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Arithmetic properties arising from Ramanujan's theta functions

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Abstract We prove some interesting arithmetic properties of theta function identities that are analogous to q-series identities obtained by Michael D. Hirschhorn. In addition, we find infinite family of congruences modulo powers of 2 for representations of a non-negative integer n as $\triangle_1 + 4\triangle_2$ and $\triangle + k\square$.

1 Introduction

Let $a_k(n)$ be the number of representations of n as $\triangle_1 + 2k\triangle_2$ and $b_k(n)$ be the number of representations of n as $\triangle + k\square$. Then

$$\sum_{n=0}^{\infty} a_k(n) q^n = \psi(q)\psi(q^{2k}),$$

$$\sum_{n=0}^{\infty} b_k(n) q^n = \psi(q)\varphi(q^k).$$

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Hirschhorn and Sellers [3] used some interesting q-series identities to obtain results of congruences modulo 5 for 4-colored Frobenius partitions, and Hirschhorn [2] proved equivalent results of q-series identities that are used in [3]: If

$$\sum_{n\geq 0} g(n)q^n = \frac{(q^2; q^2)_\infty^2 (q^4; q^4)_\infty^3}{(q; q)_\infty} \text{ and } \sum_{n\geq 0} h(n)q^n = \frac{(q; q)_\infty^3 (q^2; q^2)_\infty^2}{(q^4; q^4)_\infty},$$

where $(a; q)_\infty = \prod_{n\geq 0} (1 - aq^n)$, then

$$\sum_{n\geq 0} g(5n)q^n = \sum_{n\geq 0} h(n)q^n - 5 \sum_{n\geq 0} g(n)q^{5n+3}$$

and

$$\sum_{n\geq 0} h(5n + 3)q^n = 16 \sum_{n\geq 0} g(n)q^n - 5 \sum_{n\geq 0} h(n)q^{5n}.$$

Motivated by these works, in Sect. 3, we establish arithmetic properties of representations of a number $\triangle_1 + 2k\triangle_2$ and $\triangle + k\square$. In addition, we also find certain congruences modulo powers of 2 for number of representations of a non-negative integer n as $\triangle_1 + 4\triangle_2$ and $\triangle + k\square$. We list our main results in the following theorems and corollaries.

Theorem 1.1 *Let $a_k(n)$ and $b_k(n)$ be the number of representations of n as $\triangle_1 + 2k\triangle_2$ and $\triangle + k\square$, respectively. Then*

$$\sum_{n=0}^{\infty} a_k((2k + 1)n)q^n = \psi(q)\varphi(q^k) - q^{\frac{k(k+1)}{2}}\psi(q^{2k+1})\psi(q^{2k(2k+1)}), \quad (1.1)$$

$$\sum_{n=0}^{\infty} b_1(3n + 1)q^n = 4\psi(q)\psi(q^2) - \psi(q^3)\varphi(q^3), \quad (1.2)$$

$$\sum_{n=0}^{\infty} b_2(5n + 3)q^n = 4\psi(q)\psi(q^4) - \psi(q^5)\varphi(q^{10}), \quad (1.3)$$

$$\sum_{n=0}^{\infty} b_3(7n + 6)q^n = 4\psi(q)\psi(q^6) - \psi(q^7)\varphi(q^{21}), \quad (1.4)$$

where k is a positive integer so that $2k + 1$ is odd prime.

Theorem 1.2 *Let $a_2(n)$ and $b_2(n)$ be the number of representations of n as $\triangle_1 + 4\triangle_2$ and $\triangle + 2\square$, respectively. Then*

$$a_2(625n + 390) - 2a_2(25n + 15) + a_2(n) = 0, \quad (1.5)$$

$$b_2(625n + 78) - 2b_2(25n + 3) + b_2(n) = 0. \quad (1.6)$$

The identities (1.5) and (1.6) are analogous to the Hirschhorn [2] identities

$$g(625n + 390) - 6g(25n + 15) + 25g(n) = 0,$$
$$h(625n + 78) - 6h(25n + 3) + 25h(n) = 0.$$

Corollary 1.1 *Let $b_k(n)$ be the number of representations of n as $\triangle + k\square$. Then for each n, $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} b_1\left(3^{2\alpha}n + \frac{3^{2\alpha} - 1}{8}\right) q^n \equiv 3^\alpha \psi(q)\varphi(q) \quad (\mathrm{mod}\ 2^2), \qquad (1.7)$$

$$\sum_{n=0}^{\infty} b_2\left(5^{2\alpha}n + \frac{5^{2\alpha} - 1}{8}\right) q^n \equiv 3^\alpha \psi(q)\varphi(q^2) \quad (\mathrm{mod}\ 2^2), \qquad (1.8)$$

$$\sum_{n=0}^{\infty} b_3\left(7^{2\alpha}n + \frac{7^{2\alpha} - 1}{8}\right) q^n \equiv 3^\alpha \psi(q)\varphi(q^3) \quad (\mathrm{mod}\ 2^2). \qquad (1.9)$$

