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# The average number of divisors of the Euler function

**Sungjin Kim**[1,2]

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**Abstract** The upper bound and the lower bound of the average number of divisors of Euler Phi function and Carmichael Lambda function were obtained by Luca and Pomerance (see Publ Math Debr 70(1–2):125–148, 2007). We improve the lower bound and provide a heuristic argument which suggests that the upper bound given by Luca and Pomerance (Publ Math Debr 70(1–2):125–148, 2007) is indeed close to the truth.

**Keywords** Euler · Carmichael · Number of divisors · Average

**Mathematics Subject Classification** 11A25

## 1 Introduction

Let $n \geq 1$ be an integer. Denote by $\phi(n)$ and $\lambda(n)$ the Euler Phi function and the Carmichael Lambda function, which output the order and the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$, respectively. We use $p$(or $p_i$) and $q$(or $q_i$) to denote the prime divisors of $n$ and $\phi(n)$, respectively. Then it is clear that $\lambda(n)|\phi(n)$ and the set of prime divisors $q$ of $\phi(n)$ and that of $\lambda(n)$ are identical. Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of $n$. Then we can compute $\phi(n)$ and $\lambda(n)$ as follows:

$$\phi(n) = \prod_{i=1}^{r} \phi(p_i^{e_i}), \text{ and } \lambda(n) = \operatorname{lcm}\left(\lambda(p_1^{e_1}), \ldots, \lambda(p_r^{e_r})\right),$$

✉ Sungjin Kim
707107@gmail.com

1 Concordia University Irvine, Irvine, CA, USA

2 Santa Monica College, Santa Monica, CA, USA

where $\phi(p_i^{e_i}) = p_i^{e_i-1}(p_i - 1)$ and $\lambda(p_i^{e_i}) = \phi(p_i^{e_i})$ if $p_i > 2$ or $p_i = 2$ and $e_i = 1, 2$, and $\lambda(2^e) = 2^{e-2}$ if $e \geq 3$.

From the work of Hardy and Ramanujan [4], it is well known that the normal order of $\tau(n)$ is $(\log n)^{\log 2 + o(1)}$. On the other hand, the average order $\frac{1}{x}\sum_{n\leq x}\tau(n)$ is known to be $\log x + O(1)$ which is somewhat larger than the normal order. For $\tau(\lambda(n))$ and $\tau(\phi(n))$, the normal orders of these follow from [2] that they are $2^{(\frac{1}{2}+o(1))(\log\log n)^2}$. On the contrary, the work of Luca and Pomerance [5] showed that their average order is significantly larger than the normal order. Define $F(x) = \exp\left(\sqrt{\frac{\log x}{\log\log x}}\right)$. In [5, Theorem 1,2], they proved that

$$F(x)^{b_1+o(1)} \leq \frac{1}{x}\sum_{n\leq x}\tau(\lambda(n)) \leq \frac{1}{x}\sum_{n\leq x}\tau(\phi(n)) \leq F(x)^{b_2+o(1)}$$

as $x \to \infty$, where $b_1 = \frac{1}{7}e^{-\gamma/2}$ and $b_2 = 2\sqrt{2}e^{-\gamma/2}$.

In this paper, we are able to raise the constant $b_1$ so that it is almost $b_2$, differing only by a factor $\sqrt{2}$. Here, we take advantage of the inequalities of Bombieri–Vinogradov type regarding primes in arithmetic progression (see [1, Theorem 9], also [3, Theorem 2.1]). In this paper, we apply the following version which can be obtained from [3, Theorem 2.1]: For $(a, n) = 1$, we write $E(x; n, a) := \pi(x; n, a) - \frac{\pi(x)}{\phi(n)}$. Let $0 < \lambda < 1/10$. Let $R \leq x^{\lambda}$. For some $B = B(A) > 0$, $M = \log^B x$, and $Q = x/M$,

$$\sum_{\substack{r\leq R\\(r,a)=1}}\left|\sum_{\substack{q\leq\frac{Q}{r}\\(q,a)=1}} E(x; qr, a)\right| \ll_{A,\lambda} x\log^{-A} x.$$

In fact, [3, Theorem 2.1] builds on [1, Theorem 9] and obtains a more accurate estimate, but we only need the above form for our purpose. Note that one of the important differences between [1, Theorem 9] and [3, Theorem 2.1] is the presence of $\frac{Q}{r}$ in the inner sum. This will be essential in the proof of our lemmas (see Lemmas 2.2 and 2.3).

It is interesting to note that one of these improvements is related to a Poisson distribution that we can obtain from prime numbers. Another point of improvement comes from the idea in the proof of Gauss' Circle Problem.

**Theorem 1.1** *As $x \to \infty$, we have*

$$\sum_{n\leq x}\tau(\phi(n)) \geq \sum_{n\leq x}\tau(\lambda(n)) \geq x\exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

It is clear from $\lambda(n)|\phi(n)$ that $\sum_{n\leq x}\tau(\lambda(n)) \leq \sum_{n\leq x}\tau(\phi(n))$. A natural question to ask is how large is the latter compared to the former. Luca and Pomerance proved in [5, Theorem 2] that

$$\frac{1}{x}\sum_{n\le x}\tau(\lambda(n)) = o\left(\max_{y\le x}\frac{1}{y}\sum_{n\le y}\tau(\phi(n))\right).$$

Moreover, they mentioned that a stronger statement

$$\frac{1}{x}\sum_{n\le x}\tau(\lambda(n)) = o\left(\frac{1}{x}\sum_{n\le x}\tau(\phi(n))\right)$$

is probably true, but they did not have the proof. Here, we prove that this statement is indeed true. As in the proof of [5, Theorem 2], we take advantage of the fact that prime 2 appears rarely in the factorization of $\lambda(n)$ than in the factorization of $\phi(n)$.

**Theorem 1.2** *As $x \to \infty$, we have*

$$\sum_{n\le x}\tau(\lambda(n)) = o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

Finally, we provide a heuristic argument suggesting that the constant in the upper bound is indeed optimal. Here, we try to extend the method in the proof of Theorem 1.1 by devising a binomial distribution model. However, we were unable to prove it. The main difficulty is due to the short range of $u$ ($u < \log^{A_1} x$) in the lemmas (see Lemmas 2.1, 2.3 and Corollaries 2.1, 2.2).

**Conjecture 1.1** *As $x \to \infty$, we have*

$$\sum_{n\le x}\tau(\lambda(n)) = x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

Throughout this paper, $x$ is a positive real number, $n, k$ are positive integers, and $p, q$ are prime numbers. We use Landau symbols $O$ and $o$. Also, we write $f(x) \asymp g(x)$ for positive functions $f$ and $g$, if $f(x) = O(g(x))$ and $g(x) = O(f(x))$. We will also use Vinogradov symbols $\ll$ and $\gg$. We write the iterated logarithms as $\log_2 x = \log\log x$ and $\log_3 x = \log\log\log x$. The notations $(a, b)$ and $[a, b]$ mean the greatest common divisor and the least common multiple of $a$ and $b$, respectively. We write $P_z = \prod_{p\le z} p$. We also use the following restricted divisor functions:

$$\tau_z(n) := \prod_{\substack{p^e\|n\\ p>z}}\tau(p^e),\quad \tau_{z,w}(n) := \prod_{\substack{p^e\|n\\ z<p\le w}}\tau(p^e),\quad \text{and}\quad \tau_z'(n) := \prod_{\substack{p^e\|n\\ p\le z}}\tau(p^e).$$

Moreover, for $n > 1$, denote by $p(n)$ the smallest prime factor of $n$.

## 2 Lemmas

The following lemma is [5, Lemma 3] with a slightly relaxed $z$, and it is essential toward proving the theorem. This is stated and proved with the Chebyshev functions

$\psi(x) := \sum_{n\le x}\Lambda(n)$ and $\psi(x;q,a) := \sum_{n\le x,\ n\equiv a \bmod q}\Lambda(n)$ in [6]. Here, we use the prime counting functions $\pi(x) := \sum_{p\le x} 1$ and $\pi(x;q,a) := \sum_{p\le x,\ p\equiv a \bmod q} 1$ instead. We are allowed to do these replacements by applying the partial summation.

