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Some asymptotic expansions on hyperfactorial functions and generalized Glaisher–Kinkelin constants

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Abstract In this paper, by the Bernoulli numbers and the exponential complete Bell polynomials, we establish two general asymptotic expansions on the hyperfactorial functions $\prod_{k=1}^{n} k^{k^q}$ and the generalized Glaisher–Kinkelin constants A_q, where the coefficient sequences in the expansions can be determined by recurrences. Moreover, the explicit expressions of the coefficient sequences are presented and some special asymptotic expansions are discussed. It can be found that some well-known or recently published asymptotic expansions on the factorial function $n!$, the classical hyperfactorial function $\prod_{k=1}^{n} k^k$, and the classical Glaisher–Kinkelin constant A_1 are special cases of our results, so that we give a unified approach to dealing with such asymptotic expansions.

Keywords Asymptotic expansions · Hyperfactorial functions · Generalized Glaisher–Kinkelin constants · Bell polynomials · Bernoulli numbers

Mathematics Subject Classification 41A60 · 11Y60

1 Introduction

In 1933, Bendersky [4] studied the product $\prod_{k=1}^{n} k^{k^q}$ for $q = 0, 1, 2, \ldots$, which reduces to the classical factorial function $n!$ when $q = 0$ and the classical hyperfactorial

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function $H(n) = \prod_{k=1}^{n} k^k$ when $q = 1$. He examined the logarithm of the product and determined the first five values of the limits

$$\ln(A_q) = \lim_{n\to\infty} \ln(A_q(n)) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^q \ln k - P_q(n) \right\}, \tag{1.1}$$

where

$$\begin{aligned} P_q(n) &= \frac{n^q}{2} \ln n + \frac{n^{q+1}}{q+1} \left(\ln n - \frac{1}{q+1} \right) \\ &\quad + q! \sum_{j=1}^{q} \frac{n^{q-j} B_{j+1}}{(j+1)!(q-j)!} \left\{ \ln n + (1 - \delta_{q,j}) \sum_{l=1}^{j} \frac{1}{q-l+1} \right\}, \end{aligned}$$

B_n are the Bernoulli numbers, and $\delta_{q,j}$ is the Kronecker delta function defined by $\delta_{q,j} = 0$ for $j \neq q$ and $\delta_{q,j} = 1$ for $j = q$. In 1995 and 1998, Choudhury [16] and Adamchik [2] showed independently that the constants A_q can be expressed in terms of the derivatives of the Riemann zeta function $\zeta(s)$

$$A_q = \exp \left\{ \frac{B_{q+1} H_q}{q+1} - \zeta'(-q) \right\},$$

where H_n are the harmonic numbers.

From (1.1), it follows that

$$\begin{aligned} \ln A_0 &= \lim_{n\to\infty} \ln(A_0(n)) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \ln k - \left(n + \frac{1}{2} \right) \ln n + n \right\}, \\ \ln A_1 &= \lim_{n\to\infty} \ln(A_1(n)) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \right\}, \end{aligned}$$

which indicate that $A_0 = \sqrt{2\pi}$ and A_1 is the Glaisher–Kinkelin constant. The Glaisher–Kinkelin constant $A_1 = 1.2824271291\ldots$ is closely related to the Barnes G-function $G(z)$ by the limit

$$A_1 = \lim_{n\to\infty} \frac{(2\pi)^{\frac{n}{2}} n^{\frac{n^2}{2} - \frac{1}{12}} \mathrm{e}^{-\frac{3n^2}{4} + \frac{1}{12}}}{G(n+1)},$$

and satisfies many beautiful formulas; see Finch's book [20, Sect. 2.15].

Moreover, for $q = 2, 3$, Eq. (1.1) gives

$$\begin{aligned} \ln A_2 &= \lim_{n\to\infty} \ln(A_2(n)) \\ &= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \right\}, \end{aligned}$$

$$\ln A_3 = \lim_{n\to\infty} \ln(A_3(n))$$
$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \right\},$$

where $A_2 = 1.0309167521\ldots$ and $A_3 = 0.9795555269\ldots$. According to Finch's book [20, Sect. 2.15] and the On-Line Encyclopedia of Integer Sequences (OEIS), the constants A_q should be called the generalized Glaisher–Kinkelin constants or the Bendersky constants.

The generalized Glaisher–Kinkelin constants have been used in the closed-form evaluation of some series involving zeta functions and in calculation of some integrals of multiple gamma functions; see Choi and Srivastava's works [12,14,15,36]. Recently, many researches are devoted to establishing asymptotic expansions on these constants and the related hyperfactorial functions, and the readers are referred to the papers [5,7,10,11,13,22,23,26,29,38].

In particular, Chen [5] presented in 2012 the asymptotic expansions of $\ln A_1(n)$, $\ln A_2(n)$, and $\ln A_3(n)$ by using the Euler–Maclaurin summation formula. For example, the expansion of $\ln A_1(n)$ is

$$\ln A_1(n) \sim \ln A_1 - \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)} \frac{1}{n^{2k}}, \quad n \to \infty.$$

Substituting the values of B_n and using the expression of $\ln A_1(n)$, the above expansion can be written as

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \exp\left(\frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \frac{691}{3603600n^{10}} - \frac{1}{1872n^{12}} + \cdots \right), \quad n \to \infty. \tag{1.2}$$

Mortici [26] established (1.2) and gave a recurrence relation to compute the coefficients of the series in the formula. Using (1.2), Chen and Lin [10] obtained a general asymptotic expansion

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\check{\alpha}_k}{n^k} \right)^{\frac{1}{r}}, \quad n \to \infty, \tag{1.3}$$

and presented the expression of $(\check{\alpha}_k)$. Wang and Liu [38] gave two general expansions

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\alpha_k}{(n+h)^k} \right)^{\frac{1}{r}}, \tag{1.4}$$

