Ramanujan J (2017) 43:69–82
DOI 10.1007/s11139-015-9765-x

Some new approximations and inequalities of the sequence $(1 + 1/n)^n$ and improvements of Carleman's inequality

Dawei Lu[1] · **Lixin Song**[1] · **Zhen Liu**[1]

Received: 30 June 2015 / Accepted: 6 December 2015 / Published online: 26 July 2016

Abstract In this paper, using the polynomial approximation and the continued fraction approximation, we present some sharp inequalities for the sequence $(1 + 1/n)^n$ and some applications to Carleman's inequality. For demonstrating the superiority of our new inequalities over the classical one, some proofs and numerical computations are provided.

Keywords Polynomial approximation · Continued fraction · Constant e · Carleman's inequality · Double inequalities

Mathematics Subject Classification 26A09 · 33B10 · 26D99

1 Introduction

There has been considerable discussion concerning the following well-known double inequalities:

The research of the first author was supported by the National Natural Science Foundation of China (Grant No. 11571058) and the Fundamental Research Funds for the Central Universities (Grant No. DUT15LK19). The second author was supported by the National Natural Science Foundation of China (Grant No. 11371077).

✉ Lixin Song
lxsong1966@sina.com

Dawei Lu
ludawei_dlut@163.com

Zhen Liu
liuzhendlut@126.com

[1] School of Mathematical Sciences, Dalian University of Technology, Dalian 116023, China

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}, \quad n \geq 1. \tag{1.1}$$

Since these are often used to improve inequalities of Hardy–Carleman type, there has been considerable interest in extending these inequalities in the recent past. See for example [2, 10, 13–15].

These inequalities (1.1) are equivalent to

$$\frac{2n}{2n+1} < \frac{1}{e}\left(1 + \frac{1}{n}\right)^n < \frac{2n+1}{2n+2}. \tag{1.2}$$

Mortici and Hu [9] presented the best form approximation of (1.2) as follows:

$$\frac{1}{e}\left(1 + \frac{1}{n}\right)^n \approx \frac{n + 5/12}{n + 11/12}. \tag{1.3}$$

Based on (1.3), double inequalities

$$u_0(x) < \frac{1}{e}\left(1 + \frac{1}{x}\right)^x < v_0(x) \tag{1.4}$$

hold for every real number $x \in [1, \infty)$, where

$$u_0(x) = \frac{x + 5/12}{x + 11/12} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41472x^5},$$
$$v_0(x) = \frac{x + 5/12}{x + 11/12} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41472x^5} + \frac{300901}{3483648x^6}.$$

In the asymptotic theory, there are many methods to obtain better approximations. First, the polynomial approximation is a very useful method to give superior increasing approximations as for example the Stirling series [1]:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\right).$$

Recently, the polynomial approximation method was used by Lu [3] to provide some more general convergent sequences for Euler's constant. Using the polynomial approximation, Lu et al. [5] also obtained the extension of Windschitl's formula. Second, the continued fraction approximation is also a very useful method to give superior increasing approximations. For example, Mortici [8] provided a new continued fraction approximation starting from the Nemes' formula as follows:

$$\Gamma(x+1) \approx \sqrt{2\pi x}e^{-x}\left(x+\cfrac{1}{12x-\cfrac{1}{10x+\cfrac{a}{x+\cfrac{b}{x+\cfrac{c}{x+\cfrac{d}{x+\ddots}}}}}}\right)^x,$$

where

$$a = -\frac{2369}{252}, \quad b = \frac{2117009}{1193976}, \quad c = \frac{393032191511}{1324011300744},$$
$$d = \frac{33265896164277124002451}{14278024104089641878840}\cdots.$$

Recently, the continued fraction approximation was used by Lu and Wang [4] to provide a new asymptotic expansion for the gamma function. Lu et al. [6] also obtained some new continued fraction approximations of Euler's constant.

It is their works that motivated our study. In this paper, we give some polynomial and continued fraction approximations for the constant e in Sect. 2.

