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Summation formulae of products of the Apostol–Bernoulli and Apostol–Euler polynomials

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Abstract In this paper, a further investigation for the Apostol–Bernoulli and Apostol–Euler polynomials is performed, and some summation formulae of products of the Apostol–Bernoulli and Apostol–Euler polynomials are established by applying some summation transform techniques. Some illustrative special cases as well as immediate consequences of the main results are also considered.

Keywords Apostol–Bernoulli numbers and polynomials · Apostol–Euler numbers and polynomials · Summation methods · Recurrence formulae

Mathematics Subject Classification 11B68 · 05A19

1 Introduction

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are usually defined by means of the following exponential generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi)\quad\text{and}\quad\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad(|t|<\pi).\tag{1.1}$$

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In particular, the rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called the classical Bernoulli numbers and Euler numbers, respectively. These numbers and polynomials appear in many different areas of mathematics such as number theory, combinatorics, special functions, and analysis. Numerous interesting properties for them can be found in many books; see, for example, [9,34,35,42].

Some analogues of the classical Bernoulli and Euler polynomials are the Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ given by means of the following exponential generating functions by Luo and Srivastava [27–29], as follows:

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \quad (|t + \log\lambda| < 2\pi) \tag{1.2}$$

and

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\frac{t^n}{n!} \quad (|t + \log\lambda| < \pi). \tag{1.3}$$

Moreover, $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$ and $\mathcal{E}_n(\lambda) = 2^n\mathcal{E}_n(1/2;\lambda)$ are called the Apostol–Bernoulli numbers and Apostol–Euler numbers, respectively. Obviously $\mathcal{B}_n(x;\lambda)$ and $\mathcal{E}_n(x;\lambda)$ reduce to $B_n(x)$ and $E_n(x)$ when $\lambda = 1$. It is worth mentioning that the Apostol–Bernoulli polynomials were firstly introduced by Apostol [4] (see also Srivastava [41] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. Since the definitions of the above Apostol–Bernoulli and Apostol–Euler polynomials and numbers appeared, some arithmetic properties for them have been well investigated by many authors. For example, in [40], Srivastava and Todorov gave the closed formula for the Apostol–Bernoulli polynomials in terms of the Gaussian hypergeometric function and the Stirling numbers of the second kind; see also [28] for the corresponding closed formula for the Apostol–Euler polynomials. More recently, Luo [30] obtained some multiplication formulae for the Apostol–Bernoulli and Apostol–Euler polynomials to generalize some related results on the classical Bernoulli and Euler polynomials and numbers. Furtherly, Luo [31] considered the Fourier expansions for the Apostol–Bernoulli and Apostol–Euler polynomials by applying the Lipschitz summation formula, and derived some explicit formulae at rational arguments for these polynomials in terms of the Hurwitz zeta function. We also refer to [6,12,15,23,25,36,38,44] for another elegant results and nice methods on these type polynomials and numbers.

In [34], Nielsen presented three formulae of products of the classical Bernoulli and Euler polynomials $B_m(x)B_n(x)$, $E_m(x)E_n(x)$, and $B_m(x)E_n(x)$ for non-negative integers m, n. After that, Carlitz [7] rediscovered the expression of $B_m(x)B_n(x)$, by virtue of which he established a reciprocity formula for Rademacher's Dedekind sums in [8]; see also [26] for another application to deal with the discrete mean value of products of two Dirichlet L-functions at integral arguments. Recently, by establishing three formulae associated with the Nielsen's formulae on the classical Bernoulli and Euler polynomials, He and Zhang [18] reobtained the extensions of the famous Miki's and Woodcock's identities on the classical Bernoulli numbers due to Pan and Sun [37]. We also mention [19–22,43] for further discoveries of the Nielsen's formula on the classical Bernoulli polynomials following the work of Agoh and Dilcher [2,3] on the classical Bernoulli numbers.

In the present paper, we shall be concerned with some summation formulae of products of the Apostol–Bernoulli and Apostol–Euler polynomials. The idea stems from the expressions of some formulae of products of the Apostol–Bernoulli and Apostol–Euler polynomials. By making use of some summation transform techniques, we establish some summation formulae of products of the Apostol–Bernoulli and Apostol–Euler polynomials, by virtue of which some known results related to the famous Miki's and Matiyasevich's identities on the classical Bernoulli numbers are deduced as easy consequences.

2 The statement of results

For convenience, in this section, we always denote by $\delta_{1,\lambda}$ the Kronecker symbol given by $\delta_{1,\lambda} = 0$ or 1 according to $\lambda \neq 1$ or $\lambda = 1$, and denote by H_n the harmonic number given by $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ for positive integer n.

