x]$ (see (1.8)). Let $K = \mathbb{F}_q(x)$ be the field of rational functions over $\mathbb{F}_q$, and K_∞ be the completion of K with respect

to the infinite place. Then every $\alpha \in K_\infty$ can be expressed in a unique way in Laurent series of the form

$$\alpha = \sum_{i=n}^{+\infty} a_i \left(\frac{1}{x}\right)^i, \quad n \in \mathbb{Z}, \quad a_i \in \mathbb{F}_q \quad \text{and} \quad a_n \neq 0.$$

Let $\mathrm{res}_\infty(\alpha) = a_1$, which is said to be the residue of α at the infinite place. Let H be a fixed polynomial in $\mathbb{F}_q[x]$, $H \neq 0$, and G be in $\mathbb{F}_q[x]$ with $\deg(G) < \deg(H)$. We may define an additive character $E(G, H)$ on K_∞^+ by (see also Sect. 3 of [25])

$$E(G, H)(\alpha) = \lambda \left(\mathrm{res}_\infty \left(\frac{G\alpha}{H}\right)\right),$$

where K_∞^+ is the additive group of the local field K_∞, and λ is a fixed non-principal character on $\mathbb{F}_q$ as before. It is easy to see that the restriction of $E(G, H)$ on $\mathbb{F}_q[x]$ is an additive character modulo H on $\mathbb{F}_q[x]$. By this definition, we see that Lemma 2.2 is trivial.

Next, we give a simple proof of (1.14) and (1.15). In order to prove the Kluyver and Hölder's formula in the polynomial case, we first make some slight modification of the Möbius inversion formula on $\mathbb{F}_q[x]$.

Definition 2.1 A non-zero mapping δ from $\mathbb{F}_q[x]$ to the complex plane is said to be an arithmetical function on $\mathbb{F}_q[x]$, if $\delta(aA) = \delta(A)$ for any $a \in \mathbb{F}_q^*$ and $A \in \mathbb{F}_q[x]$. It is said to be a multiplicative function, if $\delta(AB) = \delta(A) \cdot \delta(B)$, whenever $(A, B) = 1$, and a complex multiplicative function, if $\delta(AB) = \delta(A) \cdot \delta(B)$ for any A, B in $\mathbb{F}_q[x]$.

We start with the following a few important examples of the arithmetical functions on $\mathbb{F}_q[x]$.

Möbius function $\mu(H)$: Let $\mu(0) = 0$, and

$$\mu(H) = \begin{cases} 1, & \text{if } H \in \mathbb{F}_q^*, \\ 0, & \text{if there exists a } P \text{ such that } P^2 | H, \\ (-1)^t, & \text{if } H = P_1 P_2 \cdots P_t, \text{ where } P_j \text{ are different.} \end{cases} \tag{2.11}$$

It is easy to see that $\mu(H)$ is a multiplicative function on $\mathbb{F}_q[x]$. Moreover, we have the following identities that

$$\sum_{D|H} \mu(D) = \begin{cases} 1, & \text{if } \deg(H) = 0, \\ 0, & \text{if } \deg(H) \geq 1, \end{cases} \tag{2.12}$$

and

$$\sum_{D^k|H} \mu(D) = \begin{cases} 1, & \text{if } D \text{ is } k\text{th power divisors-free,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

Euler totient function $\phi(H)$: If $H \neq 0$, we define $\phi(H)$ to be the number of polynomials of degree less than $\deg(H)$ that are coprime to H, and $\phi(0) = 0$.

Jordan totient function $\phi_k(H)$: If $H \neq 0$, $\phi_k(H)$ is the number of polynomials of degree less than $k \cdot \deg(H)$ that have no common kth power divisors other than one with H^k.

Clearly, $\phi_1(H) = \phi(H)$, and $\phi_k(H)$ is a multiplicative function on $\mathbb{F}_q[x]$. The following equalities are easy to verify:

$$\sum_{D|H} \phi(D) = |H| \quad \text{and} \quad \sum_{D|H} \phi_k(D) = |H|^k. \tag{2.14}$$

Mangoldt function $\Lambda(H)$: We define the Mangoldt function $\Lambda(H)$ on $\mathbb{F}_q[x]$ by

$$\Lambda(H) = \begin{cases} \log |P|, \text{if } H = P^t, t \geq 1, \\ 0, \qquad \text{otherwise.} \end{cases} \tag{2.15}$$

It is easy to see that

$$\sum_{D|H} \Lambda(D) = \log |D|. \tag{2.16}$$

Lemma 2.5 (Möbius Inversion Formula) *If $\Delta(H)$ and $\delta(H)$ are two arithmetical functions on $\mathbb{F}_q[x]$, then*

$$\Delta(H) = \sum_{D|H} \delta(D), \quad \text{for all } H \neq 0, \tag{2.17}$$

if and only if

$$\delta(H) = \sum_{D|H} \mu(D) \Delta\left(\frac{H}{D}\right), \quad \text{for all } H \neq 0. \tag{2.18}$$

Proof The proof is similar to the classical case; from (2.12), we have the lemma immediately. □

As a direct consequence of Lemma 2.5, from (2.14) and (2.16), we have

$$\phi(H) = \sum_{D|H} \mu(D) \left|\frac{H}{D}\right| = |H| \prod_{P|H} \left(1 - \frac{1}{|P|}\right), \tag{2.19}$$

$$\phi_k(H) = \sum_{D|H} \mu(D) \left|\frac{H}{D}\right|^k = |H|^k \prod_{P|H} \left(1 - \frac{1}{|P|^k}\right), \tag{2.20}$$

and

$$\Lambda(H) = \sum_{D|H} \log |D| \cdot \mu\left(\frac{H}{D}\right). \tag{2.21}$$

We give a simple proof of (1.14) and (1.15) now. The method here is an analogue of Anderson and Apostol [3], or see Apostol [4].

Lemma 2.6 (Kluyver's formula) *Let* $\eta_k(G, H)$ *be the polynomial Ramanujan sums given by* (1.11)*, and* $H \neq 0$*, then we have*

$$\eta_k(G, H) = \sum_{D|H, D^k|G} |D|^k \mu\left(\frac{H}{D}\right). \tag{2.22}$$

Proof By (2.13), we have

$$\eta_k(G, H) = \sum_{R \bmod H^k} E(G, H^k)(R) \sum_{D^k|(R, H^k)_k} \mu(D)$$

$$= \sum_{D|H} \mu(D) \sum_{\substack{R \bmod H^k \\ D^k|R}} E(G, H^k)(R)$$

$$= \sum_{D|H} \mu(D) \sum_{\substack{R \bmod H^k \\ D^k|R}} E(R, H^k)(G). \tag{2.23}$$

We write $R = A \cdot D^k$, and note that $E(R, H^k) = E(A, \left(\frac{H}{D}\right)^k)$ by Lemma 2.2. It follows that

$$\eta_k(G, H) = \sum_{D|H} \mu(D) \sum_{A \bmod \left(\frac{H}{D}\right)^k} E\left(A, \left(\frac{H}{D}\right)^k\right)(G)$$

$$= \sum_{D|H} \mu\left(\frac{H}{D}\right) \sum_{A \bmod D^k} E(A, D^k)(G)$$

$$= \sum_{\substack{D|H \\ D^k|G}} \mu\left(\frac{H}{D}\right) |D|^k. \tag{2.24}$$

We complete the proof of Lemma 2.6.   □

Theorem 2.1 (Hölder formula) *For any G and H in* $\mathbb{F}_q[x]$ *and* $H \neq 0$*, we have*

$$\eta_k(G, H) = \phi_k(H)\mu(N)\phi_k^{-1}(N), \quad where \ N^k = \frac{H^k}{(G, H^k)_k}. \tag{2.25}$$

Proof Let $(G, H^k)_k = A^k$, then $D|H$ and $D^k|G$ if and only if $D|A$. Let $N = \frac{H}{A}$, then by Lemma 2.6, we have

$$\eta_k(G, H) = \sum_{D|A} |D|^k \mu\left(\frac{NA}{D}\right) = \sum_{D|A} \mu(ND)\left|\frac{A}{D}\right|^k$$

$$= \mu(N) \sum_{\substack{D|A \\ (D,N)=1}} \mu(D)\left|\frac{A}{D}\right|^k$$

$$=\mu(N)|A|^k \prod_{\substack{P|A \\ P\nmid N}} (1 - |P|^{-k})$$

$$=\frac{\mu(N)|A|^k \prod_{P|A\cdot N}(1 - |P|^{-k})}{\prod_{P|N}(1 - |P|^{-k})}$$

$$=\frac{\mu(N)|H|^k \prod_{P|H}(1 - |P|^{-k})}{|N|^k \prod_{P|N}(1 - |P|^{-k})}$$

$$=\phi_k(H)\mu(N)\phi_k^{-1}(N), \qquad (2.26)$$

where $N^k = \frac{H^k}{A^k} = \frac{H^k}{(G,H^k)_k}$. We complete the proof of Theorem 2.1. □

By (2.25), (1.15) and (1.14) follow immediately; in particular, we have

$$\eta_k(G, H) = \phi_k(H) \text{ if } H^k|G, \qquad (2.27)$$

and

$$\eta_k(G, H) = \mu(H) \text{ if } (G, H^k)_k = 1. \qquad (2.28)$$

3 Reciprocity formula

In this section, we give two reciprocity formulas for the polynomial Ramanujan sums. We start with the following lemmas.