To obtain the main results in this paper, we need the following lemma which is very useful for constructing asymptotic expansions:

Lemma 1 *If $(x_n)_{n\geq 1}$ is convergent to zero and the limit*

$$\lim_{n\to\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{1.5}$$

exists for $s > 1$, then

$$\lim_{n\to\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{1.6}$$

Lemma 1 was first proved by Mortici in [7]. From Lemma 1, we can see that the speed of convergence of the sequence $(x_n)_{n\geq 1}$ increases with the value s satisfying (1.5).

The rest of the paper is organized as follows: In Sect. 2, the main results and their proofs are provided. In Sect. 3, we give some comparisons to demonstrate the superiority of inequalities (2.10) and (2.11) over the inequalities (1.4) in Mortici and Hu [9]. Finally, in Sect. 4, some applications to Carleman's inequality are presented.

2 Main results

Theorem 1 *For (1.3), using the polynomial approximation, we have*

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx 1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \frac{a_4}{n^4} + \frac{a_5}{n^5} + \frac{a_6}{n^6} + \cdots, \tag{2.1}$$

where

$$a_1 = -\frac{1}{2}, \quad a_2 = \frac{11}{24}, \quad a_3 = -\frac{7}{16}, \quad a_4 = \frac{2447}{5760},$$
$$a_5 = -\frac{959}{2304}, \quad a_6 = \frac{238043}{580608}, \cdots .$$

Proof Let $(x_i)_{i\geq 1}$ be a polynomial sequence which converges to $\frac{1}{e}\left(1+\frac{1}{n}\right)^n$, where

$$x_1 = 1 + \frac{a_1}{n}, \quad x_2 = 1 + \frac{a_1}{n} + \frac{a_2}{n^2}, \cdots, \quad x_i = 1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_i}{n^i}, \cdots . \tag{2.2}$$

To measure the accuracy of this approximation, we define a sequence $(t_i)_{i\geq 1}$,

$$t_i(n) = \frac{1}{e}\left(1+\frac{1}{n}\right)^n - x_i. \tag{2.3}$$

Then, x_i converges to $\frac{1}{e}\left(1+\frac{1}{n}\right)^n$ is equivalent to t_i converges to 0. Using (2.2) and (2.3), we have

$$t_1(n) - t_1(n+1) = \frac{-1-2a_1}{2n^2} + \frac{17+12a_1}{12n^3} + O(n^{-4}). \tag{2.4}$$

From Lemma 1, we know that the speed of convergence $(t_i)_{i\geq 1}$ is even higher as the value s satisfying (1.5). Thus, using Lemma 1, we have the following:

(i) If $a_1 \neq -2^{-1}$, then the rate of the sequence $t_1(n)$ is n^{-1}, since

$$\lim_{n\to\infty} n t_1(n) = \frac{-1-2a_1}{2} \neq 0.$$

(ii) If $a_1 = -2^{-1}$, then from (2.4), we have

$$t_1(n) - t_1(n+1) = \frac{11}{12n^3},$$

and the rate of convergence of the sequence $t_1(n)$ is n^{-2}, since

$$\lim_{n\to\infty} n^2 t_1(n) = \frac{11}{24}.$$

We know that the fastest possible sequence $t_1(n)$ is obtained only for $a_1 = -2^{-1}$.

Using the same method, we have

$$a_2 = \frac{11}{24}, a_3 = -\frac{7}{16}, a_4 = \frac{2447}{5760}, a_5 = -\frac{959}{2304}, a_6 = \frac{238043}{580608}, \cdots .$$

□

Theorem 2 *For (1.3), using the continued fraction approximation, we have*

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx 1+\cfrac{b_1}{n+\cfrac{b_2 n}{n+\cfrac{b_3 n}{n+\cfrac{b_4 n}{n+\cfrac{b_5 n}{n+\cfrac{b_6 n}{n+\cdots}}}}}}, \tag{2.5}$$

where

$$b_1=-\frac{1}{2},\quad b_2=\frac{11}{12},\quad b_3=\frac{5}{132},\quad b_4=\frac{457}{1100},$$
$$b_5=\frac{5291}{45700},\quad b_6=\frac{19753835}{55393884},\cdots.$$

