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Dyson's partition ranks and their multiplicative extensions

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Abstract We study the Dyson rank function $N(r, 3; n)$, the number of partitions of n with rank $\equiv r \pmod 3$. We investigate the convexity of these functions. We extend $N(r, 3; n)$ multiplicatively to the set of partitions, and we determine the maximum value when taken over all partitions of size n.

Keywords Dyson rank · Number theory · Partitions · Combinatorics · Asymptotics · Ramanujan

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1 Introduction

For $n \in \mathbb{N}$, a *partition* of a nonnegative integer n is a finite sequence of nonincreasing natural numbers $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. As usual, let $p(n)$ denote the number of partitions of n. The study of partitions dates back to the eighteenth century, appearing in the work of Euler. Among the most famous properties

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of partitions are the following congruences, proved by Srinivasa Ramanujan in 1919 [8]:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In the 1940s, Freeman Dyson aimed to find a combinatorial explanation for these congruences. He sought a combinatorial statistic that divides the partitions of $5n+4$ (resp. $7n+5$, $11n+6$) into 5 (resp. 7, 11) groups of equal size. He found the rank statistic [6].

The *rank* of a partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is $\lambda_1 - k$: that is, the size of the largest part minus the number of parts. Let $N(m, n)$ be the number of partitions of n with Dyson rank m. The generating function of $N(m, n)$ is the following:

$$R(w; q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n}, \quad (1.1)$$

where $(a; q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$. Understanding Ramanujan's congruences using Dyson's rank requires the following variant. Let $N(r, t; n)$ be the number of partitions of n with rank $\equiv r \pmod{t}$. Using this notion, Dyson conjectured (and Atkin and Swinnerton-Dyer later proved [2]) that for each m, we have

$$\begin{aligned} N(m, 5; 5n+4) &= \frac{1}{5} p(5n+4), \\ N(m, 7; 7n+5) &= \frac{1}{7} p(7n+5). \end{aligned}$$

This confirms that the rank statistic provides a combinatorial proof[1] of Ramanujan's congruences modulo 5 and 7. Ramanujan, in a joint work with Hardy, also proved the following asymptotic formula [1]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}. \quad (1.2)$$

Using a refinement of this asymptotic due to Lehmer [7], Bessenrodt and Ono [3] recently proved that the partition function satisfies the following convexity property. If a, b are integers with $a, b > 1$ and $a + b > 9$, then

$$p(a)p(b) > p(a+b).$$

[1] Dyson's rank does not explain Ramanujan's congruence modulo 11.

In view of this, it is natural to ask whether Dyson's rank functions $N(r, t; n)$ also satisfy convexity. We prove this for each $r = 0, 1, 2$ and $t = 3$ for all but a finite number of a and b.

Theorem 1.1 *If* $r = 0$ (*resp.* $r = 1, 2$)*, then*

$$N(r, 3; a)N(r, 3; b) > N(r, 3; a + b)$$

for all $a, b \geq 12$ (*resp.* 11*,* 11).

Remark Notice that this bound is sharp for $r = 0$ (resp. 1, 2) for $a, b = 11$ (resp. 10, 10). Namely, we have that

$$N(0, 3; 11)N(0, 3; 11) = 16 \cdot 16 < 340 = N(0, 3; 22),$$
$$N(1, 3; 10)N(1, 3; 10) = 13 \cdot 13 < 211 = N(1, 3; 20),$$
$$N(2, 3; 10)N(2, 3; 10) = 13 \cdot 13 < 211 = N(2, 3; 20).$$

Bessenrodt and Ono [3] used their convexity result to study the multiplicative extension of the partition function defined by

$$p(\lambda) := \prod_{j=1}^{k} p(\lambda_j), \tag{1.3}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition. For example, if $\lambda = (5, 3, 2)$, then $p(\lambda) = p(5)p(3)p(2) = 42$. They then studied the maximum of this function on $P(n)$, the set of all partitions of n. The maximal value is defined as

$$\max p(n) := \max(p(\lambda) : \lambda \in P(n)).$$

Their main result was a closed formula for $\max p(n)$, and they also fully characterized all partitions $\lambda \in P(n)$ that achieve this maximum. We carry out a similar analysis for the functions $N(r, t; n)$ in the case of $t = 3$. We extend each $N(r, 3; n)$ to partitions by

$$N(r, 3; \lambda) := \prod_{j=1}^{k} N(r, 3; \lambda_j). \tag{1.4}$$

We determine the maximum of each function on $P(n)$, where the maximal value is defined as

$$\mathrm{maxN}(r, 3; n) := \max(N(r, 3; \lambda) : \lambda \in P(n)). \tag{1.5}$$

We also fully characterize all $\lambda \in P(n)$ that achieve each maximum.

Theorem 1.2 *Assume the notation above. Then the following are true:*
(1) *If* $n \geq 33$*, then we have that*

$$\mathrm{maxN}(0,3;n) = \begin{cases} 7^{\frac{n}{7}} & n \equiv 0 \pmod 7 \\ 37^2 \cdot 16 \cdot 7^{\frac{n-36}{7}} & n \equiv 1 \pmod 7 \\ 37 \cdot 16 \cdot 7^{\frac{n-23}{7}} & n \equiv 2 \pmod 7 \\ 16 \cdot 7^{\frac{n-10}{7}} & n \equiv 3 \pmod 7 \\ 37^3 \cdot 7^{\frac{n-39}{7}} & n \equiv 4 \pmod 7 \\ 37^2 \cdot 7^{\frac{n-26}{7}} & n \equiv 5 \pmod 7 \\ 37 \cdot 7^{\frac{n-13}{7}} & n \equiv 6 \pmod 7, \end{cases}$$

and it is achieved at the unique partitions

$$\begin{aligned} (7,7,\ldots,7) \text{ when } n &\equiv 0 \pmod 7 \\ (13,13,10,7,\ldots,7) \text{ when } n &\equiv 1 \pmod 7 \\ (13,10,7,\ldots,7) \text{ when } n &\equiv 2 \pmod 7 \\ (10,7,\ldots,7) \text{ when } n &\equiv 3 \pmod 7 \\ (13,13,13,7,\ldots,7) \text{ when } n &\equiv 4 \pmod 7 \\ (13,13,7,\ldots,7) \text{ when } n &\equiv 5 \pmod 7 \\ (13,7,\ldots,7) \text{ when } n &\equiv 6 \pmod 7. \end{aligned}$$