Lemma 3.1 *Let $\eta(G, H)$ be the polynomial Ramanujan sum given by (1.10), and H be a square-free polynomial, then $\mu(H)\eta(G, H)$ is a multiplicative function in variable G.*

Proof Let $f(G) = \mu(H)\eta(G, H)$, $H = P_1 P_2 \cdots P_n$, and $G = G_1 \cdot G_2$, where $(G_1, G_2) = 1$. We set

$$(G, H) = \prod_{i\in\gamma} P_i, \quad (G_1, H) = \prod_{i\in\gamma_1} P_i, \quad (G_2, H) = \prod_{i\in\gamma_2} P_i, \qquad (3.1)$$

where $\gamma_1 \cup \gamma_2 = \gamma \subset \{1, 2, \ldots, n\}$, and $\gamma_1 \cap \gamma_2$ is an empty set. Then by the Hölder formula (2.25) ($k = 1$), we have

$$f(G) = (-1)^{-|\gamma|}\mu^2(H) \prod_{i\in\gamma} \phi(P_i), \qquad (3.2)$$

$$f(G_1) = (-1)^{-|\gamma_1|}\mu^2(H) \prod_{i\in\gamma_1} \phi(P_i), \qquad (3.3)$$

and

$$f(G_2) = (-1)^{-|\gamma_2|}\mu^2(H) \prod_{i\in\gamma_2} \phi(P_i). \qquad (3.4)$$

Since $|\gamma_1| + |\gamma_2| = |\gamma|$, and $\mu^2(H) = \mu^4(H)$, then $f(G) = f(G_1) \cdot f(G_2)$, and the lemma follows. $\quad\square$

Lemma 3.2 *If H is square-free, then we have*

$$\mu(H)\eta(G, H) = \sum_{D|(G,H)} |D|\mu(D). \tag{3.5}$$

Proof Since the both sides of (3.5) are multiplicative in G, it suffices to prove that when $G = P^s$. Let $H = P_1 P_2 \cdots P_n$. If $P \neq P_j$, $1 \leq j \leq n$, then the both sides of (3.5) are one, so we may let $P = P_j$. It follows that

$$\mu(H)\eta(G, H) = -\mu^2(H)\phi(P_j) = 1 - |P_j|, \tag{3.6}$$

and

$$\sum_{D|(G,H)} |D|\mu(D) = 1 - |P_j|. \tag{3.7}$$

The lemma follows at once. $\quad\square$

We note that the sum in (3.5) is symmetric in G and H, and hence as a direct consequence of Lemma 3.2, we have

Corollary 3.1 *Suppose that both G and H are square-free, then*

$$\mu(H)\eta(G, H) = \mu(G)\eta(H, G). \tag{3.8}$$

Theorem 3.1 (The first reciprocity formula) *Let $\bar{H}$ be the largest square-free divisor of H, and $H^* = \frac{H}{\bar{H}}$, then*

$$\frac{\mu(\bar{H})\eta(GH^*, H)}{|H^*|} = \frac{\mu(\bar{G})\eta(HG^*, G)}{|G^*|}. \tag{3.9}$$

Proof It is easy to verify the observation of Hardy that

$$\eta(G, H) = 0 \text{ if } H^* \nmid G \quad \text{and} \quad \phi(H) = |H^*|\phi(\bar{H}). \tag{3.10}$$

In order to prove (3.9), we first show that

$$\eta(GH^*, H) = |H^*|\eta(G, \bar{H}). \tag{3.11}$$

By (2.25)($k = 1$), one has

$$\begin{aligned}
\eta(GH^*, H) &= \phi(H)\mu\left(\frac{H}{(GH^*, H)}\right)\phi^{-1}\left(\frac{H}{(GH^*, H)}\right) \\
&= |H^*|\phi(\bar{H})\mu\left(\frac{\bar{H}}{(G, \bar{H})}\right)\phi^{-1}\left(\frac{\bar{H}}{(G, \bar{H})}\right) \\
&= |H^*|\eta(G, \bar{H}),
\end{aligned}$$

and (3.11) follows. We note that $\eta(G, \bar{H}) = \eta(\bar{G}, \bar{H})$; it follows by Corollary 3.1 that

$$
\begin{aligned}
\frac{\mu(\bar{H})\eta(GH^*, H)}{|H^*|} &= \mu(\bar{H})\eta(G, \bar{H}) = \mu(\bar{H})\eta(\bar{G}, \bar{H}) \\
&= \mu(\bar{G})\eta(\bar{H}, \bar{G}) = \mu(\bar{G})\eta(H, \bar{G}) \\
&= \frac{\mu(\bar{G})\eta(HG^*, G)}{|G^*|}.
\end{aligned}
$$

We complete the proof of Theorem 3.1. $\square$

The next reciprocity property first appeared in [20, Theorem 3.8] in the rational case, and here, we present an analogue in the polynomial case.

Theorem 3.2 (The second reciprocity formula) *If $D_1|H$, and $D_2|H$, then we have*

$$
\phi(D_2)\eta\left(\frac{H}{D_2}, D_1\right) = \phi(D_1)\eta\left(\frac{H}{D_1}, D_2\right). \tag{3.12}
$$

Proof By (2.25), then

$$
\eta\left(\frac{H}{D_2}, D_1\right) = \phi(D_1)\mu\left(\frac{D_1}{\left(\frac{H}{D_2}, D_1\right)}\right)\phi^{-1}\left(\frac{D_1}{\left(\frac{H}{D_2}, D_1\right)}\right) \tag{3.13}
$$

and

$$
\eta\left(\frac{H}{D_1}, D_2\right) = \phi(D_2)\mu\left(\frac{D_2}{\left(\frac{H}{D_1}, D_2\right)}\right)\phi^{-1}\left(\frac{D_2}{\left(\frac{H}{D_1}, D_2\right)}\right). \tag{3.14}
$$

We write $D = D_1 D_2$, then

$$
\frac{D_1}{\left(\frac{H}{D_2}, D_1\right)} = \frac{D}{(H, D)} = \frac{D_2}{\left(\frac{H}{D_1}, D_2\right)}, \tag{3.15}
$$

and the theorem follows at once. $\square$

The importance of (3.12) lies in the fact that it is equivalent to the Hölder formula (2.25) ($k = 1$). If we take $D_1 = H$, and $D_2 = \frac{H}{(G,H)}$ in (3.12), and note that

$$
\eta(G, H) = \eta((G, H), H), \tag{3.16}
$$

and $\eta(1, H) = \mu(H)$, then

$$
\eta(G, H) = \phi(H)\mu\left(\frac{H}{(G, H)}\right)\phi^{-1}\left(\frac{H}{(G, H)}\right), \tag{3.17}
$$

which is the Hölder formula of $\eta(G, H)$.

The generalized polynomial Ramanujan sum $\eta_k(G, H)$ seemingly cannot share the first reciprocity formula when $k > 1$, since G and H are not symmetric in a generalized version of (3.5). But, indeed, $\eta_k(G, H)$ shares the second reciprocity formula.

