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Congruences modulo 4 for broken k-diamond partitions

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Abstract The notion of broken k-diamond partitions was introduced by Andrews and Paule. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Recently, a number of parity results satisfied by $\Delta_k(n)$ for small values of k have been proved by Radu and Sellers and others. However, congruences modulo 4 for $\Delta_k(n)$ are unknown. In this paper, we will prove five congruences modulo 4 for $\Delta_5(n)$, four infinite families of congruences modulo 4 for $\Delta_7(n)$ and one congruence modulo 4 for $\Delta_{11}(n)$ by employing theta function identities. Furthermore, we will prove a new parity result for $\Delta_2(n)$.

Keywords Broken k-Diamond partition · Congruence · Theta function

Mathematics Subject Classification 11P83 · 05A17

1 Introduction

The aim of this paper is to establish congruences modulo 4 for broken 5-diamond, broken 7-diamond and broken 11-diamond partitions.

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Let us begin with some notation and terminology on q-series and partitions. We use the standard notation

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right)$$

and often write

$$(a_1,a_2,\ldots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty.$$

Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}. \tag{1.1}$$

Jacobi's triple product identity states that

$$f(a,b) = (-a,-b,ab;ab)_\infty. \tag{1.2}$$

Three special cases of (1.1) are defined by

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.3}$$

$$\psi(q) = f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \tag{1.4}$$

and

$$f(-q) = f\left(-q,-q^2\right) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}. \tag{1.5}$$

For any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$f_n = \left(q^n;q^n\right)_\infty = \prod_{k=1}^{\infty}\left(1-q^{nk}\right).$$

By (1.2)–(1.5),

$$f(-q) = f_1, \qquad \varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}. \tag{1.6}$$

MacMahon's partition analysis guided Andrews and Paule [2] to introduce broken k-diamond partitions. For a fixed positive integer k, let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. Andrews and Paule [2] discovered the following generating function for $\Delta_k(n)$:

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{f_2 f_{2k+1}}{f_1^3 f_{4k+2}}. \tag{1.7}$$

Various authors have obtained parity results for broken k-diamond partitions. See, Ahmed and Baruah [1], Chan [5], Cui and Gu [6], Hirschhorn and Sellers [8], Lin [10], Radu and Sellers [11,12], Wang [14], Xia [15] and Yao [18].

However, Ramanujan-type congruences modulo 4 for $\Delta_k(n)$ are unknown. With this motivation, we will prove five congruences modulo 4 for $\Delta_5(n)$, four infinite families of congruences modulo 4 for $\Delta_7(n)$ and one congruence modulo 4 for $\Delta_{11}(n)$. The main results of this paper can be stated as follows.

Theorem 1.1 *For $n \geq 0$,*

$$\Delta_5(44n + j) \equiv 0 \pmod 4, \tag{1.8}$$

where $j \in \{2,\ 14,\ 30,\ 34,\ 38\}$.

Theorem 1.2 *For $n,\ \alpha \geq 0$,*

$$\Delta_7\left(16 \times 5^{2\alpha+1} n + \frac{4 \times 5^{2\alpha} + 2}{3}\right) \equiv 0 \pmod 4, \tag{1.9}$$

$$\Delta_7\left(16 \times 5^{2\alpha+1} n + \frac{196 \times 5^{2\alpha} + 2}{3}\right) \equiv 0 \pmod 4, \tag{1.10}$$

$$\Delta_7\left(16 \times 5^{2\alpha+2} n + \frac{460 \times 5^{2\alpha} + 2}{3}\right) \equiv 0 \pmod 4, \tag{1.11}$$

$$\Delta_7\left(16 \times 5^{2\alpha+2} n + \frac{940 \times 5^{2\alpha} + 2}{3}\right) \equiv 0 \pmod 4. \tag{1.12}$$

Theorem 1.3 *For $n, \alpha \geq 0$,*

$$\Delta_{11}(4 \times 23^\alpha n + 1) - \Delta_{11}(2 \times 23^\alpha n + 1) \equiv \Delta_{11}(4n + 1) - \Delta_{11}(2n + 1) \pmod 4. \tag{1.13}$$

By Theorem 1.3 and the facts that $\Delta_{11}(5) \equiv \Delta_{11}(3) \pmod 4$, $\Delta_{11}(9) - \Delta_{11}(5) \equiv -1 \pmod 4$, $\Delta_{11}(13) - \Delta_{11}(7) \equiv 2 \pmod 4$ and $\Delta_{11}(65) - \Delta_{11}(33) \equiv 1 \pmod 4$, we obtain the following corollary:

Corollary 1.4 *For $\alpha \geq 0$,*

$$\begin{aligned}
\Delta_{11}(4 \times 23^\alpha + 1) - \Delta_{11}(2 \times 23^\alpha + 1) &\equiv 0 \pmod 4,\\
\Delta_{11}(8 \times 23^\alpha + 1) - \Delta_{11}(4 \times 23^\alpha + 1) &\equiv -1 \pmod 4,\\
\Delta_{11}(12 \times 23^\alpha + 1) - \Delta_{11}(6 \times 23^\alpha + 1) &\equiv 2 \pmod 4,\\
\Delta_{11}(64 \times 23^\alpha + 1) - \Delta_{11}(32 \times 23^\alpha + 1) &\equiv 1 \pmod 4.
\end{aligned}$$

Moreover, we will prove the following congruence modulo 2 for $\Delta_2(n)$ in Sect. 5.

