

Functional equations for the Stieltjes constants

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Abstract The Stieltjes constants $\gamma_k(a)$ appear as the coefficients in the regular part of the Laurent expansion of the Hurwitz zeta function $\zeta(s, a)$ about $s = 1$. We present the evaluation of $\gamma_1(a)$ and $\gamma_2(a)$ at rational arguments, this being of interest to theoretical and computational analytic number theory and elsewhere. We give multiplication formulas for $\gamma_0(a)$, $\gamma_1(a)$, and $\gamma_2(a)$, and point out that these formulas are cases of an addition formula previously presented. We present certain integral evaluations generalizing Gauss' formula for the digamma function at rational argument. In addition, we give the asymptotic form of $\gamma_k(a)$ as $a \rightarrow 0$ as well as a novel technique for evaluating integrals with integrands with $\ln(-\ln x)$ and rational factors.

Keywords Stieltjes constants · Riemann zeta function · Hurwitz zeta function · Laurent expansion · Digamma function · Polygamma function

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1 Introduction and statement of results

The Stieltjes (or generalized Euler) constants $\gamma_k(a)$ appear as expansion coefficients in the Laurent series for the Hurwitz zeta function $\zeta(s, a)$ about its simple pole at $s = 1$ [5–7, 10, 18, 22, 26, 28, 31],

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a) (s-1)^n. \quad (1.1)$$

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These constants are important in analytic number theory and elsewhere, where they appear in various estimations and as a result of asymptotic analyses, being given by the limit relation

$$\gamma_k(a) = \lim_{N \rightarrow \infty} \left[\sum_{j=1}^N \frac{\ln^k(j+a)}{j} - \frac{\ln^{k+1}(N+a)}{k+1} \right].$$

In particular, $\gamma_0(a) = -\psi(a)$, where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, with $\Gamma(z)$ as the Gamma function. With γ as the Euler constant and $\gamma_1 = \gamma_1(1)$ and $\gamma_2 = \gamma_2(1)$, we recall the connection with sums of reciprocal powers of the nontrivial zeros ρ of the Riemann zeta function,

$$\sum_{\rho} \frac{1}{\rho^2} = 1 - \frac{\pi^2}{8} + 2\gamma_1 + \gamma^2, \quad \sum_{\rho} \frac{1}{\rho^3} = 1 - \frac{7}{8}\zeta(3) + \gamma^3 + 3\gamma\gamma_1 + \frac{3}{2}\gamma_2;$$

such relations follow from the Hadamard factorization [20, Ch. 1.3].

An effective asymptotic expression for γ_k [23] and $\gamma_k(a)$ [24] for $k \gg 1$ has recently been given. From this expression, previously known results on sign changes within the sequence of Stieltjes constants follow.

In this paper, we first evaluate the first and second Stieltjes constants at rational arguments. These decompositions are effectively Fourier series, thus implying many extensions and applications, and they supplement the relations presented in [9]. We then present multiplication formulas for the zeroth, first, and second Stieltjes constants, and certain log–log integrals with integer parameters. The latter integral evaluations also provide explicit expressions for the differences $\gamma_1(j/m) - \gamma_1$ and $\gamma_2(j/m) - \gamma_2$. Besides elaborating on a multiplication formula for the Stieltjes constants, the Discussion section provides examples of integrals evaluating in terms of differences of the first and second of these constants. In addition, presented there is a novel method of determining log–log integrals with a certain polynomial denominator integrand.

We recall the connection of differences of Stieltjes constants with logarithmic sums,

$$\gamma_{\ell}(a) - \gamma_{\ell}(b) = \sum_{n=0}^{\infty} \left[\frac{\ln^{\ell}(n+a)}{n+a} - \frac{\ln^{\ell}(n+b)}{n+b} \right]. \tag{1.2}$$

Very recently an evaluation of $\gamma_1(j/m) - \gamma_1$ was also performed [4]. However, the method of proof is circuitous—integrals are used in addition to multiple applications of functional equations.

