Ramanujan J (2016) 40:389–403
DOI 10.1007/s11139-015-9692-x

Infinite families of congruences modulo 7 for broken 3-diamond partitions

Ernest X. W. Xia[1]

Received: 12 December 2014 / Accepted: 1 April 2015 / Published online: 9 July 2015

Abstract The notion of broken k-diamond partitions was introduced by Andrews and Paule. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Recently, Paule and Radu conjectured that $\Delta_3(343n+82) \equiv \Delta_3(343n+278) \equiv \Delta_3(343n+327) \equiv 0 \pmod 7$. Jameson confirmed this conjecture and proved that $\Delta_3(343n+229) \equiv 0 \pmod 7$ by using the theory of modular forms. In this paper, we prove several infinite families of Ramanujan-type congruences modulo 7 for $\Delta_3(n)$ by establishing a recurrence relation for a sequence related to $\Delta_3(7n+5)$. In the process, we also give new proofs of the four congruences due to Paule and Radu, and Jameson.

Keywords Broken k-diamond partition · Congruence · Theta function

Mathematics Subject Classification 11P83 · 05A17

1 Introduction

The aim of this paper is to establish several infinite families of Ramanujan-type congruences modulo 7 for broken 3-diamond partitions. In the process, we also present new proofs of four congruences modulo 7 for broken 3-diamond partitions due to Paule and Radu [13], and Jameson [9].

This work was supported by the National Natural Science Foundation of China (11201188).

✉ Ernest X. W. Xia
ernestxwxia@163.com

1 Department of Mathematics, Jiangsu University, Zhenjiang 212013, Jiangsu, People's Republic of China

Let us begin with some notation and terminology on q-series and partitions. We use the standard notation

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right) \tag{1.1}$$

and often write

$$(a_1,a_2,\ldots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty. \tag{1.2}$$

Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \tag{1.3}$$

where $|ab|<1$. The Jacobi triple product identity can be restated as

$$f(a,b) = (-a,-b,ab;ab)_\infty. \tag{1.4}$$

Two special cases of (1.3) are defined by

$$\psi(q) = f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \tag{1.5}$$

and

$$f(-q) = f\left(-q,-q^2\right) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}. \tag{1.6}$$

In this paper, for any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$f_n = \left(q^n;q^n\right)_\infty = \prod_{k=1}^{\infty}\left(1-q^{nk}\right). \tag{1.7}$$

By (1.4), (1.5) and (1.6), we have

$$f(-q) = f_1, \qquad \psi(q) = \frac{f_2^2}{f_1}. \tag{1.8}$$

A combinatorial study guided by MacMahon's Partition Analysis led Andrews and Paule [2] to the construction of a new class of directed graphs called broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Andrews and Paule [2] established the following generating function of $\Delta_k(n)$:

$$\sum_{n=0}^{\infty} \Delta_k(n)q^n = \frac{f_2 f_{2k+1}}{f_1^3 f_{4k+2}}. \tag{1.9}$$

Employing generating function manipulations, Andrews and Paule [2] proved that for all integers $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}. \tag{1.10}$$

They also gave three conjectures modulo 2, 5 and 25 for $\Delta_k(n)$. Since then, a number of congruences satisfied by $\Delta_k(n)$ for small values of k have been proved. Hirschhorn and Sellers [8] established an explicit representation of the generating function for $\Delta_1(2n+1)$ which implied (1.10). Mortenson [12] reproved (1.10) by developing a statistic on the partitions enumerated by $\Delta_1(2n+1)$ which naturally breaks these partitions into three subsets of equal size. In addition, Hirschhorn and Sellers [8] also provided elementary proofs of four congruences modulo 2 for $\Delta_1(n)$ and $\Delta_2(n)$ and one of which was a conjecture due to Andrews and Paule [2]. Radu and Sellers [15] established several infinite families of congruences modulo 3 for $\Delta_2(n)$. Lin and Wang [11] presented elementary proofs of some results of Radu and Sellers [15]. Chen, Fan and Yu [6] discovered two infinite families of congruences for $\Delta_2(n)$ modulo 3. Chan [5] found two infinite families of congruences modulo 5 for broken 2-diamond partitions. Radu and Sellers [14] have given numerous beautiful congruence properties for broken k-diamond partitions. Radu and Sellers [16] provided an extensive analysis of the parity of the function $\Delta_3(n)$, including a number of Ramanujan-like congruences modulo 2. Lin [10] gave elementary proofs of the results due to Radu and Sellers [16]. Cui and Gu [7] proved several infinite families of congruences modulo 2 for $\Delta_3(n)$. Xia [17] considered congruences modulo 4 for $\Delta_3(n)$ and proved a conjecture of Radu and Sellers [16]. Yao [19] proved several infinite families of congruences modulo 2 for $\Delta_{11}(n)$ and generalized some results due to Radu and Sellers [14]. Ahmed and Baruah [1] discovered some parity results for broken 5-diamond, 7-diamond and 11-diamond partitions. Paule and Radu [13] discovered two non-standard infinite families of congruences for broken 2-diamond partitions. They also presented four conjectures related to $\Delta_3(n)$ and $\Delta_5(n)$. Xiong [18] proved the following congruence which was a conjecture of Paule and Radu [13]:

