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Overpartition function modulo powers of 2

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Abstract Let $\overline{p}(n)$ denote the number of overpartitions of n. Recently, congruences modulo powers of 2 for $\overline{p}(n)$ were widely studied. In this paper, we prove several new infinite families of congruences modulo powers of 2 for $\overline{p}(n)$. For example, for $\alpha \geq 1$ and $n \geq 0$,

$$\overline{p}(8 \cdot 3^{4\alpha+4}n + 5 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}.$$

Keywords Partition · Overpartition · Congruence

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1 Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. An overpartition of n is a partition of n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n, and we assume that $\overline{p}(0) = 1$. From [6], we know that the generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{1.1}$$

where

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}), \quad |q| < 1.$$

The arithmetic properties of $\overline{p}(n)$ were widely studied in the literature. Fortin, Jacob and Mathieu [9], and Hirschhorn and Sellers [10] established the 2-, 3-, and 4-dissections of the generating function for $\overline{p}(n)$, from which some congruences modulo 4 and 8 are obtained. In particular, they obtained the following three Ramanujan type identities:

$$\sum_{n=0}^{\infty} \overline{p}(2n+1)q^n = 2\frac{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2}{(q;q)_\infty^4 (q^4;q^4)_\infty}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \overline{p}(4n+3)q^n = 8\frac{(q^2;q^2)_\infty (q^4;q^4)_\infty^6}{(q;q)_\infty^8}, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \overline{p}(8n+7)q^n = 64\frac{(q^2;q^2)_\infty^{22}}{(q;q)_\infty^{23}}. \tag{1.4}$$

Hirschhorn and Sellers [10] also conjectured that if p is an odd prime and r is a quadratic nonresidue modulo p,

$$\overline{p}(pn+r) \equiv \begin{cases} 0 \pmod{4} & \text{if } p \equiv \pm 3 \pmod{8}, \\ 0 \pmod{8} & \text{if } p \equiv \pm 1 \pmod{8}. \end{cases}$$

The above conjecture later was confirmed by Kim [14]. Mahlburg [16] conjectured that for all positive integers k, $\overline{p}(n) \equiv 0 \pmod{2^k}$ holds for a set of integers of arithmetic density 1 and proved the case $k = 6$. In [13], Kim confirmed the case $k = 7$, and the conjecture is still open. Recently, Ramanujan type congruences modulo 16 and 32 have been considered by several authors, see [5,20,21], for example. For congruences modulo 5 for $\overline{p}(n)$, we refer the reader to [3,4,8,15,17,18]. For modulo powers of 3, see [11,19,21].

The aim of this paper is to derive infinite families of congruences for $\overline{p}(n)$ modulo powers of 2. Here we list our main results in the following theorems.

Theorem 1.1 *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\overline{p}(8 \cdot 5^{8\alpha+2}n + 31 \cdot 5^{8\alpha+1}) \equiv 0 \pmod{2^5}, \tag{1.5}$$
$$\overline{p}(8 \cdot 5^{8\alpha+2}n + 39 \cdot 5^{8\alpha+1}) \equiv 0 \pmod{2^5}, \tag{1.6}$$
$$\overline{p}(8 \cdot 5^{8\alpha+4}n + 31 \cdot 5^{8\alpha+3}) \equiv 0 \pmod{2^6}, \tag{1.7}$$
$$\overline{p}(8 \cdot 5^{8\alpha+4}n + 39 \cdot 5^{8\alpha+3}) \equiv 0 \pmod{2^6}, \tag{1.8}$$
$$\overline{p}\left(8 \cdot 5^{8\alpha+8}n + (8i+7) \cdot 5^{8\alpha+7}\right) \equiv 0 \pmod{2^6}, \tag{1.9}$$

where $i = 0, 2, 3, 4$.

