Ramanujan J (2017) 42:689–712
DOI 10.1007/s11139-016-9800-6

The Dedekind η-function, a Hauptmodul for $\Gamma_0(13)$, and invariant theory

Lei Yang[1]

Received: 13 June 2015 / Accepted: 7 April 2016 / Published online: 26 July 2016

Abstract We give an explicit formula for the Hauptmodul $\left(\frac{\eta(\tau)}{\eta(13\tau)}\right)^2$ of the level-13 Hecke modular group $\Gamma_0(13)$ as a quotient of theta constants, together with some related explicit formulas. Similar results for primes $p = 2, 3, 5, 7$ (the other p for which $\Gamma_0(p)$ has genus zero) are well known, and date back to Klein and Ramanujan. Moreover, we find an exotic modular equation, i.e., it has the same form as Ramanujan's modular equation of degree 13, but with different kinds of modular parameterizations.

Keywords Theta constants · Hauptmodul · Modular equations

Mathematics Subject Classification 11F20 · 11F27 · 14H42 · 11G18 · 14G35 · 14H45

1 Introduction

In many applications of elliptic modular functions to number theory, the Dedekind eta function plays a central role. It is defined in the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$ by $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ with $q = e^{2\pi i z}$. The Dedekind eta function is closely related to the partition function. A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. The partition function $p(n)$ has the well-known generating function $\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1} = q^{\frac{1}{24}}/\eta(z)$.

✉ Lei Yang
yanglei@math.pku.edu.cn

1 Department of Mathematics, Peking University, Beijing 100871, People's Republic of China

In his ground-breaking works [20–22], Ramanujan found the following famous Ramanujan partition congruences: $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$. His proofs are quite ingenious:

$$q^{\frac{19}{24}} \sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{\eta(5z)^5}{\eta(z)^6}, \quad q^{\frac{17}{24}} \sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{\eta(7z)^3}{\eta(z)^4} + 49\frac{\eta(7z)^7}{\eta(z)^8}. \tag{1.1}$$

In [29], Zuckerman found an identity in the spirit of (1.1) (see also [4,26]):

$$\begin{aligned} q^{\frac{11}{24}} \sum_{n=0}^{\infty} p(13n+6)q^n = {} & 11\frac{\eta(13z)}{\eta(z)^2} + 36 \cdot 13\frac{\eta(13z)^3}{\eta(z)^4} + 38 \cdot 13^2\frac{\eta(13z)^5}{\eta(z)^6} + \\ & + 20 \cdot 13^3\frac{\eta(13z)^7}{\eta(z)^8} + 6 \cdot 13^4\frac{\eta(13z)^9}{\eta(z)^{10}} + 13^5\frac{\eta(13z)^{11}}{\eta(z)^{12}} \\ & + 13^5\frac{\eta(13z)^{13}}{\eta(z)^{14}}. \end{aligned} \tag{1.2}$$

Rademacher (see [19]) pointed out that (1.1) can be rewritten as

$$\begin{aligned} \sum_{\lambda=0}^{4} \eta(5z)\eta\left(\frac{z+24\lambda}{5}\right)^{-1} &= 5^2\left(\frac{\eta(5z)}{\eta(z)}\right)^6, \\ \sum_{\lambda=0}^{6} \eta(7z)\eta\left(\frac{z+24\lambda}{7}\right)^{-1} &= 7^2\left(\frac{\eta(7z)}{\eta(z)}\right)^4 + 7^3\left(\frac{\eta(7z)}{\eta(z)}\right)^8. \end{aligned} \tag{1.3}$$

In fact, (1.2) can also be rewritten as

$$\begin{aligned} \sum_{\lambda=0}^{12} \eta(13z)\eta\left(\frac{z+24\lambda}{13}\right)^{-1} = {} & 11 \cdot 13\left(\frac{\eta(13z)}{\eta(z)}\right)^2 + 36 \cdot 13^2\left(\frac{\eta(13z)}{\eta(z)}\right)^4 \\ & + 38 \cdot 13^3\left(\frac{\eta(13z)}{\eta(z)}\right)^6 + 20 \cdot 13^4\left(\frac{\eta(13z)}{\eta(z)}\right)^8 + \\ & + 6 \cdot 13^5\left(\frac{\eta(13z)}{\eta(z)}\right)^{10} + 13^6\left(\frac{\eta(13z)}{\eta(z)}\right)^{12} \\ & + 13^6\left(\frac{\eta(13z)}{\eta(z)}\right)^{14}. \end{aligned} \tag{1.4}$$

Let $X = X_0(p)$ be the compactification of $\mathbf{H}/\Gamma_0(p)$, where $\Gamma_0(p)$ is the level-p Hecke modular group and p is prime. The complex function field of X consists of the modular functions $f(z)$ for $\Gamma_0(p)$ which are meromorphic on the extended upper half-plane. A function f lies in the rational function field $\mathbf{Q}(X)$ if and only if the

Fourier coefficients in its expansion at ∞: $f(z) = \sum a_n q^n$ are all rational numbers. The field $\mathbf{Q}(X)$ is known to be generated over $\mathbf{Q}$ by the classical j-functions

$$\begin{cases} j &= j(z) = q^{-1} + 744 + 196884q + \cdots, \\ j_p &= j\left(\frac{-1}{pz}\right) = j(pz) = q^{-p} + 744 + \cdots. \end{cases}$$

A further element in the function field $\mathbf{Q}(X) = \mathbf{Q}(j, j_p)$ is the modular unit $u = \Delta(z)/\Delta(pz)$ with divisor $(p-1)\{(0) - (\infty)\}$, where $\Delta(z)$ is the discriminant. If $m = \gcd(p-1, 12)$, then an mth root of u lies in $\mathbf{Q}(X)$. This function has the Fourier expansion

$$t = \sqrt[m]{u} = q^{(1-p)/m} \prod_{n\geq 1} \left(\frac{1-q^n}{1-q^{np}}\right)^{24/m} = \left(\frac{\eta(z)}{\eta(pz)}\right)^{24/m}.$$

When $p-1$ divides 12, so $m = p-1$, the function t is a Hauptmodul for the curve X which has genus zero (see [11]). It is well known that the genus of the modular curve X for prime p is zero if and only if $p = 2, 3, 5, 7, 13$. In his paper [13], Klein studied the modular equations of orders 2, 3, 5, 7, 13 with degrees 3, 4, 6, 8, 14, respectively. They are uniformized, i.e., parametrized, by the so-called Hauptmoduln (principal moduli). A Hauptmodul is a function J_Γ that is a modular function for some subgroups of $\Gamma(1) = PSL(2, \mathbf{Z})$, with any other modular function expressible as a rational function of it. In this case, $\Gamma = \Gamma_0(p)$. Hence, (1.3) and (1.4) are intimately related to the Hauptmoduln for modular curves $X_0(5)$, $X_0(7)$, and $X_0(13)$.

The modular curves $X_0(5)$ and $X_0(7)$ were studied by Klein in his pioneering work (see [12–14, 16, 17]) and by Ramanujan (see [24–26]). It is well known that the celebrated Rogers–Ramanujan identities

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$
$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

are intimately associated with the Rogers–Ramanujan continued fraction

$$R(q) := \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}},$$

namely, they satisfy that $R(q) = q^{\frac{1}{5}} \frac{H(q)}{G(q)}$. In [23], Ramanujan found an algebraic relation between $G(q)$ and $H(q)$: $G^{11}(q)H(q) - q^2 G(q) H^{11}(q) = 1 + 11q G^6(q) H^6(q)$, which is equivalent to one of the most important formulas for $R(q)$ (see also [1]):

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \left(\frac{\eta(z)}{\eta(5z)}\right)^6. \tag{1.5}$$

In his celebrated work on elliptic modular functions (see [16, p. 640], [17, p. 73], and [7]), Klein showed that $R(q) = \frac{a(z)}{b(z)}$, where $a(z) = e^{-\frac{3\pi i}{10}} \theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, 5z)$ and $b(z) = e^{-\frac{\pi i}{10}} \theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, 5z)$ are theta constants of order 5. In fact, $a(z) = q^{\frac{1}{60}} \eta(5z)/G(q)$ and $b(z) = q^{-\frac{11}{60}} \eta(5z)/H(q)$. Hence, (1.5) is equivalent to the following formula:

$$\left(\frac{\eta(z)}{\eta(5z)}\right)^6 = -\frac{f(a(z), b(z))}{a(z)^6 b(z)^6}, \tag{1.6}$$

where $f(z_1, z_2) = z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10})$ is an invariant of degree 12 associated to the icosahedron, i.e., it is invariant under the action of the simple group $PSL(2, 5) = \Gamma(1)/\Gamma(5)$, where $\Gamma(5)$ is the principal modular group of level 5. Also $z_1^2 z_2^2$ is invariant under the action of the image of a Borel subgroup of $PSL(2, 5)$, i.e., a maximal subgroup of order 10 of $PSL(2, 5)$, which is simply $\Gamma_0(5)/\Gamma(5)$ (see [12] for more details). We call (1.6) the invariant decomposition formula for the icosahedron.

In his work on elliptic modular functions (see [16, p. 746]), Klein also obtained the invariant decomposition formula for the simple group $PSL(2, 7) = \Gamma(1)/\Gamma(7)$ of order 168:

$$\left(\frac{\eta(z)}{\eta(7z)}\right)^4 = \frac{\Phi_6(a(z), b(z), c(z))}{a(z)^2 b(z)^2 c(z)^2}, \tag{1.7}$$

where $a(z) = -e^{-\frac{5\pi i}{14}} \theta \begin{bmatrix} \frac{5}{7} \\ 1 \end{bmatrix} (0, 7z)$, $b(z) = e^{-\frac{3\pi i}{14}} \theta \begin{bmatrix} \frac{3}{7} \\ 1 \end{bmatrix} (0, 7z)$, and $c(z) = e^{-\frac{\pi i}{14}} \theta \begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix} (0, 7z)$ are theta constants of order 7. Here $\Phi_6(x, y, z) = xy^5 + yz^5 + zx^5 - 5x^2 y^2 z^2$ is an invariant of degree 6 associated to $PSL(2, 7)$, and xyz is invariant under the action of the image of a Borel subgroup of $PSL(2, 7)$, i.e., a maximal subgroup of order 21 of $PSL(2, 7)$, which is simply $\Gamma_0(7)/\Gamma(7)$ (see [14] for more details). Independently, Ramanujan also gave the same formula (see [25, p. 300], [3, p. 174], [8, 18] for more details).

For the modular curve $X_0(13)$, in his monograph (see [17, p. 73]), Klein showed that $(\eta(z)/\eta(13z))^2$ is a Hauptmodul for $\Gamma_0(13)$. However, neither Klein nor Ramanujan could obtain the invariant decomposition formula for $PSL(2, 13) = \Gamma(1)/\Gamma(13)$ in the spirit of (1.6) and (1.7) (see the end of Sect. 4 for more details). The following facts about the modular subgroups of level 13 should be noted. One has that $\Gamma(13) < \Gamma_0(13) < \Gamma(1)$, the respective subgroup indices being $78 = 6 \cdot 13$ and $14 = 13 + 1$. The quotient $\Gamma(1)/\Gamma(13)$ is of order $1092 = 78 \cdot 14$, and $\Gamma_0(13)/\Gamma(13)$ is a subgroup of order 78, which is isomorphic to a semidirect product of $\mathbf{Z}_{13}$ by $\mathbf{Z}_6$. The respective quotients of $\mathbf{H}$ by these three groups (compactified) are the modular curves $X(13)$, $X_0(13)$, $X(1)$, and the coverings $X(13) \to X_0(13) \to X(1)$ are, respectively, 78-sheeted and 14-sheeted. The curve $X(13)$ is of genus 50, despite $X_0(13)$ like $X(1)$ being of genus zero.