$$\sum_{n=0}^{\infty} \Delta_3(7n+5)q^n \equiv 6 f_1^4 f_2^6 \pmod{7}. \tag{1.11}$$

Employing the theory of modular forms, Jameson [9] proved the following theorem:

Theorem 1.1 *For $n \geq 0$,*

$$\Delta_3(343n+82) \equiv 0 \pmod{7}, \tag{1.12}$$
$$\Delta_3(343n+229) \equiv 0 \pmod{7}, \tag{1.13}$$
$$\Delta_3(343n+278) \equiv 0 \pmod{7}, \tag{1.14}$$
$$\Delta_3(343n+327) \equiv 0 \pmod{7}. \tag{1.15}$$

Congruences (1.12), (1.14) and (1.15) were conjectured by Paule and Radu [13] and congruence (1.13) was discovered by Jameson [9].

In this paper, we prove several infinite families of Ramanujan-type congruences modulo 7 for $\Delta_3(n)$ by establishing a recurrence relation for the coefficients of $f_1^4 f_2^6$. Furthermore, we give a new proof of Theorem 1.1. Our proof mainly relies on (1.11) and some identities involving theta functions due to Ramanujan. The main results of this paper can be stated as follows.

Theorem 1.2 *For $n \geq 0$ and $k \geq 0$, we have*

$$\Delta_3\left(7 \times 4^{7k+4}n + \frac{77 \times 4^{7k+3}+1}{3}\right) \equiv 0 \pmod 7 \tag{1.16}$$

and

$$\Delta_3\left(14 \times 4^{7k+7}n + \frac{14 \times 4^{7k+6}+1}{3}\right) \equiv 0 \pmod 7. \tag{1.17}$$

In order to state the following theorem, we introduce the Legendre symbol. Let $p \geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } p | a. \end{cases} \tag{1.18}$$

Theorem 1.3 *Let $p \geq 5$ be a prime such that $\left(\frac{-7}{p}\right) = -1$. For $n \geq 0$, $j \geq 0$ and $k \geq 1$, we have*

$$\Delta_3\left(7 \times 4^k p^{2j}(7n+s) + \frac{77 \times 4^{k-1}p^{2j}+1}{3}\right) \equiv 0 \pmod 7, \tag{1.19}$$

where $s \in \{2, 3, 4, 6\}$.

Theorem 1.4 *Let $p \geq 5$ be a prime such that $\left(\frac{-7}{p}\right) = -1$ and let $n \geq 0$, $j \geq 0$ and $k \geq 1$ be integers. If $p \nmid n$, then*

$$\Delta_3\left(7 \times 4^k p^{2j+1}n + \frac{77 \times 4^{k-1}p^{2j+2}+1}{3}\right) \equiv 0 \pmod 7. \tag{1.20}$$

This paper is organized as follows: In Sect. 2, we present a new proof of Theorem 1.1 based on (1.11). In Sect. 3, we establish a recurrence relation for $a(n)$ where the generating function of $a(n)$ is $f_1^4 f_2^6$. Moreover, we also derive some generating functions of $a(An+B)$ modulo 7 for some values of A and B. In Sect. 4, we prove Theorem 1.2 by using the recurrence relation given in Sect. 3. In Sect. 5, we prove Theorems 1.3 and 1.4 by employing the generating functions of $a(An+B)$ modulo 7.