Theorem 3.3 *If $D_1^k | H$ and $D_2^k | H$, then*

$$\phi_k(D_2)\eta_k\left(\frac{H}{D_2^k}, D_1\right) = \phi_k(D_1)\eta_k\left(\frac{H}{D_1^k}, D_2\right). \qquad (3.18)$$

Proof By (2.25), we have

$$\eta_k\left(\frac{H}{D_2^k}, D_1\right) = \phi_k(D_1)\mu(N_1)\phi_k^{-1}(N_1) \qquad (3.19)$$

and

$$\eta_k\left(\frac{H}{D_1^k}, D_2\right) = \phi_k(D_2)\mu(N_2)\phi_k^{-1}(N_2), \qquad (3.20)$$

where $N_1^k = \frac{D_1^k}{(\frac{H}{D_2^k}, D_1^k)_k}$ and $N_2^k = \frac{D_2^k}{(\frac{H}{D_1^k}, D_2^k)_k}$.

We write $D = D_1 D_2$, then

$$N_1^k = \frac{D^k}{(H, D^k)_k} = N_2^k. \qquad (3.21)$$

It follows that $N_1 = N_2$, and we have Theorem 3.3. $\qquad \square$

As the case of $k = 1$, (3.18) is equivalent to (2.25). If we replace H by H^k in (3.18), then if $D_1 | H$ and $D_2 | H$, we have

$$\phi_k(D_2)\eta_k\left(\frac{H^k}{D_2^k}, D_1\right) = \phi_k(D_1)\eta_k\left(\frac{H^k}{D_1^k}, D_2\right). \qquad (3.22)$$

As an analogue of the above equality in the rational case, one may see Lemma 1 of [12]. Taking $D_1 = H$, and $D_2 = \frac{H}{A}$, where $A^k = (G, H^k)_k$ in (3.22), we note that

$$\eta_k(G, H) = \eta_k(A^k, H) \quad \text{and} \quad \eta_k(1, H) = \mu(H). \qquad (3.23)$$

It follows that

$$\eta_k(G, H) = \phi_k(H)\mu(N)\phi_k^{-1}(N),$$

where $N = \frac{H}{A}$, and $N^k = \frac{H^k}{A^k} = \frac{H^k}{(G, H^k)_k}$, we have (2.25) at once.

To make applications of the first reciprocity formula, one may follow Johnson [26] to consider the C-series representations and the C'-series representation for the arithmetical function on $\mathbb{F}_q[x]$, and show that the two classes of representation are

equivalent under certain conditions. Here we consider a special Dirichlet series and derive its values by using Theorem 3.1. We set

$$\sigma(s, H) = \sum_{\substack{G \in \mathbb{A} \\ G \text{ square-free}}} \frac{\eta(G, H)}{|G|^s}, \tag{3.24}$$

where $\mathbb{A}$ is the set of monic polynomials of $\mathbb{F}_q[x]$, and s is a complex variable. If H is square-free, and $\mathrm{Re}(s) > 1$, we have the following formula:

$$\mu(H)\sigma(s, H) = \zeta_{\mathbb{A}}(s)\zeta_{\mathbb{A}}^{-1}(2s) \prod_{P|H} \frac{1 - |P| + |P|^s}{1 + |P|^s}, \tag{3.25}$$

where $\zeta_{\mathbb{A}}(s)$ is the zeta function on $\mathbb{F}_q[x]$ given by (1.24). If $\mathrm{Re}(s) > 1$, then $\zeta_{\mathbb{A}}(s)$ has the following Euler product formula:

$$\zeta_{\mathbb{A}}(s) = \prod_{P \in \mathbb{A}} (1 - |P|^{-s})^{-1}. \tag{3.26}$$

To prove (3.25), by (3.8), we have

$$\mu(H)\sigma(s, H) = \sum_{\substack{G \in \mathbb{A}, \\ G \text{ square-free}}} \frac{\mu(G)\eta(H, G)}{|G|^s}, \tag{3.27}$$

where $f(G) = \mu(G)\eta(H, G)$ is a multiplicative function in G, so we may make use of Euler product and obtain that

$$\mu(H)\sigma(s, H) = \prod_{P \in \mathbb{A}} \left(1 - \frac{\eta(H, P)}{|P|^s}\right).$$

It is easy to see that $\eta(H, P) = \mu(P)$, if $P \nmid H$, and $\eta(H, P) = \phi(P)$, if $P|H$. Then, we have

$$\mu(H)\sigma(s, H) = \prod_{\substack{P \in \mathbb{A}, \\ P \nmid H}} \left(1 + \frac{1}{|P|^s}\right) \prod_{P|H} \left(1 - \frac{\phi(P)}{|P|^s}\right)$$

$$= \zeta_{\mathbb{A}}(s)\zeta_{\mathbb{A}}^{-1}(2s) \prod_{P|H} \frac{1 - |P| + |P|^s}{1 + |P|^s},$$

which is (3.25).

An application of the second reciprocity formula appears in the next section (See Lemma 4.4).

4 Orthogonality relation

In this section, we derive some more complicated orthogonal formula and a number of corollaries for the polynomial Ramanujan sums. In the rational case, one may find the analogues in Cohen [11,12]. We begin by proving

Lemma 4.1 *If $D_1|H$ and $D_2|H$, then for any G in $\mathbb{F}_q[x]$, we have*

$$\sum_{A+B\equiv G \pmod{H^k}} \eta_k(A, D_1)\eta_k(B, D_2) = \begin{cases} |H|^k\eta_k(G, D), & \text{if } D_1 = D_2 = D, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

where the summation extends over a complete residue system modulo H^k.

Proof By Lemma 2.4, the left-hand side of (4.1) is

$$\sum_{(X,D_1^k)_k=1} \sum_{(Y,D_2^k)_k=1} \sum_{A+B\equiv G \pmod{H^k}} E(A, D_1^k)(X)E(B, D_2^k)(Y)$$

$$= |H|^k \sum_{(X,D^k)=1} E(X, D^k)(G) = |H|^k\eta_k(G, D), \tag{4.2}$$

if $D_1 = D_2 = D$. Otherwise it is zero. We have the lemma immediately. □

Definition 4.1 A k-reduced residue system modulo H is that D ranges over modulo H^k such that $(D, H^k)_k = 1$.

Lemma 4.2 *A complete residue system modulo H^k is given by $A = R\left(\frac{H}{D}\right)^k$, where D ranges over the divisors of H, and for each D, R ranges over a k-reduced residue system modulo D.*

Proof See Cohen [13, Lemma 4]. □

The next lemma is a more generalized form of the second reciprocity formula of $\eta_k(G, H)$.

Lemma 4.3 *If $D_1^k|H$, and $D_2^k|H$, then for any R in $\mathbb{F}_q[x]$, we have*

$$\phi_k(D_2)\eta_k\left(\frac{RH}{D_2^k}, D_1\right) = \phi_k(D_1)\eta_k\left(\frac{RH}{D_1^k}, D_2\right). \tag{4.3}$$

Proof It follows directly from Theorem 3.3. □

If we replace H by H^k in the above equality, then we have the following corollary.

Corollary 4.1 *If $D_1|H$, and $D_2|H$, then for any polynomial R in $\mathbb{F}_q[x]$ we have (comparing with [12, Lemma 1]) that*

$$\phi_k(D_2)\eta_k\left(R\left(\frac{H}{D_2}\right)^k, D_1\right) = \phi_k(D_1)\eta_k\left(R\left(\frac{H}{D_1}\right)^k, D_2\right). \tag{4.4}$$

Lemma 4.4 *If $D|H$, $D_1|H$, and $R \in \mathbb{F}_q[x]$ such that $(R, D^k)_k = 1$, then we have*

$$\eta_k \left(R \left(\frac{H}{D} \right)^k, D_1 \right) = \eta_k \left(\left(\frac{H}{D} \right)^k, D_1 \right). \tag{4.5}$$

Proof By (4.3), then

$$\eta_k \left(R \left(\frac{H}{D} \right)^k, D_1 \right) = \frac{\phi_k(D_1)}{\phi_k(D)} \eta_k \left(R \left(\frac{H}{D_1} \right)^k, D \right). \tag{4.6}$$

Because of $(R, D^k)_k = 1$, it is easy to see by (2.25) that

$$\eta_k \left(R \left(\frac{H}{D_1} \right)^k, D \right) = \eta_k \left(\left(\frac{H}{D_1} \right)^k, D \right). \tag{4.7}$$

By (4.3) once again, we have

$$\eta_k \left(\left(\frac{H}{D_1} \right)^k, D \right) = \frac{\phi_k(D)}{\phi_k(D_1)} \eta_k \left(\left(\frac{H}{D} \right)^k, D_1 \right), \tag{4.8}$$

and the lemma follows at once. $\square$

Now we state and prove the main result of this section.