In the present paper, we establish the invariant theory for $PSL(2, 13)$. Combining with theta constants of order 13, we obtain an invariant decomposition formula for $PSL(2, 13)$ in the spirit of (1.6) and (1.7). Let

$$\begin{cases} a_1(z) := e^{-\frac{11\pi i}{26}}\theta\begin{bmatrix}\frac{11}{13}\\ 1\end{bmatrix}(0, 13z) = q^{\frac{121}{104}}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(13n^2+11n)},\\ a_2(z) := e^{-\frac{7\pi i}{26}}\theta\begin{bmatrix}\frac{7}{13}\\ 1\end{bmatrix}(0, 13z) = q^{\frac{49}{104}}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(13n^2+7n)},\\ a_3(z) := e^{-\frac{5\pi i}{26}}\theta\begin{bmatrix}\frac{5}{13}\\ 1\end{bmatrix}(0, 13z) = q^{\frac{25}{104}}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(13n^2+5n)},\\ a_4(z) := -e^{-\frac{3\pi i}{26}}\theta\begin{bmatrix}\frac{3}{13}\\ 1\end{bmatrix}(0, 13z) = -q^{\frac{9}{104}}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(13n^2+3n)},\\ a_5(z) := e^{-\frac{9\pi i}{26}}\theta\begin{bmatrix}\frac{9}{13}\\ 1\end{bmatrix}(0, 13z) = q^{\frac{81}{104}}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(13n^2+9n)},\\ a_6(z) := e^{-\frac{\pi i}{26}}\theta\begin{bmatrix}\frac{1}{13}\\ 1\end{bmatrix}(0, 13z) = q^{\frac{1}{104}}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(13n^2+n)} \end{cases} \tag{1.8}$$

be theta constants of order 13. Set

$$\Phi_{12}(z_1, z_2, z_3, z_4, z_5, z_6) := -\frac{1}{26}(7\cdot 13^2\mathbf{G}_0^2 + \mathbf{G}_1\mathbf{G}_{12} + \mathbf{G}_2\mathbf{G}_{11} + \cdots + \mathbf{G}_6\mathbf{G}_7), \tag{1.9}$$

where

$$\begin{cases} \mathbf{G}_0 = \mathbf{D}_0^2 + \mathbf{D}_\infty^2,\\ \mathbf{G}_1 = -\mathbf{D}_7^2 + 2\mathbf{D}_0\mathbf{D}_1 + 10\mathbf{D}_\infty\mathbf{D}_1 + 2\mathbf{D}_2\mathbf{D}_{12} +\\ \qquad -2\mathbf{D}_3\mathbf{D}_{11} - 4\mathbf{D}_4\mathbf{D}_{10} - 2\mathbf{D}_9\mathbf{D}_5,\\ \mathbf{G}_2 = -2\mathbf{D}_1^2 - 4\mathbf{D}_0\mathbf{D}_2 + 6\mathbf{D}_\infty\mathbf{D}_2 - 2\mathbf{D}_4\mathbf{D}_{11} +\\ \qquad +2\mathbf{D}_5\mathbf{D}_{10} - 2\mathbf{D}_6\mathbf{D}_9 - 2\mathbf{D}_7\mathbf{D}_8,\\ \mathbf{G}_3 = -\mathbf{D}_8^2 + 2\mathbf{D}_0\mathbf{D}_3 + 10\mathbf{D}_\infty\mathbf{D}_3 + 2\mathbf{D}_6\mathbf{D}_{10} +\\ \qquad -2\mathbf{D}_9\mathbf{D}_7 - 4\mathbf{D}_{12}\mathbf{D}_4 - 2\mathbf{D}_1\mathbf{D}_2,\\ \mathbf{G}_4 = -\mathbf{D}_2^2 + 10\mathbf{D}_0\mathbf{D}_4 - 2\mathbf{D}_\infty\mathbf{D}_4 + 2\mathbf{D}_5\mathbf{D}_{12} +\\ \qquad -2\mathbf{D}_9\mathbf{D}_8 - 4\mathbf{D}_1\mathbf{D}_3 - 2\mathbf{D}_{10}\mathbf{D}_7,\\ \mathbf{G}_5 = -2\mathbf{D}_9^2 - 4\mathbf{D}_0\mathbf{D}_5 + 6\mathbf{D}_\infty\mathbf{D}_5 - 2\mathbf{D}_{10}\mathbf{D}_8 +\\ \qquad +2\mathbf{D}_6\mathbf{D}_{12} - 2\mathbf{D}_2\mathbf{D}_3 - 2\mathbf{D}_{11}\mathbf{D}_7,\\ \mathbf{G}_6 = -2\mathbf{D}_3^2 - 4\mathbf{D}_0\mathbf{D}_6 + 6\mathbf{D}_\infty\mathbf{D}_6 - 2\mathbf{D}_{12}\mathbf{D}_7 +\\ \qquad +2\mathbf{D}_2\mathbf{D}_4 - 2\mathbf{D}_5\mathbf{D}_1 - 2\mathbf{D}_8\mathbf{D}_{11},\\ \mathbf{G}_7 = -2\mathbf{D}_{10}^2 + 6\mathbf{D}_0\mathbf{D}_7 + 4\mathbf{D}_\infty\mathbf{D}_7 - 2\mathbf{D}_1\mathbf{D}_6 +\\ \qquad -2\mathbf{D}_2\mathbf{D}_5 - 2\mathbf{D}_8\mathbf{D}_{12} - 2\mathbf{D}_9\mathbf{D}_{11},\\ \mathbf{G}_8 = -2\mathbf{D}_4^2 + 6\mathbf{D}_0\mathbf{D}_8 + 4\mathbf{D}_\infty\mathbf{D}_8 - 2\mathbf{D}_3\mathbf{D}_5 +\\ \qquad -2\mathbf{D}_6\mathbf{D}_2 - 2\mathbf{D}_{11}\mathbf{D}_{10} - 2\mathbf{D}_1\mathbf{D}_7,\\ \mathbf{G}_9 = -\mathbf{D}_{11}^2 + 2\mathbf{D}_0\mathbf{D}_9 + 10\mathbf{D}_\infty\mathbf{D}_9 + 2\mathbf{D}_5\mathbf{D}_4 +\\ \qquad -2\mathbf{D}_1\mathbf{D}_8 - 4\mathbf{D}_{10}\mathbf{D}_{12} - 2\mathbf{D}_3\mathbf{D}_6, \end{cases}$$

$$\begin{cases} \mathbf{G}_{10} = -\mathbf{D}_5^2 + 10\mathbf{D}_0\mathbf{D}_{10} - 2\mathbf{D}_\infty\mathbf{D}_{10} + 2\mathbf{D}_6\mathbf{D}_4 + \\ \qquad -2\mathbf{D}_3\mathbf{D}_7 - 4\mathbf{D}_9\mathbf{D}_1 - 2\mathbf{D}_{12}\mathbf{D}_{11}, \\ \mathbf{G}_{11} = -2\mathbf{D}_{12}^2 + 6\mathbf{D}_0\mathbf{D}_{11} + 4\mathbf{D}_\infty\mathbf{D}_{11} - 2\mathbf{D}_9\mathbf{D}_2 + \\ \qquad -2\mathbf{D}_5\mathbf{D}_6 - 2\mathbf{D}_7\mathbf{D}_4 - 2\mathbf{D}_3\mathbf{D}_8, \\ \mathbf{G}_{12} = -\mathbf{D}_6^2 + 10\mathbf{D}_0\mathbf{D}_{12} - 2\mathbf{D}_\infty\mathbf{D}_{12} + 2\mathbf{D}_2\mathbf{D}_{10} + \\ \qquad -2\mathbf{D}_1\mathbf{D}_{11} - 4\mathbf{D}_3\mathbf{D}_9 - 2\mathbf{D}_4\mathbf{D}_8 \end{cases} \tag{1.10}$$

are the senary sextic forms (sextic forms in six variables). Here,

$$\begin{cases} \mathbf{D}_0 = z_1z_2z_3, \\ \mathbf{D}_1 = 2z_2z_3^2 + z_2^2z_6 - z_4^2z_5 + z_1z_5z_6, \\ \mathbf{D}_2 = -z_6^3 + z_2^2z_4 - 2z_2z_5^2 + z_1z_4z_5 + 3z_3z_5z_6, \\ \mathbf{D}_3 = 2z_1z_2^2 + z_1^2z_5 - z_4z_6^2 + z_3z_4z_5, \\ \mathbf{D}_4 = -z_2^2z_3 + z_1z_6^2 - 2z_4^2z_6 - z_1z_3z_5, \\ \mathbf{D}_5 = -z_4^3 + z_3^2z_5 - 2z_3z_6^2 + z_2z_5z_6 + 3z_1z_4z_6, \\ \mathbf{D}_6 = -z_5^3 + z_1^2z_6 - 2z_1z_4^2 + z_3z_4z_6 + 3z_2z_4z_5, \\ \mathbf{D}_7 = -z_2^3 + z_3z_4^2 - z_1z_3z_6 - 3z_1z_2z_5 + 2z_1^2z_4, \\ \mathbf{D}_8 = -z_1^3 + z_2z_6^2 - z_2z_3z_5 - 3z_1z_3z_4 + 2z_3^2z_6, \\ \mathbf{D}_9 = 2z_1^2z_3 + z_3^2z_4 - z_5^2z_6 + z_2z_4z_6, \\ \mathbf{D}_{10} = -z_1z_3^2 + z_2z_4^2 - 2z_4z_5^2 - z_1z_2z_6, \\ \mathbf{D}_{11} = -z_3^3 + z_1z_5^2 - z_1z_2z_4 - 3z_2z_3z_6 + 2z_2^2z_5, \\ \mathbf{D}_{12} = -z_1^2z_2 + z_3z_5^2 - 2z_5z_6^2 - z_2z_3z_4, \\ \mathbf{D}_\infty = z_4z_5z_6 \end{cases} \tag{1.11}$$

are the senary cubic forms (cubic forms in six variables). Our main result is as follows:

Theorem 1.1 (Main Theorem 1) *The invariant decomposition formula for the simple group $PSL(2, 13)$ of order 1092 is given as follows:*

$$\left[\frac{\eta^2(z)}{\eta^2(13z)}\right]^5 = \frac{\Phi_{12}(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))}{(a_1(z)a_2(z)a_3(z)a_4(z)a_5(z)a_6(z))^2}, \tag{1.12}$$

where $\Phi_{12}(z_1, z_2, z_3, z_4, z_5, z_6)$ is an invariant of degree 12 associated to $PSL(2, 13)$, and $(z_1z_2z_3z_4z_5z_6)^2$ is invariant under the action of the image of a Borel subgroup of $PSL(2, 13)$, i.e., a maximal subgroup of order 78 of $PSL(2, 13)$, which can be viewed as $\Gamma_0(13)/\Gamma(13)$.

It should be pointed out that (1.12) is a formula not for the Hauptmodul of $\Gamma_0(13)$, but for its fifth power. The appearance of the fifth power is due to the denominator $(a_1(z)a_2(z)a_3(z)a_4(z)a_5(z)a_6(z))^2$ having a factor $\eta(13z)^{10}$. In [13], Klein obtained the modular equation of degree 14 for $\Gamma_0(13)$ (see [5, p. 267] for the modular relation for $\Gamma_0(13)$). Combining with Theorem 1.1, we give a new expression of the classical j-function in terms of theta constants of order 13.