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\varphi_k}{(n+h)^k} \right)^{\frac{n}{r}+q}, \tag{1.5}$$

as $n \to \infty$, and studied systematically the recurrences and the explicit expressions of (α_k) and (φ_k). Moreover, Choi [13] presented the expression

$$\begin{aligned} \ln A_q(n) = {} & \sum_{k=1}^{n} k^q \ln k \\ & - \left\{ \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \sum_{r=1}^{[\frac{q+1}{2}]} \frac{B_{2r}}{(2r)!} \left(\prod_{j=1}^{2r-1} (q-j+1) \right) n^{q+1-2r} \right\} \ln n \\ & + \frac{n^{q+1}}{(q+1)^2} \\ & - \sum_{r=1}^{[\frac{q+1}{2}]+\frac{(-1)^q-1}{2}} \frac{B_{2r}}{(2r)!} \left\{ \prod_{j=1}^{2r-1} (q-j+1) \sum_{j=1}^{2r-1} \frac{1}{q-j+1} \right\} n^{q+1-2r} \\ = {} & \sum_{k=1}^{n} k^q \ln k - U_{q+1}(n) \ln n + V_{q+1}(n), \end{aligned} \tag{1.6}$$

and gave the general asymptotic expansion

$$\ln A_q(n) \sim \ln A_q + (-1)^q q! \sum_{r=[\frac{q+1}{2}]+1}^{\infty} \frac{B_{2r}}{(2r)!} \cdot \frac{(2r-q-2)!}{n^{2r-q-1}}, \quad n \to \infty, \tag{1.7}$$

which reduces to Stirling's formula of $n!$ when $q = 0$ and Chen's results in [5] when $q = 1, 2, 3$. Further results may be found in Chen [7], Cheng and Chen [11], Lin [22], Lu and Mortici [23], and Mortici [29].

Inspired by these works, we present in this paper the next two general asymptotic expansions on the hyperfactorial functions and the generalized Glaisher–Kinkelin constants

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\alpha_k(q;h,r)}{(n+h)^k} \right)^{\frac{1}{r}}, \tag{1.8}$$

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\varphi_k(q;h,r,s)}{(n+h)^k} \right)^{\frac{n}{r}+s}, \tag{1.9}$$

as $n \to \infty$, where the polynomials $U_{q+1}(n)$ and $V_{q+1}(n)$ are defined in (1.6). We give recurrences and explicit expressions of the coefficient sequences in the expansions by the exponential complete Bell polynomials, and discuss some special cases of these two expansions.

In particular, when $q = 0$, our results reduce to the asymptotic expansions of $n!$, including as special cases some well-known formulas due to Laplace, Wehmeier, and Ramanujan, and some recent results due to Batir [3], Chen [6,8,9], Mortici [28,30], Nemes [31,32], et al. Additionally, when $q = 1$, our results reduce to the asymptotic expansions on the classical hyperfactorial function $\prod_{k=1}^{n} k^k$ and the classical Glaisher–Kinkelin constant A_1, including those presented by Chen and Lin [10] and Wang and Liu [38].

The paper is organized as follows: Sections 2 and 3 are devoted to the first general asymptotic expansion (1.8), and Sects. 4 and 5 are devoted to the second one (1.9).

2 The first general asymptotic expansion

Define the exponential complete Bell polynomials Y_n by

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = \sum_{n=0}^{\infty} Y_n(x_1, x_2, \ldots, x_n)\frac{t^n}{n!}\,; \tag{2.1}$$

see [18, Sect. 3.3] and [34, Sect. 2.8]. Then $Y_0 = 1$ and

$$Y_n(x_1, x_2, \ldots, x_n) = \sum_{c_1+2c_2+\cdots+nc_n=n} \frac{n!}{c_1!c_2!\cdots c_n!}\left(\frac{x_1}{1!}\right)^{c_1}\left(\frac{x_2}{2!}\right)^{c_2}\cdots\left(\frac{x_n}{n!}\right)^{c_n}. \tag{2.2}$$

According to [34, Eq. (2.44)] (see also [17, Eq. (3.6)] and [35, Theorem 1]), the polynomials Y_n satisfy the recurrence

$$Y_n(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{n-1}\binom{n-1}{j} x_{n-j} Y_j(x_1, x_2, \ldots, x_j), \quad n \geq 1. \tag{2.3}$$

Using the definition and recurrence of the Bell polynomials, the following general asymptotic expansion can be obtained.

Theorem 2.1 *Let h, r be real numbers such that $r \neq 0$. Define the sequence $(\beta_k)_{k\geq 1}$ by*

$$\beta_k = (-1)^q r(k-1)!\binom{k+q}{q}^{-1}\frac{B_{k+q+1}}{k+q+1}. \tag{2.4}$$

Then

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)}\left(\sum_{k=0}^{\infty}\frac{\alpha_k(q;h,r)}{(n+h)^k}\right)^{\frac{1}{r}}, \tag{2.5}$$

as $n \to \infty$, *where* $(\alpha_k(q;h,r))_{k\ge 0}$ *is determined by*

$$\alpha_0(q;h,r)=1,\quad \alpha_k(q;h,r)=\frac{y_k}{k!}-\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k-1}{k-j}h^{k-j}\alpha_j(q;h,r),\quad k\ge 1, \tag{2.6}$$

and $(y_k)_{k\ge 0}$ *is determined by*

$$y_0=1,\quad y_k=\sum_{j=0}^{k-1}\binom{k-1}{j}\beta_{k-j}y_j,\quad k\ge 1. \tag{2.7}$$

Proof Define the falling factorials $(x)_n$ by $(x)_0=1$ and $(x)_n=x(x-1)\cdots(x-n+1)$ for $n=1,2,\ldots$. The asymptotic expansion (1.7) can be rewritten as

$$\prod_{k=1}^{n}k^{k^q}\sim A_q\cdot n^{U_{q+1}(n)}\mathrm{e}^{-V_{q+1}(n)}\exp\left\{\sum_{k=1}^{\infty}\frac{(-1)^q q!B_{k+q+1}}{(k+q+1)_{q+2}}\frac{1}{n^k}\right\},\quad n\to\infty. \tag{2.8}$$