Theorem 4.1 *If $D_1|H$, and $D_2|H$, then*

$$\sum_{D|H} \eta_k \left(\left(\frac{H}{D} \right)^k, D_1 \right) \eta_k \left(\left(\frac{H}{D} \right)^k, D \right) = \begin{cases} |H|^k, & \text{if } D_1 = D_2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

Proof By Lemma 4.2, $A \bmod H^k$ is given by $A = R\left(\frac{H}{D}\right)^k$, where D ranges over the divisors of H, and for each D, R ranges over a k-reduced residue system modulo D. Therefore, by (4.5), the left side of (4.1) may be written as

$$\sum_{A \bmod H^k} \eta_k(A, D_1) \eta_k(G - A, D_2)$$

$$= \sum_{D|H} \eta_k \left(\left(\frac{H}{D} \right)^k, D_1 \right) \sum_{\substack{R \bmod D^k \\ (R, D^k)_k = 1}} \eta_k \left(G - R \left(\frac{H}{D} \right)^k, D_2 \right). \tag{4.10}$$

We consider the inner sum of the right side of (4.10) separately, and denote this sum by S, then

$$S = \sum_{\substack{R \bmod D^k \\ (R,D^k)_k=1}} \sum_{\substack{A \bmod D_2^k \\ (A,D_2^k)_k=1}} E\left(G - R\left(\frac{H}{D}\right)^k, D_2^k\right)(A)$$

$$= \sum_{\substack{A \bmod D_2^k \\ (A,D_2^k)_k=1}} E(G, D_2^k)(A) \sum_{\substack{R \bmod D^k \\ (R,D^k)_k=1}} E\left(-A\left(\frac{H}{D}\right)^k, D_2^k\right)(R). \qquad (4.11)$$

We write $D_2 \cdot B = H$, by (2.6) of Lemma 2.2, then

$$E(-A\left(\frac{H}{D}\right)^k, D_2^k) = E\left(-A\left(\frac{H}{D}\right)^k B^k, H^k\right)$$

$$= E\left(-A\left(\frac{H}{D_2}\right)^k \left(\frac{H}{D}\right)^k, H^k\right)$$

$$= E\left(-A\left(\frac{H}{D_2}\right)^k, D^k\right). \qquad (4.12)$$

It follows that

$$S = \sum_{\substack{A \bmod D_2^k \\ (A,D_2^k)_k=1}} E(G, D_2^k)(A) \sum_{\substack{R \bmod D^k \\ (R,D^k)_k=1}} E\left(-A\left(\frac{H}{D_2}\right)^k, D^k\right)(R)$$

$$= \sum_{\substack{A \bmod D_2^k \\ (A,D_2^k)_k=1}} E(G, D_2^k)(A)\eta_k\left(-A\left(\frac{H}{D_2}\right)^k, D\right)$$

$$= \sum_{\substack{A \bmod D_2^k \\ (A,D_2^k)_k=1}} E(G, D_2^k)(A)\eta_k\left(\left(\frac{H}{D_2}\right)^k, D\right)$$

$$= \eta_k(G, D_2)\eta_k\left(\left(\frac{H}{D_2}\right)^k, D\right). \qquad (4.13)$$

By Lemma 4.1 and (4.10), we have

$$\eta_k(G, D_2) \sum_{D|H} \eta_k\left(\left(\frac{H}{D}\right)^k, D_1\right)\eta_k\left(\left(\frac{H}{D_2}\right)^k, D\right)$$

$$= \begin{cases} |H|^k\eta_k(G, D), & \text{if } D_1 = D_2 = D, \\ 0, & \text{otherwise.} \end{cases} \qquad (4.14)$$

If we take $G = 0$ in the above equality, and note that $\eta_k(0, D_2) = \phi_k(D_2)$ (see (2.27)), then $\eta_k(0, D_2) \neq 0$, and we have

$$\sum_{D|H} \eta_k\left(\left(\frac{H}{D}\right)^k, D_1\right) \eta_k\left(\left(\frac{H}{D_2}\right)^k, D\right) = \begin{cases} |H|^k, & \text{if } D_1 = D_2 = D, \\ 0, & \text{otherwise.} \end{cases}$$

(4.15)

We complete the proof of Theorem 4.1. □

Next, we deduce a number of the arithmetical relations, most of which are the straightforward consequences of (4.1) and (4.9).

Corollary 4.2

$$\sum_{G \bmod H^k} \eta_k(G, H) = \begin{cases} 1, & \text{if } \deg(H) = 0, \\ 0, & \text{if } \deg(H) \geq 1. \end{cases}$$

(4.16)

Proof Let $D_1 = H$, and $D_2 = 1$ in (4.1), and we have this corollary at once. □

Corollary 4.3

$$\sum_{D|H} \eta_k(G, D) = \begin{cases} |H|^k, \text{if } G \equiv 0 \pmod{H^k}, \\ 0, \quad \text{otherwise.} \end{cases}$$

(4.17)

Proof Taking $D_1 = 1$ in Theorem 4.1, we have (4.17) immediately. □

Corollary 4.4

$$\sum_{D|H} \eta_k\left(\left(\frac{H}{D}\right)^k, H\right) \eta_k(G, D) = \begin{cases} |H|^k, \text{if}(G, H^k)_k = 1, \\ 0, \quad \text{otherwise.} \end{cases}$$

(4.18)

Proof Let $D_1 = H$ in (4.9), and by (4.5) we have this corollary. □

Corollary 4.5 *If $D_1|H$, then we have*

$$\sum_{D|H} \eta_k\left(\left(\frac{H}{D}\right)^k, D_1\right) \phi_k(D) = \begin{cases} |H|^k, \text{if } D_1 = 1, \\ 0, \quad \text{otherwise.} \end{cases}$$

(4.19)

Proof If we take $D_2 = 1$ in (4.9), and note that $\eta_k(H^k, D) = \phi_k(D)$, then (4.19) follows immediately. □

Using (4.3) with $R = 1$, we may reformulate Theorem 4.1 as follows.

Corollary 4.6 *If $D_1|H$ and $D_2|H$, then*

$$\sum_{D|H} \frac{1}{\phi_k(D)} \eta_k\left(\left(\frac{H}{D_1}\right)^k, D\right) \eta_k\left(\left(\frac{H}{D_2}\right)^k, D\right) = \begin{cases} \frac{|H|^k}{\phi_k(D')}, \text{if } D_1 = D_2 = D', \\ 0, \quad \text{otherwise.} \end{cases}$$

(4.20)

In particular, for any $D_1|H$, we have

$$\sum_{D|H} \frac{1}{\phi_k(D)} \eta_k \left(\left(\frac{H}{D_1} \right)^k, D \right)^2 = \frac{|H|^k}{\phi_k(D_1)}. \tag{4.21}$$

If we make use of the Hölder formula in the above equality, it yields

Corollary 4.7 *If $D_1 D_1' = H$, then*

$$\sum_{\substack{D|H \\ N(D_1',D)=D}} \phi_k(D) \left(\frac{\mu(N)}{\phi_k(N)} \right)^2 = \frac{|H|^k}{\phi_k(D_1)}. \tag{4.22}$$

The special cases of $D_1 = 1$ and $D_1 = H$ lead to the following corollaries, respectively

Corollary 4.8 (see (2.14))

$$\sum_{D|H} \phi_k(D) = |H|^k \tag{4.23}$$

and

Corollary 4.9

$$\sum_{D|H} \frac{|\mu(D)|}{\phi_k(D)} = \frac{|H|^k}{\phi_k(H)}. \tag{4.24}$$

In fact, we have the following more generalized conclusions.