Theorem 1.5 *For $n \geq 0$,*

$$\Delta_2(2n+1) \equiv \Delta_2(10n+4) \pmod 2. \tag{1.14}$$

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first prove three lemmas.

Lemma 2.1 *We have*

$$\begin{aligned}\frac{f_2^2 f_{22}^2}{f_1 f_{11}} = &\frac{f_{24}^2 f_{132}^5}{f_{12} f_{66}^2 f_{264}^2}\\ &+ q\frac{f_4^2 f_6 f_{24} f_{88} f_{132}^2}{f_2 f_8 f_{12} f_{44} f_{264}}\\ &+ q^6\frac{f_8 f_{12}^2 f_{44}^2 f_{66} f_{264}}{f_4 f_{22} f_{24} f_{88} f_{132}} + q^{15}\frac{f_{12}^5 f_{264}^2}{f_6^2 f_{24}^2 f_{132}}.\end{aligned} \tag{2.1}$$

Proof From (36.8) in Berndt's book [3, p. 69], we see that if μ is even, then

$$\begin{aligned}\psi\left(q^{\mu+\nu}\right)\psi\left(q^{\mu-\nu}\right) =&\varphi\left(q^{\mu(\mu^2-\nu^2)}\right)\psi\left(q^{2\mu}\right) + \sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m}\\ &\times f\left(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}\right) f\left(q^{2\nu m}, q^{2\mu-2\nu m}\right)\\ &+ q^{\mu^3/4-\mu\nu/2}\psi\left(q^{2\mu(\mu^2-\nu^2)}\right) f\left(q^{\mu\nu}, q^{2\mu-\mu\nu}\right).\end{aligned} \tag{2.2}$$

Setting $\mu = 6$ and $\nu = 5$ in (2.2), we get

$$\begin{aligned}\psi(q)\psi\left(q^{11}\right) =&\varphi\left(q^{66}\right)\psi\left(q^{12}\right) + qf\left(q^{88}, q^{44}\right) f\left(q^{10}, q^2\right)\\ &+ q^{14} f\left(q^{110}, q^{22}\right) f\left(q^{20}, q^{-8}\right) + q^{39}\psi\left(q^{132}\right) f\left(q^{30}, q^{-18}\right).\end{aligned} \tag{2.3}$$

By (1.2),

$$f\left(q^{88}, q^{44}\right) = \frac{f_{88} f_{132}^2}{f_{44} f_{264}}, \tag{2.4}$$

$$f\left(q^{10}, q^2\right) = \frac{f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}}, \tag{2.5}$$

$$f\left(q^{20}, q^{-8}\right) = q^{-8}\frac{f_8 f_{12}^2}{f_4 f_{24}}, \tag{2.6}$$

and

$$f\left(q^{30}, q^{-18}\right) = q^{-24}\frac{f_{12}^5}{f_6^2 f_{24}^2}. \tag{2.7}$$

Substituting (1.6), (2.4)–(2.7) into (2.3), we arrive at (2.1). This completes the proof. □

Lemma 2.2 *Define*

$$\sum_{n=0}^{\infty} a(n)q^n := \frac{f_1^2 f_3^2}{f_2 f_6}. \tag{2.8}$$

Then for $n \geq 0$,

$$a(11n+i) \equiv 0 \pmod 4, \tag{2.9}$$

where $i \in \{2,\ 6,\ 7,\ 8,\ 10\}$.

Proof It is well known that

$$\frac{f_1^2}{f_2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}$$

Thus, combining (2.8) and the above identity yields

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n &= \left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)\left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{3n^2}\right)\\ &\equiv 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} + 2\sum_{n=1}^{\infty}(-1)^n q^{3n^2} \pmod 4.\end{aligned} \tag{2.10}$$

It is easy to check that for any integer n,

$$n^2 \equiv 0,\ 1,\ 3,\ 4,\ 5,\ 9 \pmod{11} \tag{2.11}$$

and

$$3n^2 \equiv 0,\ 1,\ 3,\ 4,\ 5,\ 9 \pmod{11}. \tag{2.12}$$

Congruence (2.9) follows from (2.10), (2.11) and (2.12). The proof is complete. □

Lemma 2.3 *Define*

$$\sum_{n=0}^{\infty} b(n)q^n := \frac{f_3^3}{f_1}. \tag{2.13}$$

Then for $n \geq 0$,

$$b(11n+k) \equiv 0 \pmod{2}, \tag{2.14}$$

where $k \in \{2,\ 3,\ 4,\ 6,\ 9\}$.