2 A new proof of Theorem 1.1

By the binomial theory, it is easy to see that for any positive integer k,

$$f_k^7 \equiv f_{7k} \pmod 7. \tag{2.1}$$

Thanks to (1.11) and (2.1),

$$\sum_{n=0}^{\infty} \Delta_3(7n+5)q^n \equiv 6 f_1^4 f_2^6 \equiv 6 f_7 \psi^3(q) \pmod 7, \tag{2.2}$$

where $\psi(q)$ is defined by (1.8). From Entry 17 (iv) on page 303 in Berndt's book [3], we have the 7-dissection

$$\psi(q) = A + qB + q^3C + q^6\psi(q^{49}), \tag{2.3}$$

where

$$A = f(q^{21}, q^{28}), \qquad B = f(q^{14}, q^{35}), \qquad C = f(q^7, q^{42}). \tag{2.4}$$

Therefore, combining (2.2) and (2.3), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Delta_3(7n+5)q^n \\
&\equiv 6 f_7 \left(A + qB + q^3C + q^6\psi(q^{49})\right)^3 \\
&\equiv 6 f_7 \Big(A^3 + 3qA^2B + 3q^2AB^2 + q^3B^3 + 3q^3A^2C + 6q^4ABC \\
&\quad + 3q^5B^2C + 3q^6AC^2 + 3q^6A^2\psi(q^{49}) + 6q^7AB\psi(q^{49}) + 3q^7BC^2 \\
&\quad + 3q^8B^2\psi(q^{49}) + q^9C^3 + 6q^9AC\psi(q^{49}) + 6q^{10}BC\psi(q^{49}) \\
&\quad + 3q^{12}C^2\psi(q^{49}) + 3q^{12}A\psi^2(q^{49}) + 3q^{13}B\psi^2(q^{49}) \\
&\quad + 3q^{15}C\psi^2(q^{49}) + q^{18}\psi^3(q^{49})\Big) \pmod 7.
\end{aligned} \tag{2.5}$$

Extracting the terms in (2.5) that involves q^{7n+4}, dividing the resulting identity by q^4 and then replacing q^7 by q, we deduce that

$$\sum_{n=0}^{\infty} \Delta_3(49n+33)q^n \equiv f_1 f(q^3, q^4) f(q^2, q^5) f(q, q^6) + 6q^2 f_1 \psi^3(q^7) \pmod 7. \tag{2.6}$$

By (1.4), it is easy to check that

$$f(q^3, q^4) f(q^2, q^5) f(q, q^6) = \frac{f_2 f_7^4}{f_1 f_{14}}. \tag{2.7}$$

Thanks to (1.8), (2.6) and (2.7),

$$\sum_{n=0}^{\infty}\Delta_3(49n+33)q^n \equiv \frac{f_2 f_7^4}{f_{14}} + 6q^2 f_1 \frac{f_{14}^6}{f_7^3} \pmod 7. \tag{2.8}$$

From Entry 17 (v) on page 303 in [3], we have the 7-dissection

$$f_1 = f_{49}\frac{f(-q^{14},-q^{35})}{f(-q^7,-q^{42})} - q f_{49}\frac{f(-q^{21},-q^{28})}{f(-q^{14},-q^{35})} - q^2 f_{49} + q^5 f_{49}\frac{f(-q^7,-q^{42})}{f(-q^{21},-q^{28})}. \tag{2.9}$$

Replacing q by q^2 in (2.9), we get

$$\begin{aligned} f_2 = \ & f_{98}\frac{f(-q^{28},-q^{70})}{f(-q^{14},-q^{84})} - q^2 f_{98}\frac{f(-q^{42},-q^{56})}{f(-q^{28},-q^{70})} - q^4 f_{98} \\ & + q^{10} f_{98}\frac{f(-q^{14},-q^{84})}{f(-q^{42},-q^{56})}. \end{aligned} \tag{2.10}$$

If we substitute (2.9) and (2.10) into (2.8) and compare relevant powers of q, we obtain (1.12), (1.14), (1.15) and

$$\sum_{n=0}^{\infty}\Delta_3(343n+229)q^n \equiv 6\frac{f_1^4 f_{14}}{f_2} + \frac{f_2^6 f_7}{f_1^3} \pmod 7. \tag{2.11}$$

Congruence (1.13) follows from (2.1) and (2.11). This completes the proof. □

3 A recurrence relation of a sequence

In this section, we establish a recurrence relation of $a(n)$, where $a(n)$ is related to $\Delta_3(7n+5)$ and the generating function of $a(n)$ is $f_1^4 f_2^6$. The recurrence relation plays an important role in this paper. In the process, we also find generating functions for $a(mn+k)$ for some small values of m and k by using an iterative method, which are helpful in proving Theorems 1.2, 1.3 and 1.4.