Lemma 4.5 *If $R|H$, then we have*

$$\alpha_k(R, H) = \sum_{\substack{D|\frac{H}{R} \\ (D,R)=1}} \frac{|\mu(D)|}{\phi_k(D)} = \frac{\phi_k(R)}{\phi_k(H)} \left| \frac{H}{R} \right|^k. \tag{4.25}$$

Proof The left side of (4.25) has the following product expression:

$$\alpha_k(R, H) = \prod_{\substack{P|\frac{H}{R} \\ P \nmid R}} \left(1 + \frac{1}{\phi_k(P)} \right)$$

$$= \prod_{P|H} \left(1 + \frac{1}{\phi_k(P)} \right) \prod_{P|R} \left(1 + \frac{1}{\phi_k(P)} \right)^{-1}$$

$$= \frac{\phi_k(R)}{\phi_k(H)} \left| \frac{H}{R} \right|^k,$$

and the lemma follows. $\square$

Next, we generalize the Jordan totient function $\phi_k(H)$ to $\phi_s(H)$, where s is a complex number variable. We define (see (1.5) in the rational case)

$$\phi_s(H) = \sum_{D|H} |D|^s \mu\left(\frac{H}{D}\right) = |H|^s \prod_{P|H}\left(1 - \frac{1}{|P|^s}\right).\qquad(4.26)$$

Lemma 4.6 *If s is an arbitrary complex number, then*

$$\sum_{D|H} \eta_k\left(\left(\frac{H}{D}\right)^k, H\right)\phi_{ks}(D) = |H|^{ks}\phi_{k(1-s)}(H).\qquad(4.27)$$

Proof By (2.25) and (4.26), we have

$$\sum_{D|H} \eta_k\left(\left(\frac{H}{D}\right)^k, H\right)\phi_{ks}(D) = \phi_k(H)\sum_{D|H}\frac{\mu(D)}{\phi_k(D)}\phi_{ks}(D)$$

$$= \phi_k(H)\sum_{D|H}\frac{\mu(D)}{\phi_k(D)}\sum_{R|D}|R|^{ks}\mu\left(\frac{D}{R}\right)$$

$$= \phi_k(H)\sum_{D|H}\sum_{\substack{R\cdot E=D\\(R,E)=1}}\frac{\mu^2(E)|R|^{ks}\mu(R)}{\phi_k(R)\phi_k(E)}$$

$$= \phi_k(H)\sum_{R|H}\frac{|R|^{ks}\mu(R)}{\phi_k(R)}\sum_{\substack{E|\frac{H}{R}\\(E,R)=1}}\frac{|\mu(E)|}{\phi_k(E)}$$

$$= |H|^{ks}\phi_{k(1-s)}(H),\qquad(4.28)$$

and the lemma follows. □

We may obtain more orthogonal relation formulas from (4.1) and (4.9). For example, if we take $G = 0$ in (4.1), and note that $\eta_k(-B, D_2) = \eta_k(B, D_2)$, if follows that

Corollary 4.10 *If $D_1|H$ and $D_2|H$, then*

$$\sum_{A \bmod H^k}\eta_k(A, D_1)\eta_k(A, D_2) = \begin{cases} |H|^k\phi_k(D), & \text{if } D_1 = D_2 = D,\\ 0, & \text{otherwise.} \end{cases}\qquad(4.29)$$

In particular, if $D|H$, then

$$\sum_{A \bmod H^k}\eta_k(A, D)^2 = |H|^k\phi_k(D).\qquad(4.30)$$

5 The series $\delta_k(s, G)$

In this section, we prove Theorem 1.1. Recall that the Dirichlet series $\delta_k(s, G)$ is given by (see (1.18))

$$\delta_k(s, G) = \sum_{H \in \mathbb{A}} \frac{\eta_k(G, H)}{|H|^s}. \tag{5.1}$$

Lemma 5.1 *If* $\mathrm{Re}(s) > k + 1$, $G \neq 0$, *then we have*

$$\delta_k(s, G) = (1 - q^{1-s}) \sum_{D^k | G} |D|^{k-s}. \tag{5.2}$$

Proof Since $\mathrm{Re}(s) > k + 1$, (5.1) is absolutely convergent, and thus by (2.22) we have

$$
\begin{aligned}
\delta_k(s, G) &= \sum_{H \in \mathbb{A}} \frac{1}{|H|^s} \sum_{D | H, D^k | G} |D|^k \mu\left(\frac{H}{D}\right) \\
&= \sum_{D^k | G} |D|^k \sum_{\substack{H \in \mathbb{A} \\ D | H}} \frac{\mu\left(\frac{H}{D}\right)}{|H|^s} \\
&= \sum_{D^k | G} |D|^{k-s} \sum_{H \in \mathbb{A}} \frac{\mu(H)}{|H|^s}.
\end{aligned} \tag{5.3}
$$

If $\mathrm{Re}(s) > 1$, we have (see [45, Proposition 2.6])

$$\sum_{H \in \mathbb{A}} \frac{\mu(H)}{|H|^s} = \zeta_{\mathbb{A}}^{-1}(s) = (1 - q^{1-s}), \tag{5.4}$$

and the lemma follows at once. $\square$

Proof of Theorem 1.1 We first show that the series $\sum_{n=0}^{+\infty} A(n) q^{-ns}$ in (1.18) converges for all s to an entire function, in fact a polynomial in q^{-s}. We let

$$A(n) = \sum_{\substack{H \in \mathbb{A} \\ \deg(H) = n}} \eta_k(G, H). \tag{5.5}$$

By definition (1.18), we have

$$\delta_k(s, G) = \sum_{n=0}^{+\infty} A(n) q^{-ns} = \sum_{n=0}^{+\infty} A(n) u^n, \tag{5.6}$$

where $u = q^{-s}$. On the other hand, if $G \neq 0$, and $\mathrm{Re}(s) > k + 1$, by Lemma 5.1,

$$\delta_k(s, G) = (1 - q^{1-s}) \sum_{D^k | G} q^{dk} q^{-ds}$$

$$= (1 - qu) \sum_{D^k | G} q^{dk} u^d$$

$$= \sum_{D^k | G} q^{dk} u^d - \sum_{D^k | G} q^{dk+1} u^{d+1}, \tag{5.7}$$

where $d = \deg(D)$. We set

$$\gamma(G, n) = \sum_{\substack{D^k | G \\ \deg(D) = n}} 1. \tag{5.8}$$

By (5.7), we have

$$\delta_k(s, G) = \sum_{n=0}^{+\infty} q^{nk} \gamma(G, n) u^n - \sum_{n=0}^{+\infty} q^{nk+1} \gamma(G, n) u^{n+1}$$

$$= \sum_{n=0}^{+\infty} q^{nk} \gamma(G, n) u^n - \sum_{n=1}^{+\infty} q^{(n-1)k+1} \gamma(G, n - 1) u^n$$

$$= \sum_{n=0}^{+\infty} q^{nk} \left(\gamma(G, n) - q^{1-k} \gamma(G, n - 1) \right) u^n. \tag{5.9}$$

Comparing the coefficients of u^n of (5.6) and (5.9), we have

$$A(n) = q^{nk} \left(\gamma(G, n) - q^{1-k} \gamma(G, n - 1) \right). \tag{5.10}$$

By the definition of $\gamma(G, n)$, if $n - 1 > \frac{\deg(G)}{k}$, it is easy to see that $\gamma(G, n) = \gamma(G, n - 1) = 0$, and it follows that

$$A(n) = 0, \text{ whenever } n - 1 > \frac{\deg(G)}{k}. \tag{5.11}$$

We note that the definition of $A(n)$ is independent on the choice of s, and thus for any s, we have

$$\delta_k(s, G) = \sum_{0 \leq n \leq \frac{\deg(G)}{2} + 1} A(n) u^n = \sum_{\substack{H \in \mathbb{A} \\ \deg(H) \leq \frac{\deg(G)}{2} + 1}} \frac{\eta_k(G, H)}{|H|^s}, \tag{5.12}$$

which indicates that $\delta_k(s, G)$ is, indeed, a finite summand; therefore, $\delta_k(s, G)$ is an entire function, and on the whole complex plane, we have

$$\delta_k(s, G) = (1 - q^{1-s}) \sum_{D^k | G} |D|^{k-s}. \tag{5.13}$$

In particular, if $s = 1$, then we have

$$\delta_k(1, G) = \sum_{H \in \mathbb{A}} \frac{\eta_k(G, H)}{|H|} = 0.$$

To complete the proof of Theorem 1.1, next we show the square mean values estimate (1.22). If c and T are two given real numbers and $T > 0$, by (5.13), we have

$$\int_{-T}^{T} |\delta_k(c + it, G)|^2 dt = \sum_{D_1^k | G, D_2^k | G} q^{(d_1 + d_2)(k-c)}$$

$$\times \int_{-T}^{T} (1 - q^{1-c-it})(1 - q^{1-c+it}) q^{(d_2 - d_1)it} dt, \quad (5.14)$$

where $d_1 = \deg(D_1)$ and $d_2 = \deg(D_2)$. We denote the inner integral of (5.14) by $S(d_1, d_2)$,

$$S(d_1, d_2) = \int_{-T}^{T} (1 - q^{1-c-it})(1 - q^{1-c+it}) q^{(d_2 - d_1)it} dt. \tag{5.15}$$