Corollary 1.2 *The following formulas hold:*

$$j(z) = \frac{(\tau^2 + 5\tau + 13)(\tau^4 + 247\tau^3 + 3380\tau^2 + 15379\tau + 28561)^3}{\tau^{13}} \tag{1.13}$$

and

$$j(13z) = \frac{(\tau^2+5\tau+13)(\tau^4+7\tau^3+20\tau^2+19\tau+1)^3}{\tau}, \tag{1.14}$$

where the Hauptmodul

$$\tau = \left(\frac{\eta(z)}{\eta(13z)}\right)^2 = \sqrt[5]{\frac{\Phi_{12}(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))}{(a_1(z)a_2(z)a_3(z)a_4(z)a_5(z)a_6(z))^2}}. \tag{1.15}$$

Note that in the right-hand side of modular equations (1.13) and (1.14), we have the following factorizations over $\mathbf{Q}(\sqrt{13})$:

$$\begin{aligned}
&\tau^4 + 247\tau^3 + 3380\tau^2 + 15379\tau + 28561 \\
&\quad= \left(\tau^2 + \frac{247+65\sqrt{13}}{2}\tau + \frac{1859+507\sqrt{13}}{2}\right) \\
&\qquad \times \left(\tau^2 + \frac{247-65\sqrt{13}}{2}\tau + \frac{1859-507\sqrt{13}}{2}\right), \\
&\tau^4 + 7\tau^3 + 20\tau^2 + 19\tau + 1 \\
&\quad= \left(\tau^2 + \frac{7+\sqrt{13}}{2}\tau + \frac{11+3\sqrt{13}}{2}\right)\left(\tau^2 + \frac{7-\sqrt{13}}{2}\tau + \frac{11-3\sqrt{13}}{2}\right).
\end{aligned}$$

P. Deligne (Letter to the author, July 29, 2014. Private communication) gave a modular interpretation of why such factorizations exist. More generally, P. Deligne (Letter to the author, August 7, 2014. Private communication) showed that for $p = 3, 5, 7, 13$, the corresponding modular equations of degrees 4, 6, 8, 14 have two relatively prime conjugate factors over some real quadratic fields. All of these factorizations have nice geometric interpretations.

In his notebooks (see [24, p. 326], [25, p. 244]), Ramanujan (see also [9] or [2, pp. 373–375]) obtained the following modular equation of degree 13:

$$1 + G_1(z)G_5(z) + G_2(z)G_3(z) + G_4(z)G_6(z) = -\frac{\eta^2(z/13)}{\eta^2(z)} \tag{1.16}$$

with

$$G_1(z)G_5(z)G_2(z)G_3(z)G_4(z)G_6(z) = -1, \tag{1.17}$$

where $G_m(z) := G_{m,p}(z) := (-1)^m q^{m(3m-p)/(2p^2)} \frac{f(-q^{2m/p}, -q^{1-2m/p})}{f(-q^{m/p}, -q^{1-m/p})}$. Here, m is a positive integer, $p = 13$, $q = e^{2\pi i z}$, and Ramanujan's general theta function $f(a, b)$ is given by $f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$, $|ab| < 1$. We call (1.16) the standard modular equation of degree 13. In contrast with it, we find the following invariant decomposition formula (exotic modular equation) which has the same form as (1.16), but with different kinds of modular parametrizations. Let us define the following senary quadratic forms (quadratic forms in six variables):

$$\begin{cases} \mathbf{A}_0 = z_1 z_4 + z_2 z_5 + z_3 z_6, \\ \mathbf{A}_1 = z_1^2 - 2 z_3 z_4, \\ \mathbf{A}_2 = -z_5^2 - 2 z_2 z_4, \\ \mathbf{A}_3 = z_2^2 - 2 z_1 z_5, \\ \mathbf{A}_4 = z_3^2 - 2 z_2 z_6, \\ \mathbf{A}_5 = -z_4^2 - 2 z_1 z_6, \\ \mathbf{A}_6 = -z_6^2 - 2 z_3 z_5. \end{cases} \tag{1.18}$$

Theorem 1.3 (Main Theorem 2) *The following invariant decomposition formula (exotic modular equation) holds*:

$$\frac{\Psi_2(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))}{\mathbf{A}_0(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))^2} = 0, \tag{1.19}$$

where the quadric

$$\Psi_2(z_1, z_2, z_3, z_4, z_5, z_6) := \mathbf{A}_0^2 + \mathbf{A}_1 \mathbf{A}_5 + \mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_4 \mathbf{A}_6 \tag{1.20}$$

is an invariant associated to $PSL(2, 13)$ *and* $\mathbf{A}_0^2$ *is invariant under the action of the image of a Borel subgroup of* $PSL(2, 13)$*, i.e., a maximal subgroup of order* 78 *of* $PSL(2, 13)$*, which can be viewed as* $\Gamma_0(13)/\Gamma(13)$.

Note that (1.19) has the same form as (1.16). However, in contrast with (1.17), it can be proved that (see Sect. 3, (3.8))

$$\prod_{j=1}^{6} \frac{\mathbf{A}_j(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))}{\mathbf{A}_0(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))} \neq -1. \tag{1.21}$$

Moreover, the invariant quadric (1.20) is closely related to the exceptional Lie group G_2 (see [27] for more details). As an application, we obtain the following quartic four-fold $\Phi_4(z_1, z_2, z_3, z_4, z_5, z_6) = 0$, where

$$\begin{aligned} \Phi_4 := &(z_3 z_4^3 + z_1 z_5^3 + z_2 z_6^3) - (z_6 z_1^3 + z_4 z_2^3 + z_5 z_3^3) \\ &+ 3(z_1 z_2 z_4 z_5 + z_2 z_3 z_5 z_6 + z_3 z_1 z_6 z_4), \end{aligned} \tag{1.22}$$

which is just the quadric (1.20) up to a constant. It is a higher dimensional counterpart of the Klein quartic curve (see [14]) and the Klein cubic three-fold (see [15]). Its significance comes from the following:

Corollary 1.4 *The coordinates* $(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))$ *map* $X(13)$ *into the quartic four-fold* $\Phi_4(z_1, z_2, z_3, z_4, z_5, z_6) = 0$ *in* $\mathbf{CP}^5$.

We remark that in our preprint [28], the full invariant theory of $PSL(2, 13)$, i.e., the determination of all polynomial invariants (not merely Φ_4, Φ_{12}) is worked out to give the exotic structure associated with the equation of the E_8 singularity.

This paper consists of four sections. In Sect. 2, we give a six-dimensional representation of $PSL(2, 13)$ defined over $\mathbf{Q}(e^{\frac{2\pi i}{13}})$ and the transformation formulas for theta constants associated with $\Gamma(13)$. In Sect. 3, we give a seven-dimensional representation of $PSL(2, 13)$ which is deduced from our six-dimensional representation. As an application, we obtain the exotic modular equation of degree 13. Thus, we give the proof of Theorem 1.3. In Sect. 4, we give a 14-dimensional representation of $PSL(2, 13)$ which is also deduced from our six-dimensional representation. By this representation, we find the exotic modular equation of degree 14. Using it, we give the proof of Theorem 1.1.

2 Six-dimensional representations of $PSL(2, 13)$ and transformation formulas for theta constants

In this section, we will study the six-dimensional representation of the simple group $PSL(2, 13)$ of order 1092, which acts on the five-dimensional projective space $\mathbf{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbf{C}(i = 1, 2, 3, 4, 5, 6)\}$.

Let $\zeta = \exp(2\pi i/13)$, $\theta_1 = \zeta + \zeta^3 + \zeta^9$, $\theta_2 = \zeta^2 + \zeta^6 + \zeta^5$, $\theta_3 = \zeta^4 + \zeta^{12} + \zeta^{10}$, and $\theta_4 = \zeta^8 + \zeta^{11} + \zeta^7$. We find that

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 + \theta_4 = -1, \\ \theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4 = 2, \\ \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 = 4, \\ \theta_1\theta_2\theta_3\theta_4 = 3. \end{cases}$$

Hence, θ_1, θ_2, θ_3, and θ_4 satisfy the quartic equation $z^4 + z^3 + 2z^2 - 4z + 3 = 0$, which can be decomposed as two quadratic equations

$$\left(z^2 + \frac{1+\sqrt{13}}{2}z + \frac{5+\sqrt{13}}{2}\right)\left(z^2 + \frac{1-\sqrt{13}}{2}z + \frac{5-\sqrt{13}}{2}\right) = 0$$

over the real quadratic field $\mathbf{Q}(\sqrt{13})$. Therefore, the four roots are given as follows:

$$\begin{cases} \theta_1 = \frac{1}{4}\left(-1 + \sqrt{13} + \sqrt{-26 + 6\sqrt{13}}\right), \\ \theta_2 = \frac{1}{4}\left(-1 - \sqrt{13} + \sqrt{-26 - 6\sqrt{13}}\right), \\ \theta_3 = \frac{1}{4}\left(-1 + \sqrt{13} - \sqrt{-26 + 6\sqrt{13}}\right), \\ \theta_4 = \frac{1}{4}\left(-1 - \sqrt{13} - \sqrt{-26 - 6\sqrt{13}}\right). \end{cases}$$

Moreover, we find that

$$\begin{cases} \theta_1 + \theta_3 + \theta_2 + \theta_4 = -1, \\ \theta_1 + \theta_3 - \theta_2 - \theta_4 = \sqrt{13}, \\ \theta_1 - \theta_3 - \theta_2 + \theta_4 = -\sqrt{-13 + 2\sqrt{13}}, \\ \theta_1 - \theta_3 + \theta_2 - \theta_4 = \sqrt{-13 - 2\sqrt{13}}. \end{cases}$$

Let $S = -\frac{1}{\sqrt{13}} \begin{pmatrix} -M & N \\ N & M \end{pmatrix}$ and $T = \text{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5)$, where

$$M = \begin{pmatrix} \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 \\ \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} \\ \zeta^9 - \zeta^4 & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} \end{pmatrix},$$

$$N = \begin{pmatrix} \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 \\ \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\ \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} \end{pmatrix}.$$

Then $MN = NM = -\sqrt{13}I$, $M^2 + N^2 = -13I$ and $S^2 = I$.

Theorem 2.1 *Let $G = \langle S, T \rangle$. Then $G \cong PSL(2, 13)$.*

The proof is elementary: Magma will confirm it immediately.

Recall that the theta functions with characteristic $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbf{R}^2$ are defined by the following series which converges uniformly and absolutely on compact subsets of $\mathbf{C} \times \mathbf{H}$ (see [10, p. 73]):

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau) = \sum_{n \in \mathbf{Z}} \exp \left\{ 2\pi i \left[\frac{1}{2} \left(n + \frac{\epsilon}{2} \right)^2 \tau + \left(n + \frac{\epsilon}{2} \right) \left(z + \frac{\epsilon'}{2} \right) \right] \right\}.$$

We introduce the modified theta constants (see [10], p. 215) $\varphi_l(\tau) := \theta[\chi_l](0, k\tau)$, where the characteristic $\chi_l = \begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix}$, $l = 0, \ldots, \frac{k-3}{2}$, for odd k and $\chi_l = \begin{bmatrix} \frac{2l}{k} \\ 0 \end{bmatrix}$, $l = 0, \cdots, \frac{k}{2}$, for even k. We have the following:

Theorem 2.2 (See [10, p. 236]) *For each odd integer $k \geq 5$, the map $\Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \ldots, \varphi_{\frac{k-5}{2}}(\tau), \varphi_{\frac{k-3}{2}}(\tau))$ from $\mathbf{H} \cup \mathbf{Q} \cup \{\infty\}$ to $\mathbf{C}^{\frac{k-1}{2}}$ defines a holomorphic mapping from $\overline{\mathbf{H}/\Gamma(k)}$ into $\mathbf{CP}^{\frac{k-3}{2}}$.*

In our case, the map $\Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau), \varphi_4(\tau), \varphi_5(\tau))$ gives a holomorphic mapping from the modular curve $X(13) = \overline{\mathbf{H}/\Gamma(13)}$ into $\mathbf{CP}^5$, which corresponds to our six-dimensional representation, i.e., up to the constants, $z_1, \ldots, z_6$ are just modular forms $\varphi_0(\tau), \ldots, \varphi_5(\tau)$.