Proof Hirschhorn and Sellers [9] proved that if

$$3n+1 = \prod_{p_i \equiv 1 \pmod{3}} p_i^{\alpha_i} \prod_{q_i \equiv 2 \pmod{3}} q_i^{\beta_i} \tag{2.15}$$

is the prime factorization of $3n+1$, then

$$b(n) = \begin{cases} \prod(\alpha_i + 1), & \text{if all } \beta_i \text{ are even}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.16}$$

where $b(n)$ is defined by (2.13). By (2.15) and (2.16), we find that for $n \geq 0$, $b(n)$ is odd if and only if $3n+1$ is a square of an integer. From (2.11), we know that $33n+7$, $33n+10$, $33n+13$, $33n+19$ and $33n+28$ are not squares, which implies that for $n \geq 0$,

$$b(11n+k) \equiv 0 \pmod{2},$$

where $k \in \{2,\ 3,\ 4,\ 6,\ 9\}$. This completes the proof. □

Now, we turn to prove Theorem 1.1.

Setting $k=5$ in (1.7), we have

$$\sum_{n=0}^{\infty} \Delta_5(n) q^n = \frac{f_2 f_{11}}{f_1^3 f_{22}}. \tag{2.17}$$

By the binomial theorem, for positive integers u and v,

$$\left(q^u; q^v\right)_\infty^2 \equiv \left(q^{2u}; q^{2v}\right)_\infty \pmod{2} \tag{2.18}$$

and

$$\left(q^u; q^v\right)_\infty^4 \equiv \left(q^{2u}; q^{2v}\right)_\infty^2 \pmod{4}. \tag{2.19}$$

In particular,

$$f_1^2 \equiv f_2 \pmod{2} \tag{2.20}$$

and

$$f_1^4 \equiv f_2^2 \pmod{4}. \tag{2.21}$$

Thanks to (2.17) and (2.21),

$$\sum_{n=0}^{\infty} \Delta_5(n)q^n \equiv \frac{f_1 f_{11}}{f_2 f_{22}} \pmod 4. \tag{2.22}$$

Replacing q by $-q$ in (2.1) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}, \tag{2.23}$$

then multiplying $\frac{1}{f_4 f_{44}}$ on both sides, we obtain

$$\begin{aligned}\frac{f_1 f_{11}}{f_2 f_{22}} =& \frac{f_{24}^2 f_{132}^5}{f_4 f_{12} f_{44} f_{66}^2 f_{264}^2} - q\frac{f_4 f_6 f_{24} f_{88} f_{132}^2}{f_2 f_8 f_{12} f_{44}^2 f_{264}} \\ &+ q^6 \frac{f_8 f_{12}^2 f_{44} f_{66} f_{264}}{f_4^2 f_{22} f_{24} f_{88} f_{132}} - q^{15} \frac{f_{12}^5 f_{264}^2}{f_4 f_6^2 f_{24}^2 f_{44} f_{132}}.\end{aligned} \tag{2.24}$$

Combining (2.22) and (2.24) yields

$$\sum_{n=0}^{\infty} \Delta_5(2n)q^n \equiv \frac{f_{12}^2 f_{66}^5}{f_2 f_6 f_{22} f_{33}^2 f_{132}^2} + q^3 \frac{f_4 f_6^2 f_{22} f_{33} f_{132}}{f_2^2 f_{11} f_{12} f_{44} f_{66}} \pmod 4. \tag{2.25}$$

The following relation is a consequence of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [3, p. 40]:

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.26}$$

Xia and Yao [16] proved that

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{2.27}$$

By substituting (2.26) and (2.27) into (2.25),

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_5(2n)q^n \equiv& \frac{f_{12}^2 f_{66}^3 f_{264}^5}{f_2 f_6 f_{22} f_{132}^2 f_{528}^5} + q^3 \frac{f_4 f_6^2 f_{176} f_{264}^2}{f_2^2 f_{12} f_{22} f_{88} f_{528}} \\ &+ q^{14} \frac{f_4 f_6^2 f_{88}^2 f_{132} f_{528}}{f_2^2 f_{12} f_{22} f_{44} f_{176} f_{264}} + 2q^{33} \frac{f_{12}^2 f_{528}^2}{f_2 f_6 f_{22} f_{264}} \pmod 4.\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty}\Delta_5(4n+2)q^n \equiv q\frac{f_2 f_3^2 f_{88} f_{132}^2}{f_1^2 f_6 f_{11} f_{44} f_{264}} + 2q^{16}\frac{f_6^2 f_{264}^2}{f_1 f_3 f_{11} f_{132}} \pmod 4. \tag{2.28}$$