Corollary 1.2 *Let $a_2(n)$ and $b_2(n)$ be the number of representations of n as $\triangle_1 + 4\triangle_2$ and $\triangle + 2\square$, respectively. Then for all n, $\alpha \geq 0$,*

$$a_2\left(5^{4\alpha}n + \frac{5(5^{4\alpha} - 1)}{8}\right) \equiv a_2(n) \quad (\mathrm{mod}\ 2), \qquad (1.10)$$

$$b_2\left(5^{4\alpha}n + \frac{5^{4\alpha} - 1}{8}\right) \equiv b_2(n) \quad (\mathrm{mod}\ 2). \qquad (1.11)$$

2 Preliminaries

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \qquad (2.1)$$

$$= (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty . \qquad (2.2)$$

Important special cases of $f(a, b)$ are

$$\varphi(q) := f(q, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty \qquad (2.3)$$

and

$$\psi(q) := \frac{1}{2} f(1, q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{(n^2+n)/2} = \sum_{n=-\infty}^{\infty} q^{2n^2+n} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}. \qquad (2.4)$$

From (2.3) and (2.4), we can see that

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{2.5}$$

3 Proofs of main results

Proof of Theorem 1.1 Let us write

$$E_k(q) = \psi(q)\psi(q^{2k}).$$

Using (2.4) in $E_k(q)$, we obtain

$$E_k(q) = \sum_{m=-\infty}^{\infty} q^{2m^2+m} \sum_{n=-\infty}^{\infty} q^{2k(2n^2+n)}. \tag{3.1}$$

Replacing q by q^8 and then multiplying both sides of (3.1) by q^{2k+1}, we find that

$$q^{2k+1}E_k(q^8) = \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+2k(4n+1)^2}$$

$$= \sum_{x,y\equiv 1 \ (\mathrm{mod}\ 4)} q^{x^2+2ky^2}. \tag{3.2}$$

If we write

$$E_k(q) = E_k^0(q) + E_k^1(q) + \cdots + E_k^{2k}(q), \tag{3.3}$$

where $E_k^i(q)$ consists of all those terms in $E_k(q)$ in which powers of q are congruent to i modulo $2k+1$, then from (3.2) and (3.3), we see that

$$q^{2k+1}E_k^0(q^8) = \sum_{x,y\equiv 1 \ (\mathrm{mod}\ 4)} q^{x^2+2ky^2}, \tag{3.4}$$

where $x^2 + 2ky^2 \equiv 0 \ (\mathrm{mod}\ 2k+1)$. The solution of $x^2 + 2ky^2 \equiv 0 \ (\mathrm{mod}\ 2k+1)$ is $x \equiv \pm y \ (\mathrm{mod}\ 2k+1)$, and from (3.4) it follows that

$$q^{2k+1}E_k^0(q^8) = \sum_{\substack{x\equiv y\equiv 1 \ (\mathrm{mod}\ 4)\\x\equiv y \ (\mathrm{mod}\ 2k+1)}} q^{x^2+2ky^2} + \sum_{\substack{x\equiv y\equiv 1 \ (\mathrm{mod}\ 4)\\x\equiv -y \ (\mathrm{mod}\ 2k+1)}} q^{x^2+2ky^2}$$

$$- \sum_{\substack{x\equiv y\equiv 1 \ (\mathrm{mod}\ 4)\\x\equiv y\equiv 0 \ (\mathrm{mod}\ 2k+1)}} q^{x^2+2ky^2}. \tag{3.5}$$

In the first sum of (3.5), we have $x \equiv y \ (\mathrm{mod}\ 2k+1)$, $4m+1 \equiv 4n+1 \ (\mathrm{mod}\ 2k+1)$, $m-n \equiv 0 \ (\mathrm{mod}\ 2k+1)$, $m-n = (2k+1)u$, $m+2kn = (2k+1)v$, $m = 2ku+v$, $n = v-u$,

$$x = 8ku + 4v + 1, \text{ and } y = 4v - 4u + 1. \tag{3.6}$$

In the second sum of (3.5), $x \equiv -y \pmod{2k+1}$, $4m+1 \equiv -4n-1 \pmod{2k+1}$, $4(m+n) \equiv -2 \pmod{2k+1}$, $4(m+n) \equiv -2+2(2k+1) \pmod{2k+1}$, $m+n \equiv k \pmod{2k+1}$, $m+n = (2k+1)v+k$, $m-2kn = (2k+1)u+k$, $m = u+2kv+k$, $n = v - u$,

$$x = 4u + 8kv + 4k + 1, \text{ and } y = 4v - 4u + 1. \tag{3.7}$$

In the third sum of (3.5),

$$x = 4(2k+1)u + 2k + 1 \text{ and } y = 4(2k+1)v + 2k + 1. \tag{3.8}$$

Employing (3.6), (3.7), and (3.8) in (3.5), we obtain

$$q^{2k+1} E_k^0(q^8) = \sum_{u,v=-\infty}^{\infty} q^{(8ku+4v+1)^2+2k(4v-4u+1)^2}$$