Theorem 3.1 *Let $a(n)$ be defined by*

$$\sum_{n=0}^{\infty} a(n)q^n = f_1^4 f_2^6. \tag{3.1}$$

For $n \geq 0$, we have

$$a(128n+42) = -240a(8n+2) + 1024a(2n). \tag{3.2}$$

Proof The following relations are the consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [3, p.40]:

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{3.3}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{3.4}$$

Substituting (3.3) into (3.1), we have

$$\sum_{n=0}^{\infty} a(n)q^n = \left(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\right) f_2^6 = \frac{f_2^4 f_4^{10}}{f_8^4} - 4q\frac{f_2^8 f_8^4}{f_4^2}, \tag{3.5}$$

which yields

$$\sum_{n=0}^{\infty} a(2n)q^n = \frac{f_1^4 f_2^{10}}{f_4^4}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a(2n+1)q^n = -4\frac{f_1^8 f_4^4}{f_2^2}. \tag{3.7}$$

If we substitute (3.3) into (3.6) and (3.7), and then extract the even parts and the odd parts in the resulting identity, we get

$$\sum_{n=0}^{\infty} a(4n)q^n = \frac{f_1^8 f_2^6}{f_4^4}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} a(4n+2)q^n = -4\frac{f_1^{12} f_4^4}{f_2^6}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} a(4n+3)q^n = 32\frac{f_2^{12}}{f_1^2}. \tag{3.10}$$

Similarly, substituting (3.3) into (3.8) and (3.9), and then extracting the even parts and the odd parts in the resulting identity, we have

$$\sum_{n=0}^{\infty} a(8n)q^n = \frac{f_1^2 f_2^{16}}{f_4^8} + 16q\frac{f_1^{10} f_4^8}{f_2^8}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} a(8n+2)q^n = -4\frac{f_2^{34}}{f_1^{12}f_4^{12}} - 192q\frac{f_2^{10}f_4^4}{f_1^4}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} a(8n+6)q^n = 48\frac{f_2^{22}}{f_1^8 f_4^4} + 256q\frac{f_4^{12}}{f_2^2}. \tag{3.13}$$

Substituting (3.4) into (3.12) and (3.13), and then extracting the even parts and the odd parts in the resulting identity, we see that

$$\sum_{n=0}^{\infty} a(16n+2)q^n = -4\frac{f_2^{30}}{f_1^8 f_4^{12}} - 960q f_2^6 f_4^4, \tag{3.14}$$

$$\sum_{n=0}^{\infty} a(16n+10)q^n = -240\frac{f_2^{18}}{f_1^4 f_4^4} - 256q\frac{f_1^4 f_4^{12}}{f_2^6}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a(16n+14)q^n = 640\frac{f_2^{12}}{f_1^2}. \tag{3.16}$$

Substituting (3.3) and (3.4) into (3.14) and (3.15), and then extracting the even parts and the odd parts in the resulting identity, we deduce that

$$\sum_{n=0}^{\infty} a(32n+2)q^n = -4\frac{f_1^2 f_2^{16}}{f_4^8} - 64q\frac{f_1^{10}f_4^8}{f_2^8}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} a(32n+10)q^n = -240\frac{f_1^4 f_2^{10}}{f_4^4} + 1024q\frac{f_2^{10}f_4^4}{f_1^4}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} a(32n+26)q^n = -256\frac{f_2^{22}}{f_1^8 f_4^4} - 960\frac{f_1^8 f_4^4}{f_2^2}. \tag{3.19}$$

If we substitute (3.3) and (3.4) into (3.18) and (3.19), and then extract the even parts and the odd parts in the resulting identity, we find that

$$\sum_{n=0}^{\infty} a(64n+10)q^n = -240\frac{f_1^8 f_2^6}{f_4^4} + 4096q f_2^6 f_4^4, \tag{3.20}$$

$$\sum_{n=0}^{\infty} a(64n+42)q^n = 1024\frac{f_2^{18}}{f_1^4 f_4^4} + 960\frac{f_1^{12}f_4^4}{f_2^6}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} a(64n+58)q^n = 5632\frac{f_2^{12}}{f_1^2}. \tag{3.22}$$