Then (2.1) and (2.4) give

$$\begin{aligned}\left(\frac{\prod_{k=1}^{n}k^{k^q}}{A_q\cdot n^{U_{q+1}(n)}\mathrm{e}^{-V_{q+1}(n)}}\right)^r &\sim \exp\left\{\sum_{k=1}^{\infty}\frac{(-1)^q rk!q!B_{k+q+1}}{(k+q+1)_{q+2}}\frac{(\frac{1}{n})^k}{k!}\right\}\\ &=\exp\left\{\sum_{k=1}^{\infty}\beta_k\frac{(\frac{1}{n})^k}{k!}\right\}\\ &=\sum_{k=0}^{\infty}\frac{Y_k(\beta_1,\beta_2,\ldots,\beta_k)}{k!}\frac{1}{n^k},\quad n\to\infty.\end{aligned}$$

On the other hand, expansion in powers of $1/n$ yields

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{\alpha_j(q;h,r)}{(n+h)^j}&=\sum_{j=0}^{\infty}\frac{\alpha_j(q;h,r)}{n^j(1+h/n)^j}=\sum_{j=0}^{\infty}\frac{\alpha_j(q;h,r)}{n^j}\sum_{i=0}^{\infty}\binom{-j}{i}\frac{h^i}{n^i}\\ &=\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{k-j}h^{k-j}\alpha_j(q;h,r)\right\}\frac{1}{n^k}.\end{aligned}$$

Thus, it suffices to show that the system

$$\frac{Y_k(\beta_1,\beta_2,\ldots,\beta_k)}{k!}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{k-j}h^{k-j}\alpha_j(q;h,r)$$

has unique solution $(\alpha_k(q; h, r))_{k\geq 0}$. This is established next. The case $k = 0$ gives $\alpha_0(q; h, r) = 1$. For $k \geq 1$, the system gives

$$\alpha_k(q; h, r) = \frac{Y_k(\beta_1, \beta_2, \ldots, \beta_k)}{k!} - \sum_{j=0}^{k-1} (-1)^{k-j} \binom{k-1}{k-j} h^{k-j} \alpha_j(q; h, r).$$

Setting $y_k = Y_k(\beta_1, \beta_2, \ldots, \beta_k)$ gives recurrence (2.6) and shows that $(\alpha_k(q; h, r))$ can be uniquely determined. Finally, (2.3) gives (2.7) and the proof is complete. □

By specifying the parameters q, h, r in Theorem 2.1, many special asymptotic expansions on $\prod_{k=1}^{n} k^{k^q}$ and A_q can be obtained. In particular, when $h = 0$, Theorem 2.1 reduces to the following result.

Theorem 2.2 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\alpha_k(q; 0, r)}{n^k} \right)^{\frac{1}{r}}, \tag{2.9}$$

as $n \to \infty$, where $(\alpha_k(q; 0, r))_{k\geq 0}$ is determined by

$$\alpha_0(q; 0, r) = 1,$$
$$\alpha_k(q; 0, r) = \frac{(-1)^q r}{k} \sum_{j=0}^{k-1} \binom{k-j+q}{q}^{-1} \frac{B_{k-j+q+1}}{k-j+q+1} \alpha_j(q; 0, r), \quad k \geq 1.$$

The further special cases $q = 0$ and $q = 2$ are stated next.

Corollary 2.3 *Let $r \neq 0$ be a real number. Then*

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{\mathrm{e}}\right)^n \left(\sum_{k=0}^{\infty} \frac{\alpha_k(0; 0, r)}{n^k} \right)^{\frac{1}{r}}, \tag{2.10}$$

as $n \to \infty$, where $(\alpha_k(0; 0, r))_{k\geq 0}$ is determined by

$$\alpha_0(0; 0, r) = 1, \quad \alpha_k(0; 0, r) = \frac{r}{k} \sum_{j=0}^{k-1} \frac{B_{k-j+1}}{k-j+1} \alpha_j(0; 0, r), \quad k \geq 1. \tag{2.11}$$

Corollary 2.4 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k^2} \sim A_2 \cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} \mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}} \left(\sum_{k=0}^{\infty} \frac{\alpha_k(2; 0, r)}{n^k} \right)^{\frac{1}{r}}, \tag{2.12}$$

as $n \to \infty$, *where* $(\alpha_k(2; 0, r))_{k\geq 0}$ *is determined by*

$$\alpha_0(2; 0, r) = 1,$$
$$\alpha_k(2; 0, r) = \frac{2r}{k}\sum_{j=0}^{k-1}\frac{B_{k-j+3}}{(k-j+1)(k-j+2)(k-j+3)}\alpha_j(2; 0, r), \quad k \geq 1. \tag{2.13}$$

The recurrences (2.11) and (2.13) determine the coefficients $(\alpha_k(0; 0, r))$ and $(\alpha_k(2; 0, r))$, respectively. For example,

$$\alpha_0(0; 0, r) = 1, \quad \alpha_1(0; 0, r) = \frac{r}{12}, \quad \alpha_2(0; 0, r) = \frac{r^2}{288},$$
$$\alpha_3(0; 0, r) = \frac{r(-144 + 5r^2)}{51840}, \quad \alpha_4(0; 0, r) = \frac{r^2(-576 + 5r^2)}{2488320},$$

and

$$\alpha_0(2; 0, r) = 1, \quad \alpha_1(2; 0, r) = -\frac{r}{360}, \quad \alpha_2(2; 0, r) = \frac{r^2}{259200},$$
$$\alpha_3(2; 0, r) = -\frac{r(-259200 + 7r^2)}{1959552000}, \quad \alpha_4(2; 0, r) = \frac{r^2(-1036800 + 7r^2)}{2821754880000}.$$

Example 2.1 Setting $r = 1$ in Corollary 2.3 yields

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \frac{163879}{209018880n^5} + \cdots\right),$$

as $n \to \infty$, which is the famous Laplace formula, and sometimes called Stirling's formula (see [19, pp. 2–3]). Setting $r = 2$ in Corollary 2.3 gives the Wehmeier formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(1+\frac{1}{6n}+\frac{1}{72n^2}-\frac{31}{6480n^3} - \frac{139}{155520n^4} + \frac{9871}{6531840n^5} + \cdots\right)^{\frac{1}{2}},$$

as $n \to \infty$, which was recently rediscovered by Batir [3], Luschny [25], and Mortici [27]. Setting $r = 6$ gives the well-known Ramanujan formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4} + \frac{79}{26880n^5} + \cdots\right)^{\frac{1}{6}},$$

as $n \to \infty$ (see, for example, [21,33]). Batir [3] obtained the case $r = 4$, Mortici [28,30] presented the cases $r = 8, 10, 12, 14$, and Chen and Lin [6,9] gave the cases $r = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}$, and $r = -1, -2$.

