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Some sequences converging towards Ioachimescu's constant related to Ramanujan's formula

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Abstract The purpose of this paper is to give some sequences that converge quickly to Ioachimescu's constant related to Ramanujan's formula by the multiple-correction method.

Keywords Ioachimescu's constant · Multiple-correction method · Approximation · Rate of convergence

Mathematics Subject Classification 11Y60 · 11A55 · 41A25

1 Introduction

In 1895, Ioachimescu (see [1]) introduced a constant ℓ, which today bears his names, as the limit of the sequence defined by

$$I_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2(\sqrt{n} - 1), n \in \mathbb{N}.$$

The sequence $I(n)_{n\geq 1}$ has attracted much attention lately and several generalizations have been given (see, e.g., [2,3]). Recently, Chen et al. [4] have obtained the complete asymptotic expansion of Ioachimescu's sequence,

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$$I_n \sim \ell + \frac{1}{2\sqrt{n}} - \sum_{k=1}^{\infty} \frac{\mathbf{b}_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1} n^{2k-1/2}}, n \in \mathbb{N},$$

where $\mathbf{b}_n$ denotes the nth Bernoulli number.

One easily obtains the following representations of the Ioachimescu's constant:

$$\ell = \int_0^{\infty} \frac{1 - x + \lfloor x \rfloor}{2(1+x)^{3/2}} dx$$

and

$$\ell = 2 - \sum_{k=1}^{\infty} \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2 \sqrt{k}}.$$

A representation of the Ioachimescu's constant has also been given by Ramanujan (1915) [20]:

$$\ell = 2 - \left(\sqrt{2} + 1\right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}.$$

From this, one easily obtains a representation of the Ioachimescu's constant in terms of the extended ζ function

$$\ell = \zeta\left(\frac{1}{2}\right) + 2.$$

From [2], we have $\ell = 0.539645491\ldots$.

Let $a \in (0, +\infty)$ and $s \in (0, 1)$. The sequence

$$y_n(a,s) = \frac{1}{a^s} + \frac{1}{(a+1)^s} + \cdots \frac{1}{(a+n-1)^s} - \frac{1}{1-s}\left[(a+n-1)^{1-s} - a^{1-s}\right], n \in \mathbb{N},$$

is convergent [3], and its limit is a generalized Euler constant denoted by $\ell(a,s)$. Clearly, $\ell(1, 1/2) = \ell$. Furthermore, Sîntămărian has proved that

$$\lim_{n\to\infty} n^s \left(y_n(a,s) - \ell(a,s)\right) = \frac{1}{2}.$$

Also in [3], considering the sequence

$$u_n(a,s) = y_n(a,s) - \frac{1}{2(a+n-1)^s},$$

she has proved that

$$\lim_{n\to\infty} n^{s+1} \left(\ell(a,s) - u_n(a,s)\right) = \frac{s}{12}$$

and, for the sequence

$$\alpha_n(a,s) = \frac{1}{a^s} + \frac{1}{(a+1)^s} + \cdots \frac{1}{(a+n-1)^s} - \frac{1}{1-s}\left(\left(a+n-\frac{1}{2}\right)^{1-s} - a^{1-s}\right), n \in \mathbb{N},$$

she has proved that

$$\lim_{n\to\infty} n^{s+1}\big(\alpha_n(a,s) - \ell(a,s)\big) = \frac{s}{24}.$$

In [9,10], Sîntămărian has obtained some new sequences that converge to $\ell(a,s)$ with the rate of convergence of n^{-s-15}. Other results regarding $\ell(a,s)$ can be found in [6–8] and some of the references therein. In this paper, we will give some sequences that converge quickly to Ioachimescu's constant ℓ by multiple-correction method [11–13], based on the sequence

$$I(n) = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2(\sqrt{n} - 1), n \in \mathbb{N}.$$

This method could be used to solve other problems, such as Euler–Mascheroni constant, Glaisher–Kinkelin's and Bendersky–Adamchik's constants, and Somos' quadratic recurrence constant [14–17].

2 Main result

The following lemma gives a method for measuring the rate of convergence; for its proof, see Mortici [18,19].