By (2.20), (2.21) and (2.28),

$$\sum_{n=0}^{\infty}\Delta_5(4n+2)q^n \equiv q\frac{f_1^2 f_3^2 f_{88} f_{132}^2}{f_2 f_6 f_{11} f_{44} f_{264}} + 2q^{16}\frac{f_3^3 f_{264}^2}{f_1 f_{11} f_{132}} \pmod 4. \tag{2.29}$$

Therefore, we can rewrite (2.29) as

$$\sum_{n=0}^{\infty}\Delta_5(4n+2)q^n \equiv q\frac{f_{88} f_{132}^2}{f_{11} f_{44} f_{264}}\sum_{n=0}^{\infty}a(n)q^n + 2q^{16}\frac{f_{264}^2}{f_{11} f_{132}}\sum_{n=0}^{\infty}b(n)q^n \pmod 4, \tag{2.30}$$

where $a(n)$ and $b(n)$ are defined by (2.8) and (2.13), respectively. Theorem 1.1 follows from (2.9), (2.14) and (2.30). This completes the proof. □

3 Proof of Theorem 1.2

In this section, we present a proof of Theorem 1.2.

Taking $k = 7$ in (1.7), we get

$$\sum_{n=0}^{\infty}\Delta_7(n)q^n = \frac{f_2 f_{15}}{f_1^3 f_{30}}. \tag{3.1}$$

In view of (2.21) and (3.1),

$$\sum_{n=0}^{\infty}\Delta_7(n)q^n \equiv \frac{f_1 f_{15}}{f_2 f_{30}} \pmod 4. \tag{3.2}$$

Replacing q by $-q$ in (3.2) and utilizing the relation (2.23), we get

$$\sum_{n=0}^{\infty}\Delta_7(n)(-1)^n q^n \equiv \frac{\psi(q)\psi\left(q^{15}\right)}{f_4 f_{60}} \pmod 4, \tag{3.3}$$

where $\psi(q)$ is defined by (1.6). From Entry 9 in Berndt's book [3, p. 377],

$$\psi(q)\psi\left(q^{15}\right) + \psi(-q)\psi\left(-q^{15}\right) = 2\psi\left(q^6\right)\psi\left(q^{10}\right) \tag{3.4}$$

Based on (3.3) and (3.4),

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_7(n)\big(1+(-1)^n\big)q^n &\equiv \frac{\psi(q)\psi\big(q^{15}\big)+\psi(-q)\psi\big(-q^{15}\big)}{f_4 f_{60}} \\ &\equiv 2\frac{\psi\big(q^6\big)\psi\big(q^{10}\big)}{f_4 f_{60}} \pmod 4,\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} \Delta_7(2n)q^n \equiv \frac{\psi\big(q^3\big)\psi\big(q^5\big)}{f_2 f_{30}} \pmod 4. \tag{3.5}$$

From Entry 9 in Berndt's book [3, p. 377],

$$\psi\big(q^3\big)\psi\big(q^5\big)-\psi\big(-q^3\big)\psi\big(-q^5\big)=2q^3\psi\big(q^2\big)\psi\big(q^{30}\big). \tag{3.6}$$

Combining (3.5) and (3.6), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_7(2n)\big(1-(-1)^n\big)q^n &\equiv \frac{\psi\big(q^3\big)\psi\big(q^5\big)-\psi\big(-q^3\big)\psi\big(-q^5\big)}{f_2 f_{30}} \\ &\equiv 2q^3\frac{\psi\big(q^2\big)\psi\big(q^{30}\big)}{f_2 f_{30}} \pmod 4,\end{aligned}$$

which implies

$$\sum_{n=0}^{\infty} \Delta_7(4n+2)q^n \equiv q\frac{\psi(q)\psi\big(q^{15}\big)}{f_1 f_{15}} \pmod 4. \tag{3.7}$$

In view of (1.6), (2.21) and (3.7),

$$\sum_{n=0}^{\infty} \Delta_7(4n+2)q^n \equiv q\frac{f_2^2 f_{30}^2}{f_1^2 f_{15}^2} \equiv q f_1^2 f_{15}^2 \pmod 4. \tag{3.8}$$

Ramanujan [13] stated the following identity without proof:

$$f_1 = f_{25}\left(R\big(q^5\big)-q-\frac{q^2}{R\big(q^5\big)}\right), \tag{3.9}$$

where

$$R(q)=\frac{\big(q^2,q^3;q^5\big)_\infty}{\big(q,q^4;q^5\big)_\infty}. \tag{3.10}$$