The Hurwitz zeta function, initially defined by $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ for $\text{Re } s > 1$, has an analytic continuation to the whole complex plane [3, 15, 21, 29]. In the case of $a = 1$, $\zeta(s, a)$ reduces to the Riemann zeta function $\zeta(s)$ [13, 20, 27]. In this instance, by convention, the Stieltjes constants $\gamma_k(1)$ are simply denoted as γ_k [5, 18, 22, 25, 26, 32]. We recall that $\gamma_k(a+1) = \gamma_k(a) - (\ln^k a)/a$, and more generally that for $n \geq 1$ an integer

$$\gamma_k(a + n) = \gamma_k(a) - \sum_{j=0}^{n-1} \frac{\ln^k(a + j)}{a + j},$$

as follows from the functional equation $\zeta(s, a + n) = \zeta(s, a) - \sum_{j=0}^{n-1} (a + j)^{-s}$. In fact, an interval of length $1/2$ is sufficient to characterize the $\gamma_k(a)$'s [17].

Unless specified otherwise below, letters $j, k, \ell, m, n,$ and r denote positive integers. The Euler constant is given by $\gamma = -\psi(1) = \gamma_0(1)$. The polygamma functions are denoted as $\psi^{(n)}(z)$ and we note that $\psi^{(n)}(z) = (-1)^{n+1}n!\zeta(n + 1, z)$ [1, 16].

Proposition 1 For $m > 1$ and $j < m,$ (a)

$$\begin{aligned} \gamma_1\left(\frac{j}{m}\right) &= \gamma_1 + \gamma^2 + \gamma \ln 2\pi m + \ln(2\pi) \ln m + \frac{\ln^2 m}{2} + (\gamma + \ln 2\pi m)\psi\left(\frac{j}{m}\right) \\ &+ \pi \sum_{r=1}^{m-1} \sin \frac{2\pi jr}{m} \ln \Gamma\left(\frac{r}{m}\right) + \sum_{r=1}^{m-1} \cos \frac{2\pi jr}{m} \zeta''\left(0, \frac{r}{m}\right) \end{aligned} \tag{1.3}$$

and (b)

$$\begin{aligned} \frac{1}{2}\gamma_2\left(\frac{j}{m}\right) &= \frac{\gamma_2}{2} - \gamma_1 \ln m + \frac{\gamma}{8}\pi^2 + \frac{\pi^2}{8} \ln m + \frac{\gamma}{2} \ln^2 m + \frac{\ln^3 m}{6} \\ &+ A + \sum_{r=1}^{m-1} \cos \frac{2\pi jr}{m} \left\{ \left[\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2 \right] \zeta'\left(0, \frac{r}{m}\right) \right. \\ &\left. - (\gamma + \ln 2\pi m)\zeta''\left(0, \frac{r}{m}\right) + \frac{1}{3}\zeta'''\left(0, \frac{r}{m}\right) \right\}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} A &= \frac{\pi^3}{24} \cot \frac{\pi j}{m} + \frac{\pi^2}{8} \psi\left(\frac{j}{m}\right) \\ &- \frac{\pi}{2} \sum_{r=1}^{m-1} \sin \frac{2\pi jr}{m} \left\{ -\zeta\left(0, \frac{r}{m}\right) \left[\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2 \right] \right. \\ &\left. + 2(\gamma + \ln 2\pi m)\zeta'\left(0, \frac{r}{m}\right) - \zeta''\left(0, \frac{r}{m}\right) \right\}, \end{aligned}$$

and ' indicates differentiation with respect to the first argument.

Both parts (a) and (b) may be written in many alternative forms. For instance, for (b), the well-known relation $\zeta'(0, a) = \ln \Gamma(a) - \ln(2\pi)/2$ may be used, and as well, the right member may be modified by introducing $\gamma_1(j/m)$.

Various summation results are known for the Stieltjes constants, including [6]

$$\sum_{r=1}^q \gamma_k\left(\frac{r}{q}\right) = q(-1)^k \frac{\ln^{k+1} q}{k + 1} + q \sum_{j=0}^k (-1)^j \binom{k}{j} (\ln^j q)\gamma_{k-j}.$$

As we briefly indicate, the $k = 1$ and 2 cases follow from Proposition 1.