Let $a_i(z)$ $(1 \le i \le 6)$ be given as in (1.8). We use the standard notation $(a)_\infty := (a;q) := \prod_{k=0}^{\infty}(1-aq^k)$ for $a \in \mathbf{C}^\times$ and $|q|<1$ so that $\eta(z) = q^{1/24}(q;q)$. By the Jacobi triple product identity, we have that

$$\theta\left[\begin{matrix}\frac{l}{k}\\ 1\end{matrix}\right](0,kz) = \exp\left(\frac{\pi i l}{2k}\right) q^{\frac{l^2}{8k}}(q^{\frac{k-l}{2}};q^k)(q^{\frac{k+l}{2}};q^k)(q^k;q^k),$$

where k is odd and $l = 1, 3, 5, \ldots, k-2$. Hence,

$$\begin{cases} a_1(z) = q^{\frac{121}{104}}(q;q^{13})(q^{12};q^{13})(q^{13};q^{13}),\\ a_2(z) = q^{\frac{49}{104}}(q^3;q^{13})(q^{10};q^{13})(q^{13};q^{13}),\\ a_3(z) = q^{\frac{25}{104}}(q^9;q^{13})(q^4;q^{13})(q^{13};q^{13}),\\ a_4(z) = -q^{\frac{9}{104}}(q^5;q^{13})(q^8;q^{13})(q^{13};q^{13}),\\ a_5(z) = q^{\frac{81}{104}}(q^2;q^{13})(q^{11};q^{13})(q^{13};q^{13}),\\ a_6(z) = q^{\frac{1}{104}}(q^6;q^{13})(q^7;q^{13})(q^{13};q^{13}).\end{cases} \tag{2.1}$$

It is known that Ramanujan's general theta functions are given as follows:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

In his notebooks (see [2, p. 372], [24, p. 326], and [25, p. 244]), Ramanujan obtained his modular equations of degree 13.

Theorem 2.3 *Define*

$$\mu_1 = \frac{f(-q^4,-q^9)}{q^{7/13} f(-q^2,-q^{11})},\ \mu_2 = \frac{f(-q^6,-q^7)}{q^{6/13} f(-q^3,-q^{10})},\ \mu_3 = \frac{f(-q^2,-q^{11})}{q^{5/13} f(-q,-q^{12})},$$

$$\mu_4 = \frac{f(-q^5,-q^8)}{q^{2/13} f(-q^4,-q^9)},\ \mu_5 = \frac{q^{5/13} f(-q^3,-q^{10})}{f(-q^5,-q^8)},\ \mu_6 = \frac{q^{15/13} f(-q,-q^{12})}{f(-q^6,-q^7)}.$$

Then

$$1 + \frac{f^2(-q)}{qf^2(-q^{13})} = \mu_1\mu_2 - \mu_3\mu_5 - \mu_4\mu_6, \tag{2.2}$$

$$-4 - \frac{f^2(-q)}{qf^2(-q^{13})} = \frac{1}{\mu_1\mu_2} - \frac{1}{\mu_3\mu_5} - \frac{1}{\mu_4\mu_6}, \tag{2.3}$$

$$3 + \frac{f^2(-q)}{qf^2(-q^{13})} = \mu_2\mu_3\mu_4 - \mu_1\mu_5\mu_6, \tag{2.4}$$

where $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6 = 1$.

Let $G_m(z) := G_{m,p}(z) := (-1)^m q^{m(3m-p)/(2p^2)} \frac{f(-q^{2m/p},-q^{1-2m/p})}{f(-q^{m/p},-q^{1-m/p})}$ where m is a positive integer, $p = 13$, and $q = e^{2\pi iz}$. Then the above three formulas (2.2), (2.3), and (2.4) are equivalent to the following (see [9] or [2, pp. 373–375])

$$1 + G_1(z)G_5(z) + G_2(z)G_3(z) + G_4(z)G_6(z) = -\frac{\eta^2(z/13)}{\eta^2(z)}, \tag{2.5}$$

$$\frac{1}{G_1(z)G_5(z)} + \frac{1}{G_2(z)G_3(z)} + \frac{1}{G_4(z)G_6(z)} = 4 + \frac{\eta^2(z/13)}{\eta^2(z)}, \tag{2.6}$$

$$G_1(z)G_3(z)G_4(z) - \frac{1}{G_1(z)G_3(z)G_4(z)} = 3 + \frac{\eta^2(z/13)}{\eta^2(z)}, \tag{2.7}$$

where $G_1(z)G_2(z)G_3(z)G_4(z)G_5(z)G_6(z) = -1$. Moreover, there is the following formula (see [9] or [2, pp. 375–376]): for $t = q^{1/13}$,

$$\frac{1}{(t^2)_\infty(t^3)_\infty(t^{10})_\infty(t^{11})_\infty} + \frac{t}{(t^4)_\infty(t^6)_\infty(t^7)_\infty(t^9)_\infty} = \frac{1}{(t)_\infty(t^5)_\infty(t^8)_\infty(t^{12})_\infty}, \tag{2.8}$$

which is equivalent to, for $p = 13$, $G_1^{-1}(z)G_5^{-1}(z) + G_4(z)G_6(z) = 1$.

It is known that (see [9]) $G(m; z) = (-1)^m F(2m/p; z)/F(m/p; z)$, where

$$F(u; z) = -i \sum_{k=-\infty}^{\infty} (-1)^k q^{(k+u+1/2)^2/2}$$
$$= -iq^{(u+1/2)^2/2} \prod_{m=1}^{\infty} (1 - q^{m+u})(1 - q^{m-1-u})(1 - q^m)$$

satisfies that $F(u + 1; z) = -F(u; z)$ and $F(-u; z) = -F(u; z)$. We have that

$$G_1(z)G_5(z) = \frac{F(\frac{2}{13}; z)F(\frac{3}{13}; z)}{F(\frac{1}{13}; z)F(\frac{5}{13}; z)} = \frac{(t^3)_\infty(t^{10})_\infty(t^2)_\infty(t^{11})_\infty}{(t)_\infty(t^{12})_\infty(t^5)_\infty(t^8)_\infty},$$

where $(x)_\infty = \prod_{m=0}^{\infty}(1 - xq^m)$ and $t = q^{\frac{1}{13}}$. Similarly,

$$G_2(z)G_3(z) = -\frac{(t^9)_\infty(t^4)_\infty(t^6)_\infty(t^7)_\infty}{t(t^3)_\infty(t^{10})_\infty(t^2)_\infty(t^{11})_\infty},$$

$$G_4(z)G_6(z) = \frac{t(t)_\infty(t^{12})_\infty(t^5)_\infty(t^8)_\infty}{(t^9)_\infty(t^4)_\infty(t^6)_\infty(t^7)_\infty}.$$

Note that $(t^k)_\infty = (t^k; t^{13})$ for $1 \le k \le 13$. The above formula (2.5) is equivalent to

$$\begin{aligned}
&t(t^3;t^{13})^2(t^{10};t^{13})^2(t^2;t^{13})^2(t^{11};t^{13})^2(t^9;t^{13})(t^4;t^{13})(t^6;t^{13})(t^7;t^{13})\\
&\quad-(t^9;t^{13})^2(t^4;t^{13})^2(t^6;t^{13})^2(t^7;t^{13})^2(t;t^{13})(t^{12};t^{13})(t^5;t^{13})(t^8;t^{13})\\
&\quad+t^2(t;t^{13})^2(t^{12};t^{13})^2(t^5;t^{13})^2(t^8;t^{13})^2(t^3;t^{13})(t^{10};t^{13})(t^2;t^{13})(t^{11};t^{13})\\
&=[-1-\eta^2(z/13)/\eta^2(z)]t(t;t^{13})(t^2;t^{13})\cdots(t^{12};t^{13}),\quad t=q^{1/13}=e^{\frac{2\pi iz}{13}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
a_2(z)^2a_5(z)^2a_3(z)a_6(z) = q^{\frac{11}{4}}&(q^3;q^{13})^2(q^{10};q^{13})^2(q^2;q^{13})^2(q^{11};q^{13})^2\\
&\times(q^9;q^{13})(q^4;q^{13})(q^6;q^{13})(q^7;q^{13})(q^{13};q^{13})^6,
\end{aligned}$$

$$\begin{aligned}
a_3(z)^2a_6(z)^2a_1(z)a_4(z) = -q^{\frac{7}{4}}&(q^9;q^{13})^2(q^4;q^{13})^2(q^6;q^{13})^2(q^7;q^{13})^2\\
&\times(q;q^{13})(q^{12};q^{13})(q^5;q^{13})(q^8;q^{13})(q^{13};q^{13})^6,
\end{aligned}$$

$$\begin{aligned}
a_1(z)^2a_4(z)^2a_2(z)a_5(z) = q^{\frac{15}{4}}&(q;q^{13})^2(q^{12};q^{13})^2(q^5;q^{13})^2(q^8;q^{13})^2\\
&\times(q^3;q^{13})(q^{10};q^{13})(q^2;q^{13})(q^{11};q^{13})(q^{13};q^{13})^6.
\end{aligned}$$

This implies that

$$\begin{aligned}
&a_1(z)^2a_4(z)^2a_2(z)a_5(z)+a_2(z)^2a_5(z)^2a_3(z)a_6(z)+a_3(z)^2a_6(z)^2a_1(z)a_4(z)\\
&\quad=[-1-\eta^2(z)/\eta^2(13z)]\eta(z)\eta(13z)^5. \qquad (2.9)
\end{aligned}$$

Similarly, the other three formulas (2.6), (2.7), and (2.8) are equivalent to the following:

$$\begin{aligned}
&a_1(z)^2a_4(z)^2a_3(z)a_6(z)+a_2(z)^2a_5(z)^2a_1(z)a_4(z)+a_3(z)^2a_6(z)^2a_2(z)a_5(z)\\
&\quad=[4+\eta^2(z)/\eta^2(13z)]\eta(z)\eta(13z)^5, \qquad (2.10)
\end{aligned}$$

$$a_4(z)^2a_5(z)^2a_6(z)^2-a_1(z)^2a_2(z)^2a_3(z)^2=[3+\eta^2(z)/\eta^2(13z)]\eta(z)\eta(13z)^5, \qquad (2.11)$$

and

$$\frac{1}{a_1(z)a_4(z)}+\frac{1}{a_2(z)a_5(z)}+\frac{1}{a_3(z)a_6(z)}=0, \qquad (2.12)$$

where $a_1(z)a_2(z)a_3(z)a_4(z)a_5(z)a_6(z) = -\eta(z)\eta(13z)^5$.