2.1 Extensions of Miki's identity

Theorem 2.1 *Let n be a positive integer with $n \geq 2$. Then*

$$\begin{aligned}
&\sum_{k=1}^{n-1} \frac{\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(y;\mu)}{k(n-k)} \\
&\quad -\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\mathcal{B}_k(x-y;\lambda)\mathcal{B}_{n-k}(y;\lambda\mu)+\mathcal{B}_k(y-x;\mu)\mathcal{B}_{n-k}(x;\lambda\mu)}{k^2} \\
&= \frac{H_{n-1}\big(\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu)+\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)\big)}{n} + \frac{\mathcal{B}_n(x;\lambda\mu)-\mathcal{B}_n(y;\lambda\mu)}{n(x-y)}. \qquad (2.1)
\end{aligned}$$

Proof We recall the formula of products of the Apostol–Bernoulli polynomials stated in [19,20,43], as follows:

$$\begin{aligned}
\mathcal{B}_m(x;\lambda)\mathcal{B}_n(y;\mu) &= n\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{B}_{m-k}\Big(y-x;\frac{1}{\lambda}\Big)\frac{\mathcal{B}_{n+k}(y;\lambda\mu)}{n+k} \\
&\quad + m\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\frac{\mathcal{B}_{m+k}(x;\lambda\mu)}{m+k} \\
&\quad + \delta_{1,\lambda\mu}\frac{(-1)^{m-1}m!\cdot n!}{(m+n)!}\mathcal{B}_{m+n}\Big(y-x;\frac{1}{\lambda}\Big), \qquad (2.2)
\end{aligned}$$

where m, n are positive integers. Since the Apostol–Bernoulli polynomials obey the symmetric distribution $\lambda\mathcal{B}_n(1-x;\lambda) = (-1)^n\mathcal{B}_n(x;1/\lambda)$ and the difference equation $\lambda\mathcal{B}_n(x+1;\lambda) - \mathcal{B}_n(x;\lambda) = nx^{n-1}$ for non-negative integer n (see, e.g., [27]), then for non-negative integer n,

$$(-1)^n\mathcal{B}_n\left(x;\frac{1}{\lambda}\right) = \lambda\mathcal{B}_n(1-x;\lambda) = \mathcal{B}_n(-x;\lambda) + n(-x)^{n-1}. \tag{2.3}$$

We now multiply $1/mn$ in both sides of (2.2). With the help of (2.3) and $\frac{m-k}{m}\binom{m}{k} = \binom{m-1}{k}$ for positive integer m and non-negative integer k, we obtain

$$\begin{aligned}\frac{\mathcal{B}_m(x;\lambda)\mathcal{B}_n(y;\mu)}{mn} &= \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\mathcal{B}_k(x-y;\lambda)\frac{\mathcal{B}_{m+n-k}(y;\lambda\mu)}{m+n-k}\\ &+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(y-x;\mu)\frac{\mathcal{B}_{m+n-k}(x;\lambda\mu)}{m+n-k}\\ &+\sum_{k=0}^{m-1}\binom{m-1}{k}(x-y)^k\frac{\mathcal{B}_{m+n-1-k}(y;\lambda\mu)}{m+n-1-k}\\ &+(-1)^{m-1}\delta_{1,\lambda\mu}\frac{(m-1)!\cdot(n-1)!}{(m+n)!}\mathcal{B}_{m+n}\left(y-x;\frac{1}{\lambda}\right).\end{aligned} \tag{2.4}$$

If we substitute l for m and $n-l$ for n with $1 \le l \le n-1$ and then make the summation operation $\sum_{l=1}^{n-1}$ in both sides of (2.4), we discover

$$\begin{aligned}\sum_{l=1}^{n-1}\frac{\mathcal{B}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu)}{l(n-l)} &= \sum_{l=1}^{n-1}\frac{1}{l}\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(x-y;\lambda)\frac{\mathcal{B}_{n-k}(y;\lambda\mu)}{n-k}\\ &+\sum_{l=1}^{n-1}\frac{1}{l}\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(y-x;\mu)\frac{\mathcal{B}_{n-k}(x;\lambda\mu)}{n-k}\\ &+\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}\binom{l-1}{k}(x-y)^k\frac{\mathcal{B}_{n-1-k}(y;\lambda\mu)}{n-1-k}\\ &+\delta_{1,\lambda\mu}\mathcal{B}_n(y-x;\frac{1}{\lambda})\sum_{l=1}^{n-1}(-1)^{l-1}\frac{(l-1)!\cdot(n-1-l)!}{n!}.\end{aligned} \tag{2.5}$$

Notice that for complex number x and non-negative integer n, (see, e.g., [17, Eq. (2.1)])

$$\sum_{k=0}^{n}\frac{(-1)^k}{\binom{x}{k}} = \frac{x+1}{x+2}\left\{1+\frac{(-1)^n}{\binom{x+1}{n+1}}\right\}, \tag{2.6}$$

which means for positive integer $n \geq 2$,

$$\sum_{l=1}^{n-1}(-1)^{l-1}\frac{(l-1)!\cdot(n-1-l)!}{n!} = \frac{1}{n(n-1)}\sum_{l=0}^{n-2}\frac{(-1)^l}{\binom{n-2}{l}} = \frac{1+(-1)^n}{n^2}. \quad (2.7)$$

Since $\mathcal{B}_0(x;\lambda) = 1$ when $\lambda = 1$ and $\mathcal{B}_0(x;\lambda) = 0$ when $\lambda \neq 1$ (see, e.g., [27]), then $\mathcal{B}_0(x;\lambda)$ can be rewritten as $\mathcal{B}_0(x;\lambda) = \delta_{1,\lambda}$. Hence, applying (2.7) to (2.5) gives

$$\begin{aligned}\sum_{l=1}^{n-1}\frac{\mathcal{B}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu)}{l(n-l)} &= \sum_{l=1}^{n-1}\frac{1}{l}\sum_{k=1}^{l}\binom{l}{k}\mathcal{B}_k(x-y;\lambda)\frac{\mathcal{B}_{n-k}(y;\lambda\mu)}{n-k} \\ &+\sum_{l=1}^{n-1}\frac{1}{l}\sum_{k=1}^{l}\binom{l}{k}\mathcal{B}_k(y-x;\mu)\frac{\mathcal{B}_{n-k}(x;\lambda\mu)}{n-k} \\ &+\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}\binom{l-1}{k}(x-y)^k\frac{\mathcal{B}_{n-1-k}(y;\lambda\mu)}{n-1-k} \\ &+\frac{H_{n-1}\big(\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu)+\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)\big)}{n} \\ &+\frac{\delta_{1,\lambda\mu}\mathcal{B}_n(y-x;\frac{1}{\lambda})\Big(1+(-1)^n\Big)}{n^2}. \end{aligned} \quad (2.8)$$