Corollary 1

$$\sum_{r=1}^q \gamma_1 \left(\frac{r}{q}\right) = -\frac{q}{2} \ln^2 q + q(-\gamma \ln q + \gamma_1)$$

and

$$\sum_{r=1}^q \gamma_2 \left(\frac{r}{q}\right) = \frac{q}{3} \ln^3 q + q \left(\gamma \ln^2 q - 2\gamma_1 \ln q + \gamma_2\right).$$

Proof The summation for γ_1 follows from (1.3) as the sums over the cosine and sine terms are zero, and we have the readily verified relation $\sum_{r=1}^q \psi \left(\frac{r}{q}\right) = -q(\gamma + \ln q)$. Similarly for the summation for γ_2 , the sine and cosine terms do not contribute, the just-mentioned summation of $\psi(r/q)$ holds, and $\sum_{j=1}^{q-1} \cot \frac{\pi j}{q} = 0$. \square

Corollary 2 For $\ell = 1, 2, \dots, m - 1$, (a)

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{m-1} \gamma_2 \left(\frac{j}{m}\right) \sin \frac{2\pi j \ell}{m} \\ &= \frac{\pi^3}{48} (2\ell - m) - \frac{\pi m}{4} \left\{ \left(\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2\right) \left[\zeta \left(0, 1 - \frac{\ell}{m}\right) - \zeta \left(0, \frac{\ell}{m}\right) \right] \right. \\ & \quad + 2(\gamma + \ln 2\pi m) \left[\zeta' \left(0, \frac{\ell}{m}\right) - \zeta' \left(0, 1 - \frac{\ell}{m}\right) \right] \\ & \quad \left. + \zeta'' \left(0, 1 - \frac{\ell}{m}\right) - \zeta'' \left(0, \frac{\ell}{m}\right) \right\} \end{aligned}$$

and (b)

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{m-1} \gamma_2 \left(\frac{j}{m}\right) \cos \frac{2\pi j \ell}{m} = - \left[\frac{\gamma_2}{2} - \gamma_1 \ln m + \frac{\gamma}{8} \pi^2 + \frac{\pi^2}{8} \ln m + \frac{\gamma}{2} \ln^2 m + \frac{1}{6} \ln^3 m \right] \\ & \quad + \frac{\pi^2}{8} \left[m \ln \left(2 \sin \frac{\pi \ell}{m}\right) + \gamma \right] - \sum_{r=1}^{m-1} \left\{ \left(\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2\right) \zeta' \left(0, \frac{r}{m}\right) \right. \\ & \quad \left. - (\gamma + \ln 2\pi m) \zeta'' \left(0, \frac{r}{m}\right) + \frac{1}{3} \zeta''' \left(0, \frac{r}{m}\right) \right\} \\ & \quad + \frac{m}{2} \left\{ \left(\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2\right) \left[\zeta' \left(0, \frac{\ell}{m}\right) + \zeta' \left(0, 1 - \frac{\ell}{m}\right) \right] \right. \\ & \quad \left. - (\gamma + \ln 2\pi m) \left[\zeta'' \left(0, \frac{\ell}{m}\right) + \zeta'' \left(0, 1 - \frac{\ell}{m}\right) \right] \right. \\ & \quad \left. + \frac{1}{3} \left[\zeta''' \left(0, \frac{\ell}{m}\right) + \zeta''' \left(0, 1 - \frac{\ell}{m}\right) \right] \right\}. \end{aligned}$$

Part (b) may be rewritten by using the three sums

$$\sum_{\ell=1}^{m-1} \zeta' \left(0, \frac{\ell}{m} \right) = -\frac{\ln m}{2}, \quad \sum_{\ell=1}^{m-1} \zeta'' \left(0, \frac{\ell}{m} \right) = -\frac{1}{2} \ln^2 m - (\ln m) \ln(2\pi),$$

and

$$\begin{aligned} \sum_{\ell=1}^{m-1} \zeta''' \left(0, \frac{\ell}{m} \right) &= -\frac{\ln^3 m}{2} - \frac{3}{2} \ln^2 m \ln(2\pi) \\ &\quad + 3(\ln m) \left(\frac{\gamma^2}{2} - \frac{\pi^2}{24} - \frac{1}{2} \ln^2(2\pi) + \gamma_1 \right). \end{aligned}$$