(2) *If* $n \geq 22$*, then we have that*

$$\mathrm{maxN}(1,3;n) = \mathrm{maxN}(2,3;n)$$
$$= \begin{cases} 46^{\frac{n}{14}} & \text{when } n \equiv 0 \pmod{14} \\ 59 \cdot 46^{\frac{n-15}{14}} & \text{when } n \equiv 1 \pmod{14} \\ 59^2 \cdot 46^{\frac{n-30}{14}} & \text{when } n \equiv 2 \pmod{14} \\ 101 \cdot 46^{\frac{n-17}{14}} & \text{when } n \equiv 3 \pmod{14} \\ 101 \cdot 59 \cdot 46^{\frac{n-32}{14}} & \text{when } n \equiv 4 \pmod{14} \\ 20^3 \cdot 46^{\frac{n-33}{14}} & \text{when } n \equiv 5 \pmod{14} \\ 26 \cdot 20^2 \cdot 46^{\frac{n-34}{14}} & \text{when } n \equiv 6 \pmod{14} \\ 26^2 \cdot 20 \cdot 46^{\frac{n-35}{14}} & \text{when } n \equiv 7 \pmod{14} \\ 20^2 \cdot 46^{\frac{n-22}{14}} & \text{when } n \equiv 8 \pmod{14} \\ 26 \cdot 20 \cdot 46^{\frac{n-23}{14}} & \text{when } n \equiv 9 \pmod{14} \\ 26^2 \cdot 46^{\frac{n-24}{14}} & \text{when } n \equiv 10 \pmod{14} \\ 20 \cdot 46^{\frac{n-11}{14}} & \text{when } n \equiv 11 \pmod{14} \\ 26 \cdot 46^{\frac{n-12}{14}} & \text{when } n \equiv 12 \pmod{14} \\ 59 \cdot 26 \cdot 46^{\frac{n-27}{14}} & \text{when } n \equiv 13 \pmod{14}, \end{cases}$$

and it is achieved at the unique partitions

$$\begin{aligned}
(14, 14, \ldots, 14) &\textit{ when } n \equiv 0 \pmod{14}\\
(15, 14, \ldots, 14) &\textit{ when } n \equiv 1 \pmod{14}\\
(15, 15, 14, \ldots, 14) &\textit{ when } n \equiv 2 \pmod{14}\\
(17, 14, \ldots, 14) &\textit{ when } n \equiv 3 \pmod{14}\\
(17, 15, 14, \ldots, 14) &\textit{ when } n \equiv 4 \pmod{14}\\
(11, 11, 11, 14, \ldots, 14) &\textit{ when } n \equiv 5 \pmod{14}\\
(12, 11, 11, 14, \ldots, 14) &\textit{ when } n \equiv 6 \pmod{14}\\
(12, 12, 11, 14, \ldots, 14) &\textit{ when } n \equiv 7 \pmod{14}\\
(11, 11, 14, \ldots, 14) &\textit{ when } n \equiv 8 \pmod{14}\\
(12, 11, 14, \ldots, 14) &\textit{ when } n \equiv 9 \pmod{14}\\
(12, 12, 14, \ldots, 14) &\textit{ when } n \equiv 10 \pmod{14}\\
(11, 14, \ldots, 14) &\textit{ when } n \equiv 11 \pmod{14}\\
(12, 14, \ldots, 14) &\textit{ when } n \equiv 12 \pmod{14}\\
(15, 12, 14, \ldots, 14) &\textit{ when } n \equiv 13 \pmod{14}.
\end{aligned}$$

In Sect. 2, we prove Theorem 1.1 by finding explicit upper and lower bounds for $N(r, 3; n)$ using the work of Lehmer [7] and Bringmann [4]. In Sect. 3, we prove Theorem 1.2 by applying the convexity property together with combinatorial arguments. In Sect. 4, we discuss potential extensions of our results to other values of t.

2 Proof of Theorem 1.1

Theorem 1.1 states that

$$N(r, 3; a)N(r, 3; b) > N(r, 3; a+b) \tag{2.1}$$

for $r = 0$ (resp. 1, 2) for $a, b \geq 12$ (resp. 11, 11). Essentially, this implies the convexity of Dyson's rank functions $N(r, 3; n)$. We prove (2.1) for $a, b \geq 500$ by finding a lower bound for $N(r, 3; a)N(r, 3; b)$ and an upper bound for $N(r, 3; a+b)$. We verify the remaining cases using a computer program.

2.1 Preliminaries for the Proof of Theorem 1.1

In order to obtain bounds for $N(r, 3; n)$, we use methods in analytic number theory. We use analytic estimates due to Lehmer and Bringmann in order to study and bound $N(r, 3; n)$.

For $p(n)$, we use the explicit bounds provided by Lehmer [3].

Theorem 2.1 (Lehmer) *If n is a positive integer and $\mu = \mu(n) := \frac{\pi}{6}\sqrt{24n-1}$, then*

$$p(n) = \frac{\sqrt{12}}{24n-1}\left[\left(1-\frac{1}{\mu}\right)e^{\mu} + \left(1+\frac{1}{\mu}\right)e^{-\mu}\right] + E(n), \tag{2.2}$$

where we have that

$$|E(n)| < \frac{\pi^2}{\sqrt{3}}\left[\frac{1}{\mu^3}\sinh(\mu) + \frac{1}{6} - \frac{1}{\mu^2}\right]. \tag{2.3}$$

Now, by using the Lehmer bound, Bessenrodt and Ono [3] obtained bounds $p^L(n)$ and $p^U(n)$ on $p(n)$ that satisfy

$$p^L(n) < p(n) < p^U(n), \tag{2.4}$$

where $p^L(n)$ and $p^U(n)$ are defined as follows:

$$p^L(n) := \frac{\sqrt{3}}{12n}\left(1-\frac{1}{\sqrt{n}}\right)e^{\mu},$$
$$p^U(n) := \frac{\sqrt{3}}{12n}\left(1+\frac{1}{\sqrt{n}}\right)e^{\mu}.$$

Bringmann estimates and obtains asymptotics for $R(\zeta_c^a; q)$ for positive integers $a < c$ in the case that c is odd and where $\zeta_c^a := e^{\frac{2\pi i a}{c}}$. She uses the following notation:

$$R(\zeta_c^a; q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{c}; n\right) q^n. \tag{2.5}$$

Here, we recall a precise version of her result in the special case that $a = 1$ and $c = 3$. We use the following notation:

$$\widetilde{E}_1(n) := \frac{12}{(24n-1)^{1/2}} \sum_{k=2}^{\frac{\sqrt{n}}{3}} k^{\frac{1}{2}} \cdot \sinh\left(\frac{\pi}{18k}\sqrt{24n-1}\right),$$
$$\widetilde{E}_2(n) := \frac{0.12 \cdot e^{2\pi + \frac{\pi}{24}}}{\sqrt{3}} \sum_{k=1}^{\frac{\sqrt{n}}{3}} k^{-\frac{1}{2}},$$
$$\widetilde{E}_3(n) := 1.412\sqrt{3} \cdot e^{2\pi} \sum_{1 \le k \le \sqrt{n}, 3|k} k^{-\frac{1}{2}},$$
$$\widetilde{E}_4(n) := 2\sqrt{3}e^{2\pi + \frac{\pi}{12}} \cdot n^{-1/2} \sum_{1 \le k \le \frac{\sqrt{n}}{3}} k^{\frac{1}{2}},$$

$$\widetilde{E}_5(n) := 8\pi \cdot e^{2\pi + \frac{\pi}{24}} \cdot n^{-3/4} \sum_{1 \le k \le \frac{\sqrt{n}}{3}} k,$$

$$\widetilde{E}_6(n) := 2^{\frac{1}{4}} \cdot (e + e^{-1}) \cdot e^{2\pi} \cdot n^{-1/4} \sum_{1 \le k \le \sqrt{n}} \frac{1}{k} \sum_{v=1}^{k} \left(\min \left(\left\{ \frac{v}{k} - \frac{1}{6k} + \frac{1}{3} \right\}, \left\{ \frac{v}{k} - \frac{1}{6k} - \frac{1}{3} \right\} \right) \right)^{-1}.$$

Bringmann proves the following result regarding the main term $M(n)$ and bound on the error term E_R (see pp. 18–19 of [4]):

Theorem 2.2 (Bringmann) *For $n \in \mathbb{N}$, let $M(n)$ be*

$$M(n) := -\frac{8 \sin\left(\frac{\pi}{18} - \frac{2n\pi}{3}\right) \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)}{\sqrt{24n-1}}.$$

Then we have that

$$A\left(\frac{1}{3}; n\right) = M(n) + E_R(n),$$

where

$$E_R(n) := \sum_{i=1}^{6} E_i(n),$$

and each $E_i(n)$ is bounded as follows:

$$|E_i(n)| \le \widetilde{E}_i(n).$$

Remark One can find explicit definitions of $E_1(n), \ldots, E_6(n)$ scattered throughout [4].