It is clear that for positive integers k, l, n, (see, e.g., [10])

$$\frac{1}{l}\binom{l}{k} = \frac{1}{k}\binom{l-1}{k-1} \quad \text{and} \quad \binom{n}{k} = \sum_{l=k}^{n}\binom{l-1}{k-1}, \quad (2.9)$$

from which and changing the order of the summations in right-hand side of (2.8) it follows that

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(y;\mu)}{k(n-k)} \\ &=\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{\mathcal{B}_k(x-y;\lambda)\mathcal{B}_{n-k}(y;\lambda\mu)+\mathcal{B}_k(y-x;\mu)\mathcal{B}_{n-k}(x;\lambda\mu)}{k(n-k)} \\ &+\sum_{k=0}^{n-2}\binom{n-1}{k+1}(x-y)^k\frac{\mathcal{B}_{n-1-k}(y;\lambda\mu)}{n-1-k} \\ &+\frac{H_{n-1}\big(\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu)+\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)\big)}{n} \\ &+\frac{\delta_{1,\lambda\mu}\mathcal{B}_n(y-x;\frac{1}{\lambda})\big(1+(-1)^n\big)}{n^2}. \end{aligned} \quad (2.10)$$

If we apply the combinatorial relations:

$$\frac{1}{n-k}\binom{n-1}{k}=\frac{1}{k}\binom{n-1}{k-1} \quad \text{and} \quad \frac{1}{n-1-k}\binom{n-1}{k+1}=\frac{1}{n}\binom{n}{k+1} \tag{2.11}$$

to the first summation and the second one in the right-hand side of (2.10), respectively, we get

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(y;\mu)}{k(n-k)}\\&=\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(x-y;\lambda)\mathcal{B}_{n-k}(y;\lambda\mu)+\mathcal{B}_k(y-x;\mu)\mathcal{B}_{n-k}(x;\lambda\mu)}{k^2}\\&\quad+\frac{1}{n(x-y)}\sum_{k=1}^{n-1}\binom{n}{k}(x-y)^k\mathcal{B}_{n-k}(y;\lambda\mu)\\&\quad+\frac{H_{n-1}\big(\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu)+\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)\big)}{n}\\&\quad+\frac{\delta_{1,\lambda\mu}\mathcal{B}_n(y-x;\frac{1}{\lambda})\big(1+(-1)^n\big)}{n^2}.\end{aligned} \tag{2.12}$$

Since the Apostol–Bernoulli polynomials satisfy the addition theorem (see, e.g., [27]):

$$\mathcal{B}_n(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}x^k\mathcal{B}_{n-k}(y;\lambda) \quad (n\geq 0), \tag{2.13}$$

so by $\mathcal{B}_0(x;\lambda)=\delta_{1,\lambda}$ we have

$$\sum_{k=1}^{n-1}\binom{n}{k}(x-y)^k\mathcal{B}_{n-k}(y;\lambda\mu)=\mathcal{B}_n(x;\lambda\mu)-\mathcal{B}_n(y;\lambda\mu)-\delta_{1,\lambda\mu}(x-y)^n. \tag{2.14}$$

Thus, by applying (2.3) and (2.14) to (2.12), we complete the proof Theorem 2.1. □

Corollary 2.2 *Let n be a positive integer with $n\geq 2$. Then*

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(y)}{k(n-k)}-\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x-y)B_{n-k}(y)+B_k(y-x)B_{n-k}(x)}{k^2}\\&=\frac{H_{n-1}\big(B_n(x)+B_n(y)\big)}{n}+\frac{B_n(x)-B_n(y)}{n(x-y)}.\end{aligned} \tag{2.15}$$

Proof By setting $\lambda=\mu=1$ in Theorem 2.1, the desired result follows immediately. □

The above Corollary 2.2 was firstly established by Pan and Sun [37] using the finite difference calculus and differentiation, and reobtained by He and Zhang [18].

Example 2.3 Since $B_1 = -1/2$ and $B_{2n+1} = 0$ for positive integer n (see, e.g., [1]) then the case $x = y$ and $\lambda = \mu = 1$ in (2.12) gives

$$\sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(x)}{k(n-k)} - 2\sum_{k=2}^{n} \binom{n-1}{k-1} \frac{B_k B_{n-k}(x)}{k^2} = 2H_{n-1}\frac{B_n(x)}{n} \quad (n \geq 2), \tag{2.16}$$

which appears in [37] and can be regarded as an equivalent version of the following Gessel's result (2.18) on the classical Bernoulli polynomials.

