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Inequalities and infinite product formula for Ramanujan generalized modular equation function

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Abstract We present several inequalities for the Ramanujan generalized modular equation function $\mu_a(r) = \pi F(a, 1-a; 1; 1-r^2)/\ [2\sin(\pi a)F(a, 1-a; 1; r^2)]$ with $a \in (0, 1/2]$ and $r \in (0, 1)$, and provide an infinite product formula for $\mu_{1/4}(r)$, where $F(a, b; c; x) = {}_2F_1(a, b; c; x)$ is the Gaussian hypergeometric function.

Keywords Gaussian hypergeometric function · Ramanujan generalized modular equation · Quadratic transformation · Infinite product

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1 Introduction

For real numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined by

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$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \quad (|x|<1), \tag{1.1}$$

where $(a,0)=1$ for $a \neq 0$, and (a,n) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n = 1, 2, \ldots$. $F(a,b;c;x)$ is said to be zero-balanced if $c = a+b$. Recently, the Gaussian hypergeometric function $F(a,b;c;x)$ has attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $F(a,b;c;x)$ can be found in the literature [2–4,6,7,15,18,21,22,28].

The well-known quadratic transformation formula [15, (15.8.15), (15.8.21)] for the Gaussian hypergeometric function is given by

$$F\left(\frac{1}{4},\frac{3}{4};1;1-\left(\frac{1-r}{1+3r}\right)^2\right) = \sqrt{1+3r}\,F\left(\frac{1}{4},\frac{3}{4};1;r^2\right), \tag{1.2}$$

namely,

$$F\left(\frac{1}{4},\frac{3}{4};1;\left(\frac{1-r}{1+3r}\right)^2\right) = \frac{\sqrt{1+3r}}{2}F\left(\frac{1}{4},\frac{3}{4};1;1-r^2\right). \tag{1.3}$$

Let $a \in (0,1/2]$, $r \in (0,1)$ and $p > 0$. Then the Ramanujan generalized modular equation with signature $1/a$ and degree p is given by

$$\frac{F(a,1-a;1;1-s^2)}{F(a,1-a;1;s^2)} = p\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)}. \tag{1.4}$$

Making use of the decreasing homeomorphism $\mu_a : (0,1) \to (0,\infty)$ defined by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)}\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)}, \tag{1.5}$$

(1.4) can be rewritten as

$$\mu_a(s) = p\mu_a(r). \tag{1.6}$$

The solution of (1.6) is given by

$$s \equiv \varphi_K(a,r) = {\mu_a}^{-1}(\mu_a(r)/K), \quad K = 1/p. \tag{1.7}$$

We call the function $\mu_a(r)$ defined by (1.5) the Ramanujan generalized modular equation function and $\varphi_K(a,r)$ defined by (1.7) the solution of the Ramanujan generalized modular equation with signature $1/a$ and degree $1/K$.

The Ramanujan generalized modular equation (1.4) has been developed by leading mathematicians for over a century. The classical case $a = 1/2$ was firstly studied by Jacobi in the nineteenth century. In the early 20th century, the Indian mathematical genius Ramanujan studied extensively the Gaussian hypergeometric function $F(a, b; c; x)$ and the Ramanujan generalized modular equation (1.4). Numerous algebraic identities for the solution s for some rational values of a were listed in his unpublished notebooks, but with no original proofs (see [7]). Later, Borwein and Borwein [11,12], Venkatachaliengar [23], and Berndt [5–8] et al. made great contribution to the subject. In particular, in 1995, Berndt et al. published an important paper [9] in which they studied the case p a prime. For several rational values, such as $a = 1/3, 1/4, 1/6$, and integers p (e.g. $p = 2, 3, 5, 7, 11, \ldots$) they were able to give proofs for numerous algebraic identities stated by Ramanujan. After the publication of [9], many remarkable results for the Ramanujan generalized modular equations have been established (see [13,14,19–22,24,29]).

Let $\mu(r)$ and $\varphi_K(r)$ be, respectively, the plane Grötzsch ring function and the Hersch–Pfluger distortion function in geometric function theory, that is, $\mu(r)$ is the conformal modulus of the ring consisting of the unit disk slit along the real axis from 0 to r, and $\varphi_K(r)(K \geq 1)$ gives the maximum distortion of the class of K-quasiconformal mappings of the unit disk into itself with the origin fixed. Then we clearly see that $\mu_{1/2}(r) = \mu(r)$ and $\varphi_K(1/2, r) = \varphi_K(r)$. Therefore, $\mu_a(r)$ and $\varphi_K(a, r)$ are respectively the generalized Grötzsch ring function and generalized Hersch–Pfluger distortion function, and they have been explored by Anderson, Vuorinen and Qiu et al. in [1,3,10,16,17,25,26,30,31].