Hirschhorn [7] gave a simple proof of (3.9) by using Jacobi's triple product identity. Substituting (3.9) into (3.8), we have

$$\sum_{n=0}^{\infty} \Delta_7(4n+2)q^n$$
$$\equiv f_{15}^2 f_{25}^2 \left(qR^2(q^5) - 2q^2R(q^5) - q^3 + \frac{2q^4}{R(q^5)} + \frac{q^5}{R^2(q^5)} \right) \pmod 4,$$

which yields

$$\sum_{n=0}^{\infty} \Delta_7(20n+6)q^n \equiv f_3^2 f_5^2 R^2(q) \pmod 4, \tag{3.11}$$
$$\sum_{n=0}^{\infty} \Delta_7(20n+14)q^n \equiv -f_3^2 f_5^2 \pmod 4 \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} \Delta_7(20n+2)q^n \equiv q\frac{f_3^2 f_5^2}{R^2(q)} \pmod 4, \tag{3.13}$$

where $R(q)$ is defined by (3.10). We can rewrite (3.13) as

$$\sum_{n=0}^{\infty} \Delta_7(20n+2)q^n$$
$$\equiv qf_3^2 f_5^2 \frac{(q, q^4; q^5)_\infty^2}{(q^2, q^3; q^5)_\infty^2}$$
$$\equiv q\frac{f_3^2 f_5^4 (q, q^4; q^5)_\infty^4}{f_1^2} \pmod 4. \tag{3.14}$$

By (2.19), (2.21) and (3.14),

$$\sum_{n=0}^{\infty} \Delta_7(20n+2)q^n \equiv q\frac{f_3^2 f_{10}^2 (q^2, q^8; q^{10})_\infty^2}{f_1^2} \pmod 4. \tag{3.15}$$

Xia and Yao [17] proved that

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{3.16}$$

Thanks to (2.20), (2.21) and (3.16),

$$\frac{f_3^2}{f_1^2} \equiv \frac{f_6 f_8 f_{12}^2}{f_2^5 f_{24}} + 2qf_4 f_{24} \pmod 4. \tag{3.17}$$

Substituting (3.17) into (3.15), we get

$$\sum_{n=0}^{\infty}\Delta_7(20n+2)q^n \equiv q\frac{f_6f_8f_{10}^2f_{12}^2(q^2,q^8;q^{10})_\infty^2}{f_2^5f_{24}} +2q^2f_4f_{10}^2f_{24}(q^2,q^8;q^{10})_\infty^2 \pmod 4,$$

which yields

$$\sum_{n=0}^{\infty}\Delta_7(40n+2)q^n \equiv 2qf_2f_5^2f_{12}(q,q^4;q^5)_\infty^2 \pmod 4. \tag{3.18}$$

By (2.18), (2.20) and (3.18), we obtain

$$\sum_{n=0}^{\infty}\Delta_7(40n+2)q^n \equiv 2qf_2f_{10}f_{12}(q^2,q^8;q^{10})_\infty \pmod 4,$$

which implies that for $n \geq 0$,

$$\Delta_7(80n+2) \equiv 0 \pmod 4. \tag{3.19}$$

Similarly, by (2.19), (2.21), (3.10) and (3.11),

$$\sum_{n=0}^{\infty}\Delta_7(20n+6)q^n \equiv f_3^2f_5^2\frac{(q^2,q^3;q^5)_\infty^2}{(q,q^4;q^5)_\infty^2} \equiv \frac{f_3^2f_5^4(q^2,q^3;q^5)_\infty^4}{f_1^2} \equiv \frac{f_3^2}{f_1^2}f_{10}^2(q^4,q^6;q^{10})_\infty^2 \pmod 4. \tag{3.20}$$

By substituting (3.17) into (3.20) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty}\Delta_7(40n+26)q^n \equiv 2f_2f_5^2f_{12}(q^2,q^3;q^5)_\infty^2 \pmod 4. \tag{3.21}$$

In view of (2.18), (2.20) and (3.21),

$$\sum_{n=0}^{\infty}\Delta_7(40n+26)q^n \equiv 2f_2f_{10}f_{12}(q^4,q^6;q^{10})_\infty \pmod 4,$$

which implies that for $n \geq 0$,

$$\Delta_7(80n + 66) \equiv 0 \pmod 4. \tag{3.22}$$

Substituting (3.9) into (3.12), we obtain

$$\sum_{n=0}^{\infty} \Delta_7(20n+14)q^n \equiv f_5^2 f_{75}^2 \left(-R^2(q^{15}) + 2q^3 R(q^{15}) + q^6 - \frac{2q^9}{R(q^{15})} - \frac{q^{12}}{R^2(q^{15})} \right) \pmod 4,$$

which yields

$$\sum_{n=0}^{\infty} \Delta_7(100n+14)q^n \equiv - f_1^2 f_{15}^2 R^2(q^3) \pmod 4, \tag{3.23}$$

$$\sum_{n=0}^{\infty} \Delta_7(100n+34)q^n \equiv q f_1^2 f_{15}^2 \pmod 4 \tag{3.24}$$

and

$$\sum_{n=0}^{\infty} \Delta_7(100n+54)q^n \equiv -q^2 \frac{f_1^2 f_{15}^2}{R^2(q^3)} \pmod 4. \tag{3.25}$$