2.2 Explicit bounds for error terms

In order to prove Theorem 1.1, we must effectively bound each of the error terms $\widetilde{E}_1(n), \ldots, \widetilde{E}_6(n)$. First, we obtain $L(n)$, a lower bound for $M(n)$, and $U(n)$, an upper bound for $M(n)$ by using the fact that for any integer n, $|\sin \frac{\pi}{18}| \le |\sin(\frac{\pi}{18} - \frac{2n\pi}{3})| \le 1$. Thus, we have that the following is true:

$$L(n) := \left| \frac{8 \sin\left(\frac{\pi}{18}\right) \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)}{\sqrt{24n-1}} \right| \le |M(n)| \le \left| \frac{8 \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)}{\sqrt{24n-1}} \right| =: U(n).$$

In Sects. 2.2.1–2.2.6, we prove the following bounds for each $\widetilde{E}_i(n)$:

Table 1 Explicit bounds for error term ratios

i	c_i
1	0.0065
2	0.00019
3	0.0098
4	0.0071
5	0.0072
6	0.54

Proposition 2.3 *For $i = 1, 2, \ldots, 6$ and for $n \geq 500$, we have that*

$$\frac{\widetilde{E}_i(n)}{L(n)} < c_i,$$

where all c_i are listed in Table 1.

Using Proposition 2.3, we obtain the following bound for $E_{\mathrm{R}}(n)$:

Corollary 2.4 *Assume the notation above. Then for $n \geq 500$, the following is true:*

$$|E_{\mathrm{R}}(n)| \leq 0.58L(n).$$

Proof This follows from adding the bounds for each $\widetilde{E}_i(n)$ in Proposition 2.3 and applying Theorem 2.2. □

2.2.1 Effective bounds for error $\widetilde{E}_1(n)$

We prove Proposition 2.3 for $i = 1$. We approximate the finite sum in $\widetilde{E}_1(n)$ by the number of terms multiplied by the summand evaluated at $k = 2$ (since this is the largest summand). For $n \geq 500$, we then have that

$$\widetilde{E}_1(n) \leq \frac{\sqrt{n}}{3}\left(\frac{12}{(24n-1)^{1/2}} 2^{\frac{1}{2}} \cdot \sinh\left(\frac{\pi}{36}\sqrt{24n-1}\right)\right).$$

Now, we consider the ratio of our bound of $\widetilde{E}_1(n)$ to $L(n)$. We have that

$$\begin{aligned}\frac{\widetilde{E}_1(n)}{L(n)} &\leq \frac{\frac{\sqrt{n}}{3}\left(\frac{12}{(24n-1)^{1/2}} 2^{\frac{1}{2}} \cdot \sinh\left(\frac{\pi}{36}\sqrt{24n-1}\right)\right)}{L(n)} \\ &= \frac{\sqrt{n}\sinh\left(\frac{\pi}{36}\sqrt{24n-1}\right)}{\sqrt{2}\sin\left(\frac{\pi}{18}\right)\sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)} =: F_1(n).\end{aligned}$$

It is easy to check that $F_1(n)$ is a decreasing function of n for $n \geq 500$. This means that

$$\widetilde{E}_1(n) \leq F_1(500)L(n) \leq 0.0065L(n).$$

2.2.2 Effective bounds for error $\widetilde{E}_2(n)$

We prove Proposition 2.3 for $i = 2$. We find an upper bound for $\widetilde{E}_2(n)$. Using this upper bound, for $n \geq 500$,

$$\sum_{k=1}^{\frac{\sqrt{n}}{3}} k^{-\frac{1}{2}} \leq \int_0^{\frac{\sqrt{n}}{3}} k^{-\frac{1}{2}} \mathrm{d}k.$$

For $n \geq 500$, we have that

$$\begin{aligned}\widetilde{E}_2(n) &\leq \frac{0.12 \cdot e^{2\pi + \frac{\pi}{24}}}{\sqrt{3}} \int_0^{\frac{\sqrt{n}}{3}} k^{-\frac{1}{2}} \mathrm{d}k \\ &\leq 0.08 \cdot e^{2\pi + \frac{\pi}{24}} n^{\frac{1}{4}}.\end{aligned}$$

Now, we consider the ratio of our bound of $\widetilde{E}_2(n)$ to $L(n)$. We have that

$$\frac{\widetilde{E}_2(n)}{L(n)} \leq \frac{0.08 \cdot e^{2\pi + \frac{\pi}{24}} n^{\frac{1}{4}}}{L(n)} = \frac{0.01 e^{2\pi + \frac{\pi}{24}} n^{\frac{1}{4}} \sqrt{24n-1}}{\sin\left(\frac{\pi}{18}\right) \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)} =: F_2(n).$$

It is easy to check that $F_2(n)$ is a decreasing function of n for $n \geq 500$. This means that

$$\widetilde{E}_2(n) \leq F_2(500) L(n) \leq 0.0019 L(n).$$

2.2.3 Effective bounds for error $\widetilde{E}_3(n)$

We prove Proposition 2.3 for $i = 3$. We estimate $\widetilde{E}_3(n)$ by using our method from Sect. 2.2.2. For $n \geq 500$, we have that

$$\widetilde{E}_3(n) \leq 1.412\sqrt{3} \cdot e^{2\pi} \int_0^{\sqrt{n}} k^{-\frac{1}{2}} \mathrm{d}k \leq 2.824\sqrt{3} \cdot e^{2\pi} n^{\frac{1}{4}}.$$

Now, we consider the ratio of our bound of $\widetilde{E}_3(n)$ to $L(n)$. We have that

$$\frac{\widetilde{E}_3(n)}{L(n)} \leq \frac{2.824\sqrt{3} \cdot e^{2\pi} n^{\frac{1}{4}}}{L(n)} \leq \frac{2.824\sqrt{3} \cdot e^{2\pi} n^{\frac{1}{4}} \sqrt{24n-1}}{8 \sin\left(\frac{\pi}{18}\right) \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)} =: F_3(n).$$

It is easy to check that $F_3(n)$ is a decreasing function of n for $n \geq 500$. This means that

$$\widetilde{E}_3(n) \leq F_3(500) L(n) \leq 0.0098 L(n).$$

2.2.4 Effective bounds for error $\widetilde{E}_4(n)$

We prove Proposition 2.3 for $i = 4$. We find an upper bound for $\widetilde{E}_4(n)$. Using this upper bound, for $n \geq 500$, we have that

$$\sum_{k=1}^{\frac{\sqrt{n}}{3}} k^{\frac{1}{2}} \leq \int_0^{\frac{\sqrt{n}}{2}} k^{\frac{1}{2}} \mathrm{d}k.$$