Theorem 1.2 *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\overline{p}(8 \cdot 7^{4\alpha+2}n + (8i+5)7^{4\alpha+1}) \equiv 0 \pmod{2^5}, \tag{1.10}$$

where $i = 3, 4, 6$, and

$$\overline{p}(8 \cdot 7^{4\alpha+4}n + (8i+5)7^{4\alpha+3}) \equiv 0 \pmod{2^5}, \tag{1.11}$$

where $i = 0, 1, 3, 4, 5, 6$.

Theorem 1.3 *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\overline{p}(8 \cdot 3^{4\alpha+4}n + 5 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}, \tag{1.12}$$
$$\overline{p}(8 \cdot 3^{4\alpha+4}n + 13 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}. \tag{1.13}$$

2 Preliminaries

Let $f(a, b)$ be Ramanujan's general theta function given by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows:

$$f(a, b) = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty.$$

Thus,

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}.$$

In order to prove our results, we need the following lemmas.

Lemma 2.1 [7] *For any odd prime p,*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Furthermore, we claim that for $0\le k\le (p-3)/2$,

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod{p}.$$

In particular, setting $p=5,7$ in Lemma 2.1, we have

$$\psi(q)=f(q^{10},q^{15})+qf(q^5,q^{20})+q^3\psi(q^{25}), \tag{2.1}$$
$$\psi(q)=f(q^{21},q^{28})+qf(q^{14},q^{35})+q^3f(q^7,q^{42})+q^6\psi(q^{49}). \tag{2.2}$$

For convenience, we rewrite (2.1) as the following simple form

$$\psi(q)=A_0+qA_1+q^3A_3, \tag{2.3}$$

where $A_0=f(q^{10},q^{15})$, $A_1=f(q^5,q^{20})$, $A_3=\psi(q^{25})$.

Lemma 2.2 [1, p. 26, (1.6.7)]

$$\psi^2(q)-q\psi^2(q^5)=f(q,q^4)f(q^2,q^3).$$

From (2.3) and Lemma 2.2, we see that

$$\psi^2(q^5)-q^5\psi^2(q^{25})=A_0A_1. \tag{2.4}$$

Lemma 2.3 *For integer* $n\ge 1$, *we have*

$$(q^n;q^n)_\infty^4\equiv(q^{2n};q^{2n})_\infty^2 \pmod{4},$$
$$(q^n;q^n)_\infty^8\equiv(q^{2n};q^{2n})_\infty^4 \pmod{8}.$$

3 Proof of Theorem 1.1

To prove Theorem 1.1, we first need to establish following lemma.

Lemma 3.1 *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}\left(8 \cdot 5^{8\alpha} n + 3 \cdot 5^{8\alpha}\right) q^n \equiv 8\psi^3(q) \pmod{64}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{p}(8 \cdot 5^{8\alpha+1} n + 7 \cdot 5^{8\alpha+1}) q^n \equiv 24q\psi^3(q^5) - 16\psi(q^5)\psi^2(q) \pmod{64}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \overline{p}(8 \cdot 5^{8\alpha+3} n + 7 \cdot 5^{8\alpha+3}) q^n \equiv 40q\psi^3(q^5) + 32\psi(q^5)\psi^2(q) \pmod{64}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \overline{p}\left(8 \cdot 5^{8\alpha+7} n + 7 \cdot 5^{8\alpha+7}\right) q^n \equiv 8q\psi^3(q^5) \pmod{64}. \tag{3.4}$$

Proof From (1.3), we have

$$\sum_{n=0}^{\infty} \overline{p}(4n+3) q^n \equiv 8 \frac{(q^2;q^2)_\infty (q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^4} \pmod{64},$$

and

$$\sum_{n=0}^{\infty} \overline{p}(8n+3) q^n \equiv 8\psi^3(q) \pmod{64}. \tag{3.5}$$

Define $a(n)$ as follows:

$$\sum_{n=0}^{\infty} a(n) q^n = \psi^3(q). \tag{3.6}$$

Thus,

$$\overline{p}(8n+3) \equiv 8a(n) \pmod{64}. \tag{3.7}$$

Applying (2.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(n) q^n =& A_0^3 + 3qA_0^2A_1 + 3q^2A_0A_1^2 + q^3A_1^3 + 3q^3A_0^2A_3 \\ &+ 6q^4A_0A_1A_3 + 3q^5A_1^2A_3 + 3q^6A_0A_3^2 + 3q^7A_1A_3^2 + q^9A_3^3.\end{aligned}$$