Example 2.4 Taking $x = y$ and $\lambda = \mu$ in (2.10) and then applying the second combinatorial relation of (2.11) arises

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(x;\lambda)}{k(n-k)} &= \frac{2}{n}\sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(x;\lambda^2)\frac{\mathcal{B}_{n-k}(\lambda)}{n-k} + \mathcal{B}_{n-1}(x;\lambda^2) \\ &\quad + \frac{2\delta_{1,\lambda}H_{n-1}\mathcal{B}_n(x;\lambda^2)}{n} \\ &\quad + \frac{\delta_{1,\lambda^2}\mathcal{B}_n(\frac{1}{\lambda})\big(1+(-1)^n\big)}{n^2} \quad (n \geq 2). \end{aligned} \tag{2.17}$$

Example 2.5 Since $B_{2n+1} = 0$ for positive integer n, then the case $\lambda = 1$ in (2.17) yields

$$\begin{aligned} &\frac{n}{2}\left(-B_{n-1}(x) + \sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(x)}{k(n-k)}\right) - \sum_{k=0}^{n-1}\binom{n}{k}B_k(x)\frac{B_{n-k}}{n-k} \\ &= H_{n-1}B_n(x) \quad (n \geq 2), \end{aligned} \tag{2.18}$$

which is due to Gessel [16] and was reobtained by some authors; see, for example, [5,11,24].

Remark 2.6 It is worth mentioning that the case $x = 0$ and $x = 1/2$ in (2.18) gives the famous Miki's identity (see, e.g., [13,32,33,39]):

$$\sum_{k=2}^{n-2}\frac{B_k B_{n-k}}{k(n-k)} - \sum_{k=2}^{n-2}\binom{n}{k}\frac{B_k B_{n-k}}{k(n-k)} = 2H_n\frac{B_n}{n} \quad (n \geq 4), \tag{2.19}$$

and Faber–Pandharipande–Zagier's identity (see, e.g., [14,37]):

$$\frac{n}{2}\sum_{k=2}^{n-2}\frac{\widetilde{B}_k\widetilde{B}_{n-k}}{k(n-k)}-\sum_{k=2}^{n}\binom{n}{k}\frac{B_k}{k}\widetilde{B}_{n-k}=H_{n-1}\widetilde{B}_n$$
$$\left(n\geq 4,\, \widetilde{B}_k=B_k\left(\frac{1}{2}\right)=(2^{1-k}-1)B_k\right), \tag{2.20}$$

respectively.

2.2 Extensions of Matiyasevich's identity

Theorem 2.7 *Let n be a positive integer with $n \geq 2$. Then*

$$\begin{aligned}\sum_{k=1}^{n-1}&\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(y;\mu)\\ &=\sum_{k=0}^{n}\binom{n+1}{k+1}\frac{\mathcal{B}_k(x-y;\lambda)\mathcal{B}_{n-k}(y;\lambda\mu)+\mathcal{B}_k(y-x;\mu)\mathcal{B}_{n-k}(x;\lambda\mu)}{k+2}\\ &\quad+\frac{\mathcal{B}_{n+1}(x;\lambda\mu)+\mathcal{B}_{n+1}(y;\lambda\mu)}{(x-y)^2}-\frac{2}{n+2}\cdot\frac{\mathcal{B}_{n+2}(x;\lambda\mu)-\mathcal{B}_{n+2}(y;\lambda\mu)}{(x-y)^3}\\ &\quad-\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)-\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu).\end{aligned} \tag{2.21}$$

Proof By substituting l for m and $n-l$ for n with $1 \leq l \leq n-1$ and then making the summation operation $\sum_{l=1}^{n-1}$ in both sides of (2.2), we obtain

$$\begin{aligned}\sum_{l=1}^{n-1}\mathcal{B}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu)&=\sum_{l=1}^{n-1}(n-l)\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(x-y;\lambda)\frac{\mathcal{B}_{n-k}(y;\lambda\mu)}{n-k}\\ &\quad+\sum_{l=1}^{n-1}(n-l)\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(y-x;\mu)\frac{\mathcal{B}_{n-k}(x;\lambda\mu)}{n-k}\\ &\quad+\sum_{l=1}^{n-1}l(n-l)\sum_{k=0}^{l-1}\binom{l-1}{k}(x-y)^k\frac{\mathcal{B}_{n-1-k}(y;\lambda\mu)}{n-1-k}\\ &\quad-\delta_{1,\lambda\mu}\mathcal{B}_n(y-x;\frac{1}{\lambda})\sum_{l=1}^{n-1}\frac{(-1)^l}{\binom{n}{l}}.\end{aligned} \tag{2.22}$$

Since $\mathcal{B}_0(x;\lambda)=\delta_{1,\lambda}$ and $l\binom{l-1}{k}=(k+1)\binom{l}{k+1}$ for positive integer l and non-negative integer k, so by (2.7) we can rewrite (2.22) as

$$
\begin{aligned}
&\sum_{l=1}^{n-1}\mathcal{B}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu)\\
&\quad=\sum_{l=0}^{n-1}(n-l)\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(x-y;\lambda)\frac{\mathcal{B}_{n-k}(y;\lambda\mu)}{n-k}-\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)\\
&\qquad+\sum_{l=0}^{n-1}(n-l)\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(y-x;\mu)\frac{\mathcal{B}_{n-k}(x;\lambda\mu)}{n-k}-\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu)\\
&\qquad+\sum_{l=1}^{n-1}(n-l)\sum_{k=0}^{l-1}\binom{l}{k+1}(k+1)(x-y)^k\frac{\mathcal{B}_{n-1-k}(y;\lambda\mu)}{n-1-k}\\
&\qquad+\delta_{1,\lambda\mu}\big(1+(-1)^n\big)\frac{\mathcal{B}_n(y-x;\frac{1}{\lambda})}{n+2}.
\end{aligned}
\tag{2.23}
$$