For $n \geq 500$, we have that

$$\widetilde{E}_4(n) \leq 2\sqrt{3} e^{2\pi + \frac{\pi}{12}} \cdot n^{-1/2} \int_0^{\frac{\sqrt{n}}{2}} k^{\frac{1}{2}} \mathrm{d}k \leq \frac{\sqrt{6}}{3} e^{2\pi + \frac{\pi}{12}} \cdot n^{\frac{1}{4}}.$$

Now, we consider the ratio of our bound of $\widetilde{E}_4(n)$ to $L(n)$. We have that

$$\frac{\widetilde{E}_4(n)}{L(n)} \leq \frac{\frac{\sqrt{6}}{3} e^{2\pi + \frac{\pi}{12}} \cdot n^{\frac{1}{4}}}{L(n)} \leq \frac{\sqrt{6} e^{2\pi + \frac{\pi}{12}} \cdot n^{\frac{1}{4}} \sqrt{24n - 1}}{24 \sin\left(\frac{\pi}{18}\right) \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)} =: F_4(n).$$

It is easy to check that $F_4(n)$ is a decreasing function of n for $n \geq 500$. This means that

$$\widetilde{E}_4(n) \leq F_4(500) L(n) \leq 0.0071 L(n).$$

2.2.5 Effective bounds for error $\widetilde{E}_5(n)$

We prove Proposition 2.3 for $i = 5$. We find an upper bound for $\widetilde{E}_5(n)$ by using the methods from Sect. 2.2.4. For $n \geq 500$, we have that

$$\widetilde{E}_5(n) \leq 8\pi \cdot e^{2\pi + \frac{\pi}{24}} \cdot n^{-3/4} \int_0^{\frac{\sqrt{n}}{2}} k \mathrm{d}k \leq \pi \cdot e^{2\pi + \frac{\pi}{24}} \cdot n^{\frac{1}{4}}.$$

Now, we consider the ratio of our bound of $\widetilde{E}_5(n)$ to $L(n)$. We have that

$$\frac{\widetilde{E}_5(n)}{L(n)} \leq \frac{\pi \cdot e^{2\pi + \frac{\pi}{24}} \cdot n^{\frac{1}{4}}}{L(n)} \leq \frac{\pi \cdot e^{2\pi + \frac{\pi}{24}} \cdot n^{\frac{1}{4}} \sqrt{24n - 1}}{8 \sin\left(\frac{\pi}{18}\right) \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)} =: F_5(n).$$

It is easy to check that $F_5(n)$ is a decreasing function of n for $n \geq 500$. This means that

$$\widetilde{E}_5(n) \leq F_5(500) L(n) \leq 0.0072 L(n).$$

2.2.6 Effective bounds for error $\widetilde{E}_6(n)$

We prove Proposition 2.3 for $i = 6$. First, we notice that $\left(\min\left(\left\{\frac{v}{k}-\frac{1}{6k}+\frac{1}{3}\right\}, \left\{\frac{v}{k}-\frac{1}{6k}-\frac{1}{3}\right\}\right)\right)^{-1} \leq 6k$. We estimate the sum with the methods from Sects. 2.2.4 and 2.2.5 For $n \geq 500$, we have that

$$\begin{aligned}\widetilde{E}_6(n) &\leq 2^{\frac{1}{4}} \cdot \left(e+e^{-1}\right) \cdot e^{2\pi} \cdot n^{-1/4} \int_1^{\sqrt{n}+1} 6k\mathrm{d}k \\ &\leq 2^{\frac{1}{4}} \cdot \left(e+e^{-1}\right) \cdot e^{2\pi} \cdot 3\left(n^{\frac{3}{4}}+2n^{\frac{1}{4}}\right).\end{aligned}$$

Now, we consider the ratio of our bound of $\widetilde{E}_6(n)$ to $L(n)$. We have that

$$\begin{aligned}\frac{\widetilde{E}_6(n)}{L(n)} &\leq \frac{2^{\frac{1}{4}} \cdot \left(e+e^{-1}\right) \cdot e^{2\pi} \cdot 3\left(n^{\frac{3}{4}}+2n^{\frac{1}{4}}\right)}{L(n)} \\ &\leq \frac{2^{\frac{1}{4}} \cdot \left(e+e^{-1}\right) \cdot e^{2\pi} \cdot 3(n^{\frac{3}{4}}+2n^{\frac{1}{4}})\sqrt{24n-1}}{8\sin\left(\frac{\pi}{18}\right)\sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)} =: F_6(n).\end{aligned}$$

It is easy to check that $F_6(n)$ is a decreasing function of n for $n \geq 500$. This implies that

$$\widetilde{E}_6(n) \leq F_6(500)L(n) \leq 0.54L(n).$$

2.3 Proof of Theorem 1.1

In order to prove convexity for $N(r,3;n)$, we first write $N(r,3;n)$ in terms of $p(n)$ and $A\left(\frac{1}{3};n\right)$. We have the following generating function of $N(r,t;n)$ for all t, where we use the special case that $t=3$:

Proposition 2.5 *For nonnegative integers r and t, we have that*

$$1+\sum_{n=1}^{\infty} N(r,t;n)q^n = 1+\frac{1}{t}\left[\sum_{n=1}^{\infty} p(n)q^n + \sum_{j=1}^{t-1} \zeta_t^{-rj} R(\zeta_t^j;q)\right],$$

where $\zeta_t := e^{2\pi i/t}$.

Proof For r and t as defined above, we have that

$$\begin{aligned}1+\frac{1}{t}\left[\sum_{j=0}^{t-1} \zeta_t^{-rj} R(\zeta_t^j;q)\right] &= 1+\frac{1}{t}\sum_{j=0}^{t-1}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)\zeta_t^{-rj}\zeta_t^{mj}q^n \\ &= 1+\frac{1}{t}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{j=0}^{t-1} N(m,n)\zeta_t^{(m-r)j}q^n.\end{aligned}$$

Notice that for each $m \not\equiv r \pmod{t}$, because the sum is over the complete set of tth roots of unity, the coefficient of q^n vanishes. For each $m \equiv r \pmod{t}$, the coefficient of q^n is equal to $N(m, n)$. Hence, we obtain the desired result. □

We can now determine an explicit bound for $N(r, 3; n)$:

Proposition 2.6 *For r defined as above and $n \geq 500$, we have the following bound for $N(r, 3; n)$:*

$$\frac{1}{3}\left(p^L(n) - 2\left|A\left(\frac{1}{3}; n\right)\right|\right) \leq |N(r; 3, n)| \leq \frac{1}{3}\left(p^U(n) + 2\left|A\left(\frac{1}{3}; n\right)\right|\right).$$

Proof First, we have that $A\left(\frac{1}{3}; n\right) = A\left(\frac{2}{3}; n\right)$ by the symmetry of the third roots of unity. This fact, together with Proposition 2.5, yields the following:

$$\frac{1}{3}\left(p(n) - 2\left|A\left(\frac{1}{3}; n\right)\right|\right) \leq |N(r; 3, n)| \leq \frac{1}{3}\left(p(n) + 2\left|A\left(\frac{1}{3}; n\right)\right|\right).$$