$$+ \sum_{u,v=-\infty}^{\infty} q^{(4u+8kv+4k+1)^2+2k(4v-4u+1)^2}$$

$$- \sum_{u,v=-\infty}^{\infty} q^{((4u+1)(2k+1))^2+2k((4v+1)(2k+1))^2}$$

$$= \sum_{u,v=-\infty}^{\infty} q^{16(2k+1)v^2+8(2k+1)v+16(4k^2+2k)u^2+2k+1}$$

$$+ \sum_{u,v=-\infty}^{\infty} q^{16(2k+1)u^2+8(2k+1)u+16(4k^2+2k)v^2+16(4k^2+2k)v+16k^2+10k+1}$$

$$- \sum_{u,v=-\infty}^{\infty} q^{16(2k+1)^2u^2+8(2k+1)^2u+32k(2k+1)^2v^2+16k(2k+1)^2v+(2k+1)^3}.$$

On simplification, the above equation reduces to

$$E_k^0(q) = \sum_{v=-\infty}^{\infty} q^{(2k+1)(2v^2+v)} \sum_{u=-\infty}^{\infty} q^{4k(2k+1)u^2}$$

$$+ q^{k(2k+1)} \sum_{u=-\infty}^{\infty} q^{(2k+1)(2u^2+u)} \sum_{v=-\infty}^{\infty} q^{4k(2k+1)(v^2+v)}$$

$$- q^{\frac{k(k+1)(2k+1)}{2}} \sum_{u=-\infty}^{\infty} q^{(2k+1)^2(2u^2+u)} \sum_{v=-\infty}^{\infty} q^{2k(2k+1)^2(2v^2+v)}. \tag{3.9}$$

From (2.3), (2.4), and (3.9), we have

$$E_k^0(q) = \psi(q^{2k+1}) \left(\varphi(q^{4k(2k+1)}) + 2q^{k(2k+1)}\psi(q^{8k(2k+1)}) \right)$$
$$- q^{\frac{k(k+1)(2k+1)}{2}} \psi(q^{(2k+1)^2})\psi(q^{2k(2k+1)^2}). \tag{3.10}$$

From (2.5) and (3.10), we find that

$$E_k^0(q) = \psi(q^{2k+1})\varphi(q^{k(2k+1)}) - q^{\frac{k(k+1)(2k+1)}{2}} \psi(q^{(2k+1)^2})\psi(q^{2k(2k+1)^2}). \tag{3.11}$$

The identity (3.11) can be written as

$$\sum_{n=0}^{\infty} a_k((2k+1)n)q^{(2k+1)n} = \psi(q^{2k+1})\varphi(q^{k(2k+1)})$$
$$- q^{\frac{k(k+1)(2k+1)}{2}} \psi(q^{(2k+1)^2})\psi(q^{2k(2k+1)^2}). \tag{3.12}$$

Now replacing q^{2k+1} by q in (3.12), we arrive at (1.1).

Let us write $F_2(q) = \psi(q)\varphi(q)$, and using (2.5), we find that

$$F_2(q) = \psi(q) \left(\varphi(q^4) + 2q\psi(q^8) \right)$$
$$= \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+4n^2} + q \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+4(n^2+n)}. \tag{3.13}$$

Replacing q by q^8 and then multiplying both sides of (3.13) by q, we obtain

$$qF_2(q^8) = \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+2(4n)^2} + \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+2(4n+2)^2}$$
$$= \sum_{x\equiv1, \, y\equiv0 \, (\text{mod } 4)} q^{x^2+2y^2} + \sum_{x\equiv1, \, y\equiv2 \, (\text{mod } 4)} q^{x^2+2y^2}. \tag{3.14}$$

If we write
$$F_2(q) = F_2^0(q) + F_2^1(q) + F_2^2(q), \tag{3.15}$$

where $F_2^i(q)$ consists of all those terms in $F_2(q)$ in which powers of q are congruent to i modulo 3, then from (3.14) and (3.15), we see that

$$qF_2^1(q^8) = \sum_{x\equiv1, \, y\equiv0 \, (\text{mod } 4)} q^{x^2+2y^2} + \sum_{x\equiv1, \, y\equiv2 \, (\text{mod } 4)} q^{x^2+2y^2}, \tag{3.16}$$

where, in both sums, $x^2 + 2y^2 \equiv 0$ (mod 3). The solution of $x^2 + 2y^2 \equiv 0$ (mod 3) is $x \equiv \pm y$ (mod 3), and from (3.16) it follows that

$$
\begin{aligned}
& q F_2^1(q^8) \\
& = \sum_{\substack{x \equiv 1, \, y \equiv 0 \ (\text{mod } 4) \\ x \equiv y \ (\text{mod } 3)}} q^{x^2 + 2y^2} + \sum_{\substack{x \equiv 1, \, y \equiv 0 \ (\text{mod } 4) \\ x \equiv -y \ (\text{mod } 3)}} q^{x^2 + 2y^2} - \sum_{\substack{x \equiv 1, \, y \equiv 0 \ (\text{mod } 4) \\ x \equiv y \equiv 0 \ (\text{mod } 3)}} q^{x^2 + 2y^2} \\
& + \sum_{\substack{x \equiv 1, \, y \equiv 2 \ (\text{mod } 4) \\ x \equiv y \ (\text{mod } 3)}} q^{x^2 + 2y^2} + \sum_{\substack{x \equiv 1, \, y \equiv 2 \ (\text{mod } 4) \\ x \equiv -y \ (\text{mod } 3)}} q^{x^2 + 2y^2} - \sum_{\substack{x \equiv 1, \, y \equiv 2 \ (\text{mod } 4) \\ x \equiv y \equiv 0 \ (\text{mod } 3)}} q^{x^2 + 2y^2}.
\end{aligned}
\tag{3.17}
$$