In 2013, Lu and Wang [24] studied the expansion (2.10), but they determined only the first five terms of the coefficient sequence $(\alpha_k(0; 0, r))$ and did not obtain an explicit expression nor recurrence relation for it. Chen and Lin [6,9] also presented (2.10) and established the expression for $(\alpha_k(0; 0, r))$. In 2016, Wang further gave the recurrence (2.11) for $(\alpha_k(0; 0, r))$, showed a more general expansion, and generalized Lu, Wang, Chen and Lin's results (see [37, Theorem 2.1 and Corollary 3.3]).

Example 2.2 Setting $r = 1$ in Corollary 2.4 yields

$$\prod_{k=1}^{n} k^{k^2} \sim A_2 \cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} \mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}} \left(1 - \frac{1}{360n} + \frac{1}{259200n^2} + \frac{259193}{1959552000n^3} - \frac{1036793}{2821754880000n^4} - \frac{201551328007}{5079158784000000n^5} + \cdots\right), \quad n \to \infty.$$

Other special cases of Theorems 2.1 and 2.2 and Corollaries 2.3 and 2.4 can be obtained similarly.

3 Explicit expression of the coefficient sequence in (2.5)

In this section, the method of generating functions is used to present an explicit expression for the coefficients in the asymptotic expansion (2.5).

Theorem 3.1 *The coefficient sequence* $(\alpha_k(q; h, r))$ *in* (2.5) *is given by the Bell polynomials*

$$\alpha_k(q; h, r) = \frac{1}{k!} Y_k(\delta_1, \delta_2, \ldots, \delta_k),$$

where

$$\delta_k = \frac{(-1)^q r k! q!}{(k+q+1)_{q+2}} \left\{ B_{k+q+1}(h) - \sum_{j=0}^{q+1} \binom{k+q+1}{j} B_j h^{k+q+1-j} \right\}, \quad k \geq 1,$$

and $B_n(x)$ *are the classical Bernoulli polynomials.*

Proof Introduce the notations $(\beta_k)_{k\geq 1}$, $(y_k)_{k\geq 0}$ and $(\alpha_k(q; h, r))_{k\geq 0}$ by

$$f_\beta(t) = \sum_{k=1}^{\infty} \beta_k \frac{t^k}{k!}, \quad f_y(t) = \sum_{k=0}^{\infty} y_k \frac{t^k}{k!}, \quad f_\alpha(t) = \sum_{k=0}^{\infty} \alpha_k(q; h, r) t^k.$$

The result $f_y'(t) = f_y(t) \cdot f_\beta'(t)$ comes from (2.7). Then $f_y(t) = C \cdot \mathrm{e}^{f_\beta(t)}$. The initial conditions $f_y(0) = 1$ and $f_\beta(0) = 0$ show that $C = 1$. Therefore $f_y(t) = \exp(f_\beta(t))$. On the other hand, (2.6) gives

$$
\begin{aligned}
f_y(t) &= \sum_{k=0}^{\infty} \frac{y_k}{k!} t^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} \binom{k-1}{k-j} h^{k-j} \alpha_j(q; h, r) t^k \\
&= \sum_{j=0}^{\infty} \alpha_j(q; h, r) t^j \sum_{i=0}^{\infty} \binom{-j}{i} (ht)^i \\
&= \sum_{j=0}^{\infty} \alpha_j(q; h, r) \left(\frac{t}{1+ht} \right)^j = f_\alpha \left(\frac{t}{1+ht} \right).
\end{aligned}
$$

Then

$$
f_\alpha(t) = f_y \left(\frac{t}{1-ht} \right) = \exp \left\{ f_\beta \left(\frac{t}{1-ht} \right) \right\}. \tag{3.1}
$$

From the definition of (β_k) and the identity for Bernoulli polynomials

$$
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}
$$

(see [1, Chap. 23] and [18, Sect. 1.14]), it follows that

$$
\begin{aligned}
f_\beta \left(\frac{t}{1-ht} \right) &= \sum_{k=1}^{\infty} (-1)^q r (k-1)! \binom{k+q}{q}^{-1} \frac{B_{k+q+1}}{k+q+1} \frac{(\frac{t}{1-ht})^k}{k!} \\
&= (-1)^q r q! \sum_{k=1}^{\infty} \frac{(k-1)! B_{k+q+1} t^k}{(k+q+1)!} \sum_{j=0}^{\infty} \binom{-k}{j} (-ht)^j \\
&= (-1)^q r q! \sum_{m=1}^{\infty} \sum_{k=1}^{m} \frac{(k-1)! B_{k+q+1}}{(k+q+1)!} \binom{m-1}{m-k} h^{m-k} t^m \\
&= (-1)^q r q! \sum_{m=1}^{\infty} \left\{ \frac{1}{(m+q+1)_{q+2}} \sum_{k=1}^{m} \binom{m+q+1}{k+q+1} B_{k+q+1} h^{m-k} \right\} t^m \\
&= (-1)^q r q! \sum_{m=1}^{\infty} \frac{t^m}{(m+q+1)_{q+2}} \\
&\quad \times \left\{ B_{m+q+1}(h) - \sum_{j=0}^{q+1} \binom{m+q+1}{j} B_j h^{m+q+1-j} \right\}.
\end{aligned}
$$

Define the coefficient of $t^m/m!$ in the last series by δ_m. Then

$$
f_\alpha(t) = \exp \left(\sum_{m=1}^{\infty} \delta_m \frac{t^m}{m!} \right),
$$

and the expression of (α_k) follows from here. □

The special case $h = 0$ gives

$$\tilde{\delta}_m = \frac{(-1)^q rm!q!}{(m+q+1)_{q+2}} B_{m+q+1}.$$

This produces the next corollary.