Proof Let $(y_i)_{i\geq 1}$ be a continued fraction sequence which converges to $\frac{1}{e}\left(1+\frac{1}{n}\right)^n$, where

$$y_1=1+\frac{b_1}{n},\quad y_2=1+\frac{b_1}{n+b_2},\quad y_3=1+\frac{b_1}{n+\frac{b_2 n}{n+b_3}},\cdots\quad y_i=1+\cfrac{b_1}{n+\cfrac{b_2 n}{n+\cfrac{b_3 n}{\ddots\, n+\cfrac{b_{i-1}n}{n+b_i}}}},\cdots. \tag{2.6}$$

To measure the accuracy of this approximation, we define a sequence $(s_i)_{i\geq 1}$,

$$s_i(n)=\frac{1}{e}\left(1+\frac{1}{n}\right)^n - y_i. \tag{2.7}$$

Then, y_i converges to $\frac{1}{e}\left(1+\frac{1}{n}\right)^n$ is equivalent to s_i converges to 0. Using (2.6) and (2.7), we have

$$s_1(n)-s_1(n+1)=\frac{-1-2b_1}{2n^2}+\frac{17+12b_1}{12n^3}+O(n^{-4}). \tag{2.8}$$

It is easy to see that the fastest possible sequence $s_1(n)$ is obtained only for $b_1=a_1=-2^{-1}$.

Using (2.6) and (2.7) again, we have

$$s_2(n)-s_2(n+1)=\frac{11-12b_2}{12n^3}+\frac{-43+24b_2+24b_2^2}{16n^4}+O(n^{-5}). \tag{2.9}$$

From Lemma 1, we have the following:

(i) If $b_2\neq 11/12$, then the rate of the sequence $s_2(n)$ is n^{-2}, since

$$\lim_{n\to\infty} n^2 s_2(n)=\frac{11-12b_2}{24}\neq 0.$$

(ii) If $b_2 = 11/12$, then from (2.9), we have

$$s_2(n) - s_2(n+1) = -\frac{5}{96n^4},$$

and the rate of convergence of the sequence $s_2(n)$ is n^{-3}, since

$$\lim_{n\to\infty} n^3 s_2(n) = -\frac{5}{288}.$$

We know that the fastest possible sequence $s_2(n)$ is obtained only for $b_2 = 11/12$.

Using the same method, we have

$$b_3 = \frac{5}{132}, \quad b_4 = \frac{457}{1100}, \quad b_5 = \frac{5291}{45700}, \quad b_6 = \frac{19753835}{55393884}, \cdots.$$

Using Theorem 1, we obtain the following inequalities.

Theorem 3 *For every real number $x \in [1, \infty)$, the following inequalities hold:*

$$u_1(x) < \frac{1}{e}\left(1 + \frac{1}{x}\right)^x < v_1(x), \tag{2.10}$$

where

$$u_1(x) = 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \frac{67223}{165888x^7},$$

$$v_1(x) = 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \frac{67223}{165888x^7} + \frac{559440199}{1393459200x^8}.$$

Proof The proof of inequalities (2.10) is equivalent to $f_1 > 0$ and $g_1 < 0$, as $x \in [1, \infty)$, where

$$f_1(x) = x \ln\left(1 + \frac{1}{x}\right) - 1 - \ln\left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \frac{67223}{165888x^7}\right),$$

$$g_1(x) = x \ln\left(1 + \frac{1}{x}\right) - 1 - \ln\left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \frac{67223}{165888x^7} + \frac{559440199}{1393459200x^8}\right).$$

By some calculations, we have

$$f_1''(x) = \frac{A_1(x-1)}{x^2(x+1)^2 P_1^2(x)} \quad \text{and} \quad g_1''(x) = -\frac{B_1(x-1)}{x^2(x+1)^2 Q_1^2(x)},$$

where

$$\begin{aligned}
P_1(x) &= 5806080x^7 - 2903040x^6 + 2661120x^5 - 2540160x^4 \\
&\quad +2466576x^3 - 2416680x^2 + 2380430x - 2352805, \\
Q_1(x) &= 1393459200x^8 - 696729600x^7 + 638668800x^6 - 609638400x^5 \\
&\quad +591978240x^4 - 580003200x^3 + 571303200x^2 - 564673200x \\
&\quad +559440199, \\
A_1(x) &= 10325422778525831x + 34915994344327949x^2 + 67354407138829296x^3 \\
&\quad +81069787336627560x^4 + 62331076599861888x^5 \\
&\quad +29888921547063936x^6 + 8170226471522304x^7 + 974446365182976x^8 \\
&\quad +1293233555960723, \\
B_1(x) &= 8392305752239085388 33x + 325373855354704903728 8x^2 \\
\end{aligned}$$

