Ramanujan J (2018) 45:517–520
https://doi.org/10.1007/s11139-016-9863-4

# Broken 2-diamond partitions modulo 5

**Michael D. Hirschhorn**[1]

Received: 18 May 2016 / Accepted: 27 September 2016 / Published online: 17 January 2017


**Abstract** We consider $\Delta_2(n)$, the number of broken 2-diamond partitions of $n$, and give simple proofs of two congruences given by Song Heng Chan.

**Keywords** Partitions · Broken 2-diamonds · Congruences modulo 5

**Mathematics Subject Classification** 11P83

## 1 Introduction

Andrews and Paule [1] introduced the concept of broken $k$-diamond partitions and showed that the generating function for $\Delta_k(n)$, the number of broken $k$-diamond partitions of $n$, is

$$\sum_{n\geq 0}\Delta_k(n)q^n = \frac{(q^2;q^2)_\infty(q^{2k+1};q^{2k+1})_\infty}{(q;q)^3_\infty(q^{2(2k+1)};q^{2(2k+1)})_\infty}. \tag{1.1}$$

The following congruences were proved by Chan [2] and again by Radu [4]:

$$\Delta_2(25n+14)\equiv 0 \pmod 5 \tag{1.2}$$

and

$$\Delta_2(25n+24)\equiv 0 \pmod 5. \tag{1.3}$$

✉ Michael D. Hirschhorn
m.hirschhorn@unsw.edu.au

1 School of Mathematics and Statistics, UNSW, Sydney, NSW 2052, Australia

Indeed, Chan generalised these to

$$\Delta_2\left(5^{\alpha+1}n+\frac{11\times 5^{\alpha}+1}{4}\right)\equiv 0 \pmod 5 \tag{1.4}$$

and

$$\Delta_2\left(5^{\alpha+1}n+\frac{19\times 5^{\alpha}+1}{4}\right)\equiv 0 \pmod 5. \tag{1.5}$$

The object of this note is to give as simple a proof as I can of (1.2)–(1.5).

## 2 Proofs

We start by noting that the 5–dissection of $\psi(q)=\sum_{n\geq 0} q^{(n^2+n)/2}$ is

$$\begin{aligned}\psi(q) &= 1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+q^{28}+q^{36}+q^{45}+q^{55}+q^{66}+q^{75}+\cdots\\ &= (1+q^{10}+q^{15}+q^{45}+q^{55}+\cdots)+q(1+q^5+q^{20}+q^{35}+q^{65}+\cdots)\\ &\quad +q^3(1+q^{25}+q^{75}+\cdots)\\ &= a+qb+q^3c,\end{aligned} \tag{2.1}$$

where

$$a=(-q^{10},-q^{15},q^{25};q^{25})_\infty,\quad b=(-q^5,-q^{20},q^{25};q^{25})_\infty,\quad c=\psi(q^{25}). \tag{2.2}$$

Note that by [3, (34.1.21)]

$$ab+q^5c^2=\psi(q^5)^2. \tag{2.3}$$

We have, modulo 5,

$$\begin{aligned}\sum_{n\geq 0}\Delta_2(n)q^n &= \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty}\equiv\frac{(q^2;q^2)_\infty(q;q)_\infty^5}{(q;q)_\infty^3(q^2;q^2)_\infty^5}=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^4}\\ &= \frac{1}{\psi(q)^2}=\frac{\psi(q)^3}{\psi(q)^5}\equiv\frac{\psi(q)^3}{\psi(q^5)}.\end{aligned} \tag{2.4}$$

Alternatively,

$$\sum_{n\geq 0}\Delta_2(n)q^n=\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty}=\frac{\psi(q)^3}{\psi(q^5)}\frac{(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^5}\equiv\frac{\psi(q)^3}{\psi(q^5)}, \tag{2.5}$$

or, again, by [3, (34.1.23)],

$$\sum_{n\geq 0}\Delta_2(n)q^n = \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty}$$
$$= \frac{1}{\psi(q)^2-5q\psi(q^5)^2} \equiv \frac{1}{\psi(q)^2} \equiv \frac{\psi(q)^3}{\psi(q^5)}. \tag{2.6}$$

Thus, we have

$$\sum_{n\geq 0}\Delta_2(n)q^n \equiv \frac{\psi(q)^3}{\psi(q^5)} = \frac{(a+qb+q^3c)^3}{\psi(q^5)}$$
$$= \frac{a^3+3qa^2b+3q^2ab^2+q^3b^3+3q^3a^2c+6q^4abc+3q^5b^2c+3q^6ac^2+3q^7bc^2+q^9c^3}{\psi(q^5)}. \tag{2.7}$$

It follows that

$$\sum_{n\geq 0}\Delta_2(5n+4)q^{5n} \equiv \frac{abc+q^5c^3}{\psi(q^5)} = \frac{c(ab+q^5c^2)}{\psi(q^5)}$$
$$= \frac{\psi(q^{25})\psi(q^5)^2}{\psi(q^5)} = \psi(q^{25})\psi(q^5) \tag{2.8}$$

and

$$\sum_{n\geq 0}\Delta_2(5n+4)q^n \equiv \psi(q^5)\psi(q) = \psi(q^5)(a+qb+q^3\psi(q^{25})). \tag{2.9}$$

It is now an easy induction (replace $n$ by $5n+3$) to deduce that for $\alpha \geq 1$,

$$\sum_{n\geq 0}\Delta_2\left(5^\alpha n+\frac{3\times 5^\alpha+1}{4}\right)q^n \equiv \psi(q^5)\psi(q)$$
$$= \psi(q^5)(a+qb+q^3\psi(q^{25})). \tag{2.10}$$

Since there are no terms on the right in which the power of $q$ is congruent to 2 or 4 modulo 5, we have that for $\alpha \geq 1$,

$$\Delta_2\left(5^\alpha(5n+2)+\frac{3\times 5^\alpha+1}{4}\right) \equiv 0 \tag{2.11}$$

and

$$\Delta_2\left(5^\alpha(5n+4)+\frac{3\times 5^\alpha+1}{4}\right) \equiv 0, \tag{2.12}$$

as claimed.

## References

1. Andrews, G.E., Paule, P.: MacMahon's partition analysis XI, broken diamonds and modular forms. Acta Arith. **126**(3), 281–294 (2007)
2. Chan, S.H.: Some congruences for Andrews–Paule's broken 2-diamond partitions. Discret. Math. **308**(23), 5735–5741 (2008)
3. Hirschhorn, M.D.: The power of $q$. Springer (to appear)
4. Radu, C.-S.: An algorithmic approach to Ramanujan–Kolberg identities. J. Symb. Comput. **68**(1), 225–253 (2015)