In the first sum of (3.17), we have $4m + 1 \equiv 4n$ (mod 3), $4(m - n) \equiv -1$ (mod 3), $4(m - n) \equiv -1 + 9$ (mod 3), $m - n \equiv 2$ (mod 3), $m - n \equiv -1$ (mod 3), $m + 2n \equiv -1$ (mod 3), $m - n = -3u - 1$, $m + 2n = -3v - 1$, $m = -v - 2u - 1$, $n = u - v$,

$$
x = -4v - 8u - 3, \text{ and } y = 4u - 4v. \tag{3.18}
$$

In the second sum of (3.17), $4m + 1 \equiv -4n$ (mod 3), $4(m + n) \equiv -1$ (mod 3), $4(m + n) \equiv -1 + 9$ (mod 3), $m + n \equiv 2$ (mod 3), $m + n \equiv -1$ (mod 3), $m - 2n \equiv -1$ (mod 3), $m + n = -3u - 1$, $m - 2n = -3v - 1$, $m = -v - 2u - 1$, $n = v - u$,

$$
x = -4v - 8u - 3, \text{ and } y = 4v - 4u. \tag{3.19}
$$

In the third sum of (3.17),

$$
x = 12u + 3 \text{ and } y = 12v. \tag{3.20}
$$

In the fourth sum of (3.17), $4m + 1 \equiv 4n + 2$ (mod 3), $4(m - n) \equiv 1$ (mod 3), $m - n \equiv 1$ (mod 3), $m + 2n \equiv -2$ (mod 3), $m - n = 3u + 1$, $m + 2n = -3v - 2$, $m = 2u - v$, $n = -u - v - 1$,

$$
x = 8u - 4v + 1, \text{ and } y = -4u - 4v - 2. \tag{3.21}
$$

In the fifth sum of (3.17), $4m + 1 \equiv -4n - 2$ (mod 3), $4(m + n) \equiv -3$ (mod 3), $m + n \equiv 0$ (mod 3), $m - 2n \equiv 0$ (mod 3), $m + n = 3u$, $m - 2n = -3v$, $m = 2u - v$, $n = u + v$,

$$
x = 8u - 4v + 1, \text{ and } y = 4u + 4v + 2. \tag{3.22}
$$

In the sixth sum of (3.17),

$$
x = 12u + 3 \text{ and } y = 12v + 6. \tag{3.23}
$$

Invoking (3.18), (3.19), (3.20), (3.21), (3.22), and (3.23) in (3.17), we arrive at

$$
\begin{aligned}
q F_2^1(q^8) &= \sum_{u,v=-\infty}^{\infty} q^{(-4v-8u-3)^2+2(4u-4v)^2} + \sum_{u,v=-\infty}^{\infty} q^{(-4v-8u-3)^2+2(4v-4u)^2} \\
&\quad - \sum_{u,v=-\infty}^{\infty} q^{(12u+3)^2+2(12v)^2} + \sum_{u,v=-\infty}^{\infty} q^{(8u-4v+1)^2+2(-4u-4v-2)^2} \\
&\quad + \sum_{u,v=-\infty}^{\infty} q^{(8u-4v+1)^2+2(4u+4v+2)^2} - \sum_{u,v=-\infty}^{\infty} q^{(12u+3)^2+2(12v+6)^2} \\
&= \sum_{u,v=-\infty}^{\infty} q^{48(2u^2+u)+24(2v^2+v)+9} + \sum_{u,v=-\infty}^{\infty} q^{48(2u^2+u)+24(2v^2+v)+9} \\
&\quad - \sum_{u,v=-\infty}^{\infty} q^{72(2u^2+u)+288v^2+9} + \sum_{u,v=-\infty}^{\infty} q^{48(2u^2+u)+24(2v^2+v)+9} \\
&\quad + \sum_{u,v=-\infty}^{\infty} q^{48(2u^2+u)+24(2v^2+v)+9} - \sum_{u,v=-\infty}^{\infty} q^{72(2u^2+u)+288(v^2+v)+81}.
\end{aligned}
$$