**Lemma 2.1** *Let* $0<\lambda<\frac{1}{10}$. *Assume that* $z\le \lambda\log x$. *Then, for any* $A>0$, *there is* $B=B(A)>0$ *such that, for* $M=\log^B x$ *and* $Q=\frac{x}{M}$,

$$E_z(x) := \sum_{r|P_z}\mu(r)\sum_{\substack{n\le Q\\ r|n}}\left(\pi(x;n,1)-\frac{\pi(x)}{\phi(n)}\right) \ll_{A,\lambda} \frac{x}{\log^A x}. \tag{1}$$

*Let* $0<\lambda<\frac{1}{10}$. *Assume that* $u$ *is a positive integer with* $p(u)>z$, $u<(\log x)^{A_1}$, *and* $\tau(u)<A_1$. *Then, for any* $A>0$, *there is* $B=B(A,A_1)>0$ *such, that for* $M=\log^B x$ *and* $Q=\frac{x}{M}$,

$$E_{u,z}(x) := \sum_{r|P_z}\mu(r)\sum_{\substack{n\le Q\\ r|n}}\left(\pi(x;[u,n],1)-\frac{\pi(x)}{\phi([u,n])}\right) \ll_{A,A_1,\lambda} \frac{x}{\log^A x}. \tag{2}$$

*Proof of* (1) For $(a,n)=1$, we write $E(x;n,a) := \pi(x;n,a)-\frac{\pi(x)}{\phi(n)}$. If $r|P_z$, we have by the Prime Number Theorem $r\le R := P_z = \exp(z+o(z)) \le x^{\lambda'}$ with $0<\lambda'<1/10$. By partial summation and diadically applying [3, Theorem 2.1], we have, for $B=B(A)>0$, $M=\log^B x$, and $Q=x/M$,

$$\sum_{\substack{r\le R\\ (r,a)=1}}\left|\sum_{\substack{q\le \frac{Q}{r}\\ (q,a)=1}} E(x;qr,a)\right| \ll_{A,\lambda} \frac{x}{\log^A x}. \tag{3}$$

Taking $a=1$ and $|\mu(r)|\le 1$, (1) follows. □

*Proof of* (2) Let $d\le x^{\epsilon}$ so that $dR\le x^{\lambda'}$ with $0<\lambda'<1/10$. By (3), there exists $B=B(A)>0$ such that we have, for $M=\log^B x$ and $Q=x/M$,

$$\begin{aligned}\sum_{r\le R}\left|\sum_{q\le\frac{Q}{r}} E(x;dqr,1)\right| &= \sum_{\substack{r\le dR\\ r\equiv 0 \bmod d}}\left|\sum_{q\le\frac{Q}{r}} E(x;qr,1)\right|\\ &\le \sum_{r\le dR}\left|\sum_{q\le\frac{Q}{r}} E(x;qr,1)\right| \ll_{A,\lambda} \frac{x}{\log^A x}.\end{aligned} \tag{4}$$

By $(u,r)=1$, we have $[u,n]=[u,qr]=r[u,q]=ruq/(u,q)$. We partition the set of $q\le\frac{Q}{r}$ as $\bigcup_{d|u}A_d$, where $q\in A_d$ if and only if $(u,q)=d$. Let $B_{Q,d}=\{q\le\frac{Q}{r} : q\equiv 0 \bmod d\}$. By inclusion–exclusion, we have, for any $d|u$,

$$\sum_{q\in A_d} E\left(x; \frac{ruq}{d}, 1\right) = \sum_{s|\frac{u}{d}} \mu(s) \sum_{q\in B_{Q,\mathrm{d}s}} E\left(x; \frac{ruq}{d}, 1\right).$$

It is clear that

$$\sum_{q\in B_{Q,\mathrm{d}s}} E\left(x; \frac{ruq}{d}, 1\right) = \sum_{q\in B_{\frac{uQ}{d},us}} E(x; qr, 1).$$

Since $r \le R := P_z < x^{\lambda'}$ with $\lambda' < \frac{1}{10}$, $\frac{uQ}{d} \le Q \log^{A_1} x$, and $us < \log^{2A_1} x < x^{\epsilon}$, we have, by (4),

$$\sum_{r\le R} \left| \sum_{q\in B_{\frac{uQ}{d},us}} E(x; qr, 1) \right| \ll_{A,A_1,\lambda} \frac{x}{\log^A x}$$

with a suitable choice of $B = B(A, A_1)$. Then

$$\begin{aligned}\sum_{r\le R} \left|\sum_{q\in A_d} E\left(x; \frac{ruq}{d}, 1\right)\right| &= \sum_{r\le R} \left|\sum_{s|\frac{u}{d}} \mu(s) \sum_{q\in B_{Q,\mathrm{d}s}} E\left(x; \frac{ruq}{d}, 1\right)\right| \\ &\le \sum_{s|\frac{u}{d}} \sum_{r\le R} \left|\sum_{q\in B_{Q,\mathrm{d}s}} E\left(x; \frac{ruq}{d}, 1\right)\right| \\ &\ll_{A,A_1,\lambda} \tau\left(\frac{u}{d}\right) \frac{x}{\log^A x}.\end{aligned}$$

Thus, summing over $d|u$, we have

$$\begin{aligned}\left|\sum_{r|P_z} \mu(r) \sum_{q\le\frac{Q}{r}} E(x; [u, qr], 1)\right| &\le \sum_{d|u} \sum_{r\le R} \left|\sum_{q\in A_d} E\left(x; \frac{ruq}{d}, 1\right)\right| \\ &\ll_{A,A_1,\lambda} (\tau(u))^2 \frac{x}{\log^A x} \ll_{A,A_1,\lambda} \frac{x}{\log^A x}.\end{aligned}$$

Thus, we have the result (2). □

The following is [5, Lemma 5] with a slightly relaxed $z$.

**Lemma 2.2** *Let* $0 < \lambda < \frac{1}{10}$ *and* $1 < z \le \lambda \log x$. *Let* $c_1 = e^{-\gamma}$. *Then we have*

$$R_z(x) := \sum_{p\le x} \tau_z(p-1) = c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right), \tag{5}$$

*and, for* $1 < z \le \frac{\log x}{\log_2^2 x}$,

$$S_z(x) := \sum_{p \le x} \frac{\tau_z(p-1)}{p} = c_1 \frac{\log x}{\log z} + O\left(\frac{\log x}{\log^2 z}\right). \tag{6}$$

*Proof of* (5) Take $A = 2$ and the corresponding $B(A)$ and $M$ in Lemma 2.1(1). Then by inclusion–exclusion,

$$\begin{aligned} R_z(x) &= \sum_{d \in D_z(x)} \pi(x; d, 1) \\ &= \sum_{d \in D_z(\frac{x}{M})} \pi(x; d, 1) + \sum_{r | P_z} \mu(r) \sum_{\frac{x}{rM} < q \le \frac{x}{r}} \pi(x; qr, 1) = R_1 + R_2, \text{ say.} \end{aligned}$$

By [5, Lemma 4] and Lemma 2.1(1),

$$\begin{aligned} R_1 &= \sum_{d \in D_z(\frac{x}{M})} \frac{\pi(x)}{\phi(d)} + \sum_{r | P_z} \mu(r) \sum_{q \le \frac{x}{rM}} E(x; qr, 1) \\ &= c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right) + O\left(\frac{x}{\log^2 x}\right). \end{aligned}$$

By divisor-switching technique and Brun–Titchmarsh inequality as in [6], we have

$$R_2 \ll \sum_{r | P_z} \sum_{k \le M} \pi(x; rk, 1) \ll \sum_{r | P_z} \sum_{k \le M} \frac{x}{\phi(rk) \log x} \ll \frac{x \log z \log M}{\log x} \ll \frac{x}{\log^2 z}.$$

Therefore, (5) follows. □

*Proof of* (6) By partial summation,

$$S_z(x) = \frac{R_z(t)}{t}\Big|_2^x + \int_2^x \frac{R_z(t)}{t^2} \mathrm{d}t.$$

We split the integral at $z = \lambda \log t$. Then, by (4),

$$\int_{z \le \lambda \log t} \frac{R_z(t)}{t^2} dt = \int_{e^{z/\lambda}}^x \left(c_1 \frac{t}{\log z} + O\left(\frac{t}{\log^2 z}\right)\right) \frac{\mathrm{d}t}{t^2} = c_1 \frac{\log x}{\log z} + O\left(\frac{\log x}{\log^2 z}\right).$$

On the other hand, by the trivial bound $R_z(t) \ll t$,

$$\int_{z > \lambda \log t} \frac{R_z(t)}{t^2} \mathrm{d}t \ll \int_2^{e^{z/\lambda}} t \frac{\mathrm{d}t}{t^2} \ll z.$$

Since $z \log^2 z \ll \log x$, (6) follows. □

The following is [5, Lemma 6] with a wider range of $z$. This relaxes the rather severe restriction $z \le \frac{\sqrt{\log x}}{\log_2^6 x}$.

**Lemma 2.3** *Let* $1 \le u \le x$ *be any positive integer. Then*

$$R_{u,z}(x) :\ = \sum_{\substack{p \le x \\ p \equiv 1 \bmod u}} \tau_z(p-1) \ll \frac{\tau(u)}{\phi(u)} x, \quad S_{u,z}(x) :$$
$$= \sum_{\substack{p \le x \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} \ll \frac{\tau(u)}{\phi(u)} \log x, \tag{7}$$

*and* $\phi(u)$ *can be replaced by* $u$ *if* $p(u) > z$ *and* $\tau(u) < A_1$.

*Assume that* $u$ *is a positive integer with* $p(u) > z$, $u < (\log x)^{A_1}$, *and* $\tau(u) < A_1$. *Then, for* $z \le \lambda \log x$,

$$R_{u,z}(x) = \frac{\tau(u)}{u} R_z(x) \left(1 + O\left(\frac{1}{\log z}\right)\right), \tag{8}$$

*and, for* $z \le \frac{\log x}{\log_2^2 x}$,

$$S_{u,z}(x) = \frac{\tau(u)}{u} S_z(x) \left(1 + O\left(\frac{1}{\log z}\right)\right). \tag{9}$$

*Proof of* (7) This is a uniform version of [8, Lemma 3.7]. We apply Dirichlet's hyperbola method as it was done in [8, Lemma 3.7]. First, we see that

$$R_{u,z}(x) \le \sum_{\substack{p \le x \\ p \equiv 1 \bmod u}} \tau(p-1)$$
$$\le \sum_{\substack{p \le x \\ p \equiv 1 \bmod u}} \tau\left(\frac{p-1}{u}\right) \tau(u) \le 2\tau(u) \sum_{k \le \sqrt{\frac{x}{u}}} \pi(x; ku, 1).$$

Since the sum is zero for $x \le u$, we may assume that $x > u$. By Brun–Titchmarsh inequality,

$$\pi(x; ku, 1) \le \frac{2x}{\phi(ku) \log\left(\frac{x}{ku}\right)} \le \frac{4x}{\phi(u)\phi(k) \log \frac{x}{u}}.$$

Thus, summing over $k$ gives

$$\sum_{k \le \sqrt{\frac{x}{u}}} \pi(x; ku, 1) \le \frac{8x}{\phi(u)} \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d\phi(d)}.$$

Therefore, we have the result. The estimate for $S_{u,z}$ follows from partial summation.