Applying (2.4), it can be seen that

$$\begin{aligned}\sum_{n=0}^{\infty} a(5n+4)q^{5n+4} &= q^9\psi^3(q^{25}) + 6q^4\psi(q^{25})f(q^5,q^{20})f(q^{10},q^{15})\\ &= q^9\psi^3(q^{25}) + 6q^4\psi(q^{25})\left(\psi^2(q^5) - q^5\psi^2(q^{25})\right)\\ &= -5q^9\psi^3(q^{25}) + 6q^4\psi(q^{25})\psi^2(q^5),\end{aligned}$$

and

$$\sum_{n=0}^{\infty} a(5n+4)q^n \equiv 3q\psi^3(q^5) - 2\psi(q^5)\psi^2(q) \pmod 8. \tag{3.8}$$

Using (2.3) and (2.4) again, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a\,(5(5n+1)+4)\,q^{5n+1} &= \sum_{n=0}^{\infty} a\,(25n+9)\,q^{5n+1}\\ &\equiv 3q\psi^3(q^5) - 2\psi(q^5)\left(q^6A_3^2 + 2qA_0A_1\right)\\ &\equiv 3q\psi^3(q^5) - 2\psi(q^5)\left(q^6\psi^2(q^{25}) + 2q\psi^2(q^5) - 2q^6\psi^2(q^{25})\right)\\ &\equiv -q\psi^3(q^5) + 2q^6\psi^2(q^{25})\psi(q^5) \pmod 8.\end{aligned}$$

That is,

$$\sum_{n=0}^{\infty} a\,(25n+9)\,q^n \equiv -\psi^3(q) + 2q\psi^2(q^5)\psi(q) \pmod 8.$$

Applying (2.1) and (3.8), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a\left(5^2(5n+4)+9\right)q^n &= \sum_{n=0}^{\infty} a\left(5^3n + \frac{7\cdot 5^3-3}{8}\right)q^n\\ &\equiv 5q\psi^3(q^5) + 2\psi(q^5)\psi^2(q) + 2\psi^2(q)\psi(q^5) \pmod 8\\ &= 5q\psi^3(q^5) + 4\psi(q^5)\psi^2(q).\end{aligned} \tag{3.9}$$

Using (2.3) and (2.4), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} a\left(5^3(5n+1)+\frac{7\cdot 5^3-3}{8}\right)q^{5n+1} &= \sum_{n=0}^{\infty} a\left(5^4 n+\frac{3\cdot(5^4-1)}{8}\right)q^{5n+1}\\ &\equiv 5q\psi^3(q^5)+4\psi(q^5)\left(q^6A_3^2+2qA_0A_1\right)\\ &\equiv 5q\psi^3(q^5)+4q^6\psi^2(q^{25})\psi(q^5) \pmod 8,\end{aligned}$$

namely,

$$\sum_{n=0}^{\infty} a\left(5^4 n+\frac{3\cdot(5^4-1)}{8}\right)q^n \equiv 5\psi^3(q)+4q\psi^2(q^5)\psi(q) \pmod 8.$$

Based on (2.1) and (3.8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a\left(5^4(5n+4)+\frac{3\cdot(5^4-1)}{8}\right)q^n &= \sum_{n=0}^{\infty} a\left(5^5 n+\frac{7\cdot 5^5-3}{8}\right)q^n\\ &\equiv -25q\psi^3(q^5)-10\psi^2(q)\psi(q^5)+4\psi^2(q)\psi(q^5)\\ &\equiv -q\psi^3(q^5)+2\psi^2(q)\psi(q^5) \pmod 8. \quad (3.10)\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{\infty} a\left(5^5(5n+1)+\frac{7\cdot 5^5-3}{8}\right)q^{5n+1} &= \sum_{n=0}^{\infty} a\left(5^6 n+\frac{3\cdot(5^6-1)}{8}\right)q^{5n+1}\\ &\equiv -q\psi^3(q^5)+2\psi(q^5)\left(q^6A_3^2+2qA_0A_1\right)\\ &\equiv 3q\psi^3(q^5)-2q^6\psi(q^5)\psi^2(q^{25}) \pmod 8,\end{aligned}$$