Corollary 3 (Asymptotic expressions) For $m \rightarrow \infty$,

$$\gamma_1 \left(\frac{1}{m} \right) \sim -m \ln m + \gamma_1$$

and

$$\gamma_2 \left(\frac{1}{m} \right) \sim m \ln^2 m + \gamma_2.$$

The general situation for $\gamma_k(a)$ with $a \rightarrow 0$ is more conveniently proved otherwise, and is presented in Proposition 5.

Proposition 2 For $\text{Re } z > 0$ and $0 < k < 2$,

$$\psi(kz) = \sum_{n=0}^{\infty} (k-1)^n \frac{z^n}{n!} \psi^{(n)}(z).$$

Proposition 3 For $\text{Re } z > 0$ and $0 < k < 2$, (a)

$$\begin{aligned} \gamma_1(kz) &= \gamma_1(z) + \gamma[\psi(kz) - \psi(z)] \\ &\quad - \sum_{n=1}^{\infty} (1-k)^n z^n \left[\frac{(-1)^{n+1}}{n!} \psi^{(n)}(z) \psi(n+1) + \frac{\partial \zeta(s, z)}{\partial s} \Big|_{s=n+1} \right] \end{aligned}$$

and (b)

$$\begin{aligned} \gamma_2(kz) &= \gamma_2(z) + [\gamma^2 - \zeta(2)][\psi(z) - \psi(kz)] \\ &\quad + 2 \sum_{n=1}^{\infty} (1-k)^n z^n \left[\left(\gamma + \frac{1}{2} \psi(n+1) \right) \psi(n+1) \zeta(n+1, z) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \psi'(n+1) \zeta(n+1, z) \\
 & + [\gamma + \psi(n+1)] \left. \frac{\partial \zeta(s, z)}{\partial s} \right|_{s=n+1} + \frac{1}{2} \left. \frac{\partial^2 \zeta(s, z)}{\partial s^2} \right|_{s=n+1} \Big].
 \end{aligned}$$

The following result, wherein the differences $\gamma_1(j/m) - \gamma_1$ and $\gamma_2(j/m) - \gamma_2$ appear, is a generalization of Gauss' formula for the digamma function at rational argument. For integers p and q with $0 < p < q$, we put

$$I_{pq}^k \equiv q \int_0^1 \left(\frac{x^{q-1} - x^{p-1}}{1 - x^q} \right) \ln^k(-\ln x) dx.$$

Proposition 4 *Let $\omega_k \equiv \exp(2\pi i k/q)$. Then (a)*

$$\begin{aligned}
 I_{pq}^1 &= - \sum_{k=1}^{q-1} (\omega_k^p - 1) \left[\gamma \ln \left(\frac{\omega_k - 1}{\omega_k} \right) + \sum_{n=1}^{\infty} \zeta'(1-n) \frac{\ln^n(\omega_k^{-1})}{n!} \right. \\
 &\quad \left. - \gamma \ln(-\ln \omega_k^{-1}) - \frac{1}{2} \ln^2(-\ln \omega_k^{-1}) \right] - q \left(\gamma_1 + \frac{\gamma^2}{2} + \frac{\pi^2}{12} \right) \\
 &= -(\gamma + \ln q) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] + \gamma_1 \left(\frac{p}{q} \right) - \gamma_1
 \end{aligned}$$

and (b)

$$\begin{aligned}
 I_{pq}^2 &= \sum_{k=1}^{q-1} (\omega_k^p - 1) \left[\left(\gamma^2 + \frac{\pi^2}{6} \right) \ln \left(\frac{\omega_k - 1}{\omega_k} \right) \right. \\
 &\quad \left. + 2\gamma \left. \frac{\partial Li_s}{\partial s}(\omega_k^{-1}) \right|_{s=1} - \left. \frac{\partial^2 Li_s}{\partial s^2}(\omega_k^{-1}) \right|_{s=1} \right] \\
 &= \left(\gamma^2 + \frac{\pi^2}{6} + 2\gamma \ln q + \ln^2 q \right) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] \\
 &\quad - 2(\gamma + \ln q) \left[\gamma_1 \left(\frac{p}{q} \right) - \gamma_1 \right] + \gamma_2 - \gamma_2 \left(\frac{p}{q} \right).
 \end{aligned}$$

In (b), the analytically continuable polylogarithm (or Jonquière) function $Li_s(z) = \sum_{k=1}^{\infty} z^k/k^s$. Explicit expressions for the partial derivatives appearing there are provided.