Evidently, we have $f_1''(x) > 0$, $g_1''(x) < 0$ for $x \geq 1$. Thus, g_1 is strictly concave, and f_1 is strictly convex. Combining $f_1(\infty) = g_1(\infty) = 0$, we obtain $g_1 < 0$ and $f_1 > 0$ on $[1, \infty)$. The proof of inequalities (2.10) is complete. □

Using Theorem 2, we obtain the following inequalities.

Theorem 4 *For every real number $x \in [1, \infty)$, the following inequalities hold:*

$$u_2(x) < \frac{1}{e}\left(1 + \frac{1}{x}\right)^x < v_2(x), \tag{2.11}$$

where

$$u_2(x) = 1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \frac{5291}{45700}}}}}, \quad v_2(x) = 1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \cfrac{\frac{5291}{45700}x}{x + \frac{19753835}{55393884}}}}}}.$$

Proof The proof of inequalities (2.11) is equivalent to $f_2 > 0$ and $g_2 < 0$, as $x \in [1, \infty)$, where

$$f_2(x) = x \ln\left(1 + \frac{1}{x}\right) - 1 - \ln\left(1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \frac{5291}{45700}}}}}\right),$$

$$g_2(x) = x \ln\left(1 + \frac{1}{x}\right) - 1 - \ln\left(1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \cfrac{\frac{5291}{45700}x}{x + \frac{19753835}{55393884}}}}}}\right).$$

By some calculations, we have

$$f_2''(x) = \frac{A_2(x-1)}{x^2(x+1)^2 P_2^2(x) R_2^2(x)} \quad \text{and} \quad g_2''(x) = -\frac{B_2(x-1)}{x(x+1)^2 Q_2^2(x) S_2^2(x)},$$

where

$$\begin{aligned}
P_2(x) &= 219360x^3 + 216240x^2 + 45362x - 481, \\
R_2(x) &= 109680x^2 + 162960x + 53891, \\
Q_2(x) &= 29090880x^3 + 39051120x^2 + 15041160x + 1535537, \\
S_2(x) &= 29090880x^3 + 53596560x^2 + 28506120x + 3950767, \\
A_2(x) &= 7239877975538515200(x+1)^6 + 21505720740023942400(x+1)^5 \\
&\quad +23551646037987136320(x+1)^4 + 11493402277272147840(x+1)^3 \\
&\quad +2329822406815788131(x+1)^2 + 120656484632110921x \\
&\quad +119984556789002880, \\
B_2(x) &= 1807392287181467915782963200(x+1)^6 \\
&\quad +6605509732240923396484300800(x+1)^5 \\
&\quad +9583241661488326994121772800(x+1)^4 \\
&\quad +7005074374845717299901265920(x+1)^3 \\
&\quad +2693418207575323688814043200(x+1)^2 \\
&\quad +510240050132939340975095040x \\
&\quad +54704306577249413078861568 1.
\end{aligned}$$

Evidently, we have $f_2''(x) > 0$, $g_2''(x) < 0$ for $x \geq 1$. Thus, g_2 is strictly concave, and f_2 is strictly convex. Combining $f_2(\infty) = g_2(\infty) = 0$, we obtain $g_2 < 0$ and $f_2 > 0$ on $[1, \infty)$. The proof of inequalities (2.10) is complete. □

3 Comparisons

In this section, we give some comparisons to demonstrate the superiority of inequalities (2.10) and (2.11) over the inequalities (1.4) in Mortici and Hu [9].