Corollary 3.2 *The coefficient sequence $(\alpha_k(q;0,r))$ in (2.9) has the explicit expression*

$$\begin{aligned}\alpha_k(q;0,r) = & \sum_{c_1+2c_2+\cdots+kc_k=k} \frac{((-1)^q rq!)^{c_1+c_2+\cdots+c_k}}{c_1!c_2!\cdots c_k!} \\ & \times \left(\frac{B_{q+2}}{(q+2)_{q+2}}\right)^{c_1} \left(\frac{B_{q+3}}{(q+3)_{q+2}}\right)^{c_2} \cdots \\ & \times \left(\frac{B_{q+k+1}}{(q+k+1)_{q+2}}\right)^{c_k}.\end{aligned}$$

Moreover, when $h = 0$ and q is odd, using $B_{2k+1} = 0$ for $k = 1, 2, \ldots$ gives

$$\begin{aligned}f_{\tilde{\alpha}}(t) = \sum_{k=0}^{\infty} \alpha_k(q;0,r)t^k &= \exp\left(\sum_{m=1}^{\infty} \tilde{\delta}_m \frac{t^m}{m!}\right) \\ &= \exp\left\{\sum_{m=1}^{\infty} \frac{-rq!}{(m+q+1)_{q+2}} B_{m+q+1} t^m\right\} \\ &= \exp\left\{\sum_{k=1}^{\infty} \frac{-rq!}{(2k+q+1)_{q+2}} B_{2k+q+1} t^{2k}\right\}.\end{aligned}$$

This shows that in this case $f_{\tilde{\alpha}}(t)$ is an even function and $\alpha_{2k+1}(q;0,r) = 0$ for $k = 0, 1, 2, \ldots$. Thus, Theorem 2.2 and Corollary 3.2 produce the following result.

Theorem 3.3 *Let q be an odd integer and $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\gamma_k(q;0,r)}{n^{2k}}\right)^{\frac{1}{r}}, \quad n \to \infty.$$

The sequence $(\gamma_k(q;0,r))_{k\geq 0}$ satisfies the recurrence

$$\begin{aligned}&\gamma_0(q;0,r) = 1, \\ &\gamma_k(q;0,r) = -\frac{r}{2k} \sum_{j=0}^{k-1} \binom{2k-2j+q}{q}^{-1} \frac{B_{2k-2j+q+1}}{2k-2j+q+1} \gamma_j(q;0,r), \quad k \geq 1,\end{aligned}$$

and has the explicit expression

$$\gamma_k(q;0,r) = \sum_{d_1+2d_2+\cdots+kd_k=k} \frac{(-rq!)^{d_1+d_2+\cdots+d_k}}{d_1!d_2!\cdots d_k!} \times \left(\frac{B_{q+3}}{(q+3)_{q+2}}\right)^{d_1} \left(\frac{B_{q+5}}{(q+5)_{q+2}}\right)^{d_2} \cdots \left(\frac{B_{q+2k+1}}{(q+2k+1)_{q+2}}\right)^{d_k}.$$

Proof Set $\gamma_k(q;0,r) = \alpha_{2k}(q;0,r)$. Then the theorem follows from the recurrence and expression of $\alpha_{2k}(q;0,r)$ as well as the vanishing of B_{2k+1}. □

The special cases $q = 1$ and $q = 3$ in Theorem 3.3 are stated in Corollaries 3.4 and 3.5.

Corollary 3.4 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\gamma_k(1;0,r)}{n^{2k}}\right)^{\frac{1}{r}}, \tag{3.2}$$

as $n \to \infty$, where $(\gamma_k(1;0,r))_{k\geq 0}$ is determined by

$$\gamma_0(1;0,r) = 1,$$
$$\gamma_k(1;0,r) = -\frac{r}{2k}\sum_{j=0}^{k-1} \frac{B_{2k-2j+2}}{(2k-2j+1)(2k-2j+2)}\gamma_j(1;0,r), \quad k \geq 1. \tag{3.3}$$

Corollary 3.5 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k^3} \sim A_3 \cdot n^{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}} \mathrm{e}^{-\frac{n^4}{16}+\frac{n^2}{12}} \left(\sum_{k=0}^{\infty} \frac{\gamma_k(3;0,r)}{n^{2k}}\right)^{\frac{1}{r}}, \tag{3.4}$$

as $n \to \infty$, where $(\gamma_k(3;0,r))_{k\geq 0}$ is determined by

$$\gamma_0(3;0,r) = 1, \quad \gamma_k(3;0,r) = -\frac{3r}{k}\sum_{j=0}^{k-1} \frac{B_{2k-2j+4}}{(2k-2j+4)_4}\gamma_j(3;0,r), \quad k \geq 1. \tag{3.5}$$

Using these recurrences, the coefficients $(\gamma_k(1;0,r))$ and $(\gamma_k(3;0,r))$ can be computed efficiently. For example,

$$\gamma_0(1;0,r) = 1, \quad \gamma_1(1;0,r) = \frac{r}{720}, \quad \gamma_2(1;0,r) = \frac{r(-1440+7r)}{7257600},$$
$$\gamma_3(1;0,r) = \frac{r(1555200-4320r+7r^2)}{15676416000},$$

$$\gamma_4(1;0,r)=\frac{r(-365783040000+547430400r-665280r^2+539r^3)}{3476402012160000},$$

and

$$\gamma_0(3;0,r)=1,\quad \gamma_1(3;0,r)=-\frac{r}{5040},\quad \gamma_2(3;0,r)=\frac{r(1512+r)}{50803200},$$
$$\gamma_3(3;0,r)=-\frac{r(127008000+49896r+11r^2)}{8449588224000},$$
$$\gamma_4(3;0,r)=\frac{r(35385851289600+7585171776r+1297296r^2+143r^3)}{22144680817459200000}.$$

