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Ramanujan-style proof of $p_{-3}(11n+7) \equiv 0 \pmod{11}$

Bernard L. S. Lin[1]

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Abstract In this note, we establish two identities of $(q;q)_\infty^8$ by using Jacobi's four-square theorem and two of Ramanujan's identities. As an important consequence, we present one Ramanujan-style proof of the congruence $p_{-3}(11n+7) \equiv 0 \pmod{11}$, where $p_{-3}(n)$ denotes the number of 3-color partitions of n.

Keywords Jacobi's four-square theorem · Ramanujan's identity · Partition congruence

Mathematics Subject Classification 05A17 · 11P83

1 Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers, called parts, whose sum is n. Let $p(n)$ denote the number of partitions of n. We follow the convention that $p(0) = 1$. It is well known that the generating function for $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Throughout this note, we adopt the following notation:

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✉ Bernard L. S. Lin
linlsjmu@163.com

1 School of Sciences, Jimei University, Xiamen 361021, China

$$(a;q)_\infty = \prod_{n=1}^{\infty} \left(1 - aq^{n-1}\right), \quad |q| < 1.$$

The most famous results for $p(n)$ are the so called Ramanujan's congruences: for $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Ramanujan [25, Paper 30], Atkin and Swinnerton-Dyer [1], Winquist [26], Garvan [12], Garvan and Stanton [13], Hirschhorn [15–18], and Marivani [22] have given different proofs of the congruence (1.1). It is worth mentioning that Winquist [26] found an interesting identity which plays an important role in proving Ramanujan's congruence modulo 11. In fact, Winquist used his identity to establish an identity for $(q;q)_\infty^{10}$ from which the congruence (1.1) follows easily. Later several proofs of Winquist's identity are found and new identities for $(q;q)_\infty^{10}$ are established, see [3,6,8–10,20,21,23], for example.

Recently, Hirschhorn [19] presented a most elementary, simple, beautiful proof of the congruence (1.1). Later, Gnang and Zeilberger [14] generalized and implemented Hirschhorn's amazing algorithm for proving Ramanujan-type congruences. They considered $p_{-a}(n)$, which is defined by

$$\sum_{n=0}^{\infty} p_{-a}(n)q^n = \frac{1}{(q;q)_\infty^a}. \tag{1.4}$$

There are many known Ramanujan-type congruences for $p_{-a}(n)$. Boylan [4] has found all of them for a odd and ≤ 47. For example,

$$p_{-3}(11n+7) \equiv 0 \pmod{11}. \tag{1.5}$$

Every such congruence can be checked by using impressive algorithm of Radu [24]. Although Radu's algorithm is powerful, it is not elementary. Based on this, Zeilberger said that "it is still interesting (at least to us!) to find a 'Ramanujan-style,' or 'Hirschhorn-style' proof."

In this note, we aim to give one "Ramanujan-style" proof of the congruence (1.5). To this end, we will establish two identities for $(q;q)_\infty^8$ by using Ramanujan's two identities. Although the series for $(q;q)_\infty^8$ have been considered by several authors, see [5,7,11] for example, our approach is more elementary.

2 Preliminaries

We first introduce two Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$, defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$
$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Two lemmas related to $\varphi(q)$ and $\psi(q)$ are presented as follows.

Lemma 2.1

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{2.1}$$
$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{2.2}$$
$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{2.3}$$

Proof See [2, Cor. 1.3.4] for a proof. □

Lemma 2.2

$$\varphi(-q) = \varphi(q^4) - 2q\psi(q^8). \tag{2.4}$$

Proof This identity can be derived by using series manipulations, and we omit the details here. □

The rest of this section are Jacobi's four-square theorem and two identities of Ramanujan, which are extremely useful to our later proof.