In what follows, we denote $r' = \sqrt{1 - r^2}$ for $r \in (0, 1)$,

$$\mu^*(r) = \mu_{1/4}(r) = \frac{\pi}{\sqrt{2}} \frac{F(1/4, 3/4; 1; 1 - r^2)}{F(1/4, 3/4; 1; r^2)},$$
$$\varphi_K^*(r) = \mu^{*-1}(\mu^*(r)/K).$$

Note that

$$\varphi_K^*(r) = {\varphi_{1/K}^*}^{-1}(r)$$

for all $r \in (0, 1)$ and $K > 0$.

From (1.2), (1.3), (1.5) and (1.7) one has

$$\mu_a(r)\mu_a(r') = \frac{\pi^2}{4 \sin^2(\pi a)}, \tag{1.8}$$

$$[\varphi_K(a, r)]^2 + [\varphi_{1/K}(a, r')]^2 = 1, \tag{1.9}$$

$$\mu^*(r)\mu^*\left(\frac{1 - r}{1 + 3r}\right) = \pi^2, \tag{1.10}$$

$$\mu^*(r) = 2\mu^*\left(\frac{\sqrt{8r(1 + r)}}{1 + 3r}\right), \quad \mu^*(r) = \frac{1}{2}\mu^*\left(\frac{1 - r'}{1 + 3r'}\right), \tag{1.11}$$

$$\varphi_2^*(r) = \frac{\sqrt{8r(1+r)}}{1+3r}, \quad \varphi_{1/2}^*(r) = \frac{1-r'}{1+3r'}. \tag{1.12}$$

The main purpose of this paper is to present the inequalities for the function $\mu_a(r)$ and give an infinite product formula for $\mu_{1/4}(r) = \mu^*(r)$. Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1 *Let* $a \in (0, 1/2]$, $R(a)$ *be defined by* (2.2) *and* $M^* = M^*(a) = \min\{M, 2\}$ *with*

$$M = M(a) = \left[1 + \frac{2\sin(\pi a)}{\pi}(R(a) - \log 64)\right]^2.$$

Then the following statements are true:

(1) *The function* $r \mapsto g(r)$ *defined by*

$$g(r) = 2\mu_a\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) - \mu_a(r)$$

is strictly decreasing from $(0, 1)$ *onto* $(0, R(a)/2 - 3\log 2)$ *if* $a \in (0, 1/4)$ *and strictly increasing from* $(0, 1)$ *onto* $(R(a)/2 - 3\log 2, 0)$ *if* $a \in (1/4, 1/2]$, *and* $g(r) = 0$ *if* $a = 1/4$. *Moreover, the inequalities*

$$\mu_a(r) < 2\mu_a\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) < \min\left\{\mu_a(r) + R(a)/2 - 3\log 2, M^*\mu_a(r)\right\} \tag{1.13}$$

hold for all $a \in (0, 1/4)$ *and* $r \in (0, 1)$, *and the inequalities*

$$\max\{\mu_a(r) + R(a)/2 - 3\log 2, M\mu_a(r)\} < 2\mu_a\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) < \mu_a(r) \tag{1.14}$$

hold for all $a \in (1/4, 1/2)$ *and* $r \in (0, 1)$.

(2) *The function* $r \mapsto f(r)$ *defined by*

$$f(r) = \mu_a\left(\frac{1-r}{1+3r}\right) - 2\mu_a(r')$$

is strictly decreasing from $(0, 1)$ *onto* $(3\log 2 - R(a)/2, 0)$ *if* $a \in (0, 1/4)$ *and strictly increasing from* $(0, 1)$ *onto* $(0, 3\log 2 - R(a)/2)$ *if* $a \in (1/4, 1/2]$, *and* $f(r) = 0$ *if* $a = 1/4$. *Moreover, the inequalities*

$$\max\left\{\frac{\pi^2}{2\sin^2(\pi a)}-(R(a)/2-3\log 2)\mu_a(r),\ \frac{1}{M^*}\frac{\pi^2}{2\sin^2(\pi a)}\right\}$$
$$\leq \mu_a(r)\mu_a\left(\frac{1-r}{1+3r}\right)\leq\frac{\pi^2}{2\sin^2(\pi a)} \tag{1.15}$$