By (2.19), (2.21), (3.10) and (3.23),

$$\begin{aligned}
\sum_{n=0}^{\infty} \Delta_7(100n+14)q^n &\equiv -f_1^2 f_{15}^2 \frac{(q^6, q^9; q^{15})_\infty^2}{(q^3, q^{12}; q^{15})_\infty^2} \\
&\equiv -\frac{f_1^2 f_{15}^4 (q^6, q^9; q^{15})_\infty^4}{f_3^2} \\
&\equiv -\frac{f_1^2 f_{30}^2 (q^{12}, q^{18}; q^{30})_\infty^2}{f_3^2} \\
&\equiv -\frac{f_2^2 f_3^2 f_{30}^2 (q^{12}, q^{18}; q^{30})_\infty^2}{f_1^2 f_6^2} \pmod 4.
\end{aligned} \tag{3.26}$$

By substituting (3.17) into (3.26) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty} \Delta_7(200n+114)q^n \equiv 2\frac{f_1^2 f_2 f_{12} f_{15}^2 (q^6, q^9; q^{15})_\infty^2}{f_3^2} \pmod 4. \tag{3.27}$$

Thanks to (2.18), (2.20) and (3.27),

$$\sum_{n=0}^{\infty} \Delta_7(200n+114)q^n \equiv 2f_4 f_6 f_{30}\left(q^{12}, q^{18}; q^{30}\right)_\infty \pmod 4,$$

which implies that for $n \geq 0$,

$$\Delta_7(400n+314) \equiv 0 \pmod 4. \tag{3.28}$$

It follows from (3.8) and (3.24) that for $n \geq 0$,

$$\Delta_7(100n+34) \equiv \Delta_7(4n+2) \pmod 4. \tag{3.29}$$

By (3.29) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$\Delta_7\left(4\left(5^{2\alpha}n+\frac{5^{2\alpha}-1}{3}\right)+2\right) \equiv \Delta_7(4n+2) \pmod 4. \tag{3.30}$$

By (2.19), (2.21), (3.10) and (3.25),

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_7(100n+54)q^n &\equiv -q^2 \frac{f_1^2 f_{15}^2\left(q^3, q^{12}; q^{15}\right)_\infty^2}{\left(q^6, q^9; q^{15}\right)_\infty^2} \\ &\equiv -q^2 \frac{f_1^2 f_{15}^4\left(q^3, q^{12}; q^{15}\right)_\infty^4}{f_3^2} \\ &\equiv -q^2 \frac{f_1^2 f_{30}^2\left(q^6, q^{24}; q^{30}\right)_\infty^2}{f_3^2} \\ &\equiv -q^2 \frac{f_2^2 f_3^2 f_{30}^2\left(q^6, q^{24}; q^{30}\right)_\infty^2}{f_1^2 f_6^2} \pmod 4. \end{aligned} \tag{3.31}$$

By substituting (3.17) into (3.31) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty} \Delta_7(200n+154)q^n \equiv 2q \frac{f_1^2 f_2 f_{12} f_{15}^2\left(q^3, q^{12}; q^{15}\right)_\infty^2}{f_3^2} \pmod 4. \tag{3.32}$$

Based on (2.18), (2.20) and (3.32),

$$\sum_{n=0}^{\infty} \Delta_7(200n+154)q^n \equiv 2q f_4 f_6 f_{30}\left(q^6, q^{24}; q^{30}\right)_\infty \pmod 4,$$

which implies that for $n \geq 0$,

$$\Delta_7(400n + 154) \equiv 0 \pmod 4. \tag{3.33}$$

Replacing n by $20n$ in (3.30) and using (3.19), we get (1.9). Replacing n by $20n + 16$ in (3.30) and using (3.22), we obtain (1.10). Replacing n by $100n + 38$ in (3.30) and using (3.33), we arrive at (1.11). Replacing n by $100n + 78$ in (3.30) and using (3.28), we deduce (1.12). The proof of Theorem 1.2 is complete. □

4 Proof of Theorem 1.3

Setting $k = 11$ in (1.7) and employing (2.21), we have

$$\sum_{n=0}^{\infty} \Delta_{11}(n)q^n = \frac{f_2 f_{23}}{f_1^3 f_{46}} \equiv \frac{f_1 f_{23}}{f_2 f_{46}} \pmod 4. \tag{4.1}$$

Replacing q by $-q$ in (4.1) and using (2.23) yields

$$\sum_{n=0}^{\infty} \Delta_{11}(n)(-1)^n q^n \equiv \frac{f_2^2 f_{46}^2}{f_1 f_4 f_{23} f_{92}} \pmod 4. \tag{4.2}$$

Chan and Toh [4] proved that

$$\frac{1}{f_1 f_{23}} - \frac{f_1 f_4 f_{23} f_{92}}{f_2^3 f_{46}^3} = 2q \frac{f_4 f_{92}}{f_2^2 f_{46}^2} + 2q^3 \frac{f_4^2 f_{92}^2}{f_2^3 f_{46}^3}. \tag{4.3}$$