Lemma 2.3 ([Jacobi's Four-Square Theorem])

$$\varphi(-q)^4 = 1 - 8\sum_{n=1}^{\infty}\left(\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}} - \frac{2nq^{2n}}{1+q^{2n}}\right). \tag{2.5}$$

Proof See [2, Thm. 3.3.1] for a proof of (2.5). □

Lemma 2.4

$$\sum_{n=-\infty}^{\infty} (6n+1)q^{3n^2+n} = \frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2}, \tag{2.6}$$
$$\sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n} = \frac{(q;q)_\infty^2 (q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}. \tag{2.7}$$

Proof For the proofs of (2.6) and (2.7), see [2, Cor. 1.3.21 and Cor. 1.3.22]. □

3 Ramanujan-style proof

We first establish two identities of $(q;q)_\infty^8$, either of which can be employed to produce the desired Ramanujan-style proof.

Theorem 3.1

$$3(q;q)_\infty^8 = 4\left(\sum_{m=-\infty}^{\infty}(3m+1)^3 q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right) - \left(\sum_{m=-\infty}^{\infty}(6m+1)^3 q^{3m^2+m}\right)\times\left(\sum_{n=0}^{\infty} q^{n^2+n}\right). \tag{3.1}$$

Proof Differentiating both sides of (2.6) with respect to q, we find that

$$\sum_{n=-\infty}^{\infty}(6n+1)\big(3n^2+n\big)q^{3n^2+n} = \frac{\big(q^2;q^2\big)_\infty^5}{\big(q^4;q^4\big)_\infty^2}\left(5\sum_{n=0}^{\infty}\frac{-2nq^{2n}}{1-q^{2n}} - 2\sum_{n=0}^{\infty}\frac{-4nq^{4n}}{1-q^{4n}}\right), \tag{3.2}$$

and thus,

$$\sum_{n=-\infty}^{\infty}\left((6n+1)^3-(6n+1)\right)q^{3n^2+n} = 12\frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2}\left(5\sum_{n=0}^{\infty}\frac{-2nq^{2n}}{1-q^{2n}} - 2\sum_{n=0}^{\infty}\frac{-4nq^{4n}}{1-q^{4n}}\right). \tag{3.3}$$

Applying (2.6), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(6n+1)^3 q^{3n^2+n}\times\sum_{n=0}^{\infty}q^{n^2+n} &= \sum_{n=-\infty}^{\infty}(6n+1)q^{3n^2+n}\times\sum_{n=0}^{\infty}q^{n^2+n} \\ &\quad + 12\big(q^2;q^2\big)_\infty^4\left(5\sum_{n=0}^{\infty}\frac{-2nq^{2n}}{1-q^{2n}} - 2\sum_{n=0}^{\infty}\frac{-4nq^{4n}}{1-q^{4n}}\right) \\ &= 12\big(q^2;q^2\big)_\infty^4\sum_{n=0}^{\infty}\left(\frac{-10nq^{2n}}{1-q^{2n}}+\frac{8nq^{4n}}{1-q^{4n}}\right) \\ &\quad + \big(q^2;q^2\big)_\infty^4.\end{aligned} \tag{3.4}$$

Differentiating both sides of (2.7) with respect to q, we obtain

$$\sum_{m=-\infty}^{\infty} (3m+1)\big(3m^2+2m\big)q^{3m^2+2m} = \frac{(q;q)_\infty^2\big(q^4;q^4\big)_\infty^2}{\big(q^2;q^2\big)_\infty}\sum_{n=1}^{\infty}\left(\frac{-2nq^n}{1-q^n}-\frac{8nq^{4n}}{1-q^{4n}}+\frac{2nq^{2n}}{1-q^{2n}}\right),$$

and thus

$$\sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m} = \sum_{m=-\infty}^{\infty} (3m+1)q^{3m^2+2m} + 3\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} \times\sum_{n=1}^{\infty}\left(\frac{-2nq^n}{1-q^n}-\frac{8nq^{4n}}{1-q^{4n}}+\frac{2nq^{2n}}{1-q^{2n}}\right).$$