Let $\mathbf{A}(z) = (a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))^T$. The significance of our six-dimensional representation of $PSL(2, 13)$ comes from the following:

Proposition 2.4 *If $z \in \mathbf{H}$, then the following relations hold:*

$$\mathbf{A}(z+1) = e^{-\frac{3\pi i}{4}} T\mathbf{A}(z), \quad \mathbf{A}\left(-\frac{1}{z}\right) = e^{\frac{\pi i}{4}} \sqrt{z} S\mathbf{A}(z), \tag{2.13}$$

where $T = diag(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5)$,

$$S = \frac{-1}{\sqrt{13}} \begin{pmatrix} \zeta^{12}-\zeta & \zeta^{10}-\zeta^3 & \zeta^4-\zeta^9 & \zeta^5-\zeta^8 & \zeta^2-\zeta^{11} & \zeta^6-\zeta^7 \\ \zeta^{10}-\zeta^3 & \zeta^4-\zeta^9 & \zeta^{12}-\zeta & \zeta^2-\zeta^{11} & \zeta^6-\zeta^7 & \zeta^5-\zeta^8 \\ \zeta^4-\zeta^9 & \zeta^{12}-\zeta & \zeta^{10}-\zeta^3 & \zeta^6-\zeta^7 & \zeta^5-\zeta^8 & \zeta^2-\zeta^{11} \\ \zeta^5-\zeta^8 & \zeta^2-\zeta^{11} & \zeta^6-\zeta^7 & \zeta-\zeta^{12} & \zeta^3-\zeta^{10} & \zeta^9-\zeta^4 \\ \zeta^2-\zeta^{11} & \zeta^6-\zeta^7 & \zeta^5-\zeta^8 & \zeta^3-\zeta^{10} & \zeta^9-\zeta^4 & \zeta-\zeta^{12} \\ \zeta^6-\zeta^7 & \zeta^5-\zeta^8 & \zeta^2-\zeta^{11} & \zeta^9-\zeta^4 & \zeta-\zeta^{12} & \zeta^3-\zeta^{10} \end{pmatrix},$$

and $0 < arg\sqrt{z} \leq \pi/2$.

The proof is an application of the following transformation formulas for theta constants (see [10, pp. 216–217]) to the special case $k = 13$:

$$\theta\begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix}(0, k(z+1)) = \exp\left(\frac{\pi i}{4k}\right) \exp\left(\frac{\pi i}{k}(l^2+l)\right) \theta\begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix}(0, kz)$$

and

$$\theta\begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix}(0, k(-1/z)) = \frac{\sqrt{z}}{\sqrt{ik}} \exp\left(\frac{\pi i}{2k}\right) \sum_{j=0}^{\frac{k-3}{2}} \left[\exp\left(2\pi i \frac{l(j+1)}{k}\right) + \exp\left(-\frac{\pi i}{k}\right) \exp\left(-2\pi i \frac{(l+1)j}{k}\right)\right] \theta\begin{bmatrix} \frac{2j+1}{k} \\ 1 \end{bmatrix}(0, kz),$$

where k is odd and $l = 0, 1, \ldots, \frac{k-3}{2}$.

3 Seven-dimensional representations of $PSL(2, 13)$, exotic modular equation and geometry of modular curve $X(13)$

We will construct a seven-dimensional representation of $PSL(2, 13)$ which is deduced from our six-dimensional representation. Let us study the action of ST^{ν} on the five-dimensional projective space $\mathbf{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6)\}$, where $\nu = 0, 1, \ldots, 12$. Put

$$\alpha = \zeta + \zeta^{12} - \zeta^5 - \zeta^8, \quad \beta = \zeta^3 + \zeta^{10} - \zeta^2 - \zeta^{11}, \quad \gamma = \zeta^9 + \zeta^4 - \zeta^6 - \zeta^7.$$

We find that

$$
\begin{aligned}
&13ST^{\nu}(z_1)\cdot ST^{\nu}(z_4)\\
&= \beta z_1z_4+\gamma z_2z_5+\alpha z_3z_6+\\
&\ +\gamma\zeta^{\nu}z_1^2+\alpha\zeta^{9\nu}z_2^2+\beta\zeta^{3\nu}z_3^2-\gamma\zeta^{12\nu}z_4^2-\alpha\zeta^{4\nu}z_5^2-\beta\zeta^{10\nu}z_6^2+\\
&\ +(\alpha-\beta)\zeta^{5\nu}z_1z_2+(\beta-\gamma)\zeta^{6\nu}z_2z_3+(\gamma-\alpha)\zeta^{2\nu}z_1z_3+\\
&\ +(\beta-\alpha)\zeta^{8\nu}z_4z_5+(\gamma-\beta)\zeta^{7\nu}z_5z_6+(\alpha-\gamma)\zeta^{11\nu}z_4z_6+\\
&\ -(\alpha+\beta)\zeta^{\nu}z_3z_4-(\beta+\gamma)\zeta^{9\nu}z_1z_5-(\gamma+\alpha)\zeta^{3\nu}z_2z_6+\\
&\ -(\alpha+\beta)\zeta^{12\nu}z_1z_6-(\beta+\gamma)\zeta^{4\nu}z_2z_4-(\gamma+\alpha)\zeta^{10\nu}z_3z_5.
\end{aligned}
$$

$$
\begin{aligned}
&13ST^{\nu}(z_2)\cdot ST^{\nu}(z_5)\\
&= \gamma z_1z_4+\alpha z_2z_5+\beta z_3z_6+\\
&\ +\alpha\zeta^{\nu}z_1^2+\beta\zeta^{9\nu}z_2^2+\gamma\zeta^{3\nu}z_3^2-\alpha\zeta^{12\nu}z_4^2-\beta\zeta^{4\nu}z_5^2-\gamma\zeta^{10\nu}z_6^2+\\
&\ +(\beta-\gamma)\zeta^{5\nu}z_1z_2+(\gamma-\alpha)\zeta^{6\nu}z_2z_3+(\alpha-\beta)\zeta^{2\nu}z_1z_3+\\
&\ +(\gamma-\beta)\zeta^{8\nu}z_4z_5+(\alpha-\gamma)\zeta^{7\nu}z_5z_6+(\beta-\alpha)\zeta^{11\nu}z_4z_6+\\
&\ -(\beta+\gamma)\zeta^{\nu}z_3z_4-(\gamma+\alpha)\zeta^{9\nu}z_1z_5-(\alpha+\beta)\zeta^{3\nu}z_2z_6+\\
&\ -(\beta+\gamma)\zeta^{12\nu}z_1z_6-(\gamma+\alpha)\zeta^{4\nu}z_2z_4-(\alpha+\beta)\zeta^{10\nu}z_3z_5.
\end{aligned}
$$

$$
\begin{aligned}
&13ST^{\nu}(z_3)\cdot ST^{\nu}(z_6)\\
&= \alpha z_1z_4+\beta z_2z_5+\gamma z_3z_6+\\
&\ +\beta\zeta^{\nu}z_1^2+\gamma\zeta^{9\nu}z_2^2+\alpha\zeta^{3\nu}z_3^2-\beta\zeta^{12\nu}z_4^2-\gamma\zeta^{4\nu}z_5^2-\alpha\zeta^{10\nu}z_6^2+\\
&\ +(\gamma-\alpha)\zeta^{5\nu}z_1z_2+(\alpha-\beta)\zeta^{6\nu}z_2z_3+(\beta-\gamma)\zeta^{2\nu}z_1z_3+\\
&\ +(\alpha-\gamma)\zeta^{8\nu}z_4z_5+(\beta-\alpha)\zeta^{7\nu}z_5z_6+(\gamma-\beta)\zeta^{11\nu}z_4z_6+\\
&\ -(\gamma+\alpha)\zeta^{\nu}z_3z_4-(\alpha+\beta)\zeta^{9\nu}z_1z_5-(\beta+\gamma)\zeta^{3\nu}z_2z_6+\\
&\ -(\gamma+\alpha)\zeta^{12\nu}z_1z_6-(\alpha+\beta)\zeta^{4\nu}z_2z_4-(\beta+\gamma)\zeta^{10\nu}z_3z_5.
\end{aligned}
$$

Note that $\alpha+\beta+\gamma=\sqrt{13}$, we find that

$$
\begin{aligned}
&\sqrt{13}\left[ST^{\nu}(z_1)\cdot ST^{\nu}(z_4)+ST^{\nu}(z_2)\cdot ST^{\nu}(z_5)+ST^{\nu}(z_3)\cdot ST^{\nu}(z_6)\right]\\
&=(z_1z_4+z_2z_5+z_3z_6)+(\zeta^{\nu}z_1^2+\zeta^{9\nu}z_2^2+\zeta^{3\nu}z_3^2)+\\
&\quad -(\zeta^{12\nu}z_4^2+\zeta^{4\nu}z_5^2+\zeta^{10\nu}z_6^2)-2(\zeta^{\nu}z_3z_4+\zeta^{9\nu}z_1z_5+\zeta^{3\nu}z_2z_6)\\
&\quad -2(\zeta^{12\nu}z_1z_6+\zeta^{4\nu}z_2z_4+\zeta^{10\nu}z_3z_5).
\end{aligned}
$$

Let

$$\varphi_\infty(z_1,z_2,z_3,z_4,z_5,z_6)=\sqrt{13}(z_1z_4+z_2z_5+z_3z_6) \tag{3.1}$$

and

$$\varphi_\nu(z_1,z_2,z_3,z_4,z_5,z_6)=\varphi_\infty(ST^{\nu}(z_1,z_2,z_3,z_4,z_5,z_6)) \tag{3.2}$$

for $\nu = 0, 1, \ldots, 12$. Then

$$\begin{aligned}\varphi_\nu = & (z_1z_4 + z_2z_5 + z_3z_6) + \zeta^\nu(z_1^2 - 2z_3z_4) + \zeta^{4\nu}(-z_5^2 - 2z_2z_4) \\ & + \zeta^{9\nu}(z_2^2 - 2z_1z_5) + \zeta^{3\nu}(z_3^2 - 2z_2z_6) + \zeta^{12\nu}(-z_4^2 - 2z_1z_6) \\ & + \zeta^{10\nu}(-z_6^2 - 2z_3z_5).\end{aligned} \tag{3.3}$$

This leads us to define the senary quadratic forms (quadratic forms in six variables) $\mathbf{A}_0, \ldots, \mathbf{A}_6$ given in (1.18). Hence,

$$\begin{aligned}\sqrt{13}ST^\nu(\mathbf{A}_0) = & \mathbf{A}_0 + \zeta^\nu\mathbf{A}_1 + \zeta^{4\nu}\mathbf{A}_2 + \zeta^{9\nu}\mathbf{A}_3 + \zeta^{3\nu}\mathbf{A}_4 + \zeta^{12\nu}\mathbf{A}_5 \\ & + \zeta^{10\nu}\mathbf{A}_6.\end{aligned} \tag{3.4}$$

Let $H := Q^5P^2 \cdot P^2Q^6P^8 \cdot Q^5P^2 \cdot P^3Q$, where $P = ST^{-1}S$ and $Q = ST^3$. Then

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Note that $H^6 = -I$. In the projective coordinates, this means that $H^6 = 1$. We have that $H^{-1}TH = -T^4$. Thus, $\langle H, T\rangle$ is isomorphic to the semidirect product of $\mathbf{Z}_{13}$ by $\mathbf{Z}_6$. Hence, it is a maximal subgroup of order 78 of G with index 14 (see [6]). It should be pointed out that this complicated expression for H is chosen because it represents an element of $\Gamma(1)$ which is, in fact, an element of $\Gamma_0(13)$. In consequence, the group $\Gamma_0(13)/\Gamma(13)$ is generated. We find that φ_∞^2 is invariant under the action of the maximal subgroup $\langle H, T\rangle$. Note that

$$\varphi_\infty = \sqrt{13}\mathbf{A}_0, \quad \varphi_\nu = \mathbf{A}_0 + \zeta^\nu\mathbf{A}_1 + \zeta^{4\nu}\mathbf{A}_2 + \zeta^{9\nu}\mathbf{A}_3 + \zeta^{3\nu}\mathbf{A}_4 + \zeta^{12\nu}\mathbf{A}_5 + \zeta^{10\nu}\mathbf{A}_6$$

for $\nu = 0, 1, \ldots, 12$. Let $w = \varphi^2$, $w_\infty = \varphi_\infty^2$, and $w_\nu = \varphi_\nu^2$. Then w_∞, w_ν for $\nu = 0, \ldots, 12$ form an algebraic equation of degree 14, which is just the Jacobian equation of degree 14, whose roots are these w_ν and w_∞: $w^{14} + a_1w^{13} + \cdots + a_{13}w + a_{14} = 0$. In particular, the coefficients

$$-a_1 = w_\infty + \sum_{\nu=0}^{12} w_\nu = 26(\mathbf{A}_0^2 + \mathbf{A}_1\mathbf{A}_5 + \mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_4\mathbf{A}_6). \tag{3.6}$$

This leads to an invariant quadric $\Psi_2 := \mathbf{A}_0^2 + \mathbf{A}_1\mathbf{A}_5 + \mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_4\mathbf{A}_6 = 2\Phi_4(z_1, z_2, z_3, z_4, z_5, z_6)$, where

$$\begin{aligned}\Phi_4 := & (z_3z_4^3 + z_1z_5^3 + z_2z_6^3) - (z_6z_1^3 + z_4z_2^3 + z_5z_3^3) \\ & + 3(z_1z_2z_4z_5 + z_2z_3z_5z_6 + z_3z_1z_6z_4),\end{aligned} \tag{3.7}$$

Hence, the variety $\Psi_2 = 0$ is a quartic four-fold, which is invariant under the action of the simple group G.