Example 3.1 Setting $r=1$ in Corollaries 3.4 and 3.5 gives

$$\prod_{k=1}^{n}k^k\sim A_1\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\mathrm{e}^{-\frac{n^2}{4}}\left(1+\frac{1}{720n^2}-\frac{1433}{7257600n^4}+\frac{1550887}{15676416000n^6}\right.$$
$$\left.-\frac{365236274341}{3476402012160000n^8}+\frac{31170363588856607}{1626956141690880000000n^{10}}-\cdots\right)$$

and

$$\prod_{k=1}^{n}k^{k^3}\sim A_3\cdot n^{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}}\mathrm{e}^{-\frac{n^4}{16}+\frac{n^2}{12}}$$
$$\times\left(1-\frac{1}{5040n^2}+\frac{1513}{50803200n^4}-\frac{127057907}{8449588224000n^6}\right.$$
$$\left.+\frac{7078687551763}{4428936163491840000n^8}-\frac{1626209947417109183}{5580459565999718400000000n^{10}}+\cdots\right),$$

as $n\to\infty$. Other special cases can be obtained similarly. Corollary 3.4 and some of its special cases have been presented in [38, Corollary 2.2 and Example 2.1]. See also Chen and Lin [10].

4 The second general asymptotic expansion

This section presents another general asymptotic expansion for the hyperfactorial functions.

Theorem 4.1 *Let h,r,s be real numbers such that $r\neq 0$. Define the sequence $(\psi_m)_{m\geq 1}$ by*

$$\psi_1=0,\quad \psi_m=m!\sum_{k=1}^{m-1}\frac{(-1)^{q+m-k-1}r^{m-k}s^{m-k-1}q!B_{k+q+1}}{(k+q+1)_{q+2}},\quad m\geq 2. \tag{4.1}$$

Then

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\varphi_k(q; h, r, s)}{(n+h)^k} \right)^{\frac{n}{r}+s}, \tag{4.2}$$

as $n \to \infty$, *where* $U_{q+1}(n)$ *and* $V_{q+1}(n)$ *are defined in* (1.6), $(\varphi_k(q; h, r, s))_{k \geq 0}$ *is determined by*

$$\begin{aligned} &\varphi_0(q; h, r, s) = 1, \\ &\varphi_k(q; h, r, s) = \frac{z_k}{k!} - \sum_{j=0}^{k-1} (-1)^{k-j} \binom{k-1}{k-j} h^{k-j} \varphi_j(q; h, r, s), \quad k \geq 1, \end{aligned} \tag{4.3}$$

and $(z_k)_{k \geq 0}$ *is determined by*

$$z_0 = 1, \quad z_k = \sum_{j=0}^{k-2} \binom{k-1}{j} \psi_{k-j} z_j, \quad k \geq 1. \tag{4.4}$$

Proof From (2.8) and the definition of (ψ_m), it follows that

$$\begin{aligned} &\left(\frac{\prod_{k=1}^{n} k^{k^q}}{A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)}} \right)^{\frac{r}{n+rs}} \sim \exp\left\{ \frac{r}{n+rs} \sum_{k=1}^{\infty} \frac{(-1)^q q! B_{k+q+1}}{(k+q+1)_{q+2}} \frac{1}{n^k} \right\} \\ &= \exp\left\{ \frac{r}{n} \sum_{j=0}^{\infty} \left(-\frac{rs}{n}\right)^j \sum_{k=1}^{\infty} \frac{(-1)^q q! B_{k+q+1}}{(k+q+1)_{q+2}} \frac{1}{n^k} \right\} \\ &= \exp\left\{ \sum_{m=2}^{\infty} \sum_{k=1}^{m-1} \frac{(-1)^{q+m-k-1} r^{m-k} s^{m-k-1} q! B_{k+q+1}}{(k+q+1)_{q+2}} \frac{1}{n^m} \right\} \\ &= \exp\left\{ \sum_{m=1}^{\infty} \psi_m \frac{\left(\frac{1}{n}\right)^m}{m!} \right\} = \sum_{k=0}^{\infty} \frac{Y_k(\psi_1, \psi_2, \ldots, \psi_k)}{k!} \frac{1}{n^k}, \quad n \to \infty. \end{aligned}$$

Moreover,

$$\sum_{k=0}^{\infty} \frac{\varphi_k(q; h, r, s)}{(n+h)^k} = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} (-1)^{k-j} \binom{k-1}{k-j} h^{k-j} \varphi_j(q; h, r, s) \right\} \frac{1}{n^k}.$$

Define $z_k = Y_k(\psi_1, \psi_2, \ldots, \psi_k)$. Using the same procedure as in the proof of Theorem 2.1, it can be verified that the system

$$\frac{z_k}{k!} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k-1}{k-j} h^{k-j} \varphi_j(q; h, r, s)$$

has unique solution $(\varphi_k(q; h, r, s))$. This can be computed by recurrences (4.3) and (4.4). □

In the case $h = 0$ and $s = 0$, Theorem 4.1 gives

Theorem 4.2 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\varphi_k(q; 0, r, 0)}{n^k} \right)^{\frac{n}{r}}, \tag{4.5}$$

as $n \to \infty$, where $(\varphi_k(q; 0, r, 0))_{k \geq 0}$ is determined by

$$\varphi_0(q; 0, r, 0) = 1,$$
$$\varphi_k(q; 0, r, 0) = \frac{(-1)^q r q!}{k} \sum_{j=0}^{k-2} \frac{(k-j) B_{k-j+q}}{(k-j+q)_{q+2}} \varphi_j(q; 0, r, 0), \quad k \geq 1.$$

Proof In this case, $\tilde{z}_k = k! \varphi_k(q; 0, r, 0)$ and

$$\tilde{\psi}_1 = 0, \quad \tilde{\psi}_m = \frac{(-1)^q r m! q! B_{m+q}}{(m+q)_{q+2}}, \quad m \geq 2. \tag{4.6}$$

By (4.4), the result follows. □

Setting $q = 1$ and $q = 3$ in Theorem 4.2 yields the next two corollaries.