hold for all $a\in(0,1/4]$ *and* $r\in(0,1)$, *and the inequalities*

$$\frac{\pi^2}{2\sin^2(\pi a)}\leq\mu_a(r)\mu_a\left(\frac{1-r}{1+3r}\right)$$
$$\leq\min\left\{\frac{\pi^2}{2\sin^2(\pi a)}+(3\log 2-R(a)/2)\mu_a(r),\ \frac{1}{M}\frac{\pi^2}{2\sin^2(\pi a)}\right\} \tag{1.16}$$

hold for all $a\in[1/4,1/2]$ *and* $r\in(0,1)$.
Equality is reached in each inequality of (1.15) *and* (1.16) *if and only if* $a=1/4$.

Theorem 1.2 *Let* $a\in(0,1/2]$, $r\in(0,1)$, $r_0=r'=\sqrt{1-r^2}$, $r_1=\varphi_2^*(r')=\sqrt{8r'(1+r')}/(1+3r')$, $r_2=\varphi_2^*(r_1)=\sqrt{8r_1(1+r_1)}/(1+3r_1)=\varphi_4^*(r')$ *and*

$$r_n=\varphi_2^*(r_{n-1})=\frac{\sqrt{8r_{n-1}(1+r_{n-1})}}{1+3r_{n-1}}=\varphi_{2^n}^*(r'). \tag{1.17}$$

Then the inequalities

$$\prod_{n=0}^{\infty}[(1+r_n)(1+3r_n)]^{2^{-n-1}}\leq\exp[\mu_a(r)+\log r]$$
$$\leq\frac{1}{8}\exp\left(R(a)/2\right)\prod_{n=0}^{\infty}[(1+r_n)(1+3r_n)]^{2^{-n-1}} \tag{1.18}$$

are valid for $a\in(0,1/4]$, *and the reversed inequalities of* (1.18)

$$\frac{1}{8}\exp\left(R(a)/2\right)\prod_{n=0}^{\infty}[(1+r_n)(1+3r_n)]^{2^{-n-1}}\leq\exp[\mu_a(r)+\log r]$$
$$\leq\prod_{n=0}^{\infty}[(1+r_n)(1+3r_n)]^{2^{-n-1}} \tag{1.19}$$

take place for $a\in[1/4,1/2]$. *Moreover, each equality in (1.18) and (1.19) is reached if and only if* $a=1/4$. *In particular, for all* $r\in(0,1)$ *one has*

$$\exp(\mu^*(r)+\log r)=\exp(\mu_{1/4}(r)+\log r)=\prod_{n=0}^{\infty}[(1+r_n)(1+3r_n)]^{2^{-n-1}}, \tag{1.20}$$

namely,

$$\mu^*(r) + \log r = \mu_{1/4}(r) + \log r = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \log\left[(1+r_n)(1+3r_n)\right]. \quad (1.21)$$

By Theorem 1.2, we get Corollary 1.3 immediately.

Corollary 1.3 *Let* $r \in (0, 1)$*, then*

$$\mu^*(r) \leq \mu_a^*(r) \leq \mu^*(r) + \frac{1}{2}[R(a) - \log 64] \quad (1.22)$$

if $a \in (0, 1/4]$*, and*

$$\mu^*(r) + \frac{1}{2}[R(a) - \log 64] \leq \mu_a^*(r) \leq \mu^*(r) \quad (1.23)$$

if $a \in [1/4, 1/2]$*, each equality in* (1.22) *or* (1.23) *is reached if and only if* $a = 1/4$.

2 Proofs of Theorems 1.1 and 1.2

In order to prove our main results, we introduce some basic knowledge and lemmas at first. For $x, y > 0$, the gamma function $\Gamma(x)$, the psi function $\Psi(x)$, and the beta function $B(x, y)$ are defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \mathrm{d}t, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (2.1)$$

respectively (see [28]). If x is not an integer, then the gamma function has the so-called reflection property [28, p.239]

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} = B(x, 1-x).$$

We also need the function

$$R(a) = -2\gamma - \Psi(a) - \Psi(1-a), \quad R(1/4) = \log 64, \quad R(1/2) = \log 16, \quad (2.2)$$

where γ is the Euler–Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215\ldots.$$

From [17, Lemma 2.14(2)], we know that the function $a \mapsto R(a)$ is strictly decreasing on $\in (0, 1/2]$. Thus $R(a) > \log 64$ for $a \in (0, 1/4)$, and $R(a) < \log 64$ for $a \in (1/4, 1/2]$.