Making substitution $u = q^{it}$, it follows that

$$S(d_1, d_2) = \frac{1}{i \log q} \int_{q^{-iT}}^{q^{iT}} \left(1 + q^{2(1-c)} - q^{1-c} u^{-1} - q^{1-c} u\right) u^{d_2 - d_1 - 1} du. \tag{5.16}$$

Let $n = d_2 - d_1 - 1$, if $n \neq -2, -1, 0$, then

$$S(d_1, d_2) = \frac{2}{\log q} \left[\frac{1 + q^{2(1-c)}}{n + 1} \left(q^{i(n+1)T} - q^{-i(n+1)T}\right) \right.$$

$$\left. - \frac{q^{1-c}}{n + 2} \left(q^{i(n+2)T} - q^{-i(n+2)T}\right) - \frac{q^{1-c}}{n} \left(q^{inT} - q^{-inT}\right) \right],$$

which yields the following estimate:

$$|S(d_1, d_2)| \leq \frac{4}{\log q} \left(\frac{1 + q^{2(1-c)}}{n + 1} + \frac{q^{1-c}}{n + 2} + \frac{q^{1-c}}{n} \right). \tag{5.17}$$

If $n = d_2 - d_1 - 1 = 0$, then $d_2 = 1 + d_1$, by (5.16), we have

$$
\begin{aligned}
S(d_1, d_2) &= \frac{1}{i \log q} \left[\left(1 + q^{2(1-c)}\right)\left(q^{iT} - q^{-iT}\right) - 2iTq^{1-c} \log q \right. \\
&\quad \left. - \frac{q^{1-c}}{2}\left(q^{2iT} - q^{-2iT}\right) \right] \\
&= -2Tq^{1-c} + O(1).
\end{aligned}
\tag{5.18}
$$

If $n = d_2 - d_1 - 1 = -1$, then $d_2 = d_1$, and

$$
\begin{aligned}
S(d_1, d_2) &= \frac{1}{i \log q} \left[2iT\left(1 + q^{2(1-c)}\right) \log q - 2q^{1-c}\left(q^{iT} - q^{-iT}\right) \right] \\
&= 2T\left(1 + q^{2(1-c)}\right) + O(1).
\end{aligned}
\tag{5.19}
$$

If $n = d_2 - d_1 - 1 = -2$, we also have

$$
S(d_1, d_2) = -2Tq^{1-c} + O(1).
\tag{5.20}
$$

Putting the above equalities together, by (5.14), we have

$$
\begin{aligned}
\frac{1}{2T} \int_{-T}^{T} |\delta_k(c + it, G)|^2 dt &= \left(1 + q^{2(1-c)}\right) \sum_{\substack{D_1^k | G, D_2^k | G \\ d_1 = d_2 = d}} q^{2d(k-c)} - 2q^{1-c} \\
&\quad \times \sum_{\substack{D_1^k | G, D_2^k | G \\ d_1 = d_2 + 1}} q^{(d_1 + d_2)(k-c)} + O\left(\frac{1}{T}\right) \\
&= \left(1 + q^{2(1-c)}\right) \sum_{\substack{D_1^k | G, D_2^k | G \\ d_1 = d_2 = d}} |D_1|^{2(k-c)} - 2q^{1-k} \\
&\quad \times \sum_{\substack{D_1^k | G, D_2^k | G \\ d_1 = d_2 + 1}} |D_1|^{2(k-c)} + O\left(\frac{1}{T}\right).
\end{aligned}
$$

Let

$$
\sigma_0(x, G) = \sum_{\substack{D_1^k | G, D_2^k | G \\ d_1 = d_2}} |D_1|^x \quad \text{and} \quad \sigma_1(x, G) = \sum_{\substack{D_1^k | G, D_2^k | G \\ d_1 = d_2 + 1}} |D_1|^x,
\tag{5.21}
$$

then we finally obtain

$$
\frac{1}{2T} \int_{-T}^{T} |\delta_k(c + it, G)|^2 dt = \left(1 + q^{2(1-c)}\right)\sigma_0(2(k-c)) - 2q^{1-k}\sigma_1(2(k-c)) + O\left(\frac{1}{T}\right).
$$

We complete the proof of Theorem 1.1.

6 The series $\tau_k(s, H)$

In this section, we prove Theorem 1.2. Recalling the Dirichlet series $\tau_k(s, H)$ is given by (see (1.18')) that

$$\tau_k(s, H) = \sum_{G \in \mathbb{A}} \frac{\eta_k(G, H)}{|G|^s}. \tag{6.1}$$

This series is more complicated than $\delta_k(s, G)$, because that $\eta_k(G, H)$ is not multiplicative in G, and thus we cannot make use of Euler product directly. Our treatment begins with the following auxiliary series:

$$\tau_k'(s, H) = \sum_{G \in \mathbb{A}} \frac{\chi(G)}{|G|^s}, \tag{6.2}$$

where $\chi(G) = 1$, if $(G, H^k)_k = 1$, and $\chi(G) = 0$, if $(G, H^k)_k > 1$. Since $\chi(G)$ is a multiplicative function in G, we have the following equality by using Euler product (if $\text{Re}(s) > 1$):

$$
\begin{aligned}
\tau_k'(s, H) &= \prod_{P \in \mathbb{A}} (1 - |P|^{-s})^{-1} \prod_{P | H} (1 - |P|^{-ks}) \\
&= \zeta_{\mathbb{A}}(s) \phi_{ks}(H) |H|^{-ks}.
\end{aligned} \tag{6.3}
$$

First, we show a precise analogue of Theorem 13 of [12] with the following lemma.

Lemma 6.1 *If H is positive polynomial and $\text{Re}(s) > k + 1$, then*

$$\tau_k(s, H) = (1 - q^{1-s})^{-1} \phi_{k(1-s)}(H). \tag{6.4}$$

Proof Let $A^k = (G, H^k)_k$. By (3.23), we have $\eta_k(G, H) = \eta_k(A^k, H)$, and it follows that

$$
\begin{aligned}
\tau_k(s, H) &= \sum_{A | H} \eta_k(A^k, H) \sum_{\substack{G \in \mathbb{A} \\ A^k = (G, H^k)_k}} \frac{1}{|G|^s} \\
&= \sum_{A | H} \frac{\eta_k(A^k, H)}{|A|^{sk}} \sum_{\substack{G_1 \in \mathbb{A} \\ (G_1, (\frac{H}{A})^k)_k = 1}} \frac{1}{|G_1|^s} \\
&= \zeta_{\mathbb{A}}(s) |H|^{-ks} \sum_{A | H} \eta_k(A^k, H) \phi_{ks}\left(\frac{H}{A}\right).
\end{aligned}
$$

By Lemma 4.6, we have

$$\tau_k(s, H) = \zeta_{\mathbb{A}}(s) \phi_{k(1-s)}(H) = (1 - q^{1-s})^{-1} \phi_{k(1-s)}(H).$$

We complete the proof of Lemma 6.1. $\square$

Proof of Theorem 1.2 We now return to the proof of Theorem 1.2. For any integer $n \geq 0$, we let

$$B(n) = \sum_{\substack{G \in \mathbb{A} \\ \deg(G)=n}} \eta_k(G, H). \tag{6.5}$$

By definition (1.18'), we have

$$\tau_k(s, H) = \sum_{n=0}^{+\infty} B(n) q^{-ns} = \sum_{n=0}^{+\infty} B(n) u^n, \tag{6.6}$$

where $u = q^{-s}$. If $\mathrm{Re}(s) > k + 1$, by Lemma 6.1

$$\tau_k(s, H) = (1 - q^{1-s})^{-1} \phi_{k(1-s)}(H) = (1 - q^{1-s})^{-1} \sum_{D|H} |D|^{k(1-s)} \mu\left(\frac{H}{D}\right)$$

$$= (1 - qu)^{-1} \sum_{D|H} \mu\left(\frac{H}{D}\right) q^{dk} u^{dk}, \tag{6.7}$$

where $u = q^{-s}$ and $d = \deg(D)$. Because of $|qu| < 1$, then $(1 - qu)^{-1}$ has a geometric series expression, we have

$$\tau_k(s, H) = \sum_{D|H} \mu\left(\frac{H}{D}\right) \sum_{n=0}^{+\infty} q^{n+dk} u^{n+dk}$$

$$= \sum_{D|H} \mu\left(\frac{H}{D}\right) \sum_{n=dk}^{+\infty} q^n u^n. \tag{6.8}$$