Dividing throughout by q, replacing q^8 by q, and separating the terms of u and v in the above equation, we obtain

$$
\begin{aligned}
F_2^1(q) &= 4q \sum_{u=-\infty}^{\infty} q^{12u^2+6u} \sum_{v=-\infty}^{\infty} q^{6v^2+3v} - q \sum_{u=-\infty}^{\infty} q^{18u^2+9u} \sum_{v=-\infty}^{\infty} q^{36v^2} \\
&\quad - q^{10} \sum_{u=-\infty}^{\infty} q^{18u^2+9u} \sum_{v=-\infty}^{\infty} q^{36v^2+36v} \\
&= 4q\psi(q^6)\psi(q^3) - q\psi(q^9)\varphi(q^{36}) - 2q^{10}\psi(q^9)\psi(q^{72}) \\
&= 4q\psi(q^6)\psi(q^3) - q\psi(q^9)\left(\varphi(q^{36}) + 2q^9\psi(q^{72})\right) \\
&= 4q\psi(q^6)\psi(q^3) - q\psi(q^9)\varphi(q^9).
\end{aligned}
$$

The above identity can be written as

$$
\sum_{n=0}^{\infty} b_1(3n+1)q^{3n+1} = 4q\psi(q^6)\psi(q^3) - q\psi(q^9)\varphi(q^9). \tag{3.24}
$$

Dividing throughout by q and then replacing q^3 by q in (3.24), we obtain (1.2).

Let $F_4(q) = \psi(q)\varphi(q^2)$, and using (2.5), we find that

$$F_4(q) = \psi(q)\left(\varphi(q^8) + 2q^2\psi(q^{16})\right)$$

$$= \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+8n^2} + q^2 \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+8(n^2+n)}. \quad (3.25)$$

Replacing q by q^8 and then multiplying throughout by q in the above equation, we obtain

$$q F_4(q^8) = \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+4(4n)^2} + \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+4(4n+2)^2}$$

$$= \sum_{x\equiv 1,\ y\equiv 0\ (\text{mod}\ 4)} q^{x^2+4y^2} + \sum_{x\equiv 1,\ y\equiv 2\ (\text{mod}\ 4)} q^{x^2+4y^2}. \quad (3.26)$$

Let us write

$$F_4(q) = F_4^0(q) + F_4^1(q) + F_4^2(q) + F_4^3(q) + F_4^4(q), \quad (3.27)$$

where $F_4^i(q)$ consists of all those terms in $F_4(q)$ in which powers of q are congruent to i modulo 5. From (3.26) and (3.27), we see that

$$q F_4^3(q^8) = \sum_{x\equiv 1,\ y\equiv 0\ (\text{mod}\ 4)} q^{x^2+4y^2} + \sum_{x\equiv 1,\ y\equiv 2\ (\text{mod}\ 4)} q^{x^2+4y^2}, \quad (3.28)$$

where, in both sums, $x^2 + 4y^2 \equiv 0 \pmod{5}$. The solution of $x^2 + 4y^2 \equiv 0 \pmod 5$ is $x \equiv \pm y \pmod 5$, and it follows that

$$q F_4^3(q^8)$$

$$= \sum_{\substack{x\equiv 1,\ y\equiv 0\ (\text{mod}\ 4)\\ x\equiv y\ (\text{mod}\ 5)}} q^{x^2+4y^2} + \sum_{\substack{x\equiv 1,\ y\equiv 0\ (\text{mod}\ 4)\\ x\equiv -y\ (\text{mod}\ 5)}} q^{x^2+4y^2} - \sum_{\substack{x\equiv 1,\ y\equiv 0\ (\text{mod}\ 4)\\ x\equiv y\equiv 0\ (\text{mod}\ 5)}} q^{x^2+4y^2}$$

$$+ \sum_{\substack{x\equiv 1,\ y\equiv 2\ (\text{mod}\ 4)\\ x\equiv y\ (\text{mod}\ 5)}} q^{x^2+4y^2} + \sum_{\substack{x\equiv 1,\ y\equiv 2\ (\text{mod}\ 4)\\ x\equiv -y\ (\text{mod}\ 5)}} q^{x^2+4y^2} - \sum_{\substack{x\equiv 1,\ y\equiv 2\ (\text{mod}\ 4)\\ x\equiv y\equiv 0\ (\text{mod}\ 5)}} q^{x^2+4y^2}. $$

$$(3.29)$$

In the first sum of (3.29), we have $x \equiv y \pmod 5$, $4m+1 \equiv 4n \pmod 5$, $4(m-n) \equiv -1 \pmod 5$, $m-n \equiv 1 \pmod 5$, $m+4n \equiv 1 \pmod 5$, $m-n = 5u+1$, $m+4n = 5v+1$, $m = 4u+v+1$, $n = v-u$,

$$x = 16u + 4v + 5, \text{ and } y = 4v - 4u. \quad (3.30)$$

In the second sum of (3.29), $4m + 1 \equiv -4n \pmod 5$, $4(m + n) \equiv -1 \pmod 5$, $4(m+n) \equiv -1+5 \pmod 5$, $m+n \equiv 1 \pmod 5$, $m-4n \equiv 1 \pmod 5$, $m+n = 5u+1$, $m - 4n = 5v + 1$, $m = 4u + v + 1$, $n = u - v$,