Lemma 2.1 *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent to zero and the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{2.1}$$

exists when $s > 1$, *then*

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.2}$$

Now we apply the *multiple-correction method* to study sequences with faster rate of convergence for Ioachimescu's constant.

Theorem 2.2 *For Ioachimescu's constant, we have the following convergent sequence:*

$$I_i(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \frac{a}{\sqrt[6]{n^3 + b_2 n^2 + b_1 n + b_0 + \frac{u_1}{n + v_1 + \frac{u_2}{n + v_2 + \frac{u_3}{n + v_3 + \frac{u_4}{n + v_4 + \cdots}}}}}}, \tag{2.3}$$

where

$$a = -\frac{1}{2}, \quad b_2 = \frac{1}{2}, \quad b_1 = \frac{7}{48}, \quad b_0 = \frac{1}{864}; \quad u_1 = -\frac{7}{576}, \quad v_1 = \frac{23}{42}; \quad u_2 = \frac{67483}{84672},$$
$$v_2 = -\frac{735611}{14171430};$$
$$u_3 = \frac{106772389611377}{196730868484800}, \quad v_3 = \frac{5732111704318866731}{10293315954492220130};$$
$$u_4 = \frac{3960720843020595280578879811}{2093940584692105796281439289},$$
$$v_4 = -\frac{330844832640429778096837755246211177}{537145603645654538797624944126218812220}; \ldots.$$

Proof (*Step 1*) The initial-correction. We choose $\eta_0(n) = 0$, and let

$$I_0(n) := I(n) + \eta_0(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_0(n). \tag{2.4}$$

Developing the expression (2.4) into power series expansion in $1/n$, we easily obtain

$$I_0(n) - I_0(n+1) = \frac{1}{4} \frac{1}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right). \tag{2.5}$$

By Lemma 2.1, we get the rate of convergence of the $(I_0(n) - \ell)_{n \in \mathbb{N}}$ as $n^{-\frac{1}{2}}$, since

$$\lim_{n \to \infty} n^{\frac{1}{2}} \left(I_0(n) - \ell\right) = \frac{1}{2}.$$

(*Step 2*) The first-correction. Ramanujan [20] made the claim (without proof) for the gamma function

$$\Gamma(x+1) = \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}},$$

where $\theta_x \to 1$ as $x \to +\infty$ and $\frac{3}{10} < \theta_x < 1$. This open problem was solved by Karatsuba [21]. This formula provides a more accurate estimation for the factorial function. Motivated by his idea, we let

$$\eta_1(n) = \frac{a}{\sqrt[6]{n^3 + b_2 n^2 + b_1 n + b_0}} \tag{2.6}$$

and define

$$I_1(n) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_1(n). \tag{2.7}$$

Developing (2.7) into power series expansion in $1/n$, we have

$$\begin{aligned} I_1(n) - I_1(n+1) =& \frac{1}{4}(2a+1)\frac{1}{n^{\frac{3}{2}}} - \frac{1}{8}\big(2 + a(3+2b_2)\big)\frac{1}{n^{\frac{5}{2}}} \\ &+ \frac{5}{576}\big(27 + 4a(9 - 12b_1 + 9b_2 + 7b_2^2)\big)\frac{1}{n^{\frac{7}{2}}} \\ &- \frac{7}{10368}\Big(324 + a\big(405 + 864b0 + 540b2 + 630b2^2 \\ &+ 364b2^3 - 72b1(15 + 14b2)\big)\Big)\frac{1}{n^{\frac{9}{2}}} \\ &+ \frac{7}{13824}\Big(405 + 2a\big(243 + 432b_1^2 + 405b_2 + 630b_2^2 \\ &+ 546b_2^3 + 247b_2^4 + 432b_0(3 + 2b_2) \\ &- 72b_1(15 + 21b_2 + 13b_2^2)\big)\Big)\frac{1}{n^{\frac{11}{2}}} + O\left(\frac{1}{n^{\frac{13}{2}}}\right). \end{aligned} \tag{2.8}$$