Proposition 5 (Asymptotic expression) *For $a \rightarrow 0$,*

$$\gamma_k(a) \sim \frac{1}{a} \ln^k a + \gamma_k.$$

Dirichlet L functions with characters χ_k modulo k may be written as

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} = \frac{1}{k^s} \sum_{m=1}^k \chi_k(m) \zeta\left(s, \frac{m}{k}\right), \quad \text{Re } s > 1$$

If χ_k is a nonprincipal character, then convergence additionally holds for $\text{Re } s > 0$. Such L functions may be analytically continued throughout the whole complex plane and satisfy a functional equation relating $L(s)$ to $L(1 - s)$. Then the results of this paper show that the values $L'(p/q)$ and $L'(1 - p/q)$ may be expressed in closed form.

2 Proof of Propositions

Proposition 1. We will be expanding a functional equation due to Hurwitz ([2, p. 261], [19, p. 93]),

$$\zeta\left(s, \frac{j}{m}\right) = \frac{2^s \pi^{s-1}}{m^{1-s}} \Gamma(1 - s) \sum_{r=1}^m \sin\left(\frac{\pi s}{2} + \frac{2\pi jr}{m}\right) \zeta\left(1 - s, \frac{r}{m}\right), \quad (2.1)$$

holding for $1 \leq j \leq m$, about $s = 1$. We will use two elementary trigonometric identities

$$-2 \sum_{r=1}^m \cos \frac{2\pi jr}{m} \zeta\left(0, \frac{r}{m}\right) = -2 \sum_{r=1}^m \left(\frac{1}{2} - \frac{r}{m}\right) \cos \frac{2\pi jr}{m} = 1, \quad 1 \leq j < m, \quad (2.2a)$$

and

$$\sum_{r=1}^m \sin \frac{2\pi jr}{m} \zeta\left(0, \frac{r}{m}\right) = \sum_{r=1}^m \left(\frac{1}{2} - \frac{r}{m}\right) \sin \frac{2\pi jr}{m} = \frac{1}{2} \cot \frac{\pi j}{m}, \quad 1 \leq j < m, \quad (2.2b)$$

as well as the values

$$\zeta''(0) = \gamma_1 + \frac{\gamma^2}{2} - \frac{\pi^2}{24} - \frac{\ln^2 2\pi}{2}, \quad (2.3a)$$

and

$$\zeta'''(0) = \gamma^3 + \frac{3}{2} \gamma^2 \ln 2\pi - \frac{\pi^2}{8} \ln 2\pi - \frac{1}{2} \ln^3 2\pi + 3(\gamma + \ln 2\pi) \gamma_1 + \frac{3}{2} \gamma_2 - \zeta(3). \quad (2.3b)$$

These derivative values may be obtained via (1.1) and the functional equation of the zeta function, $\zeta(s) = 2(2\pi)^{s-1} \sin(\pi s/2) \Gamma(1 - s) \zeta(1 - s)$. For (2.2) we have used

the well-known relation $\zeta(0, a) = 1/2 - a = -B_1(a)$, where $B_1(a)$ is the first-degree Bernoulli polynomial. We recall that about $s = 1$, $\Gamma(1 - s)$ has a simple pole and that

$$\Gamma(1 - s) = -\frac{1}{s - 1} - \gamma - \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right)(s - 1) - \frac{1}{6}\left(\gamma^3 + \frac{\gamma}{2}\pi^2 - \psi''(1)\right)(s - 1)^2 + O[(s - 1)^3],$$