$$x = 16u + 4v + 5, \text{ and } y = 4u - 4v. \tag{3.31}$$

In the third sum of (3.29),

$$x = 20u + 5 \text{ and } y = 20v. \tag{3.32}$$

In the fourth sum of (3.29), $4m + 1 \equiv 4n + 2 \pmod 5$, $4(m - n) \equiv 1 \pmod 5$, $m - n \equiv -1 \pmod 5$, $m + 4n \equiv -1 \pmod 5$, $m - n = -5u - 1$, $m + 4n = 5v - 1$, $m = v - 4u - 1$, $n = u + v$,

$$x = 4v - 16u - 3, \text{ and } y = 4u + 4v + 2. \tag{3.33}$$

In the fifth sum of (3.29), $4m + 1 \equiv -4n - 2 \pmod 5$, $4(m + n) \equiv -3 \pmod 5$, $m + n \equiv -2 \pmod 5$, $m - 4n \equiv 3 \pmod 5$, $m + n = -5u - 2$, $m - 4n = 5v + 3$, $m = v - 4u - 1$, $n = -v - u - 1$,

$$x = 4v - 16u - 3, \text{ and } y = -4u - 4v - 2. \tag{3.34}$$

In the sixth sum of (3.29),

$$x = 20u + 5 \text{ and } y = 20v + 10. \tag{3.35}$$

Employing (3.30), (3.31), (3.32), (3.33), (3.34), and (3.35) in (3.29), we obtain

$$
\begin{aligned}
q F_4^3(q^8) &= \sum_{u,v=-\infty}^{\infty} q^{(16u+4v+5)^2+4(4v-4u)^2} + \sum_{u,v=-\infty}^{\infty} q^{(16u+4v+5)^2+4(4u-4v)^2} \\
&\quad - \sum_{u,v=-\infty}^{\infty} q^{(20u+5)^2+4(20v)^2} + \sum_{u,v=-\infty}^{\infty} q^{(4v-16u-3)^2+4(4v+4u+2)^2} \\
&\quad + \sum_{u,v=-\infty}^{\infty} q^{(4v-16u-3)^2+4(-4v-4u-2)^2} - \sum_{u,v=-\infty}^{\infty} q^{(20u+5)^2+4(20v+10)^2} \\
&= \sum_{u,v=-\infty}^{\infty} q^{40(2v^2+v)+160(2u^2+u)+25} + \sum_{u,v=-\infty}^{\infty} q^{40(2v^2+v)+160(2u^2+u)+25} \\
&\quad - \sum_{u,v=-\infty}^{\infty} q^{200(2u^2+u)+1600v^2+25} + \sum_{u,v=-\infty}^{\infty} q^{40(2v^2+v)+160(2u^2+u)+25} \\
&\quad + \sum_{u,v=-\infty}^{\infty} q^{40(2v^2+v)+160(2u^2+u)+25} - \sum_{u,v=-\infty}^{\infty} q^{200(2u^2+u)+1600(v^2+v)+425}.
\end{aligned}
$$

Dividing throughout by q, replacing q^8 by q, and then separating the terms containing u and v in the above equation, we find that

$$F_4^3(q) = 4q^3 \sum_{v=-\infty}^{\infty} q^{10v^2+5v} \sum_{u=-\infty}^{\infty} q^{40u^2+20u} - q^3 \sum_{u=-\infty}^{\infty} q^{50u^2+25u} \sum_{v=-\infty}^{\infty} q^{200v^2}$$

$$- q^{53} \sum_{u=-\infty}^{\infty} q^{50u^2+25u} \sum_{v=-\infty}^{\infty} q^{200v^2+200v}. \tag{3.36}$$

Using (2.3) and (2.4) in (3.36), we obtain

$$F_4^3(q) = 4q^3 \psi(q^5)\psi(q^{20}) - q^3 \psi(q^{25}) \left(\varphi(q^{200}) + 2q^{50}\psi(q^{400}) \right). \tag{3.37}$$

From (2.5) and (3.37), we see that

$$F_4^3(q) = 4q^3 \psi(q^5)\psi(q^{20}) - q^3 \psi(q^{25})\varphi(q^{50}). \tag{3.38}$$

From (3.38), it follows that

$$\sum_{n=0}^{\infty} b_2\,(5n+3)\,q^{5n+3} = 4q^3 \psi(q^5)\psi(q^{20}) - q^3 \psi(q^{25})\varphi(q^{50}). \tag{3.39}$$

Dividing throughout by q^3 and then replacing q^5 by q in (3.39), we arrive at (1.3).