so it follows that

$$\sum_{n=0}^{\infty} a\left(5^6 n+\frac{3\cdot(5^6-1)}{8}\right)q^n \equiv 3\psi^3(q)-2q\psi(q)\psi^2(q^5) \pmod 8.$$

From (2.1) and (3.8), it can be seen that

$$\begin{aligned}\sum_{n=0}^{\infty} a\left(5^6(5n+4)+\frac{3\cdot(5^6-1)}{8}\right)q^n &= \sum_{n=0}^{\infty} a\left(5^7 n+\frac{7\cdot 5^7-3}{8}\right)q^n \\ &\equiv -15q\psi^3(q^5)-6\psi(q^5)\psi^2(q)-2\psi(q^5)\psi^2(q) \\ &\equiv q\psi^3(q^5) \pmod 8. \end{aligned} \tag{3.11}$$

Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} a\left(5^7(5n+1)+\frac{7\cdot 5^7-3}{8}\right)q^n &= \sum_{n=0}^{\infty} a\left(5^8 n+\frac{3\cdot(5^8-1)}{8}\right)q^n \\ &\equiv \psi^3(q) \pmod 8.\end{aligned}$$

So we have the following useful relation:

$$a\left(5^8 n+\frac{3\cdot(5^8-1)}{8}\right)\equiv a(n) \pmod 8.$$

By induction, we have

$$a\left(5^{8\alpha} n+\frac{3\cdot(5^{8\alpha}-1)}{8}\right)\equiv a(n) \pmod 8. \tag{3.12}$$

Using (3.6), (3.8), (3.9), (3.11) and (3.12), we deduce that

$$\sum_{n=0}^{\infty} a\left(5^{8\alpha} n+\frac{3\cdot(5^{8\alpha}-1)}{8}\right)q^n \equiv \psi^3(q) \pmod 8, \tag{3.13}$$

$$\sum_{n=0}^{\infty} a\left(5^{8\alpha+1} n+\frac{7\cdot 5^{8\alpha+1}-3}{8}\right)q^n \equiv 3q\psi^3(q^5)-2\psi(q^5)\psi^2(q) \pmod 8, \tag{3.14}$$

$$\sum_{n=0}^{\infty} a\left(5^{8\alpha+3} n+\frac{7\cdot 5^{8\alpha+3}-3}{8}\right)q^n \equiv 5q\psi^3(q^5)+4\psi(q^5)\psi^2(q) \pmod 8, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a\left(5^{8\alpha+7} n+\frac{7\cdot 5^{8\alpha+7}-3}{8}\right)q^n \equiv q\psi^3(q^5) \pmod 8. \tag{3.16}$$

Using the above relations and (3.7), we can easily get the desired results. □

Proof of Theorem 1.1 Applying (2.3) and (3.2), we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+1}n+7\cdot 5^{8\alpha+1})q^n\\&\equiv -40q\psi^3(q^5)-16\psi(q^5)\\&\quad\times\left(A_0^2+q^2A_1^2+q^6A_3^2+2qA_0A_1+2q^3A_0A_3+2q^4A_1A_3\right)\pmod{64}.\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+1}(5n+3)+7\cdot 5^{8\alpha+1})q^{5n+3}&=\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+2}n+31\cdot 5^{8\alpha+1})q^{5n+3}\\&\equiv -16\psi(q^5)(2q^3A_0A_3)\pmod{64},\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+1}(5n+4)+7\cdot 5^{8\alpha+1})q^{5n+4}&=\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+2}n+39\cdot 5^{8\alpha+1})q^{5n+4}\\&\equiv -16\psi(q^5)(2q^4A_1A_3)\pmod{64}.\end{aligned}$$