By (2.1) and (2.7), we deduce that

$$\begin{aligned}\sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m} \times \sum_{n=-\infty}^{\infty} q^{n^2} &= \sum_{m=-\infty}^{\infty} (3m+1)q^{3m^2+2m} \times \sum_{n=-\infty}^{\infty} q^{n^2} \\ &\quad + 3(q^2;q^2)_\infty^4 \sum_{n=1}^{\infty}\left(\frac{-2nq^n}{1-q^n}-\frac{8nq^{4n}}{1-q^{4n}}+\frac{2nq^{2n}}{1-q^{2n}}\right) \\ &= (q^2;q^2)_\infty^4 \sum_{n=1}^{\infty}\left(\frac{-6nq^n}{1-q^n}-\frac{24nq^{4n}}{1-q^{4n}}+\frac{6nq^{2n}}{1-q^{2n}}\right). \\ &\quad + (q^2;q^2)_\infty^4. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we find that

$$\begin{aligned} &4\left(\sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m}\right) \times \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right) \\ &\quad -\left(\sum_{m=-\infty}^{\infty} (6m+1)^3 q^{3m^2+m}\right) \times \left(\sum_{n=0}^{\infty} q^{n^2+n}\right) \\ &\quad = 3(q^2;q^2)_\infty^4 \left(1-8\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}+\frac{8nq^{4n}}{1-q^{4n}}-\frac{6nq^{2n}}{1-q^{2n}}\right)\right) \\ &\quad = 3(q^2;q^2)_\infty^4 \left(1-8\sum_{n=1}^{\infty}\left(\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}-\frac{2nq^{2n}}{1+q^{2n}}\right)\right), \end{aligned} \tag{3.6}$$

where the last equation follows from the following fact:

$$\begin{aligned}
&\frac{nq^n}{1-q^n}+\frac{8nq^{4n}}{1-q^{4n}}-\frac{6nq^{2n}}{1-q^{2n}}\\
&=\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}+\frac{8nq^{4n}}{1-q^{4n}}-\frac{4nq^{2n}}{1-q^{2n}}\\
&=\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}+\frac{(4n-2)q^{4n-2}}{1-q^{4n-2}}+\frac{8nq^{4n}}{1-q^{4n}}-\frac{4nq^{2n}}{1-q^{2n}}\\
&=\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}+\frac{4nq^{4n}}{1-q^{4n}}-\frac{2nq^{2n}}{1-q^{2n}}\\
&=\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}-\frac{2nq^{2n}}{1+q^{2n}}.
\end{aligned}$$

Applying (2.5) in (3.6), we conclude that

$$\begin{aligned}
&4\left(\sum_{m=-\infty}^{\infty}(3m+1)^3q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)\\
&\quad-\left(\sum_{m=-\infty}^{\infty}(6m+1)^3q^{3m^2+m}\right)\times\left(\sum_{n=0}^{\infty}q^{n^2+n}\right)\\
&\quad=3(q^2;q^2)_\infty^4\varphi^4(-q)\\
&\quad=3(q;q)_\infty^8.
\end{aligned}$$

This completes the proof. □

It is interesting to present another identity for $(q;q)_\infty^8$ which can be derived from (3.1).

Theorem 3.2

$$\begin{aligned}
3(q;q)_\infty^8&=4\left(\sum_{m=-\infty}^{\infty}(3m+1)^3q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty}(-q)^{\frac{n^2}{4}}\right)\\
&\quad-\left(\sum_{m=-\infty}^{\infty}(3m+1)^3q^{\frac{3m^2+2m}{4}}\right)\times\left(\sum_{n=0}^{\infty}q^{n^2+n}\right).
\end{aligned}\tag{3.7}$$