Substituting (3.3) and (3.4) into (3.20) and (3.21), and then extracting the even parts and the odd parts in the resulting identity, we have

$$\sum_{n=0}^{\infty} a(128n+10)q^n = -240\frac{f_1^2 f_2^{16}}{f_4^8} - 3840q\frac{f_1^{10} f_4^8}{f_2^8}, \tag{3.23}$$

$$\sum_{n=0}^{\infty} a(128n+42)q^n = 1024\frac{f_1^4 f_2^{10}}{f_4^4} + 960\frac{f_2^{34}}{f_1^{12} f_4^{12}} + 46080q\frac{f_2^{10} f_4^4}{f_1^4}, \tag{3.24}$$

$$\sum_{n=0}^{\infty} a(128n+106)q^n = 4096\frac{f_1^8 f_4^4}{f_2^2} - 11520\frac{f_2^{22}}{f_1^8 f_4^4} - 61440q\frac{f_4^{12}}{f_2^2}. \tag{3.25}$$

Theorem 3.1 follows from (3.6), (3.12) and (3.24). The proof is complete. □

4 Proof of Theorem 1.2

In this section, we present a proof of Theorem 1.2. We first prove the following two lemmas.

Lemma 4.1 *For $k \geq 1$,*

$$\sum_{n=0}^{\infty} a\left(4^k n + \frac{11 \times 4^{k-1} - 2}{3}\right) q^n = f(k)\frac{f_2^{12}}{f_1^2}, \tag{4.1}$$

where $f(1) = 32$, $f(2) = 640$, $f(3) = 5632$, $f(4) = -186368$ *and for* $k \geq 5$,

$$f(k) = -240 f(k-2) + 1024 f(k-3). \tag{4.2}$$

Proof We are ready to prove Lemma 4.1 by induction on k. Substituting (3.3) and (3.4) into (3.25), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} a(128n+106)q^n =& 4096\frac{f_4^4}{f_2^2}\left(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\right)^2 \\ & - 11520\frac{f_2^{22}}{f_4^4}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2 - 61440q\frac{f_4^{12}}{f_2^2} \\ =& -7424\frac{f_4^{24}}{f_2^6 f_8^8} - 186368q\frac{f_4^{12}}{f_2^2} - 118784q^2 f_2^2 f_8^8,\end{aligned} \tag{4.3}$$

which yields

$$\sum_{n=0}^{\infty} a(256n+234)q^n = -186368\frac{f_2^{12}}{f_1^2}. \tag{4.4}$$

By (3.10), (3.16), (3.22) and (4.4), we see that Lemma 4.1 holds when $k = 1, 2, 3, 4$. Now suppose $k \geq 5$ and that the lemma is true for $l < k$. Then

$$\sum_{n=0}^{\infty} a\left(4^{k-3}n + \frac{11 \times 4^{k-4} - 2}{3}\right) q^n = f(k-3)\frac{f_2^{12}}{f_1^2} \tag{4.5}$$

and

$$\sum_{n=0}^{\infty} a\left(4^{k-2}n + \frac{11 \times 4^{k-3} - 2}{3}\right) q^n = f(k-2)\frac{f_2^{12}}{f_1^2}. \tag{4.6}$$

If in Theorem 3.1 we replace $2n$ by $4^{k-3}n + \frac{11\times 4^{k-4}-2}{3}$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} a\left(4^{k}n + \frac{11 \times 4^{k-1} - 2}{3}\right) q^n \\
&= -240\sum_{n=0}^{\infty} a\left(4^{k-2}n + \frac{11 \times 4^{k-3} - 2}{3}\right) q^n \\
&\quad + 1024\sum_{n=0}^{\infty} a\left(4^{k-3}n + \frac{11 \times 4^{k-4} - 2}{3}\right) q^n \\
&= \Big(-240 f(k-2) + 1024 f(k-3)\Big)\frac{f_2^{12}}{f_1^2} \\
&= f(k)\frac{f_2^{12}}{f_1^2}.
\end{aligned} \tag{4.7}$$

This lemma is proved by induction. □

Employing (3.11), (3.17), (3.23) and Theorem 3.1, we can prove the following lemma:

Lemma 4.2 *For $k \geq 1$,*

$$\sum_{n=0}^{\infty} a\left(2 \times 4^{k}n + \frac{2 \times 4^{k-1} - 2}{3}\right) q^n = h(k)\left(\frac{f_1^2 f_2^{16}}{f_4^8} + 16q\frac{f_1^{10} f_4^8}{f_2^8}\right), \tag{4.8}$$

where $h(1) = 1$, $h(2) = -4$, $h(3) = -240$, $h(4) = 1984$ and for $k \geq 5$,

$$h(k) = -240h(k-2) + 1024h(k-3). \tag{4.9}$$

The proof of Lemma 4.2 is analogous to the proof of Lemma 4.1, and hence is omitted.