We remark that, for $u$ with $p(u) > z$,

$$\frac{u\phi(d)}{\phi(ud)} = \prod_{p|u,\, p\nmid d}\left(1-\frac{1}{p}\right)^{-1} = 1 + O\left(\frac{\tau(u)}{z}\right), \quad \frac{1}{\phi(u)}$$
$$= \frac{1}{u}\prod_{p|u}\left(1-\frac{1}{p}\right)^{-1} = \frac{1}{u}\left(1+O\left(\frac{\tau(u)}{z}\right)\right).$$

Therefore, $\phi(u)$ can be replaced by $u$ if $p(u) > z$ and $\tau(u) < A_1$. □

*Proof of* (8) We begin with

$$R_{u,z}(x) = \sum_{d\in D_z(x)} \pi(x; [u,d], 1).$$

Let $A > 0$ be a positive number such that $\frac{x}{\log^A x} \ll \frac{\tau(u)}{u}\frac{x}{\log^2 x}$, and $B(A)$ and $M$ be the corresponding parameters depending on $A$ in Lemma 2.1(2). By inclusion–exclusion,

$$\sum_{d\in D_z(x)} \pi(x; [u,d], 1) = \sum_{d\in D_z(\frac{x}{M})} \pi(x; [u,d], 1)$$
$$+ \sum_{r|P_z}\mu(r) \sum_{\frac{x}{rM}<q\le\frac{x}{r}} \pi(x; [u,qr], 1) = R_1 + R_2, \quad \text{say.}$$

By Lemma 2.1(2), we have

$$R_1 = \sum_{d\in D_z(\frac{x}{M})} \frac{\pi(x)}{\phi([u,d])} + \sum_{r|P_z}\mu(r)\sum_{q\le\frac{x}{rM}} E(x; [u,qr], 1)$$
$$= \sum_{d\in D_z(\frac{x}{M})} \frac{\pi(x)}{\phi([u,d])} + O\left(\frac{\tau(u)}{u}\frac{x}{\log^2 x}\right).$$

The first sum is treated as follows:

$$\sum_{d\in D_z(\frac{x}{M})} \frac{\pi(x)}{\phi([u,d])} = \sum_{d_1\in D_z(\frac{x}{uM})} \frac{\pi(x)N_{d_1}}{\phi(ud_1)} + O\left(\pi(x)\sum_{\substack{\frac{x}{uM}<d_1\le\frac{x}{M}\\ p(d_1)>z}} \frac{\tau(u)}{\phi(ud_1)}\right)$$
$$= \sum_{d_1\in D_z(\frac{x}{uM})} \frac{\pi(x)N_{d_1}}{\phi(ud_1)} + O\left(\pi(x)\frac{\tau(u)\log u}{\phi(u)\log z}\right)$$
$$= \sum_{d_1\in D_z(\frac{x}{uM})} \frac{\pi(x)N_{d_1}}{\phi(ud_1)} + O\left(\frac{\tau(u)}{u}\frac{x}{\log^2 z}\right),$$

where $N_{d_1} = \left|\{d \in D_z\left(\frac{x}{M}\right) : [u, d] = ud_1\}\right|$. Since $N_{d_1} \le \tau(u)$ and $\phi(ud_1) \ge \phi(u)\phi(d_1)$, by [5, Lemma 4],

$$\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} \le \frac{\tau(u)}{\phi(u)} \left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

Thus, we have the upper bound

$$\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} \le \frac{\tau(u)}{u} \left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

On the other hand, $N_{d_1} = \tau(u)$ if $(u, d_1) = 1$. Then, we may apply [5, Lemma 4] since $P(u) \le \log^{A_1} x$, and we obtain

$$\begin{aligned}\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} &\ge \frac{\tau(u)}{u} \left(\sum_{\substack{d_1 \in D_z\left(\frac{x}{uM}\right) \\ (u,d_1)=1}} \frac{\pi(x)}{\phi(d_1)} + O\left(\frac{x}{\log^2 z}\right)\right) \\ &\ge \frac{\tau(u)}{u}\frac{\phi(u)}{u} \left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).\end{aligned}$$

Thus, we have the lower bound

$$\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} \ge \frac{\tau(u)}{u} \left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

This shows that

$$R_1 = \frac{\tau(u)}{u} \left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

By divisor-switching technique and Brun–Titchmarsh inequality as in [6], we have

$$\begin{aligned} R_2 &\ll \sum_{r|P_z} \sum_{d|u} \sum_{s|\frac{u}{d}} \sum_{\substack{\frac{x}{rM} < q \le \frac{x}{r} \\ ds|q}} \pi\left(x; \frac{uqr}{d}, 1\right) \\ &\ll \sum_{r|P_z} \sum_{d|u} \sum_{s|\frac{u}{d}} \sum_{\frac{x}{dsrM} < q \le \frac{x}{dsr}} \pi\left(x; rusq, 1\right) \\ &\ll \sum_{r|P_z} \sum_{d|u} \sum_{s|\frac{u}{d}} \sum_{k \le \frac{dM}{u}} \pi(x; rusk, 1) \\ &\ll \sum_{r|P_z} \sum_{d|u} \sum_{s|\frac{u}{d}} \sum_{k \le \frac{dM}{u}} \frac{x}{\phi(rusk)\log x} \ll \tau(u) \frac{x \log z \log u \log M}{\phi(u) \log x} \ll \frac{\tau(u)}{u} \frac{x}{\log^2 z}.\end{aligned}$$

This completes the proof of (8). □

*Proof of* (9) We use (7) and (8) and apply partial summation as in (6). □

The following is used with inequality in [5, Lemma 7]. Here, we obtain an equality that will be used frequently in this paper.

**Lemma 2.4** *Let* $0 < \lambda < \frac{1}{10}$. *Fix* $a > 1$ *and an integer* $0 \le B < \infty$. *We use* $z = \lambda \log x$ *for the formula for* $R_B$ *and* $z = \frac{\log x}{\log_2^2 x}$ *for the formula for* $S_B$. *Let* $I_a(x) = [z, z^a]$. *Define*

$$\mathcal{U}_B = \{u\colon u \text{ is a positive square-free integer consisted of exactly } B \text{ prime divisors in } I_a(x)\}.$$

*Then we have*

$$R_B := \sum_{u\in\mathcal{U}_B} R_{u,z}(x) = \frac{(2\log a)^B}{B!} R_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right)$$

*and*

$$S_B := \sum_{u\in\mathcal{U}_B} S_{u,z}(x) = \frac{(2\log a)^B}{B!} S_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right).$$

*Proof* We apply Lemma 2.3 with $u \in \mathcal{U}_B$. Note that $u \in \mathcal{U}_B$ satisfies the conditions for $u$ in Lemma 2.3(8), (9). Then

$$\begin{aligned}
\sum_{u\in\mathcal{U}_B} R_{u,z}(x) &= \sum_{u\in\mathcal{U}_B} \frac{\tau(u)}{u} R_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right)\\
&= \left(\frac{1}{B!}\left(\sum_{p\in I_a(x)} \frac{2}{p}\right)^B\right.\\
&\quad \left. + O\left(\frac{1}{(B-2)!}\left(\sum_{p\in I_a(x)} \frac{4}{p^2}\right)\left(\sum_{p\in I_a(x)} \frac{2}{p}\right)^{B-2}\right)\right) R_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right)\\
&= \left(\frac{1}{B!}\left(\sum_{p\in I_a(x)} \frac{2}{p}\right)^B + O\left(\frac{1}{z}\right)\right) R_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right)\\
&= \frac{2^B}{B!}\left(\log\log z^a - \log\log z + O\left(\frac{1}{\log z}\right)\right)^B R_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right)\\
&= \frac{(2\log a)^B}{B!} R_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right).
\end{aligned}$$

The result for $S_B$ can be obtained similarly. □

Although we relaxed $z \leq \frac{\sqrt{\log x}}{\log_2^6 x}$ to $z \leq \frac{\log x}{\log_2^2 x}$, the range is still not enough for further use. We will see how this range can be relaxed to $\log^{\frac{1}{A}} x < z \leq \log^A x$ in Lemma 2.5. A probability mass function of a Poisson distribution comes up as certain densities.