The Ramanujan generalized modular equation function $\mu_a(r)$ satisfies the derivative formula (see [17, (2.11)], or [6, p. 86, Corollary])

$$\frac{\mathrm{d}\mu_a(r)}{\mathrm{d}r} = -\frac{1}{rr'^2 F(a, 1-a; 1; r^2)^2}. \tag{2.3}$$

Lemma 2.1 [3, Theorem 5.5(2)] *The function* $r \mapsto \mu_a(r) + \log r$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, R(a)/2)$.

Lemma 2.2 [27, Theorem 2.4] *The inequalities*

$$\begin{aligned} 0 &\le \sqrt{1+3r}F(a, 1-a; 1; r^2) - F\left(a, 1-a; 1; \frac{8r(1+r)}{(1+3r)^2}\right) \\ &\le \frac{\sin(\pi a)[R(a) - \log 64]}{\pi} \end{aligned} \tag{2.4}$$

hold for $a \in (0, 1/4]$ *and* $r \in (0, 1)$*, and the inequalities*

$$\begin{aligned} 0 &\le F\left(a, 1-a; 1; \frac{8r(1+r)}{(1+3r)^2}\right) - \sqrt{1+3r}F(a, 1-a; 1; r^2) \\ &\le \frac{\sin(\pi a)[\log 64 - R(a)]}{\pi} \end{aligned} \tag{2.5}$$

take place for $a \in [1/4, 1/2]$ *and* $r \in (0, 1)$*. Each inequality in* (2.4) *and* (2.5) *becomes equality if and only if* $a = 1/4$.

Proof of Theorem 1.1 For part (1), if $a = 1/4$, then $g(r) = 0$ by (1.11). Let $x = \sqrt{8r(1+r)}/(1+3r)$, then $x' = (1-r)/(1+3r)$ and

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{4(1-r)}{(1+3r)^2\sqrt{8r(1+r)}} = \frac{x'}{4x}(1+3x')^2. \tag{2.6}$$

It follows from $\mu_a(1^-) = 0$ and Lemma 2.1 that

$$\lim_{r \to 1^-} g(r) = 0, \tag{2.7}$$

$$\begin{aligned} \lim_{r \to 0^+} g(r) &= \lim_{r \to 0^+} \left[2\mu_a(x) + 2\log x - (\mu_a(r) + \log r) + \log r - 2\log x\right] \\ &= R(a) - \frac{R(a)}{2} - \log 8 \\ &= \frac{R(a)}{2} - 3\log 2. \end{aligned} \tag{2.8}$$

By (2.3) and (2.6), differentiating g yields

$$\begin{aligned} g'(r) &= -\frac{2}{xx'^2F(a,1-a;1;x^2)^2}\cdot\frac{x'}{4x}(1+3x')^2+\frac{1}{rr'^2F(a,1-a;1;r^2)^2}\\ &= -\frac{(1+3x')^2}{2x'x^2F(a,1-a;1;x^2)^2}+\frac{1}{rr'^2F(a,1-a;1;r^2)^2}\\ &= \frac{1}{rr'^2F(a,1-a;1;r^2)^2F(a,1-a;1;x^2)^2}\\ &\quad\times\left[F(a,1-a;1;x^2)^2-(1+3r)F(a,1-a;1;r^2)^2\right]. \end{aligned} \tag{2.9}$$

From Lemma 2.2 and (2.9), we know that g is strictly decreasing if $a \in (0, 1/4)$ and strictly increasing if $a \in (1/4, 1/2]$, while the range of g can be obtained by (2.7) and (2.8). Therefore, we get the first inequality and the first upper bound in (1.13) together with the first lower bound and the second inequality in (1.14).

Next, we prove the other inequalities in (1.13) and (1.14). If $a \in (0, 1/4]$ ($a \in [1/4, 1/2]$), then the inequality

$$\begin{aligned} &F(a,1-a;1;\left(\frac{1-r}{1+3r}\right)^2)\leq(\geq)\\ &\frac{\sqrt{1+3r}}{2}\left[F(a,1-a;1;1-r^2)+\frac{\sin(\pi a)(R(a)-\log 64)}{\pi}\right] \end{aligned} \tag{2.10}$$

holds for all $r \in (0, 1)$ by changing r to $(1-r)/(1+3r)$ in (2.4) and (2.5). Note that

$$\begin{aligned} 1+\frac{\sin(\pi a)(R(a)-\log 64)}{\pi}\geq 1+\frac{1}{\pi}\left[R\left(\frac{1}{2}\right)-\log 64\right]=1+\frac{1}{\pi}\log\frac{1}{4}\\ =0.5587\ldots \end{aligned} \tag{2.11}$$

for $a \in (0, 1/2]$.