wherein the tetragamma function value $\psi''(1) = -2\zeta(3)$, and recall the expansion $(2\pi m)^{s-1} = \sum_{\ell=0}^{\infty} \frac{\ln^{\ell} 2\pi m}{\ell!} (s - 1)^{\ell}$. For the sine factor on the right side of (2.1), we have

$$\begin{aligned} \sin\left(\frac{\pi s}{2} + \frac{2\pi jr}{m}\right) &= \cos\frac{2\pi jr}{m} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \pi^{2\ell}}{(2\ell)! 2^{2\ell}} (s - 1)^{2\ell} \\ &\quad + \sin\frac{2\pi jr}{m} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1} \pi^{2\ell+1}}{(2\ell + 1)! 2^{2\ell+1}} (s - 1)^{2\ell+1}. \end{aligned}$$

The left side of (2.1) expands as

$$\zeta\left(s, \frac{j}{m}\right) = \frac{1}{s - 1} - \psi\left(\frac{j}{m}\right) - \gamma_1\left(\frac{j}{m}\right)(s - 1) + \frac{1}{2}\gamma_2\left(\frac{j}{m}\right)(s - 1)^2 + O[(s - 1)^3].$$

The polar contributions on the two sides of (2.1) cancel due to the cosine sum (2.2a). At order $(s - 1)^0$, one finds

$$\begin{aligned} \psi\left(\frac{j}{m}\right) &= \sum_{r=1}^m \left\{ -\pi \zeta\left(0, \frac{r}{m}\right) \sin\left(\frac{2\pi jr}{m}\right) \right. \\ &\quad \left. + 2 \cos\frac{2\pi jr}{m} \left[\zeta\left(0, \frac{r}{m}\right) (\gamma + \ln 2\pi m) - \zeta'\left(0, \frac{r}{m}\right) \right] \right\}, \end{aligned} \tag{2.4}$$

where both of the sums of (2.2) apply. With $\zeta'(0, r/m) = \ln \Gamma(r/m) - \ln(2\pi)/2$,

$$\psi\left(\frac{j}{m}\right) = -\gamma - \ln 2\pi m - \frac{\pi}{2} \cot \frac{\pi j}{m} - 2 \sum_{r=1}^{m-1} \cos \frac{2\pi jr}{m} \ln \Gamma\left(\frac{r}{m}\right),$$

being a form of one of Gauss' formulas for ψ at rational argument.

At order $(s - 1)^1$, one finds

$$\begin{aligned} -\gamma_1\left(\frac{j}{m}\right) &= \sum_{r=1}^m \left\{ \frac{\pi^2}{4} \cos \frac{2\pi jr}{m} \zeta\left(0, \frac{r}{m}\right) \right. \\ &\quad \left. - \frac{\pi}{2} \sin \frac{2\pi jr}{m} \left[\zeta\left(0, \frac{r}{m}\right) (\gamma + \ln 2\pi m) + 2\zeta'\left(0, \frac{r}{m}\right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \cos \frac{2\pi jr}{m} \left[-\zeta \left(0, \frac{r}{m} \right) \left(\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2 \right) \right. \\
 & \left. + 2(\gamma + \ln 2\pi m)\zeta' \left(0, \frac{r}{m} \right) - \zeta'' \left(0, \frac{r}{m} \right) \right] \Big\}, \tag{2.5}
 \end{aligned}$$

and again the trigonometric summations of (2.2) apply. We add and subtract $(\gamma + \ln 2\pi m) \cos(2\pi jr/m)\zeta(0, r/m)$ to introduce the $\psi(j/m)$ expression (2.4). Then separating the $r = m$ terms of the sums and using $\zeta''(0)$ from (2.3a) gives part (a).