Now, we apply (2.4) to obtain the desired result. □

We use Corollary 2.4 together with Proposition 2.6 to obtain the following upper and lower bounds for $N(r, 3; n)$:

Proposition 2.7 *Assume the notation above. Then for $n \geq 500$, the following is true:*

$$\frac{1}{3}(1 - 0.01)p^L(n) < N(r, t; n) < \frac{1}{3}(1 + 0.01)p^U(n).$$

Proof By Corollary 2.4, we have that

$$\left|A\left(\frac{j}{3}; n\right)\right| \leq 1.58U(n).$$

It is easy to check that

$$\frac{U(n)}{p^L(n)} = \frac{2\sqrt{3}\sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)}{3n\sqrt{24n-1}e^{\frac{\pi\sqrt{24n-1}}{6}}\left(1 - \frac{1}{\sqrt{n}}\right)}$$

is a decreasing function in n for $n \geq 500$. As a result, we obtain the following:

$$\left|A\left(\frac{j}{3}; n\right)\right| \leq \frac{1.58U(500)}{p^L(500)}p^L(n) < 0.005p^L(n).$$

We apply this to Proposition 2.6 to obtain the desired result. □

We now use Proposition 2.7 together with an argument similar to that of Bessenrodt and Ono (see pp. 2–3 of [3]) to prove Theorem 1.1 for $a, b \geq 500$. We first define the following notation:

$$S_x(\lambda) := \frac{\left(1+\frac{1}{\sqrt{x+\lambda x}}\right)}{\left(1-\frac{1}{\sqrt{x}}\right)\left(1-\frac{1}{\sqrt{\lambda x}}\right)},$$
$$T_x(\lambda) := \frac{\pi}{6}\left(\sqrt{24x-1}+\sqrt{24\lambda x-1}-\sqrt{24(x+\lambda x)-1}\right).$$

We use the following Lemma in our proof:

Lemma 2.8 *Assume the notation above. Suppose that for a fixed* $0 < c < 1$ *and for any nonnegative integers* t *and* n*, we have that*

$$\frac{1}{t}(1-c)p^L(n) < N(r,t;n) < \frac{1}{t}(1+c)p^U(n).$$

Then we have that

$$N(r,t;a)N(r,t;b) > N(r,t;a+b)$$

for all $a, b \geq x$*, where* x *is the minimum value satisfying*

$$T_x(1) > \log\left(4x\sqrt{3}t\frac{1+c}{(1-c)^2}\right) + \log(S_x(1)).$$

Proof We may assume that $1 < a \leq b$ for convenience, so we will let $b = \lambda a$. These inequalities give us

$$N(r,t;a)N(r,t;\lambda a) > \frac{1}{48\lambda a^2}\cdot\frac{1}{t^3}(1-c)^2\left(1-\frac{1}{\sqrt{a}}\right)\left(1-\frac{1}{\sqrt{\lambda a}}\right)e^{\mu(a)+\mu(\lambda a)},$$
$$N(r,t;a+\lambda a) < \frac{\sqrt{3}}{12(a+\lambda a)}\left(1+\frac{1}{\sqrt{a+\lambda a}}\right)e^{\mu(a+\lambda a)}\frac{1}{t}(1+c).$$

For all but finitely many cases, it suffices to find conditions on $a > 1$ and $\lambda \geq 1$ for which

$$\frac{1}{48\lambda a^2}\cdot\frac{1}{t^2}(1-c)^2\left(1-\frac{1}{\sqrt{a}}\right)\left(1-\frac{1}{\sqrt{\lambda a}}\right)e^{\mu(a)+\mu(\lambda a)}$$
$$> \frac{\sqrt{3}}{12(a+\lambda a)}\left(1+\frac{1}{\sqrt{a+\lambda a}}\right)e^{\mu(a+\lambda a)}\frac{1}{t}(1+c).$$

We have that $\frac{\lambda}{\lambda+1} \leq 1$, so it suffices to consider when

$$e^{\mu(a)+\mu(\lambda a)-\mu(a+\lambda a)} > 4a\sqrt{3}t\frac{1+c}{(1-c)^2}S_a(\lambda).$$

By taking the natural log, we obtain the inequality

$$T_a(\lambda) > \log\left(4a\sqrt{3}t\frac{1+c}{(1-c)^2}\right) + \log(S_a(\lambda)).$$

Simple calculations reveal that $S_a(\lambda)$ is decreasing for $\lambda \geq 1$, while $T_a(\lambda)$ is increasing in $\lambda \geq 1$. Therefore, we consider

$$\begin{aligned} T_a(\lambda) \geq T_a(1) &> \log\left(4a\sqrt{3}t\frac{1+c}{(1-c)^2}\right) + \log(S_a(1)) \\ &\geq \log\left(4a\sqrt{3}t\frac{1+c}{(1-c)^2}\right) + \log(S_a(\lambda)). \end{aligned}$$

□

It can be verified that

$$T_x(1) > \log\left(12x\sqrt{3}\frac{1+0.01}{(1-0.01)^2}\right) + \log(S_x(1))$$

for $x \geq 500$. This means that

$$N(r,3;a)N(r,3;b) > N(r,3;a+b) \tag{2.6}$$

for $a, b \geq 500$. We used Sage to confirm (2.6) for $500 \geq \max(a,b) \geq 12$ (resp. 11, 11) for $r = 0$ (resp. $r = 1, 2$). This proves Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We compute the maximum of the multiplicative extension $N(r, 3; \lambda)$ over all partitions of n. In addition, we identify the partitions that attain these values. These results are deduced from Theorem 1.1. Since there are 3 residue classes modulo 3 and we have that[2] $N(1,3;n) = N(2,3;n)$, we split our computation into two cases: $r = 0$ and $r = 1, 2$. In Sect. 3.1, we compute maxN$(0, 3; n)$. In Sect. 3.2, we compute maxN$(1, 3; n)$.

3.1 Proof of Theorem 1.2 for $r = 0$

In Sect. 3.1.1, we prove some combinatorial properties of $N(0, 3; \lambda)$ resulting from Theorem 1.1 and the values of $N(0, 3; n)$ for small n. In Sect. 3.1.2, we use these properties to deduce Theorem 1.2 for $r = 0$.

[2] This follows immediately from considering conjugations of Ferrers diagrams.

Table 2 Maximal value partitions λ

n	$N(0, 3; n)$	maxN$(0, 3; n)$	λ
1	1	1	(1)
2	0	1	(1, 1)
3	1	1	(3), (1, 1, 1)
4	3	3	(4)
5	1	3	(4, 1)
6	3	3	(6), (4, 1, 1)
7	7	7	(7)
8	6	9	(4, 4)
9	10	10	(9)
10	16	16	(10)
11	16	21	(7, 4)
12	25	27	(4, 4, 4)
13	37	37	(13)
14	45	49	(7, 7)
15	58	63	(7, 4, 4)
16	81	81	(16), (4, 4, 4, 4)
17	95	112	(10, 7)
18	127	147	(7, 7, 4)
19	168	189	(7, 4, 4, 4)
20	205	259	(13, 7)
21	264	343	(7, 7, 7)
22	340	441	(7, 7, 4, 4)
23	413	592	(13, 10)
24	523	784	(10, 7, 7)
25	660	1029	(7, 7, 7, 4)
26	806	1369	(13, 13)
27	1002	1813	(13, 7, 7)
28	1248	2401	(7, 7, 7, 7)
29	1513	3087	(7, 7, 7, 4, 4)
30	1866	4144	(13, 10, 7)
31	2292	5488	(10, 7, 7, 7)
32	2775	7203	(7, 7, 7, 7, 4)

3.1.1 Some combinatorics for $r = 0$

We require the values of $N(0, 3; n)$ for $n \leq 32$. These values are given in the first two columns of Table 2, which were computed using Sage. We prove the correctness of the values in the last two columns over the course of this section.