(i) If $a \neq -\frac{1}{2}$, then the rate of convergence of the $(I_1(n) - \ell)_{n\in\mathbb{N}}$ is $n^{-\frac{1}{2}}$, since

$$\lim_{n\to\infty} n^{\frac{1}{2}}\left(I_1(n) - \ell\right) = \frac{1}{2}(2a+1) \neq 0.$$

(ii) If $a_1 = -\frac{1}{2}$, $b_2 = \frac{1}{2}$, $b_1 = \frac{7}{48}$, and $b_0 = \frac{1}{864}$, from (2.8) we have

$$I_1(n) - I_1(n+1) = \frac{7}{1536}\frac{1}{n^{\frac{11}{2}}} + O\left(\frac{1}{n^{\frac{13}{2}}}\right).$$

Hence the rate of convergence of the $(I_1(n) - \ell)_{n\in\mathbb{N}}$ is $n^{-\frac{9}{2}}$, since

$$\lim_{n\to\infty} n^{\frac{9}{2}}\left(I_1(n) - \ell\right) = \frac{7}{6912}.$$

(*Step 3*) The second-correction. We set the second-correction function in the form of

$$\eta_2(n) = \frac{a}{\sqrt[6]{n^3 + b_2 n^2 + b_1 n + b_0 + \frac{u_1}{n+v_1}}} \tag{2.9}$$

and define

$$I_2(n) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_2(n). \tag{2.10}$$

Developing (2.10) into power series expansion in $1/n$, we have

$$\begin{aligned} I_2(n) - I_2(n+1) = &\left(\frac{7}{1536} + \frac{3u_1}{8}\right)\frac{1}{n^{\frac{11}{2}}} - \frac{11}{41472}\left(71 + 288u_1(17 + 6v_1)\right)\frac{1}{n^{\frac{13}{2}}} \\ &+ \left(\frac{1132703}{23887872} + \frac{13}{576}u_1(141 + 80v_1 + 24v_1^2)\right)\frac{1}{n^{\frac{15}{2}}} + O\left(\frac{1}{n^{\frac{17}{2}}}\right). \end{aligned} \tag{2.11}$$

By the same method as above, we find $u_1 = -\frac{7}{576}$, $v_1 = \frac{23}{42}$.
Applying Lemma 2.1 again, one has

$$\lim_{n\to\infty} n^{\frac{15}{2}}\left(I_2(n) - I_2(n+1)\right) = -\frac{877279}{167215104}, \tag{2.12}$$

$$\lim_{n\to\infty} n^{\frac{13}{2}}\left(I_2(n) - \ell\right) = -\frac{67483}{83607552}. \tag{2.13}$$

(*Step 4*) The third-correction. Similarly, we set the third-correction function in the form of

$$\eta_3(n) = \frac{a}{\sqrt[6]{n^3 + b_2 n^2 + b_1 n + b_0 + \frac{u_1}{n+v_1+\frac{u_2}{n+v_2}}}} \tag{2.14}$$

and define

$$I_3(n) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_3(n). \tag{2.15}$$

By the same method as above, we find $u_2 = \frac{67483}{84672}$, $v_2 = -\frac{735611}{14171430}$.
Applying Lemma 1 again, one has

$$\lim_{n\to\infty} n^{\frac{19}{2}}\left(I_3(n) - I_3(n+1)\right) = \frac{259304374770487}{69639491498803200}, \tag{2.16}$$

$$\lim_{n\to\infty} n^{\frac{17}{2}}\left(I_3(n) - \ell\right) = \frac{15253198515911}{3481974574940160 0}. \tag{2.17}$$

Similarly, by repeating the above approach for Ioachimescu's constant (the details are omitted here), we can prove Theorem 2.2. □

Remark 2.3 It is worth to point out that Theorem 2.2 provides some sequences with faster rate of convergence for Ioachimescu's constant related to Ramanujan's formula by multiple-correction method. Similarly, by repeating the above approach step by step, we can get some sequences with faster rate of convergence for Ioachimescu's constant. Meanwhile, parameters that need to be calculated are also greatly increased, this will lead to dramatic increase in computing.

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