Let $F_6(q) = \psi(q)\,\varphi(q^3)$, and using (2.5), we find that

$$F_6(q) = \psi(q) \left(\varphi(q^{12}) + 2q^3\psi(q^{24}) \right)$$

$$= \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+12n^2} + q \sum_{m,n=-\infty}^{\infty} q^{2m^2+m+12(n^2+n)}. \tag{3.40}$$

Replacing q by q^8 and then multiplying both sides of (3.40) by q, we see that

$$q\,F_6(q^8) = \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+6(4n)^2} + \sum_{m,n=-\infty}^{\infty} q^{(4m+1)^2+6(4n+2)^2}$$

$$= \sum_{x\equiv 1,\ y\equiv 0\ (\text{mod } 4)} q^{x^2+6y^2} + \sum_{x\equiv 1,\ y\equiv 2\ (\text{mod } 4)} q^{x^2+6y^2}. \tag{3.41}$$

If we write

$$F_6(q) = F_6^0(q) + F_6^1(q) + \cdots + F_6^6(q), \tag{3.42}$$

where $F_6^i(q)$ consists of all those terms in $F_6(q)$ in which powers of q are congruent to i modulo 7, then from (3.41) and (3.42), we find that

$$q F_6^6(q^8) = \sum_{x \equiv 1,\ y \equiv 0 \ (\text{mod } 4)} q^{x^2+6y^2} + \sum_{x \equiv 1,\ y \equiv 2 \ (\text{mod } 4)} q^{x^2+6y^2}, \qquad (3.43)$$

where, in both sums, $x^2 + 6y^2 \equiv 0$ (mod 7). The solution of $x^2 + 6y^2 \equiv 0$ (mod 7) is $x \equiv \pm y$ (mod 7), and from (3.43), it follows that

$$
\begin{aligned}
& q F_6^6(q^8) \\
&= \sum_{\substack{x \equiv 1,\ y \equiv 0 \ (\text{mod } 4) \\ x \equiv y \ (\text{mod } 7)}} q^{x^2+6y^2} + \sum_{\substack{x \equiv 1,\ y \equiv 0 \ (\text{mod } 4) \\ x \equiv -y \ (\text{mod } 7)}} q^{x^2+6y^2} - \sum_{\substack{x \equiv 1,\ y \equiv 0 \ (\text{mod } 4) \\ x \equiv y \equiv 0 \ (\text{mod } 7)}} q^{x^2+6y^2} \\
& + \sum_{\substack{x \equiv 1,\ y \equiv 2 \ (\text{mod } 4) \\ x \equiv y \ (\text{mod } 7)}} q^{x^2+6y^2} + \sum_{\substack{x \equiv 1,\ y \equiv 2 \ (\text{mod } 4) \\ x \equiv -y \ (\text{mod } 7)}} q^{x^2+6y^2} - \sum_{\substack{x \equiv 1,\ y \equiv 2 \ (\text{mod } 4) \\ x \equiv y \equiv 0 \ (\text{mod } 7)}} q^{x^2+6y^2}.
\end{aligned}
\qquad (3.44)
$$

In the first sum of (3.44), we have $4m + 1 \equiv 4n$ (mod 7), $4(m - n) \equiv -1$ (mod 7), $4(m-n) \equiv -8 \,(\text{mod } 7), m-n \equiv -2 \,(\text{mod } 7), m+6n \equiv -2 \,(\text{mod } 7), m-n = -7v-2, m + 6n = -7u - 2, m = -u - 6v - 2, n = v - u,$

$$x = -4u - 24v - 7, \text{ and } y = 4v - 4u. \qquad (3.45)$$

In the second sum of (3.44), $4m + 1 \equiv -4n$ (mod 7), $4(m + n) \equiv -1$ (mod 7), $4(m+n) \equiv -8 \,(\text{mod } 7), m+n \equiv -2 \,(\text{mod } 7), m-6n \equiv -2 \,(\text{mod } 7), m+n = -7v-2, m - 6n = -7u - 2, m = -6v - u - 2, n = u - v,$

$$x = -24v - 4u - 7, \text{ and } y = 4u - 4v. \qquad (3.46)$$

In the third sum of (3.44),

$$x = 28u + 7 \text{ and } y = 28v. \qquad (3.47)$$

In the fourth sum of (3.44), $4m + 1 \equiv 4n + 2$ (mod 7), $4(m - n) \equiv 1$ (mod 7), $m - n \equiv 2$ (mod 7), $m + 6n \equiv -5$ (mod 7), $m - n = 7v + 2, m + 6n = -7u - 5, m = 6v - u + 1, n = -v - u - 1,$

$$x = 24v - 4u + 5, \text{ and } y = -4u - 4v - 2. \qquad (3.48)$$

In the fifth sum of (3.44), $4m + 1 \equiv -4n - 2$ (mod 7), $4(m + n) \equiv -3$ (mod 7), $m + n \equiv 1$ (mod 7), $m - 6n \equiv 1$ (mod 7), $m + n = 7v + 1, m - 6n = -7u + 1, m = 6v - u + 1, n = u + v,$

$$x = 24v - 4u + 5, \text{ and } y = 4u + 4v + 2. \qquad (3.49)$$

In the sixth sum of (3.44),

$$x = 28u + 7 \text{ and } y = 28v + 14. \tag{3.50}$$

Invoking (3.45), (3.46), (3.47), (3.48), (3.49), and (3.50) in (3.44), we find that

$$
\begin{aligned}
q F_6^6(q^8) = {} & \sum_{u,v=-\infty}^{\infty} q^{(-4u-24v-7)^2+6(4v-4u)^2} + \sum_{u,v=-\infty}^{\infty} q^{(-24v-4u-7)^2+6(4u-4v)^2} \\
& - \sum_{u,v=-\infty}^{\infty} q^{(28u+7)^2+6(28v)^2} + \sum_{u,v=-\infty}^{\infty} q^{(24v-4u+5)^2+6(-4u-4v-2)^2} \\
& + \sum_{u,v=-\infty}^{\infty} q^{(24v-4u+5)^2+6(4u+4v+2)^2} - \sum_{u,v=-\infty}^{\infty} q^{(28u+7)^2+6(28v+14)^2}.
\end{aligned}
\tag{3.51}
$$