First, comparing (2.10) with (1.4), we have

$$\begin{aligned} u_1(x) - u_0(x) &= 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} \\ &\quad - \frac{67223}{165888x^7} - \left(\frac{x + 5/12}{x + 11/12} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41472x^5} \right) \\ &= \frac{1203604(x-3)^2 + 6811838(x-3) + 4426907}{1161216x^7(12x+11)}. \end{aligned}$$

Then, $u_1(x) > u_0(x)$ for $x \in [3, \infty)$.

$$\begin{aligned} v_1(x) - v_0(x) &= 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} \\ &\quad - \frac{67223}{165888x^7} + \frac{559440199}{1393459200x^8} \\ &\quad - \left(\frac{x + 5/12}{x + 11/12} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41472x^5} + \frac{300901}{3483648x^6} \right) \\ &= -\frac{1815707600(x-3)^2 + 10392368412(x-3) + 8681894647}{1393459200x^8(12x+11)}. \end{aligned}$$

Then, $v_1(x) < v_0(x)$ for $x \in [3, \infty)$. Thus, the inequalities (2.10) in Theorem 3 are more accurate than the inequalities (1.4) in Mortici and Hu [9].

Next, comparing (2.11) with (1.4), we have

$$\begin{aligned} u_2(x) - u_0(x) &= 1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \frac{5291}{45700}}}}} \\ &\quad - \left(\frac{x + 5/12}{x + 11/12} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41472x^5} \right) = \frac{I_1(x)}{L_1(x)}, \end{aligned}$$

where

$$\begin{aligned} I_1(x) &= 23492171136x^2 + 27086459628x + 7768657105, \\ L_1(x) &= 207360x^5(109680x^2 + 162960x + 53891)(12x+11). \end{aligned}$$

Then, $u_2(x) > u_0(x)$ for $x \in (0, \infty)$.

$$v_2(x) - v_0(x) = 1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \cfrac{\frac{5291}{45700}x}{x + \frac{19753835}{55393884}}}}}} - \left(\frac{x + 5/12}{x + 11/12} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41472x^5} + \frac{300901}{3483648x^6}\right) = -\frac{I_2(x)}{L_2(x)},$$

where

$$I_2(x) = 659982630678912x^3 + 1034331228912576x^2 + 495251068622280x + 65383435758685,$$
$$L_2(x) = 17418240x^6(29090880x^3 + 53596560x^2 + 28506120x + 3950767)(12x + 11).$$

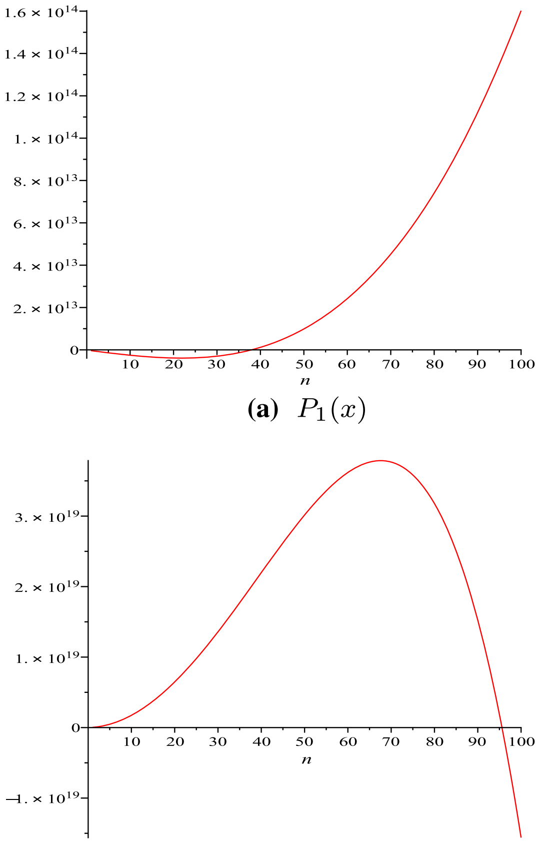
Then, $v_2(x) < v_0(x)$ for $x \in (0, \infty)$. Thus, the inequalities (2.11) in Theorem 4 are more accurate than the inequalities (1.4) in Mortici and Hu [9].