Recall that the principal congruence subgroup of level 13 is the normal subgroup $\Gamma(13)$ of $\Gamma = PSL(2, \mathbf{Z})$ defined by the exact sequence $1 \to \Gamma(13) \to \Gamma(1) \xrightarrow{f} G \to 1$, where $f(\gamma) \equiv \gamma \pmod{13}$ for $\gamma \in \Gamma = \Gamma(1)$. Then there is a representation $\rho : \Gamma \to PGL(6, \mathbf{C})$ with kernel $\Gamma(13)$ and leaving Φ_4 invariant. It is defined as follows: if $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\rho(t) = T$ and $\rho(s) = S$. To see that such a representation exists, note that Γ is defined by the presentation $\langle s, t; s^2 = (st)^3 = 1 \rangle$ satisfied by s and t and we have proved that S and T satisfy these relations. Moreover, we have proved that G is defined by the presentation $\langle S, T; S^2 = T^{13} = (ST)^3 = 1, (Q^3 P^4)^3 = 1 \rangle$. Let $p = st^{-1}s$ and $q = st^3$. Then

$$h := q^5 p^2 \cdot p^2 q^6 p^8 \cdot q^5 p^2 \cdot p^3 q = \begin{pmatrix} 4,428,249 & -10,547,030 \\ -11,594,791 & 27,616,019 \end{pmatrix}$$

satisfies that $\rho(h) = H$. The off-diagonal elements of the matrix h, which corresponds to H, are congruent to 0 mod 13. The connection to $\Gamma_0(13)$ should be obvious.

Theorem 3.1 *There is a relation between the invariant quartic four-fold* $\Phi_4(z_1, \ldots, z_6) = 0$ *and theta constants of order* $13 : \Phi_4(a_1(z), \ldots, a_6(z)) = 0$.

Proof Let $y_i(z) = \eta^3(z) a_i(z)$ $(1 \leq i \leq 6)$ and $Y(z) := (y_1(z), \ldots, y_6(z))^T$. Then $Y(z) = \eta^3(z)\mathbf{A}(z)$. Recall that $\eta(z)$ satisfies the following transformation formulas $\eta(z+1) = e^{\frac{\pi i}{12}}\eta(z)$ and $\eta\left(-\frac{1}{z}\right) = e^{-\frac{\pi i}{4}}\sqrt{z}\eta(z)$. By Proposition 2.4, we have that $Y(z+1) = e^{-\frac{\pi i}{2}}\rho(t)Y(z)$ and $Y\left(-\frac{1}{z}\right) = e^{-\frac{\pi i}{2}} z^2 \rho(s) Y(z)$. Define $j(\gamma, z) := cz + d$ if $z \in \mathbf{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Hence, $Y(\gamma(z)) = v(\gamma) j(\gamma, z)^2 \rho(\gamma) Y(z)$ for $\gamma \in \Gamma(1)$, where $v(\gamma) = \pm 1$ or $\pm i$. Since $\Gamma(13) = \ker \rho$, we have that $Y(\gamma(z)) = v(\gamma) j(\gamma, z)^2 Y(z)$ for $\gamma \in \Gamma(13)$. This means that the functions $y_1(z), \ldots, y_6(z)$ are modular forms of weight 2 for $\Gamma(13)$ with the same multiplier $v(\gamma) = \pm 1$ or $\pm i$. Thus,

$$\Phi_4(Y(\gamma(z))) = v(\gamma)^4 j(\gamma, z)^8 \Phi_4(Y(z)) = j(\gamma, z)^8 \Phi_4(Y(z)) \quad \text{for } \gamma \in \Gamma(13).$$

Moreover, for $\gamma \in \Gamma(1)$,

$$\begin{aligned} \Phi_4(Y(\gamma(z))) &= \Phi_4(v(\gamma) j(\gamma, z)^2 \rho(\gamma) Y(z)) \\ &= v(\gamma)^4 j(\gamma, z)^8 \Phi_4(\rho(\gamma) Y(z)) = j(\gamma, z)^8 \Phi_4(\rho(\gamma) Y(z)). \end{aligned}$$

Note that $\rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G$ and Φ_4 is a G-invariant polynomial, we have that

$$\Phi_4(Y(\gamma(z))) = j(\gamma, z)^8 \Phi_4(Y(z)), \quad \text{for } \gamma \in \Gamma(1).$$

This implies that $\Phi_4(Y(z))$ is a modular form of weight 8 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. A straightforward calculation gives that

$$\Phi_4(a_1(z), \ldots, a_6(z)) = q^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n q^n, \quad \text{where } a_n \in \mathbf{Z}.$$

On the other hand, $\eta(z)^{12} = q^{\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)^{12}$. We have that

$$\Phi_4(y_1(z), \ldots, y_6(z)) = q \sum_{n=0}^{\infty} a_n q^n \prod_{n=1}^{\infty}(1-q^n)^{12}$$

is a cusp form of weight 8 for the full modular group $\Gamma = PSL(2, \mathbf{Z})$, but the only such form is zero. This completes the proof of Theorem 3.1. □

Corollary 3.2 *The following invariant decomposition formula* (*exotic modular equation*) *holds*: $\Psi_2(a_1(z), \ldots, a_6(z))/\mathbf{A}_0(a_1(z), \ldots, a_6(z))^2 = 0$, *where the quadric* Ψ_2 *is an invariant associated to* $PSL(2, 13)$ *and* $\mathbf{A}_0^2$ *is invariant under the action of the image of a Borel subgroup of* $PSL(2, 13)$, *i.e., a maximal subgroup of order* 78 *of* $PSL(2, 13)$, *which can be viewed as* $\Gamma_0(13)/\Gamma(13)$.

Proof This comes from Theorem 3.1 by noting that (3.6) and (3.7). Thus, we complete the proof of Theorem 1.3.

Let $\mathbf{A}_j(a(z)) := \mathbf{A}_j(a_1(z), \ldots, a_6(z))$ for $i = 0, 1, \ldots, 6$. We will show that

$$\prod_{j=1}^{6} \mathbf{A}_j(a(z)) \neq -\mathbf{A}_0(a(z))^6. \tag{3.8}$$

We have that $\mathbf{A}_0(a(z)) = q^{\frac{1}{4}}(1+O(q))$, $\mathbf{A}_1(a(z)) = q^{\frac{34}{104}}(2+O(q))$, $\mathbf{A}_2(a(z)) = q^{\frac{58}{104}}(2+O(q))$, $\mathbf{A}_3(a(z)) = q^{\frac{98}{104}}(1+O(q))$, $\mathbf{A}_4(a(z)) = q^{\frac{50}{104}}(-1+O(q))$, $\mathbf{A}_5(a(z)) = q^{\frac{18}{104}}(-1+O(q))$, and $\mathbf{A}_6(a(z)) = q^{\frac{2}{104}}(-1+O(q))$. Thus,

$$\prod_{j=1}^{6} \mathbf{A}_j(a(z)) = q^{\frac{5}{2}}(-4+O(q)).$$

On the other hand, $-\mathbf{A}_0(a(z))^6 = q^{\frac{3}{2}}(-1+O(q))$. This gives the proof of (3.8). □

Corollary 3.3 *The coordinates* $(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))$ *map* $X(13)$ *into the quartic four-fold* $\Phi_4(z_1, z_2, z_3, z_4, z_5, z_6) = 0$ *in* $\mathbf{CP}^5$.

Proof This comes from Theorems 2.2 and 3.1. □

4 Fourteen-dimensional representations of $PSL(2, 13)$ and invariant decomposition formula

We will construct a 14-dimensional representation of $PSL(2, 13)$ which is deduced from our six-dimensional representation. It should be emphasized that our 14-dimensional representation is not a Weil representation. In contrast with this, both our six-dimensional and seven-dimensional representations of G are Weil representations, i.e., $\frac{p-1}{2}$-dimensional and $\frac{p+1}{2}$-dimensional representations of $PSL(2, p)$, respectively. In fact, what Klein used in his papers [12–15] are all Weil representations. Hence, our method is completely different from Klein's method.

To construct our 14-dimensional representation which is generated under the action of PSL by a specific vector in Sym^3(six-dimensional representation), we begin with a cubic polynomial $z_1 z_2 z_3$ and study the action of ST^{ν} (ν mod 13) on it. We have that

$$
\begin{aligned}
&-13\sqrt{13} ST^{\nu}(z_1) \cdot ST^{\nu}(z_2) \cdot ST^{\nu}(z_3)\\
&= -\sqrt{\frac{-13-3\sqrt{13}}{2}}(\zeta^{8\nu} z_1^3 + \zeta^{7\nu} z_2^3 + \zeta^{11\nu} z_3^3) +\\
&\quad -\sqrt{\frac{-13+3\sqrt{13}}{2}}(\zeta^{5\nu} z_4^3 + \zeta^{6\nu} z_5^3 + \zeta^{2\nu} z_6^3) +\\
&\quad -\sqrt{-13+2\sqrt{13}}(\zeta^{12\nu} z_1^2 z_2 + \zeta^{4\nu} z_2^2 z_3 + \zeta^{10\nu} z_3^2 z_1) +\\
&\quad -\sqrt{-13-2\sqrt{13}}(\zeta^{\nu} z_4^2 z_5 + \zeta^{9\nu} z_5^2 z_6 + \zeta^{3\nu} z_6^2 z_4) +\\
&\quad +2\sqrt{-13-2\sqrt{13}}(\zeta^{3\nu} z_1 z_2^2 + \zeta^{\nu} z_2 z_3^2 + \zeta^{9\nu} z_3 z_1^2) +\\
&\quad -2\sqrt{-13+2\sqrt{13}}(\zeta^{10\nu} z_4 z_5^2 + \zeta^{12\nu} z_5 z_6^2 + \zeta^{4\nu} z_6 z_4^2) +\\
&\quad +2\sqrt{\frac{-13-3\sqrt{13}}{2}}(\zeta^{7\nu} z_1^2 z_4 + \zeta^{11\nu} z_2^2 z_5 + \zeta^{8\nu} z_3^2 z_6) +\\
&\quad -2\sqrt{\frac{-13+3\sqrt{13}}{2}}(\zeta^{6\nu} z_1 z_4^2 + \zeta^{2\nu} z_2 z_5^2 + \zeta^{5\nu} z_3 z_6^2) +\\
&\quad +\sqrt{-13-2\sqrt{13}}(\zeta^{3\nu} z_1^2 z_5 + \zeta^{\nu} z_2^2 z_6 + \zeta^{9\nu} z_3^2 z_4) +\\
&\quad +\sqrt{-13+2\sqrt{13}}(\zeta^{10\nu} z_2 z_4^2 + \zeta^{12\nu} z_3 z_5^2 + \zeta^{4\nu} z_1 z_6^2) +\\
&\quad +\sqrt{\frac{-13+3\sqrt{13}}{2}}(\zeta^{6\nu} z_1^2 z_6 + \zeta^{2\nu} z_2^2 z_4 + \zeta^{5\nu} z_3^2 z_5) +\\
&\quad +\sqrt{\frac{-13-3\sqrt{13}}{2}}(\zeta^{7\nu} z_3 z_4^2 + \zeta^{11\nu} z_1 z_5^2 + \zeta^{8\nu} z_2 z_6^2) +\\
&\quad +[2(\theta_1-\theta_3)-3(\theta_2-\theta_4)] z_1 z_2 z_3 +
\end{aligned}
$$