Multiplying $-\frac{f_2^2 f_{46}^2}{f_4 f_{92}}$ on both sides of (4.3) yields

$$\frac{f_1 f_{23}}{f_2 f_{46}} - \frac{f_2^2 f_{46}^2}{f_1 f_4 f_{23} f_{92}} = -2q - 2q^3 \frac{f_4 f_{92}}{f_2 f_{46}}. \tag{4.4}$$

Combining (4.1), (4.2) and (4.4) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Delta_{11}(n)\big(1 - (-1)^n\big)q^n \\
&\quad \equiv \frac{f_1 f_{23}}{f_2 f_{46}} - \frac{f_2^2 f_{46}^2}{f_1 f_4 f_{23} f_{92}} \\
&\quad \equiv -2q - 2q^3 \frac{f_4 f_{92}}{f_2 f_{46}} \pmod 4.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \Delta_{11}(2n+1)q^n \equiv -1 - q\frac{f_2 f_{46}}{f_1 f_{23}} \pmod 4. \tag{4.5}$$

Replacing q by $-q$ in (4.5) and using (2.23) yields

$$\sum_{n=0}^{\infty} \Delta_{11}(2n+1)(-1)^n q^n \equiv -1 + q\frac{f_1 f_4 f_{23} f_{92}}{f_2^2 f_{46}^2} \pmod 4. \tag{4.6}$$

In view of (4.4), (4.5) and (4.6),

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_{11}(2n+1)\big(1+(-1)^n\big)q^n &\equiv -2 + q\frac{f_4 f_{92}}{f_2 f_{46}}\left(\frac{f_1 f_{23}}{f_2 f_{46}} - \frac{f_2^2 f_{46}^2}{f_1 f_4 f_{23} f_{92}}\right)\\ &\equiv -2 - 2q^2\frac{f_4 f_{92}}{f_2 f_{46}} - 2q^4\frac{f_4^2 f_{92}^2}{f_2^2 f_{46}^2} \pmod 4,\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} \Delta_{11}(4n+1)q^n \equiv -1 - q\frac{f_2 f_{46}}{f_1 f_{23}} - q^2\frac{f_2^2 f_{46}^2}{f_1^2 f_{23}^2} \pmod 4. \tag{4.7}$$

By (2.21), (4.5) and (4.7),

$$\sum_{n=0}^{\infty} \Delta_{11}(4n+1)q^n \equiv \sum_{n=0}^{\infty} \Delta_{11}(2n+1)q^n - q^2 f_1^2 f_{23}^2 \pmod 4. \tag{4.8}$$

From Berndt's book [3, Entry 31, p. 48],

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right), \tag{4.9}$$

where $U_n = a^{\frac{(n+1)n}{2}} b^{\frac{n(n-1)}{2}}$ and $V_n = a^{\frac{(n-1)n}{2}} b^{\frac{n(n+1)}{2}}$. Taking $n = 23$, $U_1 = a = -q$ and $V_1 = b = -q^2$ in (4.9), we have

$$\begin{aligned}f_1 =& f\big(-q^{782}, -q^{805}\big) - qf\big(-q^{851}, -q^{736}\big) + q^5 f\big(-q^{920}, -q^{667}\big)\\ &- q^{12} f\big(-q^{989}, -q^{598}\big) + q^{22} f_{529} - q^{35} f\big(-q^{1127}, -q^{460}\big)\end{aligned}$$

$$\begin{aligned}&+q^{51}f(-q^{1196},-q^{391})-q^{70}f(-q^{1265},-q^{322})+q^{92}f(-q^{1334},-q^{253})\\&-q^{117}f(-q^{1403},-q^{184})+q^{145}f(-q^{1472},-q^{115})-q^{176}f(-q^{1541},-q^{46})\\&-q^{187}f(-q^{23},-q^{1564})+q^{155}f(-q^{92},-q^{1495})-q^{126}f(-q^{161},-q^{1426})\\&+q^{100}f(-q^{230},-q^{1357})-q^{77}f(-q^{299},-q^{1288})+q^{57}f(-q^{368},-q^{1219})\\&-q^{40}f(-q^{437},-q^{1150})+q^{26}f(-q^{506},-q^{1081})-q^{15}f(-q^{575},-q^{1012})\\&+q^{7}f(-q^{644},-q^{943})-q^{2}f(-q^{713},-q^{874}).\end{aligned}\tag{4.10}$$

Substituting (4.10) into (4.8), extracting the terms of the form q^{23n} in both sides and then replacing q^{23} by q, we deduce that

$$\sum_{n=0}^{\infty}\Delta_{11}(92n+1)q^n\equiv\sum_{n=0}^{\infty}\Delta_{11}(46n+1)q^n-q^2f_1^2f_{23}^2 \pmod 4.\tag{4.11}$$

It follows from (4.8) and (4.11) that for $n\geq 0$,

$$\Delta_{11}(92n+1)-\Delta_{11}(46n+1)\equiv\Delta_{11}(4n+1)-\Delta_{11}(2n+1) \pmod 4.\tag{4.12}$$

Congruence (1.13) follows from (4.12) and mathematical induction. This completes the proof of Theorem 1.3.