This yields the first two congruences of the theorem. In addition, applying (2.3) and (3.3), we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+3}n+7\cdot 5^{8\alpha+3})q^n\\&\equiv 40q\psi^3(q^5)+32\psi(q^5)\\&\quad\times\left(A_0^2+q^2A_1^2+q^6A_3^2+2qA_0A_1+2q^3A_0A_3+2q^4A_1A_3\right)\pmod{64}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+3}(5n+3)+7\cdot 5^{8\alpha+3})q^{5n+3}\\&=\sum_{n=0}^{\infty}\overline{p}(8\cdot 5^{8\alpha+4}n+31\cdot 5^{8\alpha+3})q^{5n+3}\\&\equiv 32q^3\psi(q^5)(2A_0A_3)\\&\equiv 0\pmod{64},\end{aligned}\tag{3.17}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{p}(8\cdot 5^{8\alpha+3}(5n+4)+7\cdot 5^{8\alpha+3})q^{5n+4}\\
&\quad=\sum_{n=0}^{\infty} \overline{p}(8\cdot 5^{8\alpha+4}n+39\cdot 5^{8\alpha+3})q^{5n+4}\\
&\quad\equiv 32q^4\psi(q^5)(2A_1A_3)\\
&\quad\equiv 0 \pmod{64}. \qquad (3.18)
\end{aligned}$$

From (3.4), we see that for $i=0,2,3,4$,

$$\overline{p}\left(8\cdot 5^{8\alpha+7}(5n+i)+7\cdot 5^{8\alpha+7}\right)\equiv 0 \pmod{64}.$$

Therefore, we finish the proof. □

4 Proof of Theorem 1.2

Lemma 4.1 *For $\alpha\geq 0$ and $n\geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}(8\cdot 7^{4\alpha}n+3\cdot 7^{4\alpha})q^n\equiv 8\psi^3(q) \pmod{32},$$

$$\sum_{n=0}^{\infty} \overline{p}(8\cdot 7^{4\alpha+1}n+5\cdot 7^{4\alpha+1})q^n\equiv 16f_1f_{14}+8q^2\psi^3(q^7) \pmod{32}, \qquad (4.1)$$

$$\sum_{n=0}^{\infty} \overline{p}(8\cdot 7^{4\alpha+2}n+3\cdot 7^{4\alpha+2})q^n\equiv 16f_2f_7+8\psi^3(q) \pmod{32},$$

$$\sum_{n=0}^{\infty} \overline{p}(8\cdot 7^{4\alpha+3}n+5\cdot 7^{4\alpha+3})q^n\equiv 8q^2\psi^3(q^7) \pmod{32}, \qquad (4.2)$$

where $f_n:=(q^n;q^n)_\infty$.

Proof From (3.5), we have

$$\overline{p}(8n+3)\equiv 8a(n) \pmod{32}. \qquad (4.3)$$

Recalling the generating function (3.6) of $a(n)$ and the following fact

$$f(q,q^6)f(q^2,q^5)f(q^3,q^4)=\frac{f_2f_7^4}{f_1f_{14}},$$

it is not hard to see that

$$\sum_{n=0}^{\infty} a(7n+4)q^n \equiv 2\frac{f_2 f_7^4}{f_1 f_{14}} + q^2\psi^3(q^7) \pmod 4$$
$$\equiv 2f_1 f_{14} + q^2\psi^3(q^7) \pmod 4. \tag{4.4}$$

From [2, p. 303, Entry 17(v)], we have the 7-dissection

$$f_1 = f_{49}\frac{f(-q^{14},-q^{35})}{f(-q^7,-q^{42})} - qf_{49}\frac{f(-q^{21},-q^{28})}{f(-q^{14},-q^{35})}$$
$$-q^2 f_{49} + q^5 f_{49}\frac{f(-q^7,-q^{42})}{f(-q^{21},-q^{28})}. \tag{4.5}$$