Now, we turn to prove Theorem 1.2.

It is easy to check that $f(2) \equiv 3 \pmod 7$, $f(3) \equiv 4 \pmod 7$, $f(4) \equiv 0 \pmod 7$, and for $k \geq 5$, $f(k) \equiv 5f(k-2) + 2f(k-3) \pmod 7$. It follows that the sequence $\{f(k) \pmod 7\}$ is, for $k \geq 2$,

$$\begin{aligned}&3,\ 4,\ 0,\ 5,\ 1,\ 4,\ 1,\ 1,\ 6,\ 0,\ 4,\ 5,\ 6,\ 5,\ 5,\ 2,\ 0,\ 6,\ 4,\ 2,\\&4,\ 4,\ 3,\ 0,\ 2,\ 6,\ 3,\ 6,\ 6,\ 1,\ 0,\ 3,\ 2,\ 1,\ 2,\ 2,\ 5,\ 0,\ 3,\ 5,\\&3,\ 4,\ 0,\ 5,\ 1,\cdots\end{aligned}$$

which is periodic with period 42, and for any integer $k \geq 0$,

$$f(7k+4) \equiv 0 \pmod 7. \tag{4.10}$$

By Lemma 4.1 and (4.10), we see that for $k \geq 0$ and $n \geq 0$,

$$a\left(4^{7k+4}n + \frac{11 \times 4^{7k+3} - 2}{3}\right) \equiv 0 \pmod 7. \tag{4.11}$$

In view of (1.11) and (3.1), we see that for $n \geq 0$,

$$\Delta_3(7n+5) \equiv 6a(n) \pmod 7. \tag{4.12}$$

Replacing n by $4^{7k+4}n + \frac{11\times4^{7k+3}-2}{3}$ in (4.12) and utilizing (4.11), we arrive at the congruence (1.16).

Similarly, we can prove that for $k \geq 1$,

$$h(7k) \equiv 0 \pmod 7. \tag{4.13}$$

The proof of (4.13) is analogous to the proof of (4.10), and hence is omitted. It follows from (4.8) and (4.13) that for $n \geq 0$ and $k \geq 1$,

$$a\left(2 \times 4^{7k}n + \frac{2 \times 4^{7k-1} - 2}{3}\right) \equiv 0 \pmod 7. \tag{4.14}$$

Congruence (1.17) follows from (4.12) and (4.14). This completes the proof. □

5 Proofs of Theorems 1.3 and 1.4

In this section, we present proofs of Theorem 1.3 and 1.4. We first prove the following lemma:

Lemma 5.1 *Define*

$$\sum_{n=0}^{\infty} b(n)q^n = f_{14}\frac{f_2^5}{f_1^2}. \tag{5.1}$$

Let $p \geq 5$ *be a prime such that* $\left(\frac{-7}{p}\right) = -1$. *For* $n \geq 0$,

$$b\left(pn + \frac{11(p^2-1)}{12}\right) = \left(\frac{-1}{p}\right) pb(n/p). \tag{5.2}$$

Proof The following identity is Euler's pentagonal number theorem

$$f_1 = \sum_{\alpha \equiv 1 \pmod 6} (-1)^{\frac{\alpha-1}{6}} q^{\frac{\alpha^2-1}{24}}, \tag{5.3}$$

which is a direct consequence of Jacobi's triple product identity; see Corollary 1.3.5 on page 12 of Berndt's book [4]. The following identity is a consequence of the quintuple product identity:

$$\frac{f_2^5}{f_1^2} = \sum_{\beta \equiv 1 \pmod 3} (-1)^{\beta-1} \beta q^{\frac{\beta^2-1}{3}}, \tag{5.4}$$

which is Corollary 1.3.22 on page 21 of Berndt's book [4]. In view of (5.1), (5.3) and (5.4),

$$\begin{aligned} b(n) &= \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 3, \\ \frac{14(\alpha^2-1)}{24} + \frac{\beta^2-1}{3} = n}} (-1)^{\frac{\alpha-1}{6}} (-1)^{\beta-1} \beta \\ &= \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 3, \\ 7\alpha^2 + 4\beta^2 = 12n+11}} (-1)^{\frac{\alpha-1}{6}} (-1)^{\beta-1} \beta. \end{aligned} \tag{5.5}$$