**Lemma 2.5** *Let* $0 < \lambda < \frac{1}{10}$. *Fix* $a > 1$ *and an integer* $0 \leq B < \infty$. *We use* $z = \lambda \log x$ *for the formula for* $R'_B$ *and* $z = \frac{\log x}{\log_2^2 x}$ *for the formula for* $S'_B$. *Let* $I_a(x) = (z, z^a]$. *Define*

$$\tau_{z,z^a}(n) = \prod_{\substack{p^e \| n \\ p \in I_a(x)}} \tau(p^e), \quad w_{z,z^a}(n) = |\{p|n \ : \ p \in I_a(x)\}|,$$

*and*

$$R'_B := \sum_{\substack{p \leq x \\ w_{z,z^a}(p-1)=B}} \tau_z(p-1), \quad S'_B := \sum_{\substack{p \leq x \\ w_{z,z^a}(p-1)=B}} \frac{\tau_z(p-1)}{p}.$$

*Then, as* $x \to \infty$, *we have*

$$R'_B = \frac{(2\log a)^B}{B!a^2} R_z(x)(1 + o(1)), \quad S'_B = \frac{(2\log a)^B}{B!a^2} S_z(x)(1 + o(1)), \tag{10}$$

*and we have*

$$R_{z^a}(x) = \frac{1}{a} R_z(x)(1 + o(1)), \quad S_{z^a}(x) = \frac{1}{a} S_z(x)(1 + o(1)). \tag{11}$$

*Proof of* (10) We remark that by (7), (8), and (9), the contribution of primes $p$ such that $p-1$ is divisible by a square of a prime $q > z$ is negligible. In fact, those contributions to $R_z(x)$ and $S_z(x)$ are $O(R_z(x)/z)$ and $O(S_z(x)/z)$, respectively. Thus, we assume that $p-1$ is not divisible by square of any prime $q > z$. By Lemma 2.4 and inclusion–exclusion principle,

$$R'_B = R_B - \binom{B+1}{1} R_{B+1} + \binom{B+2}{2} R_{B+2} - \binom{B+3}{3} R_{B+3} + - \cdots .$$

Moreover, for any $k \geq 1$,

$$\sum_{j=0}^{2k-1} (-1)^j \binom{B+j}{j} R_{B+j} \leq R'_B \leq \sum_{j=0}^{2k} (-1)^j \binom{B+j}{j} R_{B+j}.$$

Then dividing by $R_z(x)$ gives

$$\sum_{j=0}^{2k-1}(-1)^j\binom{B+j}{j}\frac{R_{B+j}}{R_z(x)} \le \frac{R'_B}{R_z(x)} \le \sum_{j=0}^{2k}(-1)^j\binom{B+j}{j}\frac{R_{B+j}}{R_z(x)}.$$

By Lemma 2.4, we have

$$\frac{(2\log a)^B}{B!}\sum_{j=0}^{2k-1}(-1)^j\frac{(2\log a)^j}{j!}\left(1+O\left(\frac{1}{\log z}\right)\right)$$
$$\le \frac{R'_B}{R_z(x)} \le \frac{(2\log a)^B}{B!}\sum_{j=0}^{2k}(-1)^j\frac{(2\log a)^j}{j!}\left(1+O\left(\frac{1}{\log z}\right)\right).$$

Taking $x \to \infty$, we have

$$\frac{(2\log a)^B}{B!}\sum_{j=0}^{2k-1}(-1)^j\frac{(2\log a)^j}{j!} \le \liminf_{x\to\infty}\frac{R'_B}{R_z(x)}$$
$$\le \limsup_{x\to\infty}\frac{R'_B}{R_z(x)} \le \frac{(2\log a)^B}{B!}\sum_{j=0}^{2k}(-1)^j\frac{(2\log a)^j}{j!}.$$

Letting $k \to \infty$, we obtain

$$\lim_{x\to\infty}\frac{R'_B}{R_z(x)} = \frac{(2\log a)^B}{B!a^2}.$$

The result for $S'_B$ can be obtained similarly. □

*Proof of* (11) As in the proof of (10), we assume that $p-1$ is not divisible by square of any prime $q > z$. Note that $\tau_z(p-1) = \tau_{z^a}(p-1)\tau_{z,z^a}(p-1)$. Let $0 \le B < \infty$ be a fixed integer. If $w_{z,z^a}(p-1) = B$ then $\tau_{z,z^a}(p-1) = 2^B$. Then we have, by (10),

$$\sum_{\substack{p\le x\\ w_{z,z^a}(p-1)=B}}\tau_{z^a}(p-1) = \sum_{\substack{p\le x\\ w_{z,z^a}(p-1)=B}}\frac{\tau_z(p-1)}{2^B}$$
$$= \frac{R'_B}{2^B} = \frac{(\log a)^B}{B!a^2}R_z(x)(1+o(1)).$$

Then, by Lemma 2.4,

$$\frac{R_{z^a}(x)}{R_z(x)} = \sum_{j<B}\frac{(\log a)^j}{j!a^2}(1+o(1)) + \frac{1}{R_z(x)}\sum_{j\ge B}\frac{1}{2^j}\sum_{\substack{p\le x\\ w_{z,z^a}(p-1)=j}}\tau_z(p-1)$$

$$= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + O\left(\frac{1}{2^B R_z(x)} \sum_{\substack{p\le x \\ w_{z,z^a}(p-1)\ge B}} \tau_z(p-1)\right)$$

$$= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + O\left(\frac{R_B}{2^B R_z(x)}\right)$$

$$= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + O\left(\frac{(2\log a)^B}{2^B B!}\left(1+O\left(\frac{1}{\log z}\right)\right)\right).$$

Thus, both $\liminf_{x\to\infty} \frac{R_{z^a}(x)}{R_z(x)}$ and $\limsup_{x\to\infty} \frac{R_{z^a}(x)}{R_z(x)}$ are

$$\sum_{j\le B} \frac{(\log a)^j}{j!a^2} + O\left(\frac{(\log a)^B}{B!}\right)$$

and the constant implied in $O$ does not depend on $B$. Therefore, letting $B \to \infty$, we obtain

$$\lim_{x\to\infty} \frac{R_{z^a}(x)}{R_z(x)} = \frac{1}{a}.$$

The result for $S_{z^a}(x)$ can be obtained similarly. □

Lemma 2.5 allows us to have an extended range of $z$, and the same method applied to $R_{u,z}(x)$; we can also extend the range of $z$ for $R_{u,z}(x)$ and $S_{u,z}(x)$.

**Corollary 2.1** *Fix any* $A > 1$. *Let* $\log^{\frac{1}{A}} x < z \le \log^A x$. *Then, as* $x \to \infty$, *we have*

$$R_z(x) = c_1 \frac{x}{\log z}(1+o(1)), \quad S_z(x) = c_1 \frac{\log x}{\log z}(1+o(1)). \tag{12}$$

*Assume that* $u$ *is a positive integer with* $p(u) > z$, $u < (\log x)^{A_1}$, *and* $\tau(u) < A_1$. *Then, as* $x \to \infty$, *we have*

$$R_{u,z}(x) = \frac{\tau(u)}{u} R_z(x)(1+o(1)), \quad S_{u,z}(x) = \frac{\tau(u)}{u} S_z(x)(1+o(1)). \tag{13}$$

We apply Corollary 2.1 to obtain the following uniform distribution result:

**Corollary 2.2** *Let* $2 \le v \le x$ *and* $r := \left(v^{\frac{3}{2}} \log v\right)^{-1}$. *Suppose also that* $r \ge \log^{-\frac{4}{5}} x$, $0 \le \alpha \le \beta \le 1$, *and* $\beta - \alpha \ge r$. *Then, for* $z \le \frac{\log x^r}{\log_2^2 x^r}$,

$$\sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} = (\beta-\alpha) S_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right). \tag{14}$$

*For* $\log^{\frac{1}{A}} x < z \le \log^A x$*, we have, as* $x \to \infty$*,*

$$\sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} = (\beta - \alpha) S_z(x) \left(1 + o(1)\right). \tag{15}$$

*Assume that* $u$ *is a positive integer with* $p(u) > z$*,* $u < (\log x)^{A_1}$*, and* $\tau(u) < A_1$*. Then we have, for* $z \le \frac{\log x^r}{\log_2^2 x^r}$*,*

$$\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} = (\beta - \alpha) \frac{\tau(u)}{u} S_z(x) \left(1 + O\left(\frac{1}{\log z}\right)\right), \tag{16}$$

*and, for* $\log^{\frac{1}{A}} x < z \le \log^A x$*, we have, as* $x \to \infty$*,*

$$\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} = (\beta - \alpha) \frac{\tau(u)}{u} S_z(x) \left(1 + o(1)\right). \tag{17}$$

*Proof* By Lemma 2.2(5) and partial summation, we have, for $\beta - \alpha \ge r$,

$$\begin{aligned} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} &= \frac{R_z(t)}{t} \Big|_{x^\alpha}^{x^\beta} + \int_{x^\alpha}^{x^\beta} \frac{R_z(t)}{t^2} dt \\ &= c_1 (\beta - \alpha) \frac{\log x}{\log z} \left(1 + O\left(\frac{1}{\log z}\right)\right) + O\left(\frac{1}{\log^2 z}\right). \end{aligned}$$

Clearly, $r \log x \gg 1$. Thus, the second $O$-term can be included in the first $O$-term. Then (14) follows.