Thanks to (4.5), we obtain that

$$\sum_{n=0}^{\infty} a(7(7n+2)+4)q^n = \sum_{n=0}^{\infty} a(49n+18)q^n \equiv -2f_7 f_2 + \psi^3(q) \pmod 4. \tag{4.6}$$

Then in view of (4.4) and (4.5), it can be seen that

$$\sum_{n=0}^{\infty} a(49(7n+4)+18)q^n = \sum_{n=0}^{\infty} a(7^3 n+214)q^n$$
$$\equiv -2f_1 f_{14} + 2f_1 f_{14} + q^2\psi^3(q^7) \pmod 4$$
$$= q^2\psi^3(q^7), \tag{4.7}$$

and

$$\sum_{n=0}^{\infty} a(7^3(7n+2)+214)q^n = \sum_{n=0}^{\infty} a(7^4 n+900)q^n \equiv \psi^3(q) \pmod 4. \tag{4.8}$$

Thus, from (3.6) and (4.8), we see that

$$a(7^4 n+900) \equiv a(n) \pmod 4. \tag{4.9}$$

Using (3.6), (4.4), (4.6), (4.7), (4.9) and (4.3), by induction, it is easy to establish the desired results. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2 From (4.1) and (4.5), we deduce that

$$\overline{p}(8\cdot 7^{4\alpha+1}(7n+i) + 5\cdot 7^{4\alpha+1}) \equiv 0 \pmod{2^5},$$

where $i = 3, 4, 6$. In view of (4.2), we obtain

$$\overline{p}(8 \cdot 7^{4\alpha+3}(7n + i) + 5 \cdot 7^{4\alpha+3}) \equiv 0 \pmod{2^5},$$

for $i = 0, 1, 3, 4, 5, 6$. This completes the proof. □

5 Proof of Theorem 1.3

From Lemma 2.1, we have

$$\psi(q) = B_0 + B_1, \tag{5.1}$$

where $B_0 = f(q^3, q^6)$, $B_1 = q\psi(q^9)$.

To prove Theorem 1.3, we need the following three lemmas.

Lemma 5.1 [12, Lemmas 2.1 and 2.2]

$$\frac{1}{\psi(q)} = \frac{\psi(q^9)}{\psi^4(q^3)} \left(B_0^2 - B_0B_1 + B_1^2\right), \tag{5.2}$$

$$B_0^3 = \frac{\psi^4(q^3)}{\psi(q^9)} - q^3\psi^3(q^9). \tag{5.3}$$

Lemma 5.2 *Let $c(n)$ be defined by*

$$\sum_{n=0}^{\infty} c(n)q^n = \psi^4(q).$$

Then we have

$$\sum_{n=0}^{\infty} c(3n + 1)q^{3n+1} \equiv q^4\psi^4(q^9) \pmod{4}.$$

Proof Using (5.1), we have

$$\sum_{n=0}^{\infty} c(n)q^n = (B_0 + B_1)^4.$$

Then

$$\sum_{n=0}^{\infty} c(3n + 1)q^{3n+1} = B_1^4 + 4B_0^3B_1 \equiv q^4\psi^4(q^9) \pmod{4}.$$

□

Lemma 5.3 *Let $d(n)$ be defined by*

$$\sum_{n=0}^{\infty} d(n)q^n = \psi^8(q).$$

We have

$$\sum_{n=0}^{\infty} d(3n+2)q^{3n+2} \equiv q^8\psi^8(q^9) \pmod 4.$$

Proof Since

$$\sum_{n=0}^{\infty} d(n)q^n = (B_0+B_1)^8,$$

we have

$$\sum_{n=0}^{\infty} d(3n+2)q^{3n+2} = B_1^8 + \frac{8!}{3!5!}B_1^5B_0^3 + \frac{8!}{2!6!}B_1^2B_0^6$$
$$\equiv q^8\psi^8(q^9) \pmod 4.$$

□

Proof of Theorem 1.3 From (1.4), we see that

$$\frac{1}{64}\sum_{n=0}^{\infty} \overline{p}(8n+7)q^n \equiv \psi^7(q) = (B_0+B_1)^7 \pmod 4.$$