After simplification of (3.51), we obtain

$$
\begin{aligned}
q F_6^6(q^8) = {} & \sum_{u,v=-\infty}^{\infty} q^{56(2u^2+u)+336(2v^2+v)+49} + \sum_{u,v=-\infty}^{\infty} q^{56(2u^2+u)+336(2v^2+v)+49} \\
& - \sum_{u,v=-\infty}^{\infty} q^{392(2u^2+u)+4704v^2+49} + \sum_{u,v=-\infty}^{\infty} q^{56(2u^2+u)+336(2v^2+v)+49} \\
& + \sum_{u,v=-\infty}^{\infty} q^{56(2u^2+u)+336(2v^2+v)+49} - \sum_{u,v=-\infty}^{\infty} q^{392(2u^2+u)+4704(v^2+v)+1225}.
\end{aligned}
\tag{3.52}
$$

Dividing throughout by q, replacing q^8 by q, and separating the terms of u and v in (3.52), we obtain

$$
\begin{aligned}
F_6^6(q) = {} & 4q^6 \sum_{u=-\infty}^{\infty} q^{14u^2+7u} \sum_{v=-\infty}^{\infty} q^{84v^2+42v} - q^6 \sum_{u=-\infty}^{\infty} q^{98u^2+49u} \sum_{v=-\infty}^{\infty} q^{588v^2} \\
& - q^{153} \sum_{u=-\infty}^{\infty} q^{98u^2+49u} \sum_{v=-\infty}^{\infty} q^{588v^2+588v}.
\end{aligned}
\tag{3.53}
$$

From (2.3), (2.4), and (3.53), we see that

$$F_6^6(q) = 4q^6 \psi(q^7)\psi(q^{42}) - q^6 \psi(q^{49})\left(\varphi(q^{588}) + 2q^{147}\psi(q^{1176})\right). \tag{3.54}$$

From (2.5) and (3.54), we obtain

$$F_6^6(q) = 4q^6 \psi(q^7)\psi(q^{42}) - q^6 \psi(q^{49})\varphi(q^{147}). \tag{3.55}$$

The above equation can be written as

$$\sum_{n=0}^{\infty} b_3(7n+6)q^{7n+6} = 4q^6\psi(q^7)\psi(q^{42}) - q^6\psi(q^{49})\varphi(q^{147}).$$ (3.56)

Dividing throughout by q^6 and then replacing q^7 by q in (3.56), we arrive at (1.4). This completes the proof. □

Proof of Theorem 1.2 Put $k=2$ in (1.1), we find that

$$\sum_{n=0}^{\infty} a_2(5n)q^n = \sum_{n=0}^{\infty} b_2(n)q^n - \sum_{n=0}^{\infty} a_2(n)q^{5n+3},$$ (3.57)

which yields
$$a_2(n) + a_2(25n+15) = b_2(5n+3).$$ (3.58)

From (1.3), it follows that

$$\sum_{n=0}^{\infty} b_2(5n+3)q^n = 4\sum_{n=0}^{\infty} a_2(n)q^n - \sum_{n=0}^{\infty} b_2(n)q^{5n}.$$ (3.59)

Equating coefficients of q^{5n} in (3.59), we obtain

$$b_2(n) + b_2(25n+3) = 4a_2(5n).$$ (3.60)

Equations (1.5) and (1.6) follow from (3.58) and (3.60). This completes the proof. □

Proof of Corollary 1.1 From Eq. (1.2), we see that

$$\sum_{n=0}^{\infty} b_1(3n+1)q^n \equiv 3\psi(q^3)\varphi(q^3) \pmod 4.$$ (3.61)

Equating coefficients of q^{3n} on both sides of (3.61), we find that

$$\sum_{n=0}^{\infty} b_1(9n+1)q^n \equiv 3\psi(q)\varphi(q) \pmod 4.$$ (3.62)

From (3.62) and by mathematical induction, we arrive at (1.7). Similarly, we obtain (1.8) and (1.9) from (1.3) and (1.4), respectively. This completes the proof. □

Proof of Corollary 1.2 In view of Eq. (1.5), we see that

$$a_2(625n+390) \equiv a_2(n) \pmod 2.$$ (3.63)

By mathematical induction on n in (3.63), we arrive at (1.10). Similarly, we obtain (1.11) from (1.6). This completes the proof. □

References

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