We set

$$J_k(D, n) = \begin{cases} 1, & \text{if } n \geq k \deg(D), \\ 0, & \text{if } n < k \deg(D). \end{cases} \tag{6.9}$$

It follows from (6.8) that

$$\tau_k(s, H) = \sum_{n=0}^{+\infty} \left(\sum_{D|H} \mu\left(\frac{H}{D}\right) J_k(D, n) \right) q^n u^n. \tag{6.10}$$

Comparing coefficients of u^n of (6.6) and (6.10), we have

$$B(n) = \sum_{D|H} \mu\left(\frac{H}{D}\right) J_k(D, n) \cdot q^n. \tag{6.11}$$

If $n \geq k \deg(H)$, then $J_k(D, n) = 1$ for all of D that $D|H$, and hence

$$B(n) = q^n \sum_{D|H} \mu\left(\frac{H}{D}\right) = 0. \tag{6.12}$$

The last equality of (6.12) follows from (2.12) and $\deg(H) \geq 1$. We note that $B(n)$ is independent on the choice of complex number s by the definition of $B(n)$, and therefore, for any complex number s, we have $B(n) = 0$, whenever $n \geq k \cdot \deg(H)$. It follows from (6.6) that

$$\tau_k(s, H) = \sum_{n \leq k \deg(H)} B(n)u^n = \sum_{\substack{G \in \mathbb{A} \\ \deg(G) \leq k \cdot \deg(H)}} \frac{\eta_k(G, H)}{|G|^s}, \qquad (6.13)$$

which indicates that $\tau_k(s, H)$ is, indeed, a finite summand, and thus, $\tau_k(s, H)$ is an entire function on the whole complex plane. Moreover, by the principle of analytic continuation, on the whole complex plane, we have

$$\tau_k(s, H) = (1 - q^{1-s})^{-1}\phi_{k(1-s)}(H). \qquad (6.14)$$

In particular, if $s = 1$, by L'Hôpital's rule we have

$$\begin{aligned}
\tau_k(1, H) &= \lim_{s \to 1}(1 - q^{1-s})^{-1}\phi_{k(1-s)}(H) \\
&= -\frac{k \sum_{D|H} \log |D| \mu\left(\frac{H}{D}\right)}{\log q} = \frac{-k\Lambda(H)}{\log q},
\end{aligned} \qquad (6.15)$$

which is the equality (1.28), and is an analogue of Ramanujan's identity (1.2).

In order to complete the proof of Theorem 1.2, it remains to prove the square mean value estimate. If T and c are any real numbers that $T > 0$, $c \neq 1$, by (6.14) we have

$$\begin{aligned}
\int_{-T}^{T} |\tau_k(c + it, H)|^2 \mathrm{d}t &= \sum_{D_1|H, D_2|H} \mu\left(\frac{H}{D_1}\right)\mu\left(\frac{H}{D_2}\right)|D_1 D_2|^{k(1-c)} \\
&\quad \times \int_{-T}^{T} \frac{q^{k(d_2-d_1)it}\mathrm{d}t}{(1 - q^{1-c-it})(1 - q^{1-c+it})}.
\end{aligned} \qquad (6.16)$$

We denote

$$S_1(d_1, d_2) = \int_{-T}^{T} \frac{q^{k(d_2-d_1)it}\mathrm{d}t}{(1 - q^{1-c-it})(1 - q^{1-c+it})}, \qquad (6.17)$$

and make the substitution of $u = -t$, it follows that

$$S_1(d_1, d_2) = \int_{-T}^{T} \frac{q^{k(d_1-d_2)it}\mathrm{d}t}{(1 - q^{1-c-it})(1 - q^{1-c+it})}, \qquad (6.18)$$

which shows that d_1 and d_2 are symmetric in (6.16). Therefore, we may suppose that $d_2 \geq d_1$, and make the substitution $u = q^{it}$ in (6.17), then

$$S_1(d_1, d_2) = \frac{i}{q^{1-c} \log q} \int_{q^{-iT}}^{q^{iT}} \frac{u^{k(d_2-d_1)}\mathrm{d}u}{(u - q^{1-c})(u - q^{c-1})}. \qquad (6.19)$$

If $d_1 = d_2$, then we have

$$S_1(d_1, d_2) = \frac{i}{(q^{2(1-c)} - 1) \log q} \Big[\log \left(q^{iT} - q^{1-c} \right) - \log \left(q^{-iT} - q^{1-c} \right)$$
$$- \log \left(q^{iT} - q^{c-1} \right) + \log \left(q^{-iT} - q^{c-1} \right) \Big]. \tag{6.20}$$

We note that $1 - c$ and $c - 1$ are symmetric in the above: first we let $c > 1$, and then

$$\log(q^{iT} - q^{1-c}) - \log(q^{-iT} - q^{1-c})$$
$$= 2iT \log q + \log(1 - q^{-iT} q^{1-c}) - \log(1 - q^{iT} q^{1-c}). \tag{6.21}$$

Let $z = q^{1-c} q^{\pm iT}$, then $|z| = q^{1-c} < 1$ for $c > 1$, and we have the following power series expansion that

$$- \log(1 - z) = z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \cdots .$$

It follows that

$$| \log(1 - z)| \le |z| + \frac{1}{2} |z|^2 + \cdots \le \log \frac{1}{1 - |z|} = \log \frac{1}{1 - q^{1-c}}, \tag{6.22}$$

and by (6.21), we have

$$\log(q^{iT} - q^{1-c}) - \log(q^{-iT} - q^{1-c}) = 2iT \log q + O(1). \tag{6.23}$$

The remaining part of (6.20) is

$$\log(q^{iT} - q^{c-1}) - \log(q^{iT} - q^{c-1}) = \log \frac{1 - q^{1-c} q^{iT}}{1 - q^{1-c} q^{-iT}}$$
$$= \log(1 - q^{1-c} q^{iT}) - \log(1 - q^{1-c} q^{-iT}).$$

By (6.22), we have

$$\left| \log \left(1 - q^{1-c} q^{iT} \right) - \log \left(1 - q^{1-c} q^{-iT} \right) \right| \le 2 \log \frac{1}{1 - q^{1-c}}. \tag{6.24}$$

Hence, if $d_2 = d_1$, and $c > 1$, we obtain

$$S_1(d_1, d_2) = \frac{2T}{1 - q^{2(1-c)}} + O(1). \tag{6.25}$$

If $d_1 = d_2$, and $c < 1$, the same method yields the following estimate

$$S_1(d_1, d_2) = \frac{2T}{q^{2(1-c)} - 1} + O(1). \tag{6.26}$$

Therefore, if $d_1 = d_2$, we have

$$S_1(d_1, d_2) = \frac{2T}{|1 - q^{2(1-c)}|} + O(1). \tag{6.27}$$

Next, we consider $d_2 > d_1$, and let $n = (d_2 - d_1)k$. By (6.19), then

$$S_1(d_1, d_2) = \frac{i}{(q^{2(1-c)} - 1) \log q} \left[\int_{q^{-iT} - q^{1-c}}^{q^{iT} - q^{1-c}} \frac{(u + q^{1-c})^n}{u} du \right.$$
$$\left. - \int_{q^{-iT} - q^{c-1}}^{q^{iT} - q^{c-1}} \frac{(u + q^{c-1})^n}{u} du \right]. \tag{6.28}$$

We write

$$(u + q^{\pm(1-c)})^n = \sum_{j=0}^{n} \binom{n}{j} u^j q^{\pm(1-c)(n-j)}. \tag{6.29}$$

If $j \neq 0$, then it is easy to verify that the inner integral in (6.28) is $O(1)$. if $j = 0$, there is a similar argument like the case of $d_1 = d_2$, which yields

$$S_1(d_1, d_2) = \frac{q^{k(d_2-d_1)(1-c)}}{|1 - q^{2(1-c)}|} 2T + O(1). \tag{6.30}$$

By (6.16), we finally obtain

$$\frac{1}{2T} \int_{-T}^{T} |\tau_k(c + it, H)|^2 dt$$
$$= \frac{1}{|1 - q^{2(1-c)}|} \sum_{\substack{D_1|H, D_2|H \\ \deg(D_1) = \deg(D_2)}} \mu\left(\frac{H}{D_1}\right) \mu\left(\frac{H}{D_2}\right) |D_1|^{2(1-c)}$$
$$+ \frac{2}{|1 - q^{2(1-c)}|} \sum_{\substack{D_1|H, D_2|H \\ \deg(D_1) > \deg(D_2)}} \left(\mu\left(\frac{H}{D_1}\right) \mu\left(\frac{H}{D_2}\right)\right.$$
$$\left. \times |D_1|^{(k+2)(1-c)} |D_2|^{(2-k)(1-c)} \right) + O\left(\frac{1}{T}\right). \tag{6.31}$$

We complete the proof of Theorem 1.2.