Setting

$$\frac{1}{64}\sum_{n=0}^{\infty} \overline{p}(8n+7)q^n = \sum_{n=0}^{\infty} f(n)q^n, \tag{5.4}$$

we have

$$\sum_{n=0}^{\infty} f(3n+1)q^{3n+1} \equiv 7B_0^6B_1 + 35B_0^3B_1^4 + B_1^7 \pmod 4.$$

Applying (5.3), we see that

$$\sum_{n=0}^{\infty} f(3n+1)q^{3n+1} \equiv q^4\psi^3(q^9)\psi^4(q^3) + q^7\psi^7(q^9) - q\frac{\psi^8(q^3)}{\psi(q^9)} \pmod 4, \tag{5.5}$$

and

$$\sum_{n=0}^{\infty} f(3n+1)q^n \equiv q\psi^3(q^3)\psi^4(q) + q^2\psi^7(q^3) - \frac{\psi^8(q)}{\psi(q^3)} \pmod 4.$$

From Lemmas 5.2 and 5.3, it follows that

$$\sum_{n=0}^{\infty} f(3(3n+2)+1)q^{3n+2} \equiv q\psi^3(q^3)\cdot q^4\psi^4(q^9) + q^2\psi^7(q^3) - \frac{q^8\psi^8(q^9)}{\psi(q^3)} \pmod 4,$$

and

$$\sum_{n=0}^{\infty} f(9n+7)q^n \equiv q\psi^3(q)\psi^4(q^3) - \frac{q^2\psi^8(q^3)}{\psi(q)} + \psi^7(q) \pmod 4.$$

Employing (5.1), (5.5) and Lemma 5.1, it can be seen that

$$\begin{aligned}\sum_{n=0}^{\infty} f(9(3n+1)+7)q^{3n+1} \equiv& q(B_0^3+B_1^3)\psi^4(q^3) - q^2\psi^8(q^3)\frac{q^2\psi^3(q^9)}{\psi^4(q^3)} \\ &+ q^4\psi^3(q^9)\psi^4(q^3) + q^7\psi^7(q^9) - q\frac{\psi^8(q^3)}{\psi(q^9)} \pmod 4 \\ &= q^7\psi^7(q^9),\end{aligned}$$

and

$$\sum_{n=0}^{\infty} f(27n+16)q^n \equiv q^2\psi^7(q^3) \pmod 4. \tag{5.6}$$

Therefore,

$$\sum_{n=0}^{\infty} f(27(3n+2)+16)q^n = \sum_{n=0}^{\infty} f(3^4n+70)q^n \equiv \psi^7(q) \pmod 4.$$

Thus, we have

$$f(n) \equiv f(3^4n+70) \pmod 4.$$

Based on the above relation, by induction, we obtain that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} f\left(3^{4\alpha}n + \frac{7 \cdot 3^{4\alpha} - 7}{8}\right) q^n \equiv \psi^7(q) \pmod 4.$$

Combining the above relation and (5.6), we find that

$$\sum_{n=0}^{\infty} f\left(3^{4\alpha+3}n + \frac{5 \cdot 3^{4\alpha+3} - 7}{8}\right) q^n \equiv q^2\psi^7(q^3) \pmod 4. \tag{5.7}$$

From (5.4) and (5.7), we see that

$$\sum_{n=0}^{\infty} \overline{p}(8 \cdot 3^{4\alpha+3}n + 5 \cdot 3^{4\alpha+3})q^n \equiv q^2\psi^7(q^3) \pmod{2^8}. \tag{5.8}$$

Since there are no terms on the right of (5.8) in which the powers of q are congruent to 0, 1 modulo 3, we have

$$\overline{p}(8 \cdot 3^{4\alpha+3}(3n) + 5 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8},$$
$$\overline{p}(8 \cdot 3^{4\alpha+3}(3n+1) + 5 \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8}.$$

This completes the proof. □

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