Throughout this section, let λ be a partition $(\lambda_1, \lambda_2, \ldots) \in P(n)$ such that $N(0, 3; \lambda)$ is maximal. First, we bound the size of λ_1.

Proposition 3.1 *Assume the notation and hypotheses above. Then the following is true:*

$$\lambda_1 \leq 23.$$

Proof Suppose that λ has a part $k \geq 24$. Then by Theorem 1.1, replacing k with the parts $\lfloor \frac{k}{2} \rfloor$ and $\lceil \frac{k}{2} \rceil$ would yield a partition μ such that $N(0, 3; \mu) > N(0, 3; \lambda)$. This is a contradiction since $N(0, 3; \lambda)$ is maximal. □

For $i > 0$, let m_i be the multiplicity of the part i in λ. We bound each m_i for $i \neq 7$.

Proposition 3.2 *Assume the notation and hypotheses above. Then the following are true:*

$$\begin{cases} m_i = 0 & i = 2, 5, 8, 11, 12, 14, 15, i \geq 17 \\ m_i \leq 1 & i = 3, 6, 9, 10, 16 \\ m_i \leq 3 & i = 1, 13 \\ m_i \leq 4 & i = 4. \end{cases}$$

Proof If $i \geq 24$, then this follows from Proposition 3.1. If $i \leq 23$ and $i \neq 1, 3, 4, 6, 7, 9, 10, 13, 16$, then replacing i with the representation of i in the Table 2 would yield a partition μ with $N(0, 3; \mu) > N(0, 3; \lambda)$, so $m_i = 0$. For the remaining i, notice that the following replacements yield partitions μ with $N(0, 3; \mu) > N(0, 3; \lambda)$:

$$(1, 1, 1, 1) \to (4); (3, 3) \to (6); (4, 4, 4, 4, 4) \to (13, 7); (6, 6) \to (4, 4, 4);$$
$$(9, 9) \to (7, 7, 4); (10, 10) \to (13, 7); (13, 13, 13, 13) \to (10, 7, 7, 7, 7, 7, 7).$$

□

Now, we present an improved bound for m_i for $i = 3, 6, 16$.

Proposition 3.3 *Assume the notation and hypotheses above. Then the following is true:*

$$m_3 = m_6 = m_{16} = 0$$

unless $\lambda = (3)$, (6), *or* (16).

Proof If $m_a \geq 1$ for some a, by Proposition 3.2, we know that $a = 3, 4, 6, 7, 9, 10, 13, 16$ and that $m_i \leq 1$ for $i = 3, 6, 16$.

Suppose that $m_3 = 1$ (resp. $m_6 = 1, m_{16} = 1$). Then it can be verified that replacing $(a, 3)$ (resp. $(a, 6)$, $(a, 16)$) with the representation of $a + 3$ (resp. $a + 6$, $a + 16$) in Table 2 will produce a partition μ with $N(0, 3; \mu) > N(0, 3; \lambda)$. □

We now impose restrictions on the pairs of distinct integers $a, b \neq 7$ that can simultaneously be present in λ.

Proposition 3.4 *Assume the notation and hypotheses above. If* $m_a = 1$ *and* $m_b = 1$ *where* $a > b$ *and* $a, b \neq 7$*, then the following is true:*

$$(a, b) = (4, 1), (13, 10).$$

Proof By Propositions 3.2 and 3.3, we know that $a, b \in \{1, 4, 9, 10, 13\}$. It can be verified that replacing a and b with the representation of $a + b$ in Table 2 will yield a partition μ with $N(0, 3; \mu) > N(0, 3; \lambda)$ if $(a, b) \neq (4, 1), (13, 10)$. □

We now determine restrictions on the sets of natural numbers $a_1, a_2, \ldots, a_l \neq 7$ that can simultaneously be present in λ.

Proposition 3.5 *Assume the notation and hypotheses above. Suppose that* λ *contains* $a_1 \geq a_2 \geq \cdots \geq a_l$ *such that* $a_1, a_2, \ldots, a_l \neq 7$*. Then* λ *is one of the following:*

$$\begin{aligned}(a_1, a_2, \ldots a_l) = \ & (1), (1, 1), (1, 1, 1), (3),\\ & (4), (4, 1), (4, 1, 1), (4, 4), (4, 4, 4), (4, 4, 4, 4),\\ & (6), (10), (13), (13, 10), (13, 13), (13, 13, 10), (13, 13, 13), (16).\end{aligned}$$

Proof By Propositions 3.2, 3.3, and 3.4, we know that $(a_1, \ldots a_l)$ is either one of the above partitions, or it is one of the following (which we will rule out):

$$\begin{aligned}(a_1, \ldots a_l) = \ & (4, 1, 1, 1), (4, 4, 1), (4, 4, 1, 1), (4, 4, 1, 1, 1), (4, 4, 4, 1),\\ & (4, 4, 4, 1, 1), (4, 4, 4, 4, 1), (4, 4, 4, 4, 1, 1), (4, 4, 4, 4, 1, 1, 1),\\ & (13, 13, 13, 10).\end{aligned}$$

Let a_t be $\sum_{j=1}^{l} a_j$. Suppose that $(a_1, \ldots a_l) \neq (4, 1, 1), (4, 4, 4), (4, 4, 4, 4), (13, 13, 10), (13, 13, 13)$. If $a_t > 32$, then it can be verified that replacing $(a_1, \ldots a_l)$ with the representation of a_t in Theorem 1.2 will yield a partition μ with $N(0, 3; \mu) > N(0, 3; \lambda)$. If $a_t \leq 32$, then replacing $(a_1, \ldots a_l)$ with the representation of a_t in Table 2 will yield a partition μ with $N(0, 3; \mu) > N(0, 3; \lambda)$. □

Now, we will characterize the finitely many partitions λ that contain a 1 or a 4.

Proposition 3.6 *Assume the notation above. Suppose* $m_1 \geq 1$ *or* $m_4 \geq 1$*. Then* λ *is one of the following partitions:*

$$\begin{aligned}&(1), (1, 1), (1, 1, 1), (4), (4, 1), (4, 1, 1), (4, 4), (4, 4, 4),\\ &(4, 4, 4, 4), (7, 4), (7, 7, 4), (7, 4, 4, 4), (7, 7, 7, 7, 4).\end{aligned}$$

Proof Suppose that $m_1 \geq 1$ or $m_4 \geq 1$. Consider the partition λ_2 obtained by deleting any parts of size 7 from λ. Then by Proposition 3.5, we know that

$$\lambda_2 = (4, 1, 1), (4, 4, 4), (4, 4, 4, 4).$$

Now, we add back in the parts of size 7. Notice that the following operations will produce a partition μ with $N(0,3;\mu) > N(0,3;\lambda)$:

$$(7,1) \to (4,4);\ (7,7,4,4,4) \to (13,13);$$
$$(7,7,7,7,4,4) \to (13,13,10);\ (7,7,7,7,7,4) \to (13,13,13).$$

This proves the desired statement. □

We first consider the case where $n \leq 32$ and prove the third and fourth columns of Table 2.