Making use of Lemma 2.2, (2.10) and (2.11) one has

$$\begin{aligned} &2\mu_a\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right)/\mu_a(r)=2\frac{F(a,1-a;1;r^2)}{F(a,1-a;1;1-r^2)}\frac{F\left(a,1-a;1;(\frac{1-r}{1+3r})^2\right)}{F\left(a,1-a;1;\frac{8r(1+r)}{(1+3r)^2}\right)}\\ &\leq(\geq)\frac{\sqrt{1+3r}F(a,1-a;1;r^2)}{F(a,1-a;1;1-r^2)}\frac{F(a,1-a;1;1-r^2)+\sin(\pi a)(R(a)-\log 64)/\pi}{F\left(a,1-a;1;\frac{8r(1+r)}{(1+3r)^2}\right)}\\ &\leq(\geq)\left[1+\frac{\sin(\pi a)(R(a)-\log 64)}{\pi F(a,1-a;1;1-r^2)}\right]\left[1+\frac{\sin(\pi a)(R(a)-\log 64)}{\pi F\left(a,1-a;1;\frac{8r(1+r)}{(1+3r)^2}\right)}\right]\\ &\leq(\geq)\left[1+\frac{\sin(\pi a)(R(a)-\log 64)}{\pi}\right]^2 \end{aligned} \tag{2.12}$$

for $a \in (0, 1/4]$ ($a \in [1/4, 1/2]$). Equality holds in each of above inequalities if and only if $a = 1/4$. On the other hand, since $x > r$, it follows from the monotonicity of $\mu_a(r)$ with respect to $r \in (0, 1)$ that $\mu_a(x) < \mu_a(r)$. Hence, the remaining bounds in (1.13) and (1.14) follow.

For part (2), let $t = (1 - r)/(1 + 3r)$. Then $r' = \sqrt{8t(1+t)}/(1 + 3t)$ and $f(r) = -g(t)$. So that the assertion about f follows from part (1).

Equations (1.8), (1.13), and (1.14) imply that

$$\mu_a(r)\mu_a\left(\frac{1-r}{1+3r}\right) = \frac{2\pi^2}{4\sin^2(\pi a)} \cdot \frac{\mu_a(t)}{2\mu_a\left(\frac{\sqrt{8t(1+t)}}{1+3t}\right)} \geq \frac{1}{M^*}\left[\frac{\pi^2}{2\sin^2(\pi a)}\right] \tag{2.13}$$

for all $a \in (0, 1/4]$ and $r \in (0, 1)$, and the inequality

$$\mu_a(r)\mu_a\left(\frac{1-r}{1+3r}\right) \leq \frac{1}{M}\left[\frac{\pi^2}{2\sin^2(\pi a)}\right] \tag{2.14}$$

holds for $a \in [1/4, 1/2]$ and $r \in (0, 1)$, with equality of (2.13) or (2.14) if and only if $a = 1/4$.

It follows from $f(r) \leq (\geq)0$ for $a \in (0, 1/4]$ ($a \in [1/4, 1/2)$) and $\mu_a[(1-r)/(1+3r)] \leq (\geq)2\mu_a(r')$ that

$$\mu_a(r)\mu_a\left(\frac{1-r}{1+3r}\right) \leq (\geq)2\mu_a(r)\mu_a(r') = \frac{\pi^2}{2\sin^2(\pi a)} \tag{2.15}$$

for $a \in (0, 1/4]$ ($a \in [1/4, 1/2)$).

Therefore, inequalities (1.15) and (1.16) follow from (2.13)–(2.15) together with the monotonicity of f. □

Proof of Theorem 1.2 From $r_0 = r'$, $r_1 = \varphi_2^*(r_0) = \sqrt{8r'(1+r')}/(1 + 3r')$, and $r_2 = \varphi_2^*(r_1) = \varphi_4^*(r')$, we clearly see that $r' = \varphi_{1/2}^*(r_1) = (1 - r_1')/(1 + 3r_1')$, $r_1' = (1 - r')/(1 + 3r') = \varphi_{1/2}^*(r)$, and $r = \varphi_2^*(r_1')$.