$$
\begin{aligned}
&+[2(\theta_4-\theta_2)-3(\theta_1-\theta_3)]z_4z_5z_6+\\
&-\sqrt{\frac{-13-3\sqrt{13}}{2}}(\zeta^{11\nu}z_1z_2z_4+\zeta^{8\nu}z_2z_3z_5+\zeta^{7\nu}z_1z_3z_6)+\\
&+\sqrt{\frac{-13+3\sqrt{13}}{2}}(\zeta^{2\nu}z_1z_4z_5+\zeta^{5\nu}z_2z_5z_6+\zeta^{6\nu}z_3z_4z_6)+\\
&-3\sqrt{\frac{-13-3\sqrt{13}}{2}}(\zeta^{7\nu}z_1z_2z_5+\zeta^{11\nu}z_2z_3z_6+\zeta^{8\nu}z_1z_3z_4)+\\
&+3\sqrt{\frac{-13+3\sqrt{13}}{2}}(\zeta^{6\nu}z_2z_4z_5+\zeta^{2\nu}z_3z_5z_6+\zeta^{5\nu}z_1z_4z_6)+\\
&-\sqrt{-13+2\sqrt{13}}(\zeta^{10\nu}z_1z_2z_6+\zeta^{4\nu}z_1z_3z_5+\zeta^{12\nu}z_2z_3z_4)+\\
&+\sqrt{-13-2\sqrt{13}}(\zeta^{3\nu}z_3z_4z_5+\zeta^{9\nu}z_2z_4z_6+\zeta^{\nu}z_1z_5z_6).
\end{aligned}
$$

This leads us to define the senary cubic forms (cubic forms in six variables) $\mathbf{D}_0, \ldots, \mathbf{D}_{12}, \mathbf{D}_\infty$ given in (1.11). Let $r_0 = 2(\theta_1-\theta_3)-3(\theta_2-\theta_4)$, $r_\infty = 2(\theta_4-\theta_2)-3(\theta_1-\theta_3)$, $r_1 = \sqrt{-13-2\sqrt{13}}$, $r_2 = \sqrt{\frac{-13+3\sqrt{13}}{2}}$, $r_3 = \sqrt{-13+2\sqrt{13}}$, and $r_4 = \sqrt{\frac{-13-3\sqrt{13}}{2}}$. We have that

$$
\begin{aligned}
-13\sqrt{13}ST^\nu(\mathbf{D}_0) = {}& r_0\mathbf{D}_0 + r_1\zeta^\nu\mathbf{D}_1 + r_2\zeta^{2\nu}\mathbf{D}_2 + r_1\zeta^{3\nu}\mathbf{D}_3 + r_3\zeta^{4\nu}\mathbf{D}_4 +\\
&+ r_2\zeta^{5\nu}\mathbf{D}_5 + r_2\zeta^{6\nu}\mathbf{D}_6 + r_4\zeta^{7\nu}\mathbf{D}_7 + r_4\zeta^{8\nu}\mathbf{D}_8 + r_1\zeta^{9\nu}\mathbf{D}_9 +\\
&+ r_3\zeta^{10\nu}\mathbf{D}_{10} + r_4\zeta^{11\nu}\mathbf{D}_{11} + r_3\zeta^{12\nu}\mathbf{D}_{12} + r_\infty\mathbf{D}_\infty.
\end{aligned}
$$

$$
\begin{aligned}
-13\sqrt{13}ST^\nu(\mathbf{D}_\infty) = {}& r_\infty\mathbf{D}_0 - r_3\zeta^\nu\mathbf{D}_1 - r_4\zeta^{2\nu}\mathbf{D}_2 - r_3\zeta^{3\nu}\mathbf{D}_3 + r_1\zeta^{4\nu}\mathbf{D}_4 +\\
&- r_4\zeta^{5\nu}\mathbf{D}_5 - r_4\zeta^{6\nu}\mathbf{D}_6 + r_2\zeta^{7\nu}\mathbf{D}_7 + r_2\zeta^{8\nu}\mathbf{D}_8 - r_3\zeta^{9\nu}\mathbf{D}_9 +\\
&+ r_1\zeta^{10\nu}\mathbf{D}_{10} + r_2\zeta^{11\nu}\mathbf{D}_{11} + r_1\zeta^{12\nu}\mathbf{D}_{12} - r_0\mathbf{D}_\infty.
\end{aligned}
$$

Let

$$\delta_\infty(z_1, z_2, z_3, z_4, z_5, z_6) = 13^2(\mathbf{D}_0^2 + \mathbf{D}_\infty^2) \tag{4.1}$$

and

$$\delta_\nu(z_1, z_2, z_3, z_4, z_5, z_6) = \delta_\infty(ST^\nu(z_1, z_2, z_3, z_4, z_5, z_6)) \tag{4.2}$$

for $\nu = 0, 1, \ldots, 12$. Then

$$\delta_\nu = 13^2 ST^\nu(\mathbf{G}_0) = -13\mathbf{G}_0 + \zeta^\nu\mathbf{G}_1 + \zeta^{2\nu}\mathbf{G}_2 + \cdots + \zeta^{12\nu}\mathbf{G}_{12}, \tag{4.3}$$

where the senary sextic forms (i.e., sextic forms in six variables) $\mathbf{G}_0, \ldots, \mathbf{G}_{12}$ are given in (1.10). We have that $\mathbf{G}_0$ is invariant under the action of $\langle H, T \rangle$, a maximal subgroup of order 78 of G with index 14. Note that $\delta_\infty, \delta_\nu$ for $\nu = 0, \ldots, 12$ form an algebraic equation of degree 14. However, we have that $\delta_\infty + \sum_{\nu=0}^{12} \delta_\nu = 0$. Hence, it is not the Jacobian equation of degree 14. We call it exotic modular equation of degree 14. We have that

$$\delta_\infty^2 + \sum_{\nu=0}^{12} \delta_\nu^2 = 26(7 \cdot 13^2 \mathbf{G}_0^2 + \mathbf{G}_1\mathbf{G}_{12} + \mathbf{G}_2\mathbf{G}_{11} + \cdots + \mathbf{G}_6\mathbf{G}_7). \tag{4.4}$$

This leads us to define Φ_{12} as in (1.9).

Theorem 4.1 *The invariant decomposition formula for the simple group* $PSL(2, 13)$ *of order* 1092 *is given as follows:*

$$\left[\frac{\eta^2(z)}{\eta^2(13z)}\right]^5 = \frac{\Phi_{12}(a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))}{(a_1(z)a_2(z)a_3(z)a_4(z)a_5(z)a_6(z))^2}, \tag{4.5}$$

where $\Phi_{12}(z_1, z_2, z_3, z_4, z_5, z_6)$ *is an invariant of degree 12 associated to* $PSL(2, 13)$*, and* $(z_1z_2z_3z_4z_5z_6)^2$ *is invariant under the action of the image of a Borel subgroup of* $PSL(2, 13)$*, i.e., a maximal subgroup of order* 78 *of* $PSL(2, 13)$*, which can be viewed as* $\Gamma_0(13)/\Gamma(13)$.

Proof Let $x_i(z) = \eta(z)a_i(z)$ $(1 \leq i \leq 6)$ and $X(z) = (x_1(z), \ldots, x_6(z))^T$. Then $X(z) = \eta(z)\mathbf{A}(z)$. Recall that $\eta(z)$ satisfies the following transformation formulas $\eta(z+1) = e^{\frac{\pi i}{12}}\eta(z)$ and $\eta\left(-\frac{1}{z}\right) = e^{-\frac{\pi i}{4}}\sqrt{z}\eta(z)$. By Proposition 2.4, we have that $X(z+1) = e^{-\frac{2\pi i}{3}}\rho(t)X(z)$ and $X\left(-\frac{1}{z}\right) = z\rho(s)X(z)$. Hence, $X(\gamma(z)) = u(\gamma)j(\gamma, z)\rho(\gamma)X(z)$ for $\gamma \in \Gamma(1)$, where $u(\gamma) = 1, \omega$ or ω^2 with $\omega = e^{\frac{2\pi i}{3}}$. Since $\Gamma(13) = \ker \rho$, we have that $X(\gamma(z)) = u(\gamma)j(\gamma, z)X(z)$ for $\gamma \in \Gamma(13)$. This means that the functions $x_1(z), \ldots, x_6(z)$ are modular forms of weight 1 for $\Gamma(13)$ with the same multiplier $u(\gamma) = 1, \omega$ or ω^2. Thus,

$$\Phi_{12}(X(\gamma(z))) = u(\gamma)^{12} j(\gamma, z)^{12} \Phi_{12}(X(z)) = j(\gamma, z)^{12} \Phi_{12}(X(z))$$

for $\gamma \in \Gamma(13)$. Moreover, for $\gamma \in \Gamma(1)$,

$$\begin{aligned}\Phi_{12}(X(\gamma(z))) &= \Phi_{12}(u(\gamma)j(\gamma, z)\rho(\gamma)X(z)) \\ &= u(\gamma)^{12} j(\gamma, z)^{12} \Phi_{12}(\rho(\gamma)X(z)) = j(\gamma, z)^{12} \Phi_{12}(\rho(\gamma)X(z)).\end{aligned}$$

Note that $\rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G$ and Φ_{12} is a G-invariant polynomial, we have that

$$\Phi_{12}(X(\gamma(z))) = j(\gamma, z)^{12} \Phi_{12}(X(z)), \quad \gamma \in \Gamma(1).$$

This implies that $\Phi_{12}(X(z))$ is a modular form of weight 12 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. We have that

$$\begin{cases} \mathbf{D}_0 &= q^{\frac{15}{8}}(1+O(q)), \\ \mathbf{D}_\infty &= q^{\frac{7}{8}}(-1+O(q)), \\ \mathbf{D}_1 &= q^{\frac{99}{104}}(2+O(q)), \\ \mathbf{D}_2 &= q^{\frac{3}{104}}(-1+O(q)), \\ \mathbf{D}_3 &= q^{\frac{11}{104}}(1+O(q)), \\ \mathbf{D}_4 &= q^{\frac{19}{104}}(-2+O(q)), \\ \mathbf{D}_5 &= q^{\frac{27}{104}}(-1+O(q)), \end{cases} \qquad \begin{cases} \mathbf{D}_6 &= q^{\frac{35}{104}}(-1+O(q)), \\ \mathbf{D}_7 &= q^{\frac{43}{104}}(1+O(q)), \\ \mathbf{D}_8 &= q^{\frac{51}{104}}(3+O(q)), \\ \mathbf{D}_9 &= q^{\frac{59}{104}}(-2+O(q)), \\ \mathbf{D}_{10} &= q^{\frac{67}{104}}(1+O(q)), \\ \mathbf{D}_{11} &= q^{\frac{75}{104}}(-4+O(q)), \\ \mathbf{D}_{12} &= q^{\frac{83}{104}}(-1+O(q)). \end{cases}$$