Note that for non-negative integers n, r, s, (see, e.g., [17, Eq. (3.3)])

$$
\sum_{k=r}^{n-s}\binom{k}{r}\binom{n-k}{s}=\binom{n+1}{r+s+1}.
\tag{2.24}
$$

Hence, by taking $s=1$ in (2.24), we get that for positive integer n and non-negative integer k,

$$
\sum_{l=k}^{n-1}\frac{n-l}{n-k}\binom{l}{k}=\frac{1}{k+2}\binom{n+1}{k+1}.
\tag{2.25}
$$

We now change the order of the summations in the right-hand side of (2.23). In light of (2.25), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{n-1}\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(y;\mu)\\
&\quad=\sum_{k=0}^{n-1}\binom{n+1}{k+1}\frac{\mathcal{B}_k(x-y;\lambda)\mathcal{B}_{n-k}(y;\lambda\mu)+\mathcal{B}_k(y-x;\mu)\mathcal{B}_{n-k}(x;\lambda\mu)}{k+2}\\
&\qquad+\sum_{k=0}^{n-2}\binom{n+1}{k+2}\frac{k+1}{k+3}(x-y)^k\mathcal{B}_{n-1-k}(y;\lambda\mu)+\delta_{1,\lambda\mu}\big(1+(-1)^n\big)\frac{\mathcal{B}_n(y-x;\frac{1}{\lambda})}{n+2}\\
&\qquad-\delta_{1,\lambda}\mathcal{B}_n(y;\lambda\mu)-\delta_{1,\mu}\mathcal{B}_n(x;\lambda\mu).
\end{aligned}
\tag{2.26}
$$

Observe that

$$\sum_{k=0}^{n-2}\binom{n+1}{k+2}\frac{k+1}{k+3}(x-y)^k\mathcal{B}_{n-1-k}(y;\lambda\mu)$$
$$=\frac{1}{(x-y)^2}\sum_{k=2}^{n}\binom{n+1}{k}(x-y)^k\mathcal{B}_{n+1-k}(y;\lambda\mu)$$
$$-\frac{2}{(n+2)(x-y)^3}\sum_{k=3}^{n+1}\binom{n+2}{k}(x-y)^k\mathcal{B}_{n+2-k}(y;\lambda\mu). \tag{2.27}$$

It is easily seen from (2.13) and $\mathcal{B}_0(x;\lambda)=\delta_{1,\lambda}$ that

$$\frac{1}{(x-y)^2}\sum_{k=2}^{n}\binom{n+1}{k}(x-y)^k\mathcal{B}_{n+1-k}(y;\lambda\mu)$$
$$=\frac{\mathcal{B}_{n+1}(x;\lambda\mu)-\mathcal{B}_{n+1}(y;\lambda\mu)}{(x-y)^2}-\frac{(n+1)\mathcal{B}_n(y;\lambda\mu)}{x-y}-\delta_{1,\lambda\mu}(x-y)^{n-1}, \tag{2.28}$$

and

$$\frac{2}{(n+2)(x-y)^3}\sum_{k=3}^{n+1}\binom{n+2}{k}(x-y)^k\mathcal{B}_{n+2-k}(y;\lambda\mu)$$
$$=\frac{2}{n+2}\cdot\frac{\mathcal{B}_{n+2}(x;\lambda\mu)-\mathcal{B}_{n+2}(y;\lambda\mu)}{(x-y)^3}-\frac{2\mathcal{B}_{n+1}(y;\lambda\mu)}{(x-y)^2}$$
$$-\frac{(n+1)\mathcal{B}_n(y;\lambda\mu)}{x-y}-\frac{2\delta_{1,\lambda\mu}(x-y)^{n-1}}{n+2}. \tag{2.29}$$

It follows from (2.27), (2.28) and (2.29) that

$$\sum_{k=0}^{n-2}\binom{n+1}{k+2}\frac{k+1}{k+3}(x-y)^k\mathcal{B}_{n-1-k}(y;\lambda\mu)$$
$$=\frac{\mathcal{B}_{n+1}(x;\lambda\mu)+\mathcal{B}_{n+1}(y;\lambda\mu)}{(x-y)^2}-\frac{2}{n+2}\cdot\frac{\mathcal{B}_{n+2}(x;\lambda\mu)-\mathcal{B}_{n+2}(y;\lambda\mu)}{(x-y)^3}$$
$$-\frac{\delta_{1,\lambda\mu}n(x-y)^{n-1}}{n+2}. \tag{2.30}$$

Thus, in view of (2.26) and (2.30), the desired result follows by applying (2.3). □

Corollary 2.8 (Pan and Sun [37]) *Let n be a positive integer. Then*

$$\begin{aligned}\sum_{k=0}^{n} B_k(x)B_{n-k}(y) &= \sum_{k=0}^{n}\binom{n+1}{k+1}\frac{B_k(x-y)B_{n-k}(y)+B_k(y-x)B_{n-k}(x)}{k+2}+\frac{B_{n+1}(x)+B_{n+1}(y)}{(x-y)^2}\\ &\quad -\frac{2}{n+2}\cdot\frac{B_{n+2}(x)-B_{n+2}(y)}{(x-y)^3}. \end{aligned} \tag{2.31}$$