Let $a \in (0, 1/2]$ and $r \in (0, 1)$, $h(r) = \mu_a(r) + \log r$, $g(r)$ be defined as in Theorem 1.1 and $\zeta(r)$ be defined by

$$\begin{aligned}\zeta(r) &= \mu_a(r) + \frac{1}{2}\log\left(\frac{1-r'}{1+3r'}\right) \\ &= \mu_a(r) + \log r - \frac{1}{2}\log\left[(1+r_0)(1+3r_0)\right].\end{aligned} \tag{2.16}$$

Then one has

$$\begin{aligned}\zeta(r) = \mu_a(r) + \frac{1}{2}\log r_1' &= \frac{1}{2}[\mu_a(r_1') + \log r_1' + 2\mu_a(r) - \mu_a(r_1')] \\ &= \frac{1}{2}[h(r_1') + g(r_1')],\end{aligned}$$

that is

$$h(r)-\frac{1}{2}\log[(1+r_0)(1+3r_0)]=\frac{1}{2}[h(r_1')+g(r_1')]. \tag{2.17}$$

Similarly, putting $r_1=\varphi_2^*(r_0)$, we get

$$h(r_1')-\frac{1}{2}\log[(1+r_1)(1+3r_1)]=\frac{1}{2}[h(r_2')+g(r_2')], \tag{2.18}$$

and hence, by (2.17) and (2.18)

$$\begin{aligned}&h(r)-\frac{1}{2}\log[(1+r_0)(1+3r_0)]-\frac{1}{4}\log[(1+r_1)(1+3r_1)]\\&=\frac{1}{2}g(r_1')+\frac{1}{4}g(r_2')+\frac{1}{4}h(r_2').\end{aligned}$$

Generally, assume that

$$h(r)-\sum_{k=0}^{n-1}\frac{1}{2^{k+1}}\log[(1+r_k)(1+3r_k)]=\sum_{k=1}^{n}\frac{1}{2^k}g(r_k')+\frac{1}{2^n}h(r_n') \tag{2.19}$$

for $n\in\mathbb{N}$ and $n\geq 2$. Let $r_n=\varphi_2^*(r_{n-1})=\varphi_{2^n}^*(r')$ in (2.17), then

$$h(r_n')-\frac{1}{2}\log[(1+r_n)(1+3r_n)]=\frac{1}{2}[h(r_{n+1}')+g(r_{n+1}')],$$

and from (2.19) we get

$$h(r)-\sum_{k=0}^{n}\frac{1}{2^{k+1}}\log[(1+r_k)(1+3r_k)]=\sum_{k=1}^{n+1}\frac{1}{2^k}g(r_k')+\frac{1}{2^{n+1}}h(r_{n+1}'). \tag{2.20}$$

Hence, by induction, Eq. (2.20) holds for all $n\in\mathbb{N}$, $a\in(0,1/2]$, and $r\in(0,1)$.

Next, we divide the proof into two cases.

Case A $a\in(0,1/4]$. Then by Lemma 2.1, Theorem 1.1, and (2.20) we have

$$\begin{aligned}0&\leq h(r)-\sum_{k=0}^{n}\frac{1}{2^{k+1}}\log[(1+r_k)(1+3r_k)]\\&\leq\sum_{k=1}^{n+1}\frac{1}{2^k}\left[\frac{R(a)}{2}-3\log 2\right]+\frac{1}{2^{n+1}}\frac{R(a)}{2}\\&=\frac{1}{2}[R(a)-\log 64]+\frac{3\log 2}{2^{n+1}}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \log[(1+r_k)(1+3r_k)] \le h(r)$$
$$\le \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \log[(1+r_k)(1+3r_k)] + \frac{1}{2}[R(a) - \log 64]. \tag{2.21}$$

The double inequality (1.18) follows from (2.21).

Case B $a \in [1/4, 1/2]$. It follows from (2.20), Lemma 2.1, and Theorem 1.1(1) that

$$\sum_{k=1}^{n+1} \frac{1}{2^k} \left[\frac{R(a)}{2} - 3 \log 2 \right] \le h(r) - \sum_{k=0}^{n} \frac{1}{2^{k+1}} \log[(1+r_k)(1+3r_k)] \le \frac{1}{2^{n+1}} \frac{R(a)}{2}.$$

Letting $n \to \infty$, we get

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \log[(1+r_k)(1+3r_k)] + \frac{1}{2}[R(a) - \log 64] \le h(r)$$
$$\le \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \log[(1+r_k)(1+3r_k)]. \tag{2.22}$$

The double inequality (1.19) follows from (2.22), and the remaining conclusions are clear. □

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