Proof Applying (2.4) with q replaced by $q^{1/4}$, we arrive at

$$\begin{aligned}
&4\varphi(-q^{1/4})\sum_{m=-\infty}^{\infty}(3m+1)^3q^{3m^2+2m}-\psi(q^2)\\
&\times\sum_{m=-\infty}^{\infty}(3m+1)^3q^{\frac{3m^2+2m}{4}}
\end{aligned}$$

$$
\begin{aligned}
&= 4\Big(\varphi(q) - 2q^{1/4}\psi(q^2)\Big) \sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m} \\
&\quad - \psi(q^2)\left(\sum_{m=-\infty}^{\infty} (6m+1)^3 q^{3m^2+m} - 8q^{1/4} \sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m}\right) \\
&= 4\varphi(q) \sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m} - \psi(q^2) \sum_{m=-\infty}^{\infty} (6m+1)^3 q^{3m^2+m} \\
&= 3(q;q)_\infty^8,
\end{aligned}
$$

where the last equation follows from Theorem 3.1. This finishes the proof. □

Now we are ready to prove the following theorem by using Theorem 3.1.

Theorem 3.3 *For $n \geq 0$,*

$$p_{-3}(11n+7) \equiv 0 \pmod{11}. \tag{3.8}$$

Proof Let us define $a(n)$ to be

$$3(q;q)_\infty^8 = \sum_{n=0}^{\infty} a(n) q^n.$$

Then,

$$
\begin{aligned}
\sum_{n=0}^{\infty} 3p_{-3}(n) q^n &\equiv \frac{3(q;q)_\infty^8}{\left(q^{11};q^{11}\right)_\infty} \\
&= \frac{1}{\left(q^{11};q^{11}\right)_\infty} \sum_{n=0}^{\infty} a(n) q^n \pmod{11}.
\end{aligned}
$$

Extracting those terms with powers of the form $11n+7$, we conclude that

$$\sum_{n=0}^{\infty} 3p_{-3}(11n+7) q^{11n+7} \equiv \frac{1}{\left(q^{11};q^{11}\right)_\infty} \sum_{n=0}^{\infty} a(11n+7) q^{11n+7} \pmod{11}.$$

To prove $p_{-3}(11n+7) \equiv 0 \pmod{11}$, we only need to show that

$$a(11n+7) \equiv 0 \pmod{11}.$$

Consider the congruence equation

$$3m^2 + 2m + n^2 \equiv 7 \pmod{11},$$

which can be rewritten as

$$4(3m+1)^2+n^2 \equiv 0 \pmod{11}.$$

Since -4 is a quadratic nonresidue modulo 11, we see that $3m+1$ is divisible by 11.

Similarly, if we consider the congruence equation $3m^2+m+n^2+n \equiv 7 \pmod{11}$, we can deduce that $6m+1$ is divisible by 11.

By Theorem 3.1, we see that

$$a(11n+7) \equiv 0 \pmod{11^3},$$

which completes the proof. □

Remark (3.7) can also be used to prove (3.8), and we leave the details to readers. Multiplying on both sides of (3.1) or (3.7) by $(q;q)_\infty^2$, and using Ramanujan's identities (2.6) and (2.7), we obtain a new proof of the following two identities of $(q;q)_\infty^{10}$ due to Chu [8, Cor. 4.2] and Chan [6, Thm. 3.4]:

$$\begin{aligned} 3(q;q)_\infty^{10} &= 4\left(\sum_{m=-\infty}^{\infty}(3m+1)^3q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty}(6n+1)q^{3n^2+n}\right)\\ &\quad-\left(\sum_{m=-\infty}^{\infty}(3m+1)q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty}(6n+1)^3q^{3n^2+n}\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} 3(q;q)_\infty^{10} &= 4\left(\sum_{m=-\infty}^{\infty}(3m+1)^3q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty}(3n+1)q^{(3n^2+2n)/4}\right)\\ &\quad-\left(\sum_{m=-\infty}^{\infty}(3m+1)q^{3m^2+2m}\right)\times\left(\sum_{n=-\infty}^{\infty}(3n+1)^3q^{(3n^2+2n)/4}\right). \end{aligned} \tag{3.10}$$

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