Corollary 4.3 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k} \sim A_1 \cdot n^{\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\varphi_k(1; 0, r, 0)}{n^k} \right)^{\frac{n}{r}}, \tag{4.7}$$

as $n \to \infty$, where $(\varphi_k(1; 0, r, 0))_{k \geq 0}$ is determined by

$$\varphi_0(1; 0, r, 0) = 1,$$
$$\varphi_k(1; 0, r, 0) = -\frac{r}{k} \sum_{j=0}^{k-2} \frac{B_{k-j+1}}{(k-j+1)(k-j-1)} \varphi_j(1; 0, r, 0), \quad k \geq 1. \tag{4.8}$$

Corollary 4.4 *Let $r \neq 0$ be a real number. Then*

$$\prod_{k=1}^{n} k^{k^3} \sim A_3 \cdot n^{\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}} \mathrm{e}^{-\frac{n^4}{16} + \frac{n^2}{12}} \left(\sum_{k=0}^{\infty} \frac{\varphi_k(3; 0, r, 0)}{n^k} \right)^{\frac{n}{r}}, \tag{4.9}$$

as $n \to \infty$, where $(\varphi_k(3;0,r,0))_{k\geq 0}$ is determined by

$$\varphi_0(3;0,r,0)=1, \quad \varphi_k(3;0,r,0)=-\frac{6r}{k}\sum_{j=0}^{k-2}\frac{(k-j)B_{k-j+3}}{(k-j+3)_5}\varphi_j(3;0,r,0), \quad k\geq 1. \tag{4.10}$$

The first few terms of $(\varphi_k(1;0,r,0))$ are

$$\varphi_0(1;0,r,0)=1, \quad \varphi_1(1;0,r,0)=0, \quad \varphi_2(1;0,r,0)=0, \quad \varphi_3(1;0,r,0)=\frac{r}{720},$$
$$\varphi_4(1;0,r,0)=0, \quad \varphi_5(1;0,r,0)=-\frac{r}{5040}, \quad \varphi_6(1;0,r,0)=\frac{r^2}{1036800},$$

and the first few terms of $(\varphi_k(3;0,r,0))$ are

$$\varphi_0(3;0,r,0)=1, \quad \varphi_1(3;0,r,0)=0, \quad \varphi_2(3;0,r,0)=0, \quad \varphi_3(3;0,r,0)=-\frac{r}{5040},$$
$$\varphi_4(3;0,r,0)=0, \quad \varphi_5(3;0,r,0)=\frac{r}{33600}, \quad \varphi_6(3;0,r,0)=\frac{r^2}{50803200}.$$

Example 4.1 In the case $r=1$, Corollaries 4.3 and 4.4 produce

$$\prod_{k=1}^{n}k^k \sim A_1\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\mathrm{e}^{-\frac{n^2}{4}}\left(1+\frac{1}{720n^3}-\frac{1}{5040n^5}+\frac{1}{1036800n^6}\right.$$
$$\left.+\frac{1}{10080n^7}-\frac{1}{3628800n^8}-\frac{2591989}{24634368000n^9}+\cdots\right)^n$$

and

$$\prod_{k=1}^{n}k^{k^3} \sim A_3\cdot n^{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}}\mathrm{e}^{-\frac{n^4}{16}+\frac{n^2}{12}}\left(1-\frac{1}{5040n^3}+\frac{1}{33600n^5}+\frac{1}{50803200n^6}\right.$$
$$\left.-\frac{1}{66528n^7}-\frac{1}{169344000n^8}+\frac{1755250417}{109844646912000n^9}+\cdots\right)^n,$$

as $n\to\infty$. Corollary 4.3 and its special cases appear as [38, Corollary 4.2 and Example 4.1]. See also Chen [7, Remark 2].

5 Explicit expression of the coefficient sequence in (4.2)

Now define

$$f_\psi(t)=\sum_{k=1}^{\infty}\psi_k\frac{t^k}{k!}, \quad f_z(t)=\sum_{k=0}^{\infty}z_k\frac{t^k}{k!}, \quad f_\varphi(t)=\sum_{k=0}^{\infty}\varphi_k(q;h,r,s)t^k.$$

As in the discussion above,

$$f_{\varphi}(t) = f_z\left(\frac{t}{1-ht}\right) = \exp\left\{f_{\psi}\left(\frac{t}{1-ht}\right)\right\},$$
$$f_{\psi}(t) = (-1)^q rq! \sum_{k=1}^{\infty} \frac{B_{k+q+1}}{(k+q+1)_{q+2}} \cdot \frac{t^{k+1}}{1+rst},$$

which give an explicit expression of $(\varphi_k(q; h, r, s))$ in terms of the Bell polynomials. In particular, when $h = 0$ and $s = 0$,

$$f_{\tilde{\varphi}}(t) = \sum_{k=0}^{\infty} \varphi_k(q; 0, r, 0)t^k = \exp\{f_{\tilde{\psi}}(t)\} = \exp\left(\sum_{m=1}^{\infty} \tilde{\psi}_m \frac{t^m}{m!}\right),$$

where the sequence $(\tilde{\psi}_m)$ is defined in (4.6). Then the following result holds.

Theorem 5.1 *The coefficient sequence* $(\varphi_k(q; 0, r, 0))$ *in* (4.5) *is*

$$\begin{aligned}\varphi_k(q; 0, r, 0) &= \frac{1}{k!} Y_k(\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_k)\\ &= \sum_{2c_2+3c_3+\cdots+kc_k=k} \frac{((-1)^q rq!)^{c_2+c_3+\cdots+c_k}}{c_2!c_3!\cdots c_k!}\\ &\quad\times \left(\frac{B_{q+2}}{(q+2)_{q+2}}\right)^{c_2} \left(\frac{B_{q+3}}{(q+3)_{q+2}}\right)^{c_3} \cdots \left(\frac{B_{q+k}}{(q+k)_{q+2}}\right)^{c_k}.\end{aligned}$$

Moreover, when $h = 0$, $s = 0$, and q is even, we have

$$\begin{aligned}f_{\tilde{\varphi}}(t) = \sum_{k=0}^{\infty} \varphi_k(q; 0, r, 0)t^k &= \exp\left\{rq! \sum_{m=2}^{\infty} \frac{B_{m+q}}{(m+q)_{q+2}} t^m\right\}\\ &= \exp\left\{rq! \sum_{j=1}^{\infty} \frac{B_{2j+q}}{(2j+q)_{q+2}} t^{2j}\right\}.\end{aligned}$$

In this case $f_{\tilde{\varphi}}(t)$ is an even function and $\varphi_{2k+1}(q; 0, r, 0) = 0$ for $k = 0, 1, 2, \ldots$. Now, defining $\omega_k(q; 0, r, 0) = \varphi_{2k}(q; 0, r, 0)$, we obtain the next result.