Proof Setting $\lambda=\mu=1$ in Theorem 2.7 gives the desired result. □

Example 2.9 By taking $x=y$ and $\lambda=\mu$ in (2.26), we get

$$\begin{aligned}\sum_{k=1}^{n-1}\mathcal{B}_k(x;\lambda)\mathcal{B}_{n-k}(x;\lambda) &= 2\sum_{k=0}^{n-1}\binom{n+1}{k+1}\frac{\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(x;\lambda^2)}{k+2}+\frac{n(n+1)}{6}\mathcal{B}_{n-1}(x;\lambda^2)\\ &\quad +\delta_{1,\lambda^2}\left(1+(-1)^n\right)\frac{\mathcal{B}_n(\frac{1}{\lambda})}{n+2}-\delta_{1,\lambda}2\mathcal{B}_n(x;\lambda^2)\quad (n\geq 2).\end{aligned} \tag{2.32}$$

Example 2.10 Since $B_0=1$, $B_1=-1/2$ and $B_{2n+1}=0$ for positive integer n, then the case $\lambda=1$ in (2.32) arises that for positive integer $n\geq 2$, (see, e.g., [37])

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x)-2\sum_{k=2}^{n}\binom{n+1}{k+1}\frac{B_kB_{n-k}(x)}{k+2}=(n+1)B_n(x). \tag{2.33}$$

Remark 2.11 It is worth noticing that the case $x=0$ in (2.33) gives that for positive integer $n\geq 4$,

$$(n+2)\sum_{k=2}^{n-2}B_kB_{n-k}-2\sum_{k=2}^{n-2}\binom{n+2}{k}B_kB_{n-k}=n(n+1)B_n, \tag{2.34}$$

which was firstly discovered by Matiyasevich [32] using the software *Mathematica*.

2.3 Analogues of Matiyasevich's identity

Theorem 2.12 *Let n be a non-negative integer. Then*

$$\sum_{k=0}^{n}\mathcal{E}_k(x;\lambda)\mathcal{E}_{n-k}(y;\mu)$$
$$=-2\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{\mathcal{E}_k(x-y;\lambda)\mathcal{B}_{n+1-k}(y;\lambda\mu)+\mathcal{E}_k(y-x;\mu)\mathcal{B}_{n+1-k}(x;\lambda\mu)}{k+1}$$
$$+\frac{4}{n+2}\cdot\frac{\mathcal{B}_{n+2}(x;\lambda\mu)-\mathcal{B}_{n+2}(y;\lambda\mu)}{x-y}. \tag{2.35}$$

Proof Since the Apostol–Euler polynomials satisfy the symmetric distribution $\lambda\mathcal{E}_n(1-x;\lambda)=(-1)^n\mathcal{E}_n(x;1/\lambda)$ and the difference equation $\lambda\mathcal{E}_n(x+1;\lambda)+\mathcal{E}_n(x;\lambda)=2x^n$ for non-negative integer n (see, e.g., [28]), then for non-negative integer n,

$$(-1)^n\mathcal{E}_n\Big(x;\frac{1}{\lambda}\Big)=\lambda\mathcal{E}_n(1-x;\lambda)=2(-x)^n-\mathcal{E}_n(-x;\lambda). \tag{2.36}$$

Hence, by applying (2.36) to the following formula of mixed products of the Apostol–Bernoulli and Apostol–Euler polynomials stated in [19,20,43],

$$\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu)=2\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}\Big(y-x;\frac{1}{\lambda}\Big)\frac{\mathcal{B}_{n+1+k}(y;\lambda\mu)}{n+1+k}$$
$$-2\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}(y-x;\mu)\frac{\mathcal{B}_{m+1+k}(x;\lambda\mu)}{m+1+k}$$
$$+2\delta_{1,\lambda\mu}\frac{(-1)^{m-1}m!\cdot n!}{(m+n+1)!}\mathcal{E}_{m+n+1}\Big(y-x;\frac{1}{\lambda}\Big)\quad(m,n\geq 0), \tag{2.37}$$

we get that for non-negative integers m,n,

$$\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu)=4\sum_{k=0}^{m}\binom{m}{k}(x-y)^k\frac{\mathcal{B}_{m+n+1-k}(y;\lambda\mu)}{m+n+1-k}$$
$$-2\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_k(x-y;\lambda)\frac{\mathcal{B}_{m+n+1-k}(y;\lambda\mu)}{m+n+1-k}$$
$$-2\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(y-x;\mu)\frac{\mathcal{B}_{m+n+1-k}(x;\lambda\mu)}{m+n+1-k}$$
$$+2\delta_{1,\lambda\mu}\frac{(-1)^{m-1}m!\cdot n!}{(m+n+1)!}\mathcal{E}_{m+n+1}(y-x;\frac{1}{\lambda}). \tag{2.38}$$