Since $r \log x \ge \log^{\frac{1}{5}} x$, the range $\log^{\frac{1}{A}} x < z \le \log^A x$ can be obtained from taking powers of $\frac{\log x^r}{\log_2^2 x^r}$. We have by (12), as $x \to \infty$,

$$\begin{aligned} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} &= \frac{R_z(t)}{t} \Big|_{x^\alpha}^{x^\beta} + \int_{x^\alpha}^{x^\beta} \frac{R_z(t)}{t^2} dt \\ &= c_1 (\beta - \alpha) \frac{\log x}{\log z} \left(1 + o(1)\right) + o\left(\frac{1}{\log z}\right). \end{aligned}$$

Also, by $r \log x \gg 1$, the second $o$-term can be included in the first $o$-term. Therefore, (15) follows. Similarly, (16) follows from Lemma 2.3 (8) and (17) follows from (13). □

We use $p_1, p_2, \ldots, p_v$ to denote prime numbers. We define the following multiple sums for $2 \le v \le x$:

$$\mathfrak{T}_{v,z}(x) := \sum_{p_1 p_2 \cdots p_v \le x} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1 p_2 \cdots p_v},$$

and, for $\mathbf{u} = (u_1, \ldots, u_v)$ with $1 \le u_i \le x$,

$$\mathfrak{T}_{\mathbf{u},v,z}(x) := \sum_{\substack{p_1 p_2 \cdots p_v \le x \\ \forall_i,\ p_i \equiv 1 \bmod u_i}} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1 p_2 \cdots p_v}.$$

Define $\mathbb{T}_v := \{(t_1, \ldots, t_v) : \forall_i,\ t_i \in [0,1],\ t_1 + \cdots + t_v \le 1\}$. We adopt the idea from Gauss' Circle Problem. Recall that $r = (v^{\frac{3}{2}} \log v)^{-1}$. Consider a covering of $\mathbb{T}_v$ by $v$-cubes of side length $r$ of the form:

Let $s_1, \ldots, s_v$ be nonnegative integers and let

$$B_{s_1,\ldots,s_v} := \{(t_1, \ldots, t_v) : \forall_i,\ rs_i \le t_i < r(s_i+1)\}.$$

Let $M_v$ be the set of those $v$-cubes lying completely inside $\mathbb{T}_v$. Then the sum $\mathfrak{T}_{v,z}(x)$ is over the primes satisfying

$$\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \mathbb{T}_v.$$

Instead of the whole $\mathbb{T}_v$, we consider the contribution of the sum over primes satisfying

$$\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v,$$

which come from the $v$-cubes lying completely inside $\mathbb{T}_v$. We define

$$\mathfrak{S}_{v,z}(x) := \sum_{\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1 p_2 \cdots p_v},$$

and similarly, for $\mathbf{u} = (u_1, \cdots, u_v)$ with $1 \le u_i \le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x) := \sum_{\substack{\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v \\ \forall_i,\ p_i \equiv 1 \bmod u_i}} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1 p_2 \cdots p_v},$$

Let $v = \left\lfloor c\sqrt{\frac{\log x}{\log_2 x}} \right\rfloor$ for some positive constant $c$ to be determined. Then $v$ satisfies the conditions in Corollary 2.2. Then we have:

**Lemma 2.6** *Let* $\log^{\frac{1}{A}} x < z \le \log^A x$. *Then, as* $x \to \infty$,

$$\mathfrak{S}_{v,z}(x) = \frac{1}{v!} S_z(x)^v (1 + o(1))^v. \tag{18}$$

*For* $\mathbf{u} = (u_1, u_2, 1, \ldots, 1)$ *with* $1 \le u_i \le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x) \ll \frac{\tau(u_1)\tau(u_2)}{\phi(u_1)\phi(u_2)} \mathfrak{S}_{v,z}(x) \log^k z, \tag{19}$$

*where* $0 \le k \le 2$ *is the number of* $u_i$*'s that are not* 1.

*Assume that each* $u_i$, $i = 1, 2$, *is a positive integer with* $p(u_i) > z$, $u_i < (\log x)^{A_1}$ *and* $\tau(u_i) < A_1$. *Then, as* $x \to \infty$, *we have*

$$\mathfrak{S}_{\mathbf{u},v,z}(x) = \frac{\tau(u_1)\tau(u_2)}{u_1 u_2} \mathfrak{S}_{v,z}(x) \left(1 + o(1)\right). \tag{20}$$

*Proof of* (18) It is clear that

$$\mathrm{vol}\left((1 - r\sqrt{v})\mathbb{T}_v\right) \le |M_v| \mathrm{vol}(B_{0,\ldots,0}) \le \mathrm{vol}(\mathbb{T}_v).$$

We have $\mathrm{vol}(\mathbb{T}_v) = \frac{1}{v!}$, $\mathrm{vol}(B_{0,\ldots,0}) = r^v$, and $\mathrm{vol}\left((1 - r\sqrt{v})\mathbb{T}_v\right) = \frac{1}{v!}\left(1 - r\sqrt{v}\right)^v$. Also, recall that $r := (v^{\frac{3}{2}} \log v)^{-1}$. Then

$$\frac{\frac{1}{v!}\left(1 - \frac{1}{v \log v}\right)^v}{(v^{\frac{3}{2}} \log v)^{-v}} \le |M_v| \le \frac{\frac{1}{v!}}{(v^{\frac{3}{2}} \log v)^{-v}}.$$

On the other hand, by Corollary 2.2 (15), the contribution of each $v$-cube $[\alpha_1, \beta_1] \times \cdots \times [\alpha_v, \beta_v] \subseteq [0, 1]^v$ of side length $r$ to the sum is

$$\begin{aligned}\sum_{\forall_i,\ \alpha_i \le \frac{\log p_i}{\log x} < \beta_i} &\frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1) \cdots \tau_z(p_v - 1)}{p_1 p_2 \cdots p_v} \\ &= \left(\prod_{i=1}^{v} (\beta_i - \alpha_i)\right) S_z(x)^v (1 + o(1))^v \\ &= r^v S_z(x)^v (1 + o(1))^v.\end{aligned}$$

Combining this with the bounds for $|M_v|$, we obtain the result. □

*Proof of* (19), (20) Let $v$ and $r$ be as defined in Corollary 2.2. We write (15) and (17) in the form of

$$\sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p - 1)}{p} = (\beta - \alpha) S_z(x) \left(1 + f_{\alpha,\beta}(x)\right) \tag{21}$$

and

$$\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} = (\beta - \alpha)\frac{\tau(u)}{u} S_z(x)\left(1 + g_{\alpha,\beta}(x)\right). \tag{22}$$

We note that there is a function $f(x) = o(1)$ such that uniformly, for $0 \le \alpha \le \beta \le 1$ and $\beta - \alpha \ge r$,

$$\max(|f_{\alpha,\beta}(x)|, |g_{\alpha,\beta}(x)|) \le f(x).$$

Then we can write

$$\begin{aligned}
\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} &= (\beta - \alpha)\frac{\tau(u)}{u} S_z(x)\left(1 + g_{\alpha,\beta}(x)\right) \\
&= \frac{\tau(u)}{u} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} \left(\frac{1 + g_{\alpha,\beta}(x)}{1 + f_{\alpha,\beta}(x)}\right) \\
&= \frac{\tau(u)}{u} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} (1 + O(f(x))).
\end{aligned}$$

Consider any $v$-cube $[\alpha_1, \beta_1] \times \cdots \times [\alpha_v, \beta_v] \subseteq [0, 1]^v$ of side length $r$. Then, by the above observation,

$$\begin{aligned}
&\sum_{\substack{\forall_i,\ \alpha_i \le \frac{\log p_i}{\log x} < \beta_i \\ p_i \equiv 1 \bmod u_i \text{ for } i=1,\,2}} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1) \cdots \tau_z(p_v - 1)}{p_1 p_2 \cdots p_v} \\
&= \frac{\tau(u_1)\tau(u_2)}{u_1 u_2} \sum_{\forall_i,\ \alpha_i \le \frac{\log p_i}{\log x} < \beta_i} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1) \cdots \tau_z(p_v - 1)}{p_1 p_2 \cdots p_v} (1 + O(f(x)))^2.
\end{aligned}$$

This proves (20). For the proof of (19), we use instead

$$\begin{aligned}
\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} &= \frac{R_{u,z}(t)}{t}\Big|_{x^\alpha}^{x^\beta} + \int_{x^\alpha}^{x^\beta} \frac{R_{u,z}(t)}{t^2} dt \\
&\ll \frac{\tau(u)}{\phi(u)}((\beta - \alpha)\log x + O(1)) \ll \frac{\tau(u)}{\phi(u)}(\beta - \alpha)\log x \\
&\ll \frac{\tau(u)}{\phi(u)}(\beta - \alpha) S_z(x) \log z \ll \frac{\tau(u)}{\phi(u)} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} \log z,
\end{aligned}$$

which follows from Lemma 2.3(7). □

We impose some restrictions on the primes $p_1, \dots, p_v$:

R1. $p_1, \dots, p_v$ are distinct.
R2. For each $i$, $q^2 \nmid p_i - 1$ for any prime $q > z$.
R3. $q^2 \nmid \phi(p_1 \cdots p_v)$ for any prime $q > z^2$.