Finally, comparing (2.10) with (2.11), we have

$$u_1(x) - u_2(x) = 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \frac{67223}{165888x^7} - \left(1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \frac{5291}{45700}}}}}\right) = \frac{P_1(x)}{Q_1(x)},$$

$$v_1(x) - v_2(x) = 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \frac{67223}{165888x^7} + \frac{559440199}{1393459200x^8} - \left(1 + \cfrac{-\frac{1}{2}}{x + \cfrac{\frac{11}{12}x}{x + \cfrac{\frac{5}{132}x}{x + \cfrac{\frac{457}{1100}x}{x + \cfrac{\frac{5291}{45700}x}{x + \frac{19753835}{55393884}}}}}}\right) = \frac{P_2(x)}{Q_2(x)},$$

Fig. 1 Simulations for $P_1(x)$ and $P_2(x)$



(a) $P_1(x)$

(b) $P_2(x)$

where

$$\begin{aligned}
P_1(x) &= 189636816x^3 - 378081480x^2 - 255129349670x - 126795014255,\\
Q_1(x) &= 1393459200x^8(29090880x^3 + 53596560x^2 + 28506120x + 3950767),\\
P_2(x) &= -1826787697920x^4 + 4246725222720x^3 + 16144514021685840x^2\\
&\quad +13716577201173480x + 2210217876682633,\\
Q_2(x) &= 1393459200x^8(29090880x^3 + 53596560x^2 + 28506120x + 3950767).
\end{aligned}$$

By some simulations, we obtain the following figures.

We see that for $x \in [100, \infty)$, $u_1(x) > u_2(x)$ and $v_1(x) < v_2(x)$. So, the inequalities (2.10) in Theorem 3 are more accurate than the inequalities (2.11) in Theorem 4 for $x \in [100, \infty)$. And for $x \in (0, 35]$, $u_2(x) > u_1(x)$ and $v_2(x) < v_1(x)$. So we can see that for $x \in (0, 35]$, the inequalities (2.11) in Theorem 4 are more accurate than the inequalities (2.10) in Theorem 3 (Fig. 1).

Table 1 Simulations for u_0, u_1 and u_2

n	$\frac{E(n)-u_0(n)}{E(n)}$	$\frac{E(n)-u_1(n)}{E(n)}$	$\frac{E(n)-u_2(n)}{E(n)}$
20	1.3006×10^{-9}	1.5307×10^{-11}	4.0198×10^{-12}
50	5.4457×10^{-12}	1.0177×10^{-14}	1.7964×10^{-14}
100	8.5726×10^{-14}	3.9951×10^{-17}	2.8908×10^{-16}
500	5.5196×10^{-18}	1.0267×10^{-22}	1.8945×10^{-20}

Table 2 Simulations for v_0, v_1 and v_2

n	$\frac{v_0(n)-E(n)}{E(n)}$	$\frac{v_1(n)-E(n)}{E(n)}$	$\frac{v_2(n)-E(n)}{E(n)}$
20	8.2007×10^{-11}	7.5951×10^{-13}	2.9403×10^{-14}
50	1.3709×10^{-13}	2.0200×10^{-16}	5.3631×10^{-17}
100	1.0783×10^{-15}	3.9647×10^{-19}	4.3447×10^{-19}
500	1.3879×10^{-20}	2.0379×10^{-25}	5.7261×10^{-24}

Furthermore, some numerical computations are given to demonstrate the superiority of our new double inequalities over the classical ones again. Let $E(n) = \frac{1}{e}\left(1 + \frac{1}{n}\right)^n$. Combining Theorems 3 and 4, we have Tables 1 and 2.

4 Applications to Carleman's inequality

If $\sum a_n$ is a convergent series of nonnegative reals, then the following inequality

$$\sum_{n=1}^{\infty}(a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{n=1}^{\infty} a_n \tag{4.1}$$

holds. Now, it is known as the Carleman's inequality which was firstly discovered by Torsten Carleman.