Proof of Table 2. Consider the partition λ_2 obtained by deleting any parts with size 7 from λ. From Proposition 3.5 and Proposition 3.6, we can obtain all possible partitions λ_2. It can be verified that appending parts of size 7 to these λ_2 yields exactly the partitions in Table 2. It can be verified that in the case where multiple partitions λ of n remain, we have that the values $N(0,3;\lambda)$ are all equal. The values of $\mathrm{maxN}(0,3;n)$ can be deduced from this and the first two columns of Table 2. □

We now suppose that $n \geq 33$. we further limit the sets of natural numbers $a_1, a_2, \ldots, a_l \neq 7$ that can simultaneously be present in λ.

Proposition 3.7 *Assume the notation and hypotheses above. For $n \geq 33$, suppose that λ contains $a_1 \geq a_2 \geq \ldots \geq a_l$ such that $a_1, a_2, \ldots, a_l \neq 7$. Then $(a_1, a_2, \ldots a_l)$ is one of the following:*

$$(a_1, a_2, \ldots a_l) = (10), (13), (13,10), (13,13), (13,13,10), (13,13,13).$$

Proof By Proposition 3.6, we have that $m_1 = 0$ and $m_4 = 0$. Hence, the desired statement follows from Proposition 3.5. □

3.1.2 Proof of Theorem 1.2 for $r = 0$

We now use Proposition 3.7 to deduce Theorem 1.2. Assume the notation in Sect. 3.1.1.

Proof of Theorem 1.2 for $r = 0$. Consider the partition λ_2 obtained by deleting any parts with size 7 from λ. Then by Proposition 3.5, we know that

$$\lambda_2 = (10), (13), (13,10), (13,13), (13,13,10), (13,13,13).$$

These partitions cover all the classes modulo 7 of n except for $n \equiv 0 \pmod 7$ exactly once. If $n \not\equiv 0 \pmod 7$, then appending parts of size 7 to these partitions covers each n exactly once and yields the partitions λ in Theorem 1.2. If $n \equiv 0 \pmod 7$, we can deduce that $\lambda = (7,7,7,\ldots,7)$ as stated in Theorem 1.2. The values for $\mathrm{maxN}(0,3;n)$ can be deduced from this and the first two columns of Table 2. □

Table 3 Maximal value partitions λ

n	$N(1, 3; n)$	maxN$(1, 3; n)$	λ
1	0	0	(1)
2	1	1	(2)
3	1	1	(3)
4	1	1	(4), (2, 2)
5	3	3	(5)
6	4	4	(6)
7	4	4	(7)
8	8	8	(8)
9	10	10	(9)
10	13	13	(10)
11	20	20	(11)
12	26	26	(12)
13	32	32	(13)
14	46	46	(14)
15	59	59	(15)
16	75	75	(16)
17	101	101	(17)
18	129	129	(18)
19	161	161	(19)
20	211	211	(20)
21	264	264	(21)

3.2 Proof of Theorem 1.2 for $r = 1, 2$

We prove Theorem 1.2 for $r = 1, 2$ at the same time, since $N(1, 3; n) = N(2, 3; n)$. In Sect. 3.1.1, we study the combinatorial properties of $N(r, 3; \lambda)$ for $r = 1, 2$ resulting from Theorem 1.1 and the values of $N(r, 3; n)$ for $r = 1, 2$ for small n. In Sect. 3.1.2, we use these properties to deduce Theorem 1.2 for $r = 1, 2$.

3.2.1 Some combinatorics for $r = 1, 2$

For simplicity of notation, we write our propositions in terms of $N(1, 3; n)$ to denote the shared value $N(r, 3; n)$ for $r = 1, 2$. In our combinatorial arguments, we require the values of $N(1, 3; n)$ for $n \leq 21$. These values are given in the first two columns of Table 3, which were computed using Sage. We prove the correctness of the values in the last two columns over the course of this section.

Let λ be $(\lambda_1, \lambda_2, \ldots) \in P(n)$ such that $N(1, 3; \lambda)$ is maximal. First, we bound the size of λ_1.

Proposition 3.8 *Assume the notation and hypotheses above. Then the following is true:*

$$\lambda_1 \leq 21.$$

Proof Suppose that λ has a part $k \geq 22$. Then by Theorem 1.1, replacing k with the parts $\lfloor \frac{k}{2} \rfloor$ and $\lceil \frac{k}{2} \rceil$ would yield a partition μ such that $N(1, 3; \mu) > N(1, 3; \lambda)$. This is a contradiction since $N(1, 3; \lambda)$ is maximal. □

For $i > 0$, let m_i be the multiplicity of the part i in λ. We bound each m_i for $i \neq 14$.

Proposition 3.9 *Assume the notation and hypotheses above. Then the following are true:*

$$\begin{cases} m_i = 0 & i \geq 22 \\ m_i \leq 1 & 1 \leq i \leq 22 \textit{ such that } i \neq 2, 11, 12, 14, 15 \\ m_i \leq 2 & i = 2, 12, 15 \\ m_i \leq 3 & i = 11 \end{cases}$$

Proof For $i \geq 22$, this follows from Proposition 3.8.

Suppose that $m_i \geq 2$ for $1 \leq a \leq 21$, $i \neq 2, 11, 12, 14, 15$. If $2i \geq 22$ (resp. $2i < 22$), it can be verified that replacing the parts i and i with the representation of $2i$ in Theorem 1.2 (resp. Table 3) would yield a partition μ with $N(1, 3; \mu) > N(1, 3; \lambda)$. For $i = 2, 11, 12, 15$, note that the following operations yield partitions μ with $N(1, 3; \mu) > N(1, 3; \lambda)$:

$$(2, 2, 2) \to (6); (11, 11, 11, 11) \to (15, 15, 14);$$
$$(12, 12, 12) \to (14, 11, 11); (15, 15, 15) \to (12, 11, 11, 11).$$

□

Proposition 3.10 *Assume the notation and hypotheses above. If λ contains a part of size 2, then*

$$\lambda = (2), (2, 2).$$

Proof Suppose that λ contains a part of size 2. For $i \neq 2$, if $i + 2 \geq 22$ (resp. < 22), then replacing i and 2 with the representation of $i + 2$ in Theorem 1.2 (resp. Table 3) will yield a partition μ with $N(1, 3; \mu) > N(1, 3; \lambda)$. This means λ must contain only parts of size 2. By Proposition 3.9, we have that $m_2 \leq 2$. □

We now impose restrictions on the pairs of distinct integers $a, b \neq 14$ that can simultaneously be present in λ.