Recall that we chose

$$v = \left\lfloor c\sqrt{\frac{\log x}{\log_2 x}} \right\rfloor$$

for some positive constant $c$ to be determined. Let ${\mathfrak{S}_{v,z}}^{(1)}(x)$ be the contribution of primes to $\mathfrak{S}_{v,z}(x)$ not satisfying R1. Note that if R1 is not satisfied, then some primes among $p_1, \dots, p_v$ are repeated. Then, by Lemma 2.6(18),

$$\begin{aligned} {\mathfrak{S}_{v,z}}^{(1)}(x) &\ll \binom{v}{2} \left( \sum_{z<p\le x} \frac{\tau_z(p-1)^2}{p^2} \right) \mathfrak{S}_{v-2,z}(x) \\ &\ll v^2 \frac{\log^3 z}{z} \frac{v(v-1)}{S_z(x)^2} \mathfrak{S}_{v,z}(x) \\ &\ll \frac{v^4 \log^5 z}{z \log^2 x} \mathfrak{S}_{v,z}(x) \ll \frac{\log^3 z}{z} \mathfrak{S}_{v,z}(x). \end{aligned}$$

Let ${\mathfrak{S}_{v,z}}^{(2)}(x)$ be the contribution of primes to $\mathfrak{S}_{v,z}(x)$ not satisfying R2. Note that if R2 is not satisfied, then $q^2 | p_i - 1$ for some primes $p_i$ and $q > z$. Let $\mathbf{u}_{q^2} := (q^2, 1, \dots, 1)$. Suppose that $q^2 | p_i - 1$ for some $p_i$ and $q > z^2$. Then the contribution of those primes to ${\mathfrak{S}_{v,z}}^{(2)}(x)$ is, by (19),

$$\begin{aligned} &\ll \sum_{q>z^2} \binom{v}{1} \mathfrak{S}_{\mathbf{u}_{q^2},v,z}(x) \\ &\ll \sum_{q>z^2} \frac{v}{\phi(q^2)} \mathfrak{S}_{v,z}(x) \log z \\ &\ll \sum_{q>z^2} \frac{v}{q^2} \mathfrak{S}_{v,z}(x) \log z \ll \frac{v}{z^2} \mathfrak{S}_{v,z}(x). \end{aligned}$$

Suppose that $q^2 | p_i - 1$ for some $p_i$ and $z < q \le z^2$. Then we have, by (20),

$$\ll \sum_{z<q\le z^2} \binom{v}{1} \mathfrak{S}_{\mathbf{u}_{q^2},v,z}(x) \ll \sum_{z<q\le z^2} \frac{v}{q^2} \mathfrak{S}_{v,z}(x) \ll \frac{v}{z \log z} \mathfrak{S}_{v,z}(x).$$

Thus, we have

$$\mathfrak{S}_{v,z}{}^{(2)}(x) \ll \frac{v}{z \log z} \mathfrak{S}_{v,z}(x).$$

Let $\mathfrak{S}_{v,z}{}^{(3)}(x)$ be the contribution of primes to $\mathfrak{S}_{v,z}(x)$ satisfying R1 and R2, but not satisfying R3. Note that if R1, R2 are satisfied and R3 is not satisfied, then there are at least two distinct primes $p_i$, $p_j$ such that $q | p_i - 1$ and $q | p_j - 1$. Let $\mathbf{u}_{q,q} := (q, q, 1, \ldots, 1)$. Suppose first that this happens with $q > z^4$. Then, by (19), the contribution is

$$\ll \sum_{q > z^4} \binom{v}{2} \mathfrak{S}_{\mathbf{u}_{q,q},v,z}(x) \ll \sum_{q > z^4} \frac{v^2}{\phi(q)^2} \mathfrak{S}_{v,z}(x) \log^2 z \ll \frac{v^2 \log z}{z^4} \mathfrak{S}_{v,z}(x).$$

Suppose that this happens with $z^2 < q \le z^4$. Then, by (20), the contribution is

$$\ll \sum_{z^2 < q \le z^4} \binom{v}{2} \mathfrak{S}_{\mathbf{u}_{q,q},v,z}(x) \ll \sum_{z^2 < q \le z^4} \frac{v^2}{q^2} \mathfrak{S}_{v,z}(x) \ll \frac{v^2}{z^2 \log z} \mathfrak{S}_{v,z}(x).$$

Thus, we have

$$\mathfrak{S}_{v,z}{}^{(3)}(x) \ll \frac{v^2}{z^2 \log z} \mathfrak{S}_{v,z}(x).$$

We write $\mathfrak{S}_{v,z}{}^{(0)}(x)$ to denote the contribution of those primes to $\mathfrak{S}_{v,z}(x)$ satisfying all three restrictions R1, R2, and R3. By the above estimates, we have

$$\begin{aligned} \mathfrak{S}_{v,z}{}^{(0)}(x) &\ge \mathfrak{S}_{v,z}(x) - \mathfrak{S}_{v,z}{}^{(1)}(x) - \mathfrak{S}_{v,z}{}^{(2)}(x) - \mathfrak{S}_{v,z}{}^{(3)}(x) \\ &= \mathfrak{S}_{v,z}(x) \left(1 + O\left(\frac{\log^3 z}{z}\right) + O\left(\frac{v}{z \log z}\right) + O\left(\frac{v^2}{z^2 \log z}\right)\right). \end{aligned}$$

Therefore,

$$\mathfrak{S}_{v,z}{}^{(0)}(x) = \mathfrak{S}_{v,z}(x) \left(1 + O\left(\frac{\log^3 z}{z}\right) + O\left(\frac{v}{z \log z}\right) + O\left(\frac{v^2}{z^2 \log z}\right)\right). \tag{23}$$

## 3 Proof of Theorem 1.1

We set

$$v = v(x) := \left\lfloor c \sqrt{\frac{\log x}{\log_2 x}} \right\rfloor, \quad z = z(x) := \sqrt{\log x},$$

$$y := \exp\left(\sqrt{\log x}\right)$$

with a positive constant $c$ to be determined.

Consider a subset $Q_z(x)$ of primes defined by

$$Q = Q_z(x) := \{p : p \le x,\ q^2 \nmid p-1 \text{ for any prime } q > z\}.$$

We define $\mathcal{N}$, $\mathcal{M}$ by

$$\mathcal{N} = \mathcal{N}_v(x) := \{n \le x : n \text{ is square-free, } p|n \Rightarrow p \in Q,\ w(n) = v\},$$

$$\mathcal{M} = \mathcal{M}_v(x) := \{n \le x : n \in \mathcal{N},\ q^2 \nmid \phi(n) \text{ for any prime } q > z^2\}.$$

We write

$$V_{\mathcal{M}}(x) := \sum_{n \in \mathcal{M}} \frac{\tau_z(\lambda(n))}{n}, \quad \tau_z''(n) := \prod_{p|n} \tau_z(p-1).$$

We also write

$$W_{\mathcal{M}} := \sum_{n \in \mathcal{M}} \frac{\tau_z''(n)}{n}, \quad W'_{\mathcal{M}} := \sum_{n \in \mathcal{M}} \frac{\tau_{z^2}''(n)}{n}.$$

By (23), the contribution of those primes satisfying R1, R2, and R3 to $\mathfrak{S}_{v,z}(x)$, which we wrote as $\mathfrak{S}_{v,z}{}^{(0)}(x)$ satisfies

$$\begin{aligned}\mathfrak{S}_{v,z}{}^{(0)}(x) &= \mathfrak{S}_{v,z}(x)\left(1 + O\left(\frac{\log^3 z}{z}\right) + O\left(\frac{v}{z\log z}\right) + O\left(\frac{v^2}{z^2\log z}\right)\right).\\ &= \mathfrak{S}_{v,z}(x)\left(1 + O\left(\frac{1}{\log_2 x}\right)\right).\end{aligned}$$

Then, by Lemma 2.6(18) and Stirling's formula,

$$W_{\mathcal{M}} \ge \frac{1}{v!}\mathfrak{S}_{v,z}{}^{(0)}(x) \asymp \frac{1}{v}\left(\frac{e}{v}\right)^{2v}\left(c_1\frac{\log x}{\log z}\right)^v (1+o(1))^v$$

Thus,

$$W_{\mathcal{M}} \gg \exp\left(\sqrt{\frac{\log x}{\log_2 x}}\,(2c + c\log c_1 - 2c\log c + c\log 2 + o(1))\right).$$

Maximizing $2c + c\log c_1 - 2c\log c + c\log 2$ by the first derivative, we have $c = \sqrt{2}e^{-\gamma/2}$, and hence

$$W_{\mathcal{M}} \gg \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

For $W'_{\mathcal{M}}$, we have, by (23), the contribution of those primes satisfying R1, R2, and R3 to $\mathfrak{S}_{v,z^2}(x)$, say $\mathfrak{S}_{v,z^2}{}^{(0')}(x)$ satisfies

$$\mathfrak{S}_{v,z^2}{}^{(0')}(x) = \mathfrak{S}_{v,z^2}(x)\left(1 + O\left(\frac{\log^3 z}{z^2}\right) + O\left(\frac{v}{z\log z}\right) + O\left(\frac{v^2}{z^2\log z}\right)\right)$$
$$= \mathfrak{S}_{v,z^2}(x)\left(1 + O\left(\frac{1}{\log_2 x}\right)\right).$$

Then by Lemma 2.6(18) and Stirling's formula, as $x \to \infty$,

$$W'_{\mathcal{M}} \geq \frac{1}{v!}\mathfrak{S}_{v,z^2}{}^{(0')}(x) \asymp \frac{1}{v}\left(\frac{e}{v}\right)^{2v}\left(c_1\frac{\log x}{\log z^2}\right)^{v}(1+o(1))^{v}$$

Thus,

$$W'_{\mathcal{M}} \gg \exp\left(\sqrt{\frac{\log x}{\log_2 x}}\,(2c + c\log c_1 - 2c\log c + o(1))\right).$$

Maximizing $2c + c\log c_1 - 2c\log c$ by the first derivative, we have $c = e^{-\gamma/2}$, and hence, as $x \to \infty$,

$$W'_{\mathcal{M}} \gg \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

Therefore, we have just proved the lower bounds of the following:

**Theorem 3.1** *For $z = \sqrt{\log x}$, as $x \to \infty$,*

$$\sum_{n\leq x}\mu^2(n)\frac{\tau''_z(n)}{n} = \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right) \tag{24}$$

*and*

$$\sum_{n\leq x}\mu^2(n)\frac{\tau''_{z^2}(n)}{n} = \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right). \tag{25}$$

Note that the upper bounds follow from Rankin's method as in [5, Theorem 1].