Hence,

$$b\left(pn + \frac{11(p^2-1)}{12}\right) = \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 3, \\ 7\alpha^2 + 4\beta^2 = 12pn+11p^2}} (-1)^{\frac{\alpha-1}{6}} (-1)^{\beta-1} \beta. \tag{5.6}$$

Note that $7\alpha^2 + (2\beta)^2 \equiv 0 \pmod p$. Since $p \geq 5$ is a prime and $\left(\frac{-7}{p}\right) = -1$, then $\alpha \equiv \beta \equiv 0 \pmod p$. Set $\alpha = \left(\frac{-3}{p}\right) p\alpha'$ and $\beta = \left(\frac{-3}{p}\right) p\beta'$. The facts $\alpha \equiv 1 \pmod 6$ and $\beta \equiv 1 \pmod 3$ imply that $\alpha' \equiv 1 \pmod 6$ and $\beta' \equiv 1 \pmod 3$. Therefore,

$$(-1)^{\frac{\alpha-1}{6}} = \left(\frac{3}{p}\right) (-1)^{\frac{\alpha'-1}{6}} \tag{5.7}$$

and

$$(-1)^{\beta-1} = (-1)^{\beta'-1}. \tag{5.8}$$

Combining (5.5), (5.6), (5.7) and (5.8), we get

$$\begin{aligned} b\left(pn+\frac{11(p^2-1)}{12}\right) &= \sum_{\substack{\alpha'\equiv 1 \pmod 6,\ \beta'\equiv 1 \pmod 3,\\ 7p^2\alpha'^2+4p^2\beta'^2=12pn+11p^2}} \left(\frac{3}{p}\right)(-1)^{\frac{\alpha'-1}{6}}\left(\frac{-3}{p}\right)(-1)^{\beta'-1}p\beta' \\ &= \left(\frac{-1}{p}\right)p \sum_{\substack{\alpha'\equiv 1 \pmod 6,\ \beta'\equiv 1 \pmod 3,\\ 7\alpha'^2+4\beta'^2=12n/p+11}} (-1)^{\frac{\alpha'-1}{6}}(-1)^{\beta'-1}\beta' \\ &= \left(\frac{-1}{p}\right)pb(n/p). \end{aligned} \tag{5.9}$$

This completes the proof of this lemma. □

Now, we are ready to prove Theorems 1.3 and 1.4. Replacing n by np in (5.2), we have

$$b\left(p^2n+\frac{11(p^2-1)}{12}\right)=\left(\frac{-1}{p}\right)pb(n). \tag{5.10}$$

By (5.10) and mathematical induction, we find that for $j\geq 0$,

$$b\left(p^{2j}n+\frac{11(p^{2j}-1)}{12}\right)=\left(\left(\frac{-1}{p}\right)p\right)^j b(n). \tag{5.11}$$

It follows from (5.2) that if $p\nmid n$, then

$$b\left(pn+\frac{11(p^2-1)}{12}\right)=0. \tag{5.12}$$

Replacing n by $pn+\frac{11(p^2-1)}{12}$ in (5.11) and employing (5.12), we see that if $p\nmid n$, then

$$b\left(p^{2j+1}n+\frac{11\left(p^{2j+2}-1\right)}{12}\right)=0. \tag{5.13}$$

In view of (5.1) and (5.4), we find that

$$\sum_{n=0}^{\infty}b(n)q^n=f_{14}\sum_{\beta\equiv 1 \pmod 3}(-1)^{\beta-1}\beta q^{\frac{\beta^2-1}{3}}. \tag{5.14}$$

It is easy to check that if $\beta\equiv 1 \pmod 3$, then

$$\frac{\beta^2-1}{3}\equiv 0,\ 1,\ 2,\ 5 \pmod 7. \tag{5.15}$$

Thus,

$$b(7n+3)=b(7n+4)=b(7n+6)=0. \tag{5.16}$$

Furthermore, $\frac{\beta^2-1}{3} \equiv 2 \pmod 7$ holds if and only if $\beta \equiv 0 \pmod 7$. Therefore,

$$b(7n+2) \equiv 0 \pmod 7. \tag{5.17}$$

Replacing n by $7n+s$ ($s \in \{2,\ 3,\ 4,\ 6\}$) in (5.11) and using (5.16) and (5.17), we deduce that

$$b\left(p^{2j}(7n+s)+\frac{11\left(p^{2j}-1\right)}{12}\right) \equiv 0 \pmod 7. \tag{5.18}$$

By (2.1) and (5.1), we see that

$$\sum_{n=0}^{\infty} b(n)q^n \equiv \frac{f_2^{12}}{f_1^2} \pmod 7. \tag{5.19}$$