For part (b), at order $(s - 1)^2$ in (2.1) we have

$$\frac{1}{2}\gamma_2 \left(\frac{j}{m} \right) = A + B + C,$$

where

$$\begin{aligned}
 A = \sum_{r=1}^m \Big\{ & -\frac{\pi^3}{24}\zeta \left(0, \frac{r}{m} \right) \sin \frac{2\pi jr}{m} \\
 & + \frac{\pi^2}{4} \cos \frac{2\pi jr}{m} \left[\zeta \left(0, \frac{r}{m} \right) (\gamma + \ln 2\pi m) - \zeta' \left(0, \frac{r}{m} \right) \right] \\
 & - \frac{\pi}{2} \sin \frac{2\pi jr}{m} \left[-\zeta \left(0, \frac{r}{m} \right) \left(\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2 \right) \right. \\
 & \left. + 2(\gamma + \ln 2\pi m)\zeta' \left(0, \frac{r}{m} \right) - \zeta'' \left(0, \frac{r}{m} \right) \right] \Big\},
 \end{aligned}$$

$$\begin{aligned}
 B = & - \sum_{r=1}^m \cos \frac{2\pi jr}{m} \zeta \left(0, \frac{r}{m} \right) \left[\frac{1}{3} \ln^3 2\pi m - \left(\gamma^2 + \frac{\pi^2}{6} \right) \ln 2\pi m \right. \\
 & \left. - \gamma \ln^2 2\pi m + \frac{1}{3} \left(\gamma^3 - \frac{\gamma}{2} \pi^2 + 2\zeta(3) \right) \right],
 \end{aligned}$$

and

$$\begin{aligned}
 C = \sum_{r=1}^m \cos \frac{2\pi jr}{m} \Big\{ & \left[\frac{\pi^2}{6} + (\gamma + \ln 2\pi m)^2 \right] \zeta' \left(0, \frac{r}{m} \right) \\
 & - (\gamma + \ln 2\pi m)\zeta'' \left(0, \frac{r}{m} \right) + \frac{1}{3}\zeta''' \left(0, \frac{r}{m} \right) \Big\}.
 \end{aligned}$$

The sum in B is immediately evaluated with the aid of (2.2a). The expression for A is rewritten by using (2.4), making use of the values $\psi(j/m)$, and (2.2b). In the sum C , the $r = m$ terms are separated and all of the values $\zeta'(0) = -\ln(2\pi)/2$, (2.3a), and (2.3b) are used. Combining terms then yields (1.4). □

Remark The expression for C in the proof may be written in terms of $\gamma_1(j/m)$ by using (2.5). For the evaluation of general $\gamma_k(j/m)$, there will always be a sum $\sum_{r=1}^{m-1} \cos \frac{2\pi jr}{m} \zeta^{(k+1)} \left(0, \frac{r}{m} \right)$, as according to the highest derivative of $\zeta(s, a)$ present,

there will be no derivative of the sine factor in (2.1). The evaluation will then also contain a contribution of $\zeta^{(k+1)}(0, 1) = \zeta^{(k+1)}(0)$.

Corollary 2. The proof uses the discrete orthogonality of sine and cosine functions, implying

$$\sum_{j=1}^{m-1} \cot \frac{\pi j}{m} \cos \frac{2\pi j\ell}{m} = 0$$

and

$$\sum_{j=1}^{m-1} \cot \frac{\pi j}{m} \sin \frac{2\pi j\ell}{m} = m - 2\ell$$

along with

$$\sum_{j=1}^{m-1} \psi \left(\frac{j}{m} \right) \cos \frac{2\pi j\ell}{m} = m \ln \left(2 \sin \frac{\pi \ell}{m} \right) + \gamma$$

and

$$\sum_{j=1}^{m-1} \psi \left(\frac{j}{m} \right) \sin \frac{2\pi j\ell}{m} = \frac{\pi}{2} (2\ell - m).$$

The three sums following part (b) may be determined by successively differentiating the relation $\sum_{\ell=1}^{m-1} \zeta \left(s, \frac{\ell}{m} \right) = (m^s - 1)\zeta(s)$ and putting s to 0. □

Proposition 2. We will indicate four proofs. For the first, we may use a standard integral representation for the polygamma function [16, p. 943] to write

$$\begin{aligned} \sum_{n=0}^{\infty} (k-1)^n \frac{z^n}{n!} \psi^{(n)}(z) &= \psi(z) + \sum_{n=1}^{\infty} (k-1)^n \frac{z^n}{n!} \int_0^1 \frac{t^{z-1} \ln^n t}{t-1} dt \\ &= \psi(z) + \int_0^1 \frac{[1-t^{(k-1)z}]}{t-1} t^{z-1} dt \\ &= \psi(z) - \psi(kz) = \psi(kz). \end{aligned}$$