Proposition 3.11 *Assume the notation and hypotheses above. If $m_a = 1$ and $m_b = 1$ where $a > b$ and $a, b \neq 2, 14$, then the following is true:*

$$(a, b) = (12, 11), (15, 12), (17, 15).$$

Proof By Proposition 3.8, we know that $a, b \leq 21$. If $(a, b) \neq (12, 11), (15, 12), (17, 15)$, it can be verified that replacing a and b with the representation of $a + b$ in Table 2 will yield a partition μ with $N(1, 3; \mu) > N(1, 3; \lambda)$. □

We now impose restrictions on the sets of integers $a_1, a_2, \ldots, a_l \neq 14$ that can simultaneously be present in λ.

Proposition 3.12 *Assume the notation and hypotheses above. Suppose that* λ *contains* $a_1 \geq a_2 \geq \ldots \geq a_l$ *such that* $a_1, a_2, \ldots, a_l \neq 14$*. Then* $(a_1, a_2, \ldots a_l)$ *is one of the following:*

$$(a_1, a_2, \ldots a_l) = (i)(\textit{for } 1 \leq i \leq 21), (2, 2), (11, 11), (11, 11, 11), (12, 11),$$
$$(12, 11, 11), (12, 12), (12, 12, 11), (15, 12), (15, 15), (17, 15).$$

Proof By Propositions 3.9, 3.10, and 3.11, we know that $(a_1, a_2, \ldots, a_l)$ is either one of the above partitions or is one of the following (which we will rule out):

$$(a_1, a_2, \ldots, a_l) = (12, 11, 11, 11), (12, 12, 11, 11), (12, 12, 11, 11, 11),$$
$$(15, 12, 12), (15, 15, 12), (15, 15, 12, 12), (17, 15, 15).$$

Let $a_t = \sum_{j=1}^{l} a_j$. Suppose that $(a_1, a_2, \ldots a_l)$ is not one of the sets in the statement. It can be verified that replacing $(a_1, a_2, \ldots a_l)$ with the representation of a_t in Theorem 1.2 will produce a partition μ with $N(1, 3; \mu) > N(1, 3; \lambda)$. □

We first consider the case where $n \leq 21$ and prove the third and fourth columns of Table 3.

Proof of Table 3. Consider the partition λ_2 obtained by deleting any parts with size 14 from λ. From Proposition 3.12, we can obtain all possible partitions λ_2. It can be verified that appending parts of size 14 to these λ_2 yields exactly the partitions in Table 3. It can be verified that in the case where multiple partitions λ of n remain, we have that the values $N(1, 3; \lambda)$ are all equal. The values of maxN$(1, 3; n)$ can be deduced from this and the first two columns of Table 3. □

We now suppose that $n \geq 22$. we further limit the sets of integers $a_1, a_2, \ldots, a_l \neq 14$ that can simultaneously be present in λ.

Proposition 3.13 *Assume the notation and hypotheses above. For* $n \geq 22$*, suppose that* λ *contains* $a_1 \geq a_2 \geq \cdots \geq a_l$ *such that* $a_1, a_2, \ldots, a_l \neq 14$*. Then the following is true:*

$$(a_1, a_2, \ldots a_l) = (11), (11, 11), (11, 11, 11), (12), (12, 11), (12, 11, 11), (12, 12),$$
$$(12, 12, 11), (15), (15, 12), (15, 15), (17), (17, 15).$$

Proof The desired statement follows from Proposition 3.10 and Proposition 3.12. □

3.2.2 Proof of Theorem 1.2 for $r = 1, 2$

We now use Proposition 3.13 to deduce Theorem 1.2 for $r = 1, 2$ using the fact that $N(1, 3; n) = N(2, 3; n)$. Assume the notation in Sect. 3.2.1.

Proof of Theorem 1.2. for $r = 1, 2$ Since $N(1, 3; n)$ is always equal to $N(2, 3; n)$, we know that $\mathrm{maxN}(1, 3; n)$ and $\mathrm{maxN}(2, 3; n)$ are equal and are achieved at the same partitions. Consider the partition λ_2 obtained by deleting any parts with size 14 from λ. Then by Proposition 3.5, we know that

$$\begin{aligned}\lambda_2 = &(11), (11, 11), (11, 11, 11), (12), (12, 11), (12, 11, 11),\\ &(12, 12), (12, 12, 11), (15), (15, 12), (15, 15), (17), (17, 15).\end{aligned}$$

These partitions cover all the classes modulo 14 of n except for $n \equiv 0 \pmod{14}$ exactly once. If $n \not\equiv 0 \pmod{14}$, then appending parts of size 14 to these partitions covers each n exactly once and yields the partitions λ in Theorem 1.2. If $n \equiv 0 \pmod{14}$, we can deduce that $\lambda = (14, 14, \ldots, 14)$ as stated in Theorem 1.2. The values for $\mathrm{maxN}(1, 3; n)$ and $\mathrm{maxN}(2, 3; n)$ can be deduced from this fact and the first two columns of Table 3. □

4 Discussion

For general t, it is difficult to obtain effective asymptotics and effective bounds on error terms for $N(r, t; n)$. In particular, the exact formulas for $t = 2$ as an infinite series were not known until a recent work by Bringmann and Ono [5]. Using these bounds, we believe that similar methods can be used to prove the following convexity result.

Conjecture 4.1 *If $t = 2$ and $r = 0$ (resp. $r = 1$), then we have that*

$$N(r, 2; a)N(r, 2; b) > N(r, 2; a + b)$$

for all $a, b \geq 11$ (resp. 12).

This convexity result would imply the following description of $\mathrm{maxN}(r, 2; n)$:

Conjecture 4.2 *Assume the notation above. Then the following are true.*

(1) *If $n \geq 6$, then we have that*

$$\mathrm{maxN}(0, 2; n) = \begin{cases} 3^{\frac{n}{3}} & n \equiv 0 \pmod{3} \\ 11 \cdot 3^{\frac{n-7}{3}} & n \equiv 1 \pmod{3} \\ 5 \cdot 3^{\frac{n-5}{3}} & n \equiv 2 \pmod{3}, \end{cases}$$

and it is achieved at the unique partitions

$$\begin{aligned}&(3, 3, \ldots, 3) \text{ when } n \equiv 0 \pmod{3}\\ &(7, 3, \ldots, 3) \text{ when } n \equiv 1 \pmod{3}\\ &(5, 3, \ldots, 3) \text{ when } n \equiv 2 \pmod{3}.\end{aligned}$$

(2) *If $n \geq 8$, then we have that*

$$\mathrm{maxN}(1,2;n) = \begin{cases} 2^{\frac{n}{2}} & n \equiv 0 \pmod 2 \\ 12 \cdot 2^{\frac{n-9}{2}} & n \equiv 1 \pmod 2, \end{cases}$$

and it is achieved at the following classes of partitions

$$(2,2,\ldots,2) \text{ when } n \equiv 0 \pmod 2$$
$$(9,2,\ldots,2) \text{ when } n \equiv 1 \pmod 2.$$

up to any number of the following substitutions: $(2,2) \to (4)$ *and* $(2,2,2) \to (6)$.

Example For $n = 8$, we would have that $\mathrm{maxN}(1,2;8) = 16$ is achieved at the partitions $(6,2)$, $(4,4)$, $(4,2,2)$, and $(2,2,2,2)$.

We believe that a similar convexity result holds for all r, t for sufficiently large a and b.

Conjecture 4.3 *If $0 \leq r < t$ and $t \geq 2$, then*

$$N(r,t;a)N(r,t;b) > N(r,t;a+b)$$

for sufficiently large a and b.

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