In view of (4.1) and (5.19), we see that for $n \geq 0$,

$$a\left(4^k n + \frac{11 \times 4^{k-1}-2}{3}\right) \equiv f(k)b(n) \pmod 7, \tag{5.20}$$

where $f(k)$ is defined by (4.2). Replacing n by $p^{2j}(7n+s)+\frac{11(p^{2j}-1)}{12}$ ($s \in \{2,\ 3,\ 4,\ 6\}$) in (5.20) and utilizing (5.18), we see that for $n \geq 0$, $k \geq 1$ and $j \geq 0$,

$$a\left(4^k\left(p^{2j}(7n+s)+\frac{11\left(p^{2j}-1\right)}{12}\right)+\frac{11 \times 4^{k-1}-2}{3}\right) \equiv 0 \pmod 7. \tag{5.21}$$

Similarly, it follows from (5.13) and (5.20) that for $n \geq 0$, $k \geq 1$ and $j \geq 0$,

$$a\left(4^k\left(p^{2j+1}n+\frac{11\left(p^{2j+2}-1\right)}{12}\right)+\frac{11 \times 4^{k-1}-2}{3}\right) \equiv 0 \pmod 7, \qquad p \nmid n. \tag{5.22}$$

Congruences (1.19) and (1.20) follow from (4.12), (5.21) and (5.22). This completes the proof. □.

References

1. Ahmed, Z., Baruah, N.D.: Parity results for broken 5-diamond, 7-diamond and 11-diamond partitions. Int. J. Number Theory **11**, 527–542 (2015)

2. Andrews, G.E., Paule, P.: MacMahon's partition analysis XI: broken diamonds and modular forms. Acta Arith. **126**, 281–294 (2007)
3. Berndt, B.C.: Ramanujan's Notebooks Part III. Springer, New York (1991)
4. Berndt, B.C.: Number Theory in the Spirit of Ramanujan. American Mathematical Society, Providence (2006)
5. Chan, S.H.: Some congruences for Andrews–Paule's broken 2-diamond partitions. Discrete Math. **308**, 5735–5741 (2008)
6. Chen, W.Y.C., Fan, A.R.B., Yu, R.T.: Ramanujan-type congruences for broken 2-diamond partitions modulo 3. Sci. China Math. **57**, 1553–1560 (2014)
7. Cui, S.P., Gu, N.S.S.: Congruences for broken 3-diamond and 7 dots bracelet partitions. Ramanujan J. **35**, 165–178 (2014)
8. Hirschhorn, M.D., Sellers, J.A.: On recent congruence results of Andrews and Paule. Bull. Aust. Math. Soc. **75**, 121–126 (2007)
9. Jameson, M.: Congruences for broken k-diamond partitions. Ann. Comb. **17**, 333–338 (2013)
10. Lin, B.L.S.: Elementary proofs of parity results for broken 3-diamond partitions. J. Number Theory **135**, 1–7 (2014)
11. Lin, B.L.S., Wang, A.Y.Z.: Elementary proofs of Radu and Sellers' results for broken 2-diamond partitions. Ramanujan J. **37**, 291–297 (2015)
12. Mortenson, E.: On the broken 1-diamond partition. Int. J. Number Theory **4**, 199–218 (2008)
13. Paule, P., Radu, S.: Infinite families of strange partition congruences for broken 2-diamonds. Ramanujan J. **23**, 409–416 (2010)
14. Radu, S., Sellers, J.A.: Parity results for broken k-diamond partitions and $(2k + 1)$-cores. Acta Arith. **146**, 43–52 (2011)
15. Radu, S., Sellers, J.A.: Infinitely many congruences for broken 2-diamond partitions modulo 3. J. Comb. Number Theory **4**, 195–200 (2013)
16. Radu, S., Sellers, J.A.: An extensive analysis of the parity of broken 3-diamond partitions. J. Number Theory **133**, 3703–3716 (2013)
17. Xia, E.X.W.: New congruences modulo powers of 2 for broken 3-diamond partitions and 7-core partitions. J. Number Theory **141**, 119–135 (2014)
18. Xiong, X.: Two congruences involving Andrews–Paule's broken 3-diamond partitions and 5-diamond partitions. Proc. Jpn. Acad., Ser. A (Math. Sci.) **87**(5), 65–68 (2011)
19. Yao, O.X.M.: New parity results for broken 11-diamond partitions. J. Number Theory **140**, 267–276 (2014)