5 Proof of Theorem 1.5

In this section, we prove Theorem 1.5.

By setting $k=2$ in (1.7),

$$\sum_{n=0}^{\infty}\Delta_2(n)q^n=\frac{f_2f_5}{f_1^3f_{10}}.\tag{5.1}$$

In view of (2.20) and (5.1),

$$\sum_{n=0}^{\infty}\Delta_2(n)q^n\equiv\frac{f_5}{f_1f_{10}} \pmod 2.\tag{5.2}$$

Xia and Yao [17] proved that

$$\frac{f_5}{f_1}=\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}.\tag{5.3}$$

Thanks to (2.20) and (5.3),

$$\frac{f_5}{f_1}\equiv f_4+q\frac{f_{10}f_{20}}{f_2} \pmod 2.\tag{5.4}$$

By substituting (5.4) into (5.2),

$$\sum_{n=0}^{\infty} \Delta_2(n)q^n \equiv \frac{f_4}{f_{10}} + q\frac{f_{20}}{f_2} \pmod{2},$$

which yields

$$\sum_{n=0}^{\infty} \Delta_2(2n)q^n \equiv \frac{f_2}{f_5} \pmod{2} \tag{5.5}$$

and

$$\sum_{n=0}^{\infty} \Delta_2(2n+1)q^n \equiv \frac{f_{10}}{f_1} \pmod{2}. \tag{5.6}$$

Substituting (3.9) into (5.5), we obtain

$$\sum_{n=0}^{\infty} \Delta_2(2n)q^n \equiv \frac{f_{50}}{f_5}\left(R\left(q^{10}\right) - q^2 - \frac{q^4}{R\left(q^{10}\right)}\right) \pmod{2},$$

which yields

$$\sum_{n=0}^{\infty} \Delta_2(10n+4)q^n \equiv \frac{f_{10}}{f_1} \pmod{2}. \tag{5.7}$$

Congruence (1.14) follows from (5.6) and (5.7). This completes the proof of Theorem 1.5. □

References

1. Ahmed, Z., Baruah, N.D.: Parity results for broken 5-diamond, 7-diamond and 11-diamond partitions. Int. J. Number Theory **11**, 527–542 (2015)
2. Andrews, G.E., Paule, P.: MacMahon's partition analysis XI: broken diamonds and modular forms. Acta Arith. **126**, 281–294 (2007)
3. Berndt, B.C.: Ramanujan's Notebooks. Part III. Springer, New York (1991)
4. Chan, H.H., Toh, P.C.: New analogues of Ramanujan's partition identities. J. Number Theory **130**, 1898–1913 (2010)
5. Chan, S.H.: Some congruences for Andrews–Paule's broken 2-diamond partitions. Discret. Math. **308**, 5735–5741 (2008)
6. Cui, S.P., Gu, N.S.S.: Congruences for broken 3-diamond and 7 dots bracelet partitions. Ramanujan J. **35**, 165–178 (2014)
7. Hirschhorn, M.D.: Ramanujan's "most beautiful identity". Am. Math. Monthly **118**, 839–845 (2011)
8. Hirschhorn, M.D., Sellers, J.A.: On recent congruence results of Andrews and Paule. Bull. Aust. Math. Soc. **75**, 121–126 (2007)
9. Hirschhorn, M.D., Sellers, J.A.: Elementary proofs of various facts about 3-cores. Bull. Aust. Math. Soc. **79**, 507–512 (2009)

10. Lin, B.L.S.: Elementary proofs of parity results for broken 3-diamond partitions. J. Number Theory **135**, 1–7 (2014)
11. Radu, S., Sellers, J.A.: Parity results for broken k-diamond partitions and $(2k+1)$-cores. Acta Arith. **146**, 43–52 (2011)
12. Radu, S., Sellers, J.A.: An extensive analysis of the parity of broken 3-diamond partitions. J. Number Theory **133**, 3703–3716 (2013)
13. Ramanujan, S.: Collected Papers. In: Hardy, G.H., Seshu Aiyar, P.V., Wilson, B.M. (eds.) AMS Chelsea, Providence, RI (2000)
14. Wang, Y.D.: More parity results for broken 8-diamond partitions. Ramanujan J. **39**, 339–346 (2016)
15. Xia, E.X.W.: New congruences modulo powers of 2 for broken 3-diamond partitions and 7-core partitions. J. Number Theory **141**, 119–135 (2014)
16. Xia, E.X.W., Yao, O.X.M.: New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions. J. Number Theory **133**, 1932–1949 (2013)
17. Xia, E.X.W., Yao, O.X.M.: Analogues of Ramanujan's partition identities. Ramanujan J. **31**, 373–396 (2013)
18. Yao, O.X.M.: New parity results for broken 11-diamond partitions. J. Number Theory **140**, 267–276 (2014)