If we replace m by l and n by $n-l$ with $0 \le l \le n$ and make the summation operation $\sum_{l=0}^{n}$ in both sides of (2.38), we discover

$$\begin{aligned}\sum_{l=0}^{n} \mathcal{E}_l(x;\lambda)\mathcal{E}_{n-l}(y;\mu) &= 4\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}(x-y)^k \frac{\mathcal{B}_{n+1-k}(y;\lambda\mu)}{n+1-k} \\ &\quad -2\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}\mathcal{E}_k(x-y;\lambda)\frac{\mathcal{B}_{n+1-k}(y;\lambda\mu)}{n+1-k} \\ &\quad -2\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}\mathcal{E}_k(y-x;\mu)\frac{\mathcal{B}_{n+1-k}(x;\lambda\mu)}{n+1-k} \\ &\quad -2\delta_{1,\lambda\mu}\frac{\mathcal{E}_{n+1}(y-x;\frac{1}{\lambda})}{n+1}\sum_{l=0}^{n}\frac{(-1)^l}{\binom{n}{l}}. \end{aligned} \tag{2.39}$$

By changing the order of the summations in the right-hand side of (2.39) and then applying the second formula of (2.9) and the first one of (2.11), in light of (2.6), we obtain

$$\begin{aligned}&\sum_{k=0}^{n} \mathcal{E}_k(x;\lambda)\mathcal{E}_{n-k}(y;\mu) \\ &= \frac{4}{n+2}\sum_{k=0}^{n}\binom{n+2}{k+1}(x-y)^k\mathcal{B}_{n+1-k}(y;\lambda\mu) \\ &\quad -2\sum_{k=0}^{n}\binom{n+1}{k}\frac{\mathcal{E}_k(x-y;\lambda)\mathcal{B}_{n+1-k}(y;\lambda\mu)+\mathcal{E}_k(y-x;\mu)\mathcal{B}_{n+1-k}(x;\lambda\mu)}{k+1} \\ &\quad -2\delta_{1,\lambda\mu}\frac{\left(1+(-1)^n\right)\mathcal{E}_{n+1}(y-x;\frac{1}{\lambda})}{n+2}. \end{aligned} \tag{2.40}$$

It is easily seen that the formula (2.13) implies

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n+2}{k+1}(x-y)^k\mathcal{B}_{n+1-k}(y;\lambda\mu) \\ &\quad = \frac{\mathcal{B}_{n+2}(x;\lambda\mu)-\mathcal{B}_{n+2}(y;\lambda\mu)}{x-y} - \delta_{1,\lambda\mu}(x-y)^{n+1}. \end{aligned} \tag{2.41}$$

So from (2.36), (2.40), and (2.41), the desired result follows immediately. □

Corollary 2.13 (Pan and Sun [37]) *Let n be a non-negative integer. Then*

$$\sum_{l=0}^{n} E_l(x)E_{n-l}(y) = -2\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{E_k(x-y)B_{n+1-k}(y)+E_k(y-x)B_{n+1-k}(x)}{k+1} + \frac{4}{n+2}\cdot\frac{B_{n+2}(x)-B_{n+2}(y)}{x-y}. \tag{2.42}$$

Proof Setting $\lambda = \mu = 1$ in Theorem 2.12 gives the desired result. □

Example 2.14 By taking $x = y$ and $\lambda = \mu$ in (2.40), we get

$$\sum_{k=0}^{n}\mathcal{E}_k(x;\lambda)\mathcal{E}_{n-k}(x;\lambda) = 4\mathcal{B}_{n+1}(x;\lambda^2) - 4\sum_{k=0}^{n}\binom{n+1}{k}\frac{\mathcal{E}_k(0;\lambda)\mathcal{B}_{n+1-k}(x;\lambda^2)}{k+1} - 2\delta_{1,\lambda^2}\frac{(1+(-1)^n)\mathcal{E}_{n+1}(0;\frac{1}{\lambda})}{n+2} \quad (n \ge 0). \tag{2.43}$$

Example 2.15 Since $B_0(x) = E_0(x) = 1$ and $E_n(0) = (2-2^{n+2})B_{n+1}/(n+1)$ for non-negative integer n (see, e.g., [1]), so by taking $\lambda = 1$ in (2.43), in view of the first formula of (2.9), we obtain that for non-negative integer n,

$$\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = \frac{8}{n+2}\sum_{k=2}^{n+2}\binom{n+2}{k}\frac{(2^k-1)B_k}{k}B_{n+2-k}(x) \tag{2.44}$$

$$= -\frac{4}{n+2}\sum_{k=0}^{n}\binom{n+2}{k}E_{n+1-k}(0)B_k(x). \tag{2.45}$$

Remark 2.16 The identity (2.44) was discovered by Pan and Sun [37], and the identity (2.45) was also stated in [24] (but note a misprint: E_{n+1-k} should be $E_{n+1-k}(0)$).

Theorem 2.17 *Let n be a positive integer. Then*

$$\sum_{l=0}^{n-1}\mathcal{E}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu) = \frac{1}{2}\sum_{k=1}^{n}\binom{n+1}{k+1}\big(2\mathcal{B}_k(y-x;\mu)\mathcal{E}_{n-k}(x;\lambda\mu) - \mathcal{E}_{k-1}(x-y;\lambda)\mathcal{E}_{n-k}(y;\lambda\mu)\big) + \frac{\mathcal{E}_{n+1}(x;\lambda\mu)-\mathcal{E}_{n+1}(y;\lambda\mu)}{(x-y)^2} + \delta_{1,\mu}n\mathcal{E}_n(x;\lambda\mu) - \frac{(n+1)\mathcal{E}_n(y;\lambda\mu)}{x-y}. \tag{2.46}$$