7 Davenport–Hasse type formula

The polynomial Ramanujan sum $\eta(G, H)$ essentially is a special type of Gauss sum on $\mathbb{F}_q[x]$. Let χ be a multiplicative character modulo H on $\mathbb{F}_q[x]$, and $\psi_G = E(G, H)$ be the additive character modulo H given by (1.8); the Gauss sum $G(\chi, \psi_G)$ modulo

H on $\mathbb{F}_q[x]$ is defined by

$$G(\chi, \psi_G) = \sum_{D \bmod H} \chi(D)\psi_G(D), \tag{7.1}$$

where D extends over a complete residue system modulo H in $\mathbb{F}_q[x]$. Let χ_0 be the principal multiplicative character: it is $\chi_0(D) = 1$ if $(D, H) = 1$, and $\chi_0(D) = 0$ if $(D, H) > 1$, and then we see that $\eta(G, H) = G(\chi_0, \psi_G)$.

In an upcoming paper [51], we presented an analogue of Davenport–Hasse's theorem for the polynomial Gauss sums (see [51, Theorem 1.3]). To state this result, let $\mathbb{F}_{q^n}/\mathbb{F}_q$ be a finite extension over $\mathbb{F}_q$ of degree n, and $\mathrm{tr}(a)$ and $N(a)$ be the trace map and norm from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$, respectively,

$$\mathrm{tr}(a) = \sum_{i=1}^{n} \sigma^i(a) \quad \text{and} \quad N(a) = \prod_{i=1}^{n} \sigma^i(a), \tag{7.2}$$

where $\sigma(a) = a^q$ for a in $\mathbb{F}_{q^n}$ is the Frobenius of $\mathbb{F}_{q^n}$. If A is a polynomial in $\mathbb{F}_{q^n}[x]$, $A = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$, the trace map and norm can be extended to $\mathbb{F}_{q^n}[x]$ by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \sigma^i(A) \quad \text{and} \quad N(A) = \prod_{i=1}^{n} \sigma^i(A), \tag{7.3}$$

where $\sigma(A) = \sum_{i=0}^{k} \sigma(a_i)x^i$.

Let H be a polynomial in $\mathbb{F}_q[x]$ and, therefore, also a polynomial in $\mathbb{F}_{q^n}[x]$. To define a Gauss sum modulo H on $\mathbb{F}_{q^n}[x]$, for any A in $\mathbb{F}_{q^n}[x]$, we set

$$\psi_G^{(n)}(A) = \psi_G(\mathrm{tr}(A)) \quad \text{and} \quad \chi^{(n)}(A) = \chi(N(A)), \tag{7.4}$$

and thus, the Gauss sums $G(\chi^{(n)}, \psi_G^{(n)})$ modulo H on $\mathbb{F}_{q^n}[x]$ is given by

$$\psi(\chi^{(n)}, \psi_G^{(n)}) = \sum_{\substack{D \in \mathbb{F}_{q^n}[x] \\ D \bmod H}} \chi^{(n)}(D)\psi_G^{(n)}(D), \tag{7.5}$$

where the summation extends over a complete residue system modulo H in $\mathbb{F}_{q^n}[x]$.

By the above notations, we may define a polynomial Ramanujan sum $\eta^{(n)}(G, H)$ modulo H on $\mathbb{F}_{q^n}[x]$ by

$$\eta^{(n)}(G, H) = \sum_{\substack{D \bmod H \\ D \in \mathbb{F}_{q^n}[x]}} \chi_0^{(n)}(D)\psi_G^{(n)}(D), \tag{7.6}$$

and a generalized version $\eta_k^{(n)}(G, H)$ by

$$\eta_k^{(n)}(G, H) = \sum_{\substack{D \bmod H^k \\ (D, H^k)_k = 1}} \psi_G^{(n)}(D), \tag{7.7}$$

where the summation ranges over a complete residue system modulo H^k in $\mathbb{F}_{q^n}[x]$.

If χ and ψ_G are not both principal, in [51] we showed the following Davenport–Hasse type formula:

$$(-1)^{m-m_1} \frac{\phi^{(n)}(N)}{\phi^{(n)}(H)} G(\chi^{(n)}, \psi_G^{(n)}) = \left((-1)^{m-m_1} \frac{\phi(N)}{\phi(H)} G(\chi, \psi_G) \right)^n, \tag{7.8}$$

where $\phi(H)$ is the Euler totient function on $\mathbb{F}_q[x]$, $\phi^{(n)}(H)$ is the function on $\mathbb{F}_{q^n}[x]$, $N = \frac{H}{(G,H)}$, $m = \deg(H)$, and $m_1 = \deg(G, H)$.

As a direct consequence of (7.8), if $H \nmid G$, then ψ_G is not principal, and we have

$$(-1)^{m-m_1} \frac{\phi^{(n)}(N)}{\phi^{(n)}(H)} \eta^{(n)}(G, H) = \left((-1)^{m-m_1} \frac{\phi(N)}{\phi(H)} \eta(G, H) \right)^n. \tag{7.9}$$

The main purpose of this section is to show that the generalized version $\eta_k(H, G)$ also shares this kind of Davenport–Hasse type formula. We have

Theorem 7.1 *If H and G are any polynomials in $\mathbb{F}_q[x]$ such that $H^k \nmid G$ and $H \neq 0$, then*

$$(-1)^{m-m_1} \frac{\phi_k^{(n)}(N)}{\phi_k^{(n)}(H)} \eta_k^{(n)}(G, H) = \left((-1)^{m-m_1} \frac{\phi_k(N)}{\phi_k(H)} \eta_k(G, H) \right)^n, \tag{7.10}$$

where $\phi_k(H)$ is the Jordan totient function on $\mathbb{F}_q[x]$, and $\phi_k^{(n)}(H)$ is the function on $\mathbb{F}_{q^n}[x]$, $N = \frac{H}{A}$, $A^k = (G, H^k)_k$, $m = \deg(H)$, and $m_1 = \deg(A)$.

Proof If $A^k = (G, H^k)_k$ in $\mathbb{F}_q[x]$, it is easy to verify that $A^k = (G, H^k)_k$ holds in $\mathbb{F}_{q^n}[x]$. By (2.25), we have

$$\eta_k^{(n)}(G, H) = \phi_k^{(n)}(H) \mu^{(n)}(N) \left(\phi_k^{(n)}(N) \right)^{-1}, \tag{7.11}$$

and

$$\eta_k(G, H) = \phi_k(H) \mu(N) \phi_k^{-1}(N), \tag{7.12}$$

where $\mu^{(n)}(H)$ is the Möbius function on $\mathbb{F}_{q^n}[x]$. To prove (7.10), it suffices to show that

$$(-1)^{m+m_1} \mu^{(n)}(N) = \left((-1)^{m+m_1} \mu(N) \right)^n, \tag{7.13}$$

where $N = \frac{H}{A}$, and $A^k = (G, H^k)_k$.

We note that both sides of (7.13) are multiplicative in H, so it suffices to prove (7.13) when $H = P^t$, where P is an irreducible in $\mathbb{F}_q[x]$ and $t \geq 1$. If $t \geq 1$ and $A = P^{t_1}$ with $t_1 < t - 1$, then both sides of (7.13) are zero. Therefore, we may suppose $A = P^{t-1}$, and $N = P$. It is well known (see Hayes [23], for example) that P is product of exactly (h, n) irreducibles in $\mathbb{F}_{q^n}[x]$, where $h = \deg(P)$, so (7.13) becomes that

$$(-1)^{th+(t-1)h+(h,n)} = (-1)^{n(th+(t-1)h+1)}, \tag{7.14}$$

which is equivalent to

$$h + (h, n) \equiv n(h + 1) \pmod 2. \tag{7.15}$$

It is easy to verify that (7.15) is true for any positive integers n and h, and we complete the proof of Theorem 7.1. □

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