Hence,

$$\begin{cases} \mathbf{G}_0 = q^{\frac{7}{4}}(1+O(q)), \\ \mathbf{G}_1 = q^{\frac{86}{104}}(13+O(q)), \\ \mathbf{G}_2 = q^{\frac{94}{104}}(-22+O(q)), \\ \mathbf{G}_3 = q^{\frac{102}{104}}(-21+O(q)), \\ \mathbf{G}_4 = q^{\frac{6}{104}}(-1+O(q)), \\ \mathbf{G}_5 = q^{\frac{14}{104}}(2+O(q)), \\ \mathbf{G}_6 = q^{\frac{22}{104}}(2+O(q)), \end{cases} \qquad \begin{cases} \mathbf{G}_7 &= q^{\frac{30}{104}}(-2+O(q)), \\ \mathbf{G}_8 &= q^{\frac{38}{104}}(-8+O(q)), \\ \mathbf{G}_9 &= q^{\frac{46}{104}}(6+O(q)), \\ \mathbf{G}_{10} &= q^{\frac{54}{104}}(1+O(q)), \\ \mathbf{G}_{11} &= q^{\frac{62}{104}}(-8+O(q)), \\ \mathbf{G}_{12} &= q^{\frac{70}{104}}(17+O(q)). \end{cases}$$

Therefore,

$$\begin{aligned} &7\cdot 13^2\mathbf{G}_0^2+\mathbf{G}_1\mathbf{G}_{12}+\mathbf{G}_2\mathbf{G}_{11}+\cdots+\mathbf{G}_6\mathbf{G}_7 \\ &\quad = 7\cdot 13^2 q^{\frac{7}{2}}(1+O(q))+q^{\frac{3}{2}}(13\cdot 17+O(q))+q^{\frac{3}{2}}(22\cdot 8+O(q))+ \\ &\qquad + q^{\frac{3}{2}}(-21+O(q))+q^{\frac{1}{2}}(-6+O(q))+q^{\frac{1}{2}}(-16+O(q))+ \\ &\qquad + q^{\frac{1}{2}}(-4+O(q)) \\ &\quad = q^{\frac{1}{2}}(-26+O(q)). \end{aligned}$$

We have that

$$\Phi_{12}(x_1(z),\ldots,x_6(z)) = \eta(z)^{12}q^{\frac{1}{2}}(1+O(q)) = q(1+O(q))$$

is a cusp form of weight 12 for the full modular group $\Gamma(1)$. Because every $\Gamma(1)$ cusp form of weight 12 is a multiple of $\Delta(z)$, checking the q^1 coefficient, we find that $\Phi_{12}(x_1(z),\ldots,x_6(z)) = \Delta(z)$. On the other hand, we have that $x_1(z)\ldots x_6(z) = -\eta(z)^7\eta(13z)^5$. This completes the proof of Theorem 4.1. □

Note that (2.9), (2.10), and (2.11) are equivalent to the following:

$$a_1(z)^2a_4(z)^2a_2(z)a_5(z) + a_2(z)^2a_5(z)^2a_3(z)a_6(z) + \\ + a_3(z)^2a_6(z)^2a_1(z)a_4(z) - a_1(z)a_4(z)a_2(z)a_5(z)a_3(z)a_6(z) \\ = -\eta(z)^3\eta(13z)^3, \tag{4.6}$$

$$a_1(z)^2a_4(z)^2a_3(z)a_6(z) + a_2(z)^2a_5(z)^2a_1(z)a_4(z) + \\ +a_3(z)^2a_6(z)^2a_2(z)a_5(z) + 4a_1(z)a_4(z)a_2(z)a_5(z)a_3(z)a_6(z) \\ = \eta(z)^3\eta(13z)^3, \tag{4.7}$$

$$a_4(z)^2a_5(z)^2a_6(z)^2 - a_1(z)^2a_2(z)^2a_3(z)^2 + \\ + 3a_1(z)a_4(z)a_2(z)a_5(z)a_3(z)a_6(z) = \eta(z)^3\eta(13z)^3. \tag{4.8}$$

The corresponding polynomials are given by

$$f_6(z_1, z_2, z_3, z_4, z_5, z_6) := z_1^2z_4^2z_2z_5 + z_2^2z_5^2z_3z_6 + z_3^2z_6^2z_1z_4 - z_1z_2z_3z_4z_5z_6,$$

$$g_6(z_1, z_2, z_3, z_4, z_5, z_6) := z_1^2z_4^2z_3z_6 + z_2^2z_5^2z_1z_4 + z_3^2z_6^2z_2z_5 + 4z_1z_2z_3z_4z_5z_6,$$

$$h_6(z_1, z_2, z_3, z_4, z_5, z_6) := z_4^2z_5^2z_6^2 - z_1^2z_2^2z_3^2 + 3z_1z_2z_3z_4z_5z_6.$$

Similarly to the above argument in the proof of Theorem 4.1, we have that

$$f_6(X(\gamma(z))) = f_6(u(\gamma)j(\gamma, z)\rho(\gamma)X(z)) = u(\gamma)^6j(\gamma, z)^6f_6(\rho(\gamma)X(z)) \\ = j(\gamma, z)^6f_6(\rho(\gamma)X(z)) \quad \text{for } \gamma \in \Gamma(1).$$

If f_6 is a G-invariant polynomial, we have that

$$f_6(X(\gamma(z))) = j(\gamma, z)^6f_6(X(z)) \quad \text{for } \gamma \in \Gamma(1).$$

This implies that $f_6(X(z))$ is a modular form of weight 6 for the full modular group $\Gamma(1)$. On the other hand, by (4.6), we have that

$$f_6(X(z)) = -\eta(z)^9\eta(13z)^3 = -q^2\prod_{n=1}^{\infty}(1-q^n)^9\prod_{n=1}^{\infty}(1-q^{13n})^3.$$

This shows that $f_6(X(z))$ is a cusp form of weight 6 for the full modular group $\Gamma(1)$, but the only such form is zero. This leads to a contradiction! Therefore, f_6 is not a G-invariant polynomial. Similarly, we can prove that g_6 and h_6 are not G-invariant polynomials.

Acknowledgements The author thanks Pierre Deligne for his comments on an earlier version of this paper. The author thanks George Andrews and Bruce Berndt for their encouragement on the present paper. In particular, the author thanks the referee very much for his helpful comments.

References

1. Andrews, G.E., Berndt, B.C.: Ramanujan's Lost Notebook, Part I. Springer, New York (2005)
2. Berndt, B.C.: Ramanujan's Notebooks, Part III. Springer, New York (1991)
3. Berndt, B.C.: Ramanujan's Notebooks, Part IV. Springer, New York (1994)
4. Berndt, B.C., Ono, K.: Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary. In: The Andrews Festschrift, Seventeen Papers on Classical Number Theory and Combinatorics, pp. 39–110. Springer, Berlin (2001)
5. Chen, I., Yui, N.: Singular values of Thompson series, In: Groups, difference sets, and the Monster (Columbus, OH, 1993), pp. 255–326. Ohio State Univ. Math. Res. Inst. Publ., vol. 4. de Gruyter, Berlin (1996)
6. Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of Finite Groups, Maximal Subgroups and Ordinary Characters for Simple Groups. Clarendon Press, Oxford (1985)
7. Duke, W.: Continued fractions and modular functions. Bull. Am. Math. Soc. (N.S.) **42**, 137–162 (2005)
8. Elkies, N.D.: The Klein quartic in number theory. In: The Eightfold Way, The Beauty of Klein's Quartic Curve, pp. 51–101, Math. Sci. Res. Inst. Publ., vol. 35. Cambridge University Press, Cambridge (1999)
9. Evans, R.J.: Theta function identities. J. Math. Anal. Appl. **147**, 97–121 (1990)
10. Farkas, H.M., Kra, I.: Theta Constants, Riemann Surfaces and the Modular Group. An Introduction with Applications to Uniformization Theorems, Partition Identities and Combinatorial Number Theory. Graduate Studies in Mathematics, vol. 37. American Mathematical Society, Providence (2001)
11. Gross, B.H.: Heegner points and the modular curve of prime level. J. Math. Soc. Jpn. **39**, 345–362 (1987)
12. Klein, F.: Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree. Translated by G. G. Morrice, second and revised edition. Dover, New York (1956)
13. Klein, F.: Ueber die Transformation der elliptischen Funktionen und die Auflösung der Gleichungen fünften Grades, Math. Ann. 14, 111–172 (1879). Gesammelte Mathematische Abhandlungen, Bd. III, pp. 13–75. Springer, Berlin (1923)
14. Klein, F.: Ueber die Transformation siebenter Ordnung der elliptischen Funktionen, Math. Ann. 14, 428–471 (1879). Gesammelte Mathematische Abhandlungen, Bd. III, pp. 90–136. Springer, Berlin (1923)
15. Klein, F.: Ueber die Transformation elfter Ordnung der elliptischen Funktionen, Math. Ann. 15, 533–555 (1879). Gesammelte Mathematische Abhandlungen, Bd. III, pp. 140–168. Springer, Berlin (1923)
16. Klein, F., Fricke, R.: Vorlesungen über die Theorie der Elliptischen Modulfunctionen, Bd. I. Teubner, Leipzig (1890)
17. Klein, F., Fricke, R.: Vorlesungen über die Theorie der Elliptischen Modulfunctionen, Bd. II. Teubner, Leipzig (1892)
18. Lachaud, G.: Ramanujan modular forms and the Klein quartic. Mosc. Math. J. **5**, 829–856 (2005)
19. Rademacher, H.: The Ramanujan identities under modular substitutions. Trans. Am. Math. Soc. **51**, 609–636 (1942)
20. Ramanujan, S.: Some properties of $p(n)$, Proc. Cambridge Phil. Soc. 19, 207–210 (1919). In: Collected Papers of Srinivasa Ramanujan, 210–213. Cambridge (1927)
21. Ramanujan, S.: Congruence properties of partitions, Proc. London Math. Soc. 18, 19 (1920). In: Collected Papers of Srinivasa Ramanujan, 230. Cambridge (1927)
22. Ramanujan, S.: Congruence properties of partitions, Math. Z. 9, 147–153 (1921). In: Collected Papers of Srinivasa Ramanujan, 232–238. Cambridge (1927)
23. Ramanujan, S.: Algebraic relations between certain infinite products, Proc. London Math. Soc. 18, (1920). In: Collected Papers of Srinivasa Ramanujan, 231, Cambridge (1927)
24. Ramanujan, S.: Notebooks of Srinivasa Ramanujan, vol. I. Springer, Berlin (1984)
25. Ramanujan, S.: Notebooks of Srinivasa Ramanujan, vol. II. Springer, Berlin (1984)
26. Ramanujan, S.: The Lost Notebook and Other Unpublished Papers. Narosa, New Delhi (1988)
27. Yang, L.: Exotic arithmetic structure on the first Hurwitz triplet, arXiv:1209.1783v5 [math.NT] (2013)
28. Yang, L.: Icosahedron, exceptional singularities and modular forms, arXiv:1511.05278 [math.NT] (2015)
29. Zuckerman, H.S.: Identities analogous to Ramanujan's identities involving the partition function. Duke Math. J. **5**, 88–110 (1939)