Proof By applying (2.36) to another formula of mixed products of the Apostol–Bernoulli and Apostol–Euler polynomials stated in [19,20,43], namely

$$\begin{aligned}&\mathcal{E}_{m-1}(x;\lambda)\mathcal{B}_n(y;\mu)\\&=\frac{n}{2}\sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^{m-1-k}\mathcal{E}_{m-1-k}(y-x;\frac{1}{\lambda})\mathcal{E}_{n-1+k}(y;\lambda\mu)\\&\quad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\mathcal{E}_{m-1+k}(x;\lambda\mu)\quad(m,n\geq 1),\end{aligned}\tag{2.47}$$

we obtain that for positive integers m, n,

$$\begin{aligned}\mathcal{E}_{m-1}(x;\lambda)\mathcal{B}_n(y;\mu)&=n\sum_{k=0}^{m-1}\binom{m-1}{k}(x-y)^k\mathcal{E}_{m+n-2-k}(y;\lambda\mu)\\&\quad-\frac{n}{2}\sum_{k=0}^{m-1}\binom{m-1}{k}\mathcal{E}_k(x-y;\lambda)\mathcal{E}_{m+n-2-k}(y;\lambda\mu)\\&\quad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(y-x;\mu)\mathcal{E}_{m+n-1-k}(x;\lambda\mu).\end{aligned}\tag{2.48}$$

If we substitute m for l and n for $n+1-l$ and make the summation operation $\sum_{l=1}^{n}$ in both sides of (2.48), we discover

$$\begin{aligned}&\sum_{l=0}^{n-1}\mathcal{E}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu)\\&=\sum_{l=1}^{n}(n+1-l)\sum_{k=1}^{l}\binom{l-1}{k-1}(x-y)^{k-1}\mathcal{E}_{n-k}(y;\lambda\mu)\\&\quad-\frac{1}{2}\sum_{l=1}^{n}(n+1-l)\sum_{k=1}^{l}\binom{l-1}{k-1}\mathcal{E}_{k-1}(x-y;\lambda)\mathcal{E}_{n-k}(y;\lambda\mu)\\&\quad+\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_k(y-x;\mu)\mathcal{E}_{n-k}(x;\lambda\mu)-\delta_{1,\mu}\mathcal{E}_n(x;\lambda\mu).\end{aligned}\tag{2.49}$$

Taking $s=1$ and substituting $r-1$ for r in (2.24) gives that for positive integers n, k,

$$\sum_{l=k}^{n}\binom{l-1}{k-1}(n+1-l)=\binom{n+1}{k+1}.\tag{2.50}$$

By changing the order of the summations in the right-hand side of (2.49) and then applying the second formula of (2.9) and (2.50), we get

$$\sum_{l=0}^{n-1}\mathcal{E}_l(x;\lambda)\mathcal{B}_{n-l}(y;\mu)=\sum_{k=1}^{n}\binom{n+1}{k+1}(x-y)^{k-1}\mathcal{E}_{n-k}(y;\lambda\mu)$$
$$+\frac{1}{2}\sum_{k=1}^{n}\binom{n+1}{k+1}\big(2\mathcal{B}_k(y-x;\mu)\mathcal{E}_{n-k}(x;\lambda\mu)$$
$$-\mathcal{E}_{k-1}(x-y;\lambda)\mathcal{E}_{n-k}(y;\lambda\mu)\big)+\delta_{1,\mu}n\mathcal{E}_n(x;\lambda\mu). \tag{2.51}$$

Since the Apostol–Euler polynomials satisfy the addition theorem (see, e.g., [27]):

$$\mathcal{E}_n(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}x^k\mathcal{E}_{n-k}(y;\lambda)\quad(n\geq 0), \tag{2.52}$$

then

$$\sum_{k=1}^{n}\binom{n+1}{k+1}(x-y)^{k-1}\mathcal{E}_{n-k}(y;\lambda\mu)=\frac{\mathcal{E}_{n+1}(x;\lambda\mu)-\mathcal{E}_{n+1}(y;\lambda\mu)}{(x-y)^2}$$
$$-\frac{(n+1)\mathcal{E}_n(y;\lambda\mu)}{x-y}. \tag{2.53}$$

Thus, by combining (2.51) and (2.53), we complete the proof of Theorem 2.17. □

Corollary 2.18 (Pan and Sun [37]) *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}E_k(x)B_{n-k}(y)=\frac{1}{2}\sum_{k=1}^{n}\binom{n+1}{k+1}\big(2B_k(y-x)E_{n-k}(x)-E_{k-1}(x-y)E_{n-k}(y)\big)$$
$$+\frac{E_{n+1}(x)-E_{n+1}(y)}{(x-y)^2}-(n+1)\left(\frac{E_n(y)}{x-y}-E_n(x)\right). \tag{2.54}$$

Proof Taking $\lambda=\mu=1$ in Theorem 2.17 gives the desired result. □

Example 2.19 Since $E_n(0)=(2-2^{n+2})B_{n+1}/(n+1)$ for non-negative integer n, so by taking $x=y$ and $\lambda=\mu=1$ in (2.51), we obtain that for positive integer $n\geq 2$, (see, e.g., [37])

$$\sum_{k=0}^{n}E_k(x)B_{n-k}(x)=\sum_{k=2}^{n}\binom{n+1}{k+1}\frac{(2^k+k-1)B_k}{k}E_{n-k}(x)+(n+1)E_n(x). \tag{2.55}$$

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