We proceed the similar argument as in [5]. Let $\mathcal{M} = \mathcal{M}_v(x)$ be as above with the choice $c = e^{-\gamma/2}$. Now, for $n \in \mathcal{M}$, we have

$$\tau_z(\phi(n)) = \tau_{z,z^2}(\phi(n))\tau_{z^2}(\phi(n)) \geq \tau_{z^2}(\phi(n)) = \tau''_{z^2}(n),$$

$$\tau_z(\lambda(n)) = \tau_{z,z^2}(\lambda(n))\tau_{z^2}(\lambda(n)) \geq \tau_{z^2}(\lambda(n)) = \tau''_{z^2}(n).$$

Then, as $x \to \infty$,

$$V_{\mathcal{M}}(x) \geq W'_{\mathcal{M}} \gg \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

The argument proceeds as in [5]. Let $\mathcal{M}'$ be defined by

$$\mathcal{M}' := \left\{np : n \in \mathcal{M}_v(xy^{-1}), \quad p \text{ is a prime}, \quad p \leq \frac{x}{n}\right\}.$$

For those $n' = np \in \mathcal{M}'$, we have

$$\tau(\lambda(np)) \geq \tau(\lambda(n)) \geq \tau_z(\lambda(n)),$$

and a given $n' \in \mathcal{M}'$ has at most $v + 1$ decompositions of the form $n' = np$ with $n \in \mathcal{M}_v(xy^{-1})$, $p \leq \frac{x}{n}$.

Since $n \leq xy^{-1}$ for $n \in \mathcal{M}_v(xy^{-1})$, the number of $p$ in $p \leq \frac{x}{n}$ is

$$\pi\left(\frac{x}{n}\right) \gg \frac{x}{n \log x}.$$

Note that $\log y = \sqrt{\log x} = o(\log x)$. This gives

$$V_{\mathcal{M}}(xy^{-1}) \gg \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

Then

$$\begin{aligned}\sum_{n \leq x} \tau(\lambda(n)) &\geq \sum_{n \in \mathcal{M}'} \tau(\lambda(n)) \gg V_{\mathcal{M}}(xy^{-1}) \frac{x}{v \log x} \\ &\gg x \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).\end{aligned}$$

This completes the proof of Theorem 1.1.

*Remarks* 1. In the proof of Theorem 1.1, we dropped $\tau_{z,z^2}(\phi(n))$. This is where a prime $z < q \leq z^2$ can divide multiple $p_i - 1$ for $i = 1, 2, \cdots, v$, and that is the main difficulty in obtaining more precise formulas for $\sum_{n \leq x} \tau(\phi(n))$ and $\sum_{n \leq x} \tau(\lambda(n))$.

2. We will see a heuristic argument suggesting that, as $x \to \infty$,

$$\sum_{n \leq x} \tau(\lambda(n)) = x \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right),$$

and hence

$$\sum_{n\le x}\tau(\phi(n)) = x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

However, we have

$$\sum_{n\le x}\tau(\lambda(n)) = o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

We will prove this in the following section. The prime 2 plays a crucial role in the proof of Theorem 1.2.

## 4 Proof of Theorem 1.2

We put $k$ and $w$ as in [5]:

$$k = \lfloor A\log_2 x\rfloor,\quad \omega = \left\lfloor\frac{\sqrt{\log x}}{\log_2^2 x}\right\rfloor.$$

Here, $A$ is a positive constant to be determined. Also, define $\mathcal{E}_1(x)$, $\mathcal{E}_2(x)$, and $\mathcal{E}_3(x)$ in the same way:

$$\mathcal{E}_1(x) := \{n\le x : 2^k|n \text{ or there is a prime } p|n \text{ with } p\equiv 1 \bmod 2^k\},$$

$$\mathcal{E}_2(x) := \{n\le x : \omega(n)\le\omega\},$$

and

$$\mathcal{E}_3(x) := \{n\le x\} - (\mathcal{E}_1(x)\cup\mathcal{E}_2(x)).$$

We need the following lemma.

**Lemma 4.1** *For any* $2\le y\le x$*, we have*

$$\sum_{n\le\frac{x}{y}}\frac{\tau(\phi(n))}{n} \ll \frac{\log^5 x}{x}\sum_{n\le x}\tau(\phi(n)).$$

*Proof* As in the proof of [5, Theorem 1], we use the square-free kernel $k=k(n)$ (if a prime $p$ divides $n$, then $p|k$, and $k$ is a square-free positive integer which divides $n$) and the factorization $n=mk$ to rewrite the sum as

$$\sum_{n\le\frac{x}{y}} \frac{\tau(\phi(n))}{n} \le \sum_{k\le\frac{x}{y}} \mu^2(k) \sum_{m\le\frac{x}{ky}} \frac{\tau(m)\tau(\phi(k))}{mk}$$
$$\ll \sum_{k\le\frac{x}{y}} \mu^2(k)\frac{\tau(\phi(k))}{k}\log^2 x.$$

Note that we have uniformly $w(n) \ll \log x$. Find $v$ such that

$$\sum_{\substack{k\le\frac{x}{y}\\ \omega(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k}$$

is maximal. Then we have

$$\sum_{k\le\frac{x}{y}} \mu^2(k)\frac{\tau(\phi(k))}{k} \ll \log x \sum_{\substack{k\le\frac{x}{y}\\ \omega(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k}.$$

We adopt an idea from the proof of Theorem 1.1. Let $\mathcal{M} = \mathcal{M}_v(xy^{-1})$ be the set of square-free numbers $k \le xy^{-1}$ with $\omega(k) = v$. Define

$$\mathcal{M}' := \left\{kp : k \in \mathcal{M}_v(xy^{-1}),\quad p \text{ is a prime},\quad p \le \frac{x}{k}\right\}.$$

For those $n' = kp \in \mathcal{M}'$ with $k \in \mathcal{M}$, we have

$$\tau(\phi(kp)) \ge \tau(\phi(k)),$$

and any given $n' \in \mathcal{M}'$ has at most $v + 1$ decompositions of the form $n' = kp$ with $k \in \mathcal{M}$, $p \le \frac{x}{k}$.

Since the number of $p$ satisfying $p \le \frac{x}{k}$ is

$$\pi\left(\frac{x}{k}\right) \gg \frac{x}{k\log x},$$

it follows that

$$\sum_{n\le x} \tau(\phi(n)) \ge \sum_{n\in\mathcal{M}'} \tau(\phi(n)) \gg \sum_{\substack{k\le\frac{x}{y}\\ \omega(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k}\frac{x}{v\log x}.$$

Since $v \ll \log x$, we have

$$\sum_{\substack{k\le\frac{x}{y}\\ w(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k} \ll \frac{\log^2 x}{x}\sum_{n\le x}\tau(\phi(n)).$$

This gives

$$\sum_{k\le \frac{x}{y}} \mu^2(k)\frac{\tau(\phi(k))}{k} \ll \frac{\log^3 x}{x}\sum_{n\le x}\tau(\phi(n)).$$

Then the result follows. □

For $n \in \mathcal{E}_1(x)$, we have, by Lemmas 2.3 and 4.1,

$$\begin{aligned}
\sum_{n\in\mathcal{E}_1(x)} \tau(\lambda(n)) &\le x \sum_{n\in\mathcal{E}_1(x)} \frac{\tau(\phi(n))}{n} \\
&\le x\frac{\tau(2^k)}{2^k}\sum_{m\le \frac{x}{2^k}}\frac{\tau(\phi(m))}{m} + x\sum_{\substack{p\le x\\ p\equiv 1 \bmod 2^k}}\frac{\tau(p-1)}{p}\sum_{m\le\frac{x}{p}}\frac{\tau(\phi(m))}{m} \\
&\ll \log^5 x\left(\frac{\tau(2^k)}{\phi(2^k)}\log x\sum_{n\le x}\tau(\phi(n))\right) \\
&\ll \log^6 x\,\frac{A\log_2 x}{\log^{A\log 2} x}\sum_{n\le x}\tau(\phi(n)).
\end{aligned}$$

If we take $A\log 2 > 7$, then we obtain

$$\sum_{n\in\mathcal{E}_1(x)} \tau(\lambda(n)) = o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

For $n \in \mathcal{E}_2(x)$, we use the square-free kernel $k = k(n)$ and the factorization $n = mk$ as before,

$$\begin{aligned}
\sum_{n\in\mathcal{E}_2(x)} \tau(\lambda(n)) &\le \sum_{n\in\mathcal{E}_2(x)} \tau(\phi(n)) \\
&\ll \sum_{\substack{k\le x\\ \omega(k)\le\omega}} \mu^2(k)\sum_{m\le\frac{x}{k}}\tau(m)\tau(\phi(k)) \\
&\ll \sum_{\substack{k\le x\\ \omega(k)\le\omega}} \mu^2(k)\frac{x}{k}(\log x)\tau(\phi(k)) \\
&\ll x\omega\log x\left(\sum_{p\le x}\frac{\tau(p-1)}{p}\right)^{\omega} \\
&\ll x(\log x)^{\frac{3}{2}}(C\log x)^{\omega} \ll x\exp\left(2\frac{\sqrt{\log x}}{\log_2 x}\right).
\end{aligned}$$

Thus, by Theorem 1.1,

$$\sum_{n\in\mathcal{E}_2(x)} \tau(\lambda(n)) = o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

For $n \in \mathcal{E}_3(x)$, we follow the method of [5]. We have

$$\frac{\tau(\phi(n))}{\tau(\lambda(n))} \ge \frac{\omega}{k} \gg \frac{\sqrt{\log x}}{\log_2^3 x}.$$

Then

$$\sum_{n\in\mathcal{E}_3(x)} \tau(\lambda(n)) \ll \frac{\log_2^3 x}{\sqrt{\log x}} \sum_{n\in\mathcal{E}_3(x)} \tau(\phi(n)) \le \frac{\log_2^3 x}{\sqrt{\log x}} \sum_{n\le x}\tau(\phi(n)).$$

Therefore, putting these together, we have

$$\sum_{n\le x}\tau(\lambda(n)) \ll \frac{\log_2^3 x}{\sqrt{\log x}} \sum_{n\le x}\tau(\phi(n)),$$

and Theorem 1.2 follows.