The Carleman's inequality appeared in many problems from pure and applied analysis. Up to now, many researchers have made great efforts to improve it. For example, using AM-GM inequality

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{c_1 a_1 + c_2 a_2 + \cdots + c_n a_n}{n(c_1 c_2 \cdots c_n)^{1/n}},$$

where $c_1, c_2, \cdots, c_n > 0$, Pólya [11,12] obtained the following inequality:

$$\sum_{n=1}^{\infty}(a_1 a_2 \cdots a_n)^{1/n} \le \sum_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)^n a_n.$$

Using $(1 + 1/n)^n < e$, the Carleman's inequality (4.1) holds.

Almost all improvements stated in the recent past used upper bounds for $(1 + 1/n)^n$, stronger than $(1 + 1/n)^n < e$. For example, Mortici and Hu [9] used the double inequalities (1.4) to establish the following improvements of the Carleman's inequality:

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{\infty}\left(\frac{12n+5}{12n+11}\right)a_n \tag{4.2}$$

and

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{\infty}\left(\frac{12n+5}{12n+11}-\varepsilon_n\right)a_n, \tag{4.3}$$

where

$$\varepsilon_n = \frac{5}{288n^3} - \frac{343}{8640n^4} + \frac{2621}{41472n^5} - \frac{300901}{3483648n^6},$$

and $a_n > 0$ such that $\sum a_n < \infty$.

Using the same idea, combining Theorems 3 and 4, we establish the following improvements of the Carleman's inequality.

Theorem 5 *Let $a_n > 0$ such that $\sum a_n < \infty$. Then*

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\frac{2447}{5760n^4}\right)a_n, \tag{4.4}$$

and

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{\infty}\left(1+\cfrac{-\frac{1}{2}}{n+\cfrac{\frac{11}{12}n}{n+\cfrac{\frac{5}{132}n}{n+\cfrac{\frac{457}{1100}n}{n+\cfrac{\frac{5291}{45700}n}{n+\frac{19753835}{55393884}}}}}}\right)a_n. \tag{4.5}$$

Combining the comparisons in Sect. 4, it is easy to see that our upper bounds in (4.4) and (4.5) are sharper than ones in (4.2) and (4.3), respectively.

Acknowledgements Computations made in this paper were performed using Maple software.

References

1. Abramowitz, M., Stegun, I. A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Nation Bureau of Standards, Applied Mathematical Series 55, 9th printing. Dover, New York (1972)
2. Hu, Y.: A strengthened Carleman's inequality. Commun. Math. Anal. **1**(2), 115–119 (2006)
3. Lu, D.: Some new convergent sequences and inequalities of Euler's constant. J. Math. Anal. Appl. **419**, 541–552 (2014)
4. Lu, D., Wang, X.: A new asymptotic expansion and some inequalities for the Gamma function. J. Math. Anal. Appl. **140**, 314–323 (2014)

5. Lu, D., Song, L., Ma, C.: A generated approximation of the gamma function related to Windschitl's formula. J. Number Theory. **140**, 215–225 (2014)
6. Lu, D., Song, L., Yu, Y.: Some new continued fraction approximation of Euler's constant. J. Number Theory. **147**, 69–80 (2014)
7. Mortici, C.: Product approximations via asymptotic integration. Am. Math. Mon. **117**(5), 434–441 (2010)
8. Mortici, C.: A continued fraction approximation of the gamma function. J. Math. Anal. Appl. **402**, 405–410 (2013)
9. Mortici, C., Hu, Y.: On some convergences to the constant e and improvements of Carleman's inequality. Carpathian J. Math. **31**(2), 243–249 (2015)
10. Ping, Y., Sun, G.: A strengthened Carleman's inequality. J. Math. Anal. Appl. **240**, 290–293 (1999)
11. Pólya, G.: Proof of an inequality. Proc. Lond. Math. Soc. **24**(2), 55 (1925)
12. Pólya, G.: With, or without motivation? Am. Math. Mon. **56**, 684–691 (1949)
13. Pólya, G., Szegö, G.: Problems and Theorems in Analysis, vol. I. Springer, New York (1972)
14. Yang, X.: On Carleman's inequality. J. Math. Anal. Appl. **253**, 691–694 (2001)
15. Yang, B., Debnath, L.: Some inequalities involving the constant e and an application to Carleman's inequality. J. Math. Anal. Appl. **223**, 347–353 (1998)