Theorem 5.2 *Let* q *be an even integer and* $r \neq 0$ *be a real number. Then*

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\omega_k(q; 0, r, 0)}{n^{2k}}\right)^{\frac{n}{r}}, \quad n \to \infty.$$

The sequence $(\omega_k(q; 0, r, 0))_{k\geq 0}$ *satisfies the recurrence*

$$\omega_0(q;0,r,0)=1,$$
$$\omega_k(q;0,r,0)=\frac{rq!}{k}\sum_{j=0}^{k-1}\frac{(k-j)B_{2k-2j+q}}{(2k-2j+q)_{q+2}}\omega_j(q;0,r,0),\quad k\ge 1,$$

and has the explicit expression

$$\omega_k(q;0,r,0)=\sum_{d_1+2d_2+\cdots+kd_k=k}\frac{(rq!)^{d_1+d_2+\cdots+d_k}}{d_1!d_2!\cdots d_k!}\left(\frac{B_{q+2}}{(q+2)_{q+2}}\right)^{d_1}\times\left(\frac{B_{q+4}}{(q+4)_{q+2}}\right)^{d_2}\cdots\left(\frac{B_{q+2k}}{(q+2k)_{q+2}}\right)^{d_k}.$$

The special cases $q=0$ and $q=2$ in Theorem 5.2 are stated next.

Corollary 5.3 *Let $r\neq 0$ be a real number. Then*

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(\sum_{k=0}^{\infty}\frac{\omega_k(0;0,r,0)}{n^{2k}}\right)^{\frac{n}{r}},\tag{5.1}$$

as $n\to\infty$, where $(\omega_k(0;0,r,0))_{k\ge 0}$ is determined by

$$\omega_0(0;0,r,0)=1,$$
$$\omega_k(0;0,r,0)=\frac{r}{2k}\sum_{j=0}^{k-1}\frac{B_{2k-2j}}{2k-2j-1}\omega_j(0;0,r,0),\quad k\ge 1.\tag{5.2}$$

Corollary 5.4 *Let $r\neq 0$ be a real number. Then*

$$\prod_{k=1}^{n}k^{k^2}\sim A_2\cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}\mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}}\left(\sum_{k=0}^{\infty}\frac{\omega_k(2;0,r,0)}{n^{2k}}\right)^{\frac{n}{r}},\tag{5.3}$$

as $n\to\infty$, where $(\omega_k(2;0,r,0))_{k\ge 0}$ is determined by

$$\omega_0(2;0,r,0)=1,$$
$$\omega_k(2;0,r,0)=\frac{r}{k}\sum_{j=0}^{k-1}\frac{B_{2k-2j+2}}{(2k-2j+2)(2k-2j+1)(2k-2j-1)}\omega_j(2;0,r,0),\quad k\ge 1.\tag{5.4}$$

The first few terms of $(\omega_k(0;0,r,0))$ are

$$\omega_0(0;0,r,0)=1,\quad \omega_1(0;0,r,0)=\frac{r}{12},\quad \omega_2(0;0,r,0)=\frac{r(-4+5r)}{1440},$$
$$\omega_3(0;0,r,0)=\frac{r(288-84r+35r^2)}{362880},$$
$$\omega_4(0;0,r,0)=\frac{r(-51840+6096r-840r^2+175r^3)}{87091200},$$

and the first few terms of $(\omega_k(2;0,r,0))$ are

$$\omega_0(2;0,r,0)=1,\quad \omega_1(2;0,r,0)=-\frac{r}{360},\quad \omega_2(2;0,r,0)=\frac{r(240+7r)}{1814400},$$
$$\omega_3(2;0,r,0)=-\frac{r(77760+720r+7r^2)}{1959552000},$$
$$\omega_4(2;0,r,0)=\frac{r(6531840000+25850880r+110880r^2+539r^3)}{217275125760000}.$$

Example 5.1 Setting $r=1$ in Corollary 5.3 gives

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(1+\frac{1}{12n^2}+\frac{1}{1440n^4}+\frac{239}{362880n^6}-\frac{46409}{87091200n^8}\right.$$
$$\left.+\frac{9113897}{11496038400n^{10}}-\frac{695818219549}{376610217984000n^{12}}+\cdots\right)^n,\quad n\to\infty,$$

which is the Nemes formula [32]. Besides the case $r=1$, Nemes gave the case $r=\frac{4}{5}$ in [31], and Chen presented the case $r=2$ in [8]. In 2016, Wang obtained the general asymptotic expansion (5.1) and gave the recurrence for the coefficient sequence in [37, Corollary 3.5].

Example 5.2 Setting $r=1$ in Corollary 5.4 yields

$$\prod_{k=1}^{n}k^{k^2}\sim A_2\cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}\mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}}\left(1-\frac{1}{360n^2}+\frac{247}{1814400n^4}-\frac{78487}{1959552000n^6}\right.$$
$$\left.+\frac{6557802299}{217275125760000n^8}-\frac{31014318613001}{726319706112000000n^{10}}+\cdots\right)^n,\quad n\to\infty.$$

Other special cases can be obtained similarly.

6 Conclusions

In this paper, we establish two general asymptotic expansions on the hyperfactorial functions $\prod_{k=1}^{n}k^{k^q}$ and the generalized Glaisher–Kinkelin constants A_q. From these two general expansions, we can not only rediscover some asymptotic expansions that have recently appeared in the literature but also obtain new ones. It would be interesting to find more properties of the hyperfactorial functions and the generalized Glaisher–Kinkelin constants by such a unified way.

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