## 5 Heuristics

Recall that $\tau_z(\lambda(n)) = \tau_{z,z^2}(\lambda(n))\tau_{z^2}(\lambda(n))$. Let $\mathcal{M}$ be the set defined in Sect. 3 with the choice of $v = \left\lfloor \sqrt{2}e^{-\gamma/2}\sqrt{\frac{\log x}{\log_2 x}}\right\rfloor$. As in Sect. 3, we have $\tau_{z^2}(\lambda(n)) = \tau''_{z^2}(n)$ for $n \in \mathcal{M}$. It is important to note that $q^2 \nmid p_i - 1$ for any primes $p_i|n$ and $q > z$. Also, we have $q^2 \nmid \phi(n)$ for $q > z^2$. Thus, it is enough to focus on the sum $V_{\mathcal{M}}(x)$. If we could prove that $V_{\mathcal{M}}(x) = \sum_{n\in\mathcal{M}} \frac{\tau_z(\lambda(n))}{n} \gg \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right)$, then the same argument as in Theorem 1.1 would allow $\sum_{n\le x}\tau(\lambda(n)) \gg x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right)$. We need the contribution of $\tau_{z,z^2}(\lambda(n))$ over $n \in \mathcal{M}$. Let $\mathfrak{S}_{v,z}(x)$ be the sum defined in Sect. 2, and define

$$\mathfrak{U}_{v,z}(x) := \sum_{\left(\frac{\log p_1}{\log x},\ldots,\frac{\log p_v}{\log x}\right)\in\cup M_v} \frac{\tau_{z,z^2}(\mathrm{lcm}(p_1-1,p_2-1,\ldots,p_v-1))}{\tau_{z,z^2}(p_1-1)\tau_{z,z^2}(p_2-1)\cdots\tau_{z,z^2}(p_v-1)} \times \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}.$$

We have also defined in Sect. 2 that, for $\mathbf{u} = (u_1, \ldots, u_v)$ with $1 \le u_i \le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x) := \sum_{\substack{\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v \\ \forall_i,\ p_i \equiv 1 \bmod u_i}} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1)\cdots\tau_z(p_v - 1)}{p_1 p_2 \cdots p_v}.$$

We need to extend Lemma 2.6 to cover all components of $\mathbf{u}$.

**Lemma 5.1** *Let* $\log^{\frac{1}{A}} x < z \le \log^A x$. *Then, for* $\mathbf{u} = (u_1, u_2, \ldots, u_v)$ *with* $1 \le u_i \le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x) \ll \frac{\tau(u_1)\tau(u_2)\cdots\tau(u_v)}{\phi(u_1)\phi(u_2)\cdots\phi(u_v)} \mathfrak{S}_{v,z}(x)(1 + o(1))^k \log^k z, \tag{26}$$

*where* $0 \le k \le v$ *is the number of* $u_i$*'s that are not* 1.

*Assume that each* $u_i$, $1 \le i \le v$, *is either* 1 *or a positive integer with* $p(u_i) > z$, $u_i < (\log x)^{A_1}$, *and* $\tau(u_i) < A_1$. *Then*

$$\mathfrak{S}_{\mathbf{u},v,z}(x) = \frac{\tau(u_1)\tau(u_2)\cdots\tau(u_v)}{u_1 u_2 \cdots u_v} \mathfrak{S}_{v,z}(x)\,(1 + o(1))^k\,, \tag{27}$$

*where* $0 \le k \le v$ *is the number of* $u_i$*'s that are not* 1.

The same proof as in Lemma 2.6 applies with the need of considering all components of $\mathbf{u}$.

Fix a prime $z < q \le z^2$. Consider the number $X_q$ of primes $p_1, \ldots, p_v$ such that $q$ divides $p_i - 1$. By Lemma 5.1, it is natural to model $X_q$ by a binomial distribution with parameters $v$ and $\frac{2}{q}$. In fact, Lemma 5.1 implies that

**Lemma 5.2** *For any* $0 \le k \le v$, *as* $x \to \infty$,

$$\begin{aligned} P(X_q = k) &:= \frac{1}{\mathfrak{S}_{v,z}(x)} \\ &\quad \times \sum_{\substack{\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v \\ \text{Exactly } k \text{ primes } p_i \text{ satisfy } q \mid p_i - 1}} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1)\cdots\tau_z(p_v - 1)}{p_1 p_2 \cdots p_v} \\ &= \binom{v}{k}\left(\frac{2}{q}\right)^k \left(1 - \frac{2}{q}\right)^{v-k} (1 + o(1))^v. \end{aligned}$$

*Here, the functions implied in* $1 + o(1)$ *only depend on* $x$ *and do not depend on* $k$.

Denote by $A_q$ the contribution of a power of $q$ in

$$\frac{\tau_{z,z^2}(\operatorname{lcm}(p_1 - 1, p_2 - 1, \ldots, p_v - 1))}{\tau_{z,z^2}(p_1 - 1)\tau_{z,z^2}(p_2 - 1)\cdots\tau_{z,z^2}(p_v - 1)}.$$

Similarly, denote by $A_{q_1,\cdots,q_j}$ the contribution of powers of $q_1, \cdots, q_j$ in the above. Let

$$B_{z,v} := \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}.$$

We can combine the contributions of finite number of primes $q_1, \ldots, q_j$ in $(z, z^2]$. For these multiple primes, Lemma 5.2 becomes:

**Lemma 5.3** *For any* $0 \le k_1, \ldots, k_j \le v$, *as* $x \to \infty$,

$$\begin{aligned} &P(X_{q_1} = k_1, \ldots, X_{q_j} = k_j) \\ &:= \frac{1}{\mathfrak{S}_{v,z}(x)} \sum_{\substack{\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v \\ For each s=1,\ldots,j, \\ exactly k_s primes p_i satisfy q_s | p_i - 1}} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v} \\ &= \prod_{s\le j} \binom{v}{k_s} \left(\frac{2}{q_s}\right)^{k_s} \left(1-\frac{2}{q_s}\right)^{v-k_s} (1+o(1))^v. \end{aligned}$$

*Here, the functions implied in* $1 + o(1)$ *only depend on* $j$, $x$ *and they do not depend on* $k_s$.

This shows that the random variables $X_{q_i}$ behave similar as independent binomial distributions. For $z < q \le z^2$, we have $A_q = \frac{2}{2^k}$ for $k \ge 1$, and $A_q = 1$ for $k = 0$. Thus, the contribution of this prime $q$ is

$$\mathbf{E}[A_q] = \left(2\left(1-\frac{1}{q}\right)^v - \left(1-\frac{2}{q}\right)^v\right)(1+o(1))^v.$$

For distinct primes $q_1, \ldots, q_j$ in $(z, z^2]$, the contribution of these primes is

$$\mathbf{E}[A_{q_1,\ldots,q_j}] = \prod_{s\le j}\left(2\left(1-\frac{1}{q_s}\right)^v - \left(1-\frac{2}{q_s}\right)^v\right)(1+o(1))^v,$$

where the function implied in $1 + o(1)$ only depends on $j$, $x$.

Then, we conjecture that the contribution of all primes in $z < q \le z^2$ will be

**Conjecture 5.1** *As* $x \to \infty$, *we have*

$$\mathfrak{U}_{v,z}(x) = \prod_{z<q\le z^2}\left(2\left(1-\frac{1}{q}\right)^v - \left(1-\frac{2}{q}\right)^v\right)\mathfrak{S}_{v,z}(x)(1+o(1))^v.$$

It is clear that

$$2\left(1-\frac{1}{q}\right)^v - \left(1-\frac{2}{q}\right)^v = 1 + o\left(\frac{v}{q}\right).$$

Thus, we have, as $x \to \infty$,

$$\prod_{z<q\leq z^2} \left(2\left(1-\frac{1}{q}\right)^v - \left(1-\frac{2}{q}\right)^v\right) = (1+o(1))^v.$$

Therefore, we obtain the following heuristic result according to Conjecture 5.1.

**Conjecture 5.2** *As $x \to \infty$, we have*

$$\mathfrak{U}_{v,z}(x) = \mathfrak{S}_{v,z}(x)(1+o(1))^v.$$

Then Conjecture 1.1 follows from Lemma 2.6.

*Remarks* We were unable to prove Conjecture 1.1. The main difficulty is due to the short range of $u$ in Corollary 2.1. Because of the range of $u$, we could not extend Lemma 5.3 to all primes in $z < q \leq z^2$.

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