□

For the second, we may start with the expansion [16, p. 944]

$$\psi(x+1) = \psi(x) + \frac{1}{x} = -\gamma + \sum_{j=2}^{\infty} (-1)^j \zeta(j) x^{j-1}, \quad |x| < 1.$$

Then

$$\psi^{(n)}(z) = \frac{(-1)^{n+1}n!}{z^{n+1}} + \sum_{j=2}^{\infty} (-1)^j \zeta(j) \frac{(j-1)!}{(j-n-1)!} z^{j-n-1},$$

so that

$$\begin{aligned} & \sum_{n=0}^{\infty} (k-1)^n \frac{z^n}{n!} \psi^{(n)}(z) \\ &= \psi(z) + \sum_{n=1}^{\infty} (k-1)^n \frac{z^n}{n!} \left[\frac{(-1)^{n+1}n!}{z^{n+1}} + \sum_{j=2}^{\infty} (-1)^j \zeta(j) \frac{(j-1)!}{(j-n-1)!} z^{j-n-1} \right] \\ &= \psi(z) - \frac{1}{kz} + \frac{1}{z} + \sum_{j=2}^{\infty} (-1)^j \zeta(j) z^{j-1} \sum_{n=1}^{j-1} (k-1)^n \binom{j-1}{n} \\ &= \psi(z) - \frac{1}{kz} + \frac{1}{z} + \sum_{j=2}^{\infty} (-1)^j \zeta(j) z^{j-1} (k^{j-1} - 1) \\ &= \psi(z) - \frac{1}{kz} + \frac{1}{z} + \psi(kz) + \frac{1}{kz} - \psi(z) - \frac{1}{z} = \psi(kz). \end{aligned}$$

□

A third method of proof follows from the representations

$$\gamma_0(a) = -\psi(a) = -\ln a - \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \ln(\ell+a)$$

and

$$\psi^{(j)}(a) = (-1)^{j-1} \frac{(j-1)!}{a^j} + (j-1)! \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \frac{(-1)^{j-1}}{(\ell+a)^j}.$$

□

The fourth method of proof follows the first proof of Proposition 3 so we omit it. *Proposition 3.* We have the multiplication formula

$$\zeta(s, kz) = \sum_{n=0}^{\infty} \binom{s+n-1}{n} (1-k)^n z^n \zeta(s+n, z),$$

being a case of a more general result of Truesdell [30]. We then expand both sides about $s = 1$. Equating the coefficients of $(s-1)^0$ on both sides gives another means of demonstrating Proposition 2. Equating the coefficients of $(s-1)^1$ and using Proposition 2 gives part (a). Equating the coefficients of $(s-1)^2$ gives part (b). □

For a second method of proof of part (a), we may use the integral representation for $\text{Re } s > 1$ and $\text{Re } a > 0$,

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{e^t - 1} dt,$$

so that

$$\frac{\partial \zeta(s, a)}{\partial s} = -\psi(s)\zeta(s, a) + \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{e^t - 1} \ln t \, dt.$$

Then

$$\begin{aligned} & \sum_{n=1}^\infty (1-k)^n z^n \left. \frac{\partial \zeta(s, z)}{\partial s} \right|_{s=n+1} \\ &= \sum_{n=1}^\infty (1-k)^n z^n \left[-\psi(n+1)\zeta(n+1, z) + \frac{1}{n!} \int_0^\infty \frac{t^n e^{-(a-1)t}}{e^t - 1} \ln t \, dt \right] \\ &= -\sum_{n=1}^\infty (1-k)^n z^n \psi(n+1)\zeta(n+1, z) + \int_0^\infty e^{-(z-1)t} \frac{[e^{-(k-1)tz} - 1]}{e^t - 1} \ln t \, dt. \end{aligned}$$

Now

$$\int_0^\infty t^{s-1} e^{-(z-1)t} \frac{[e^{-(k-1)tz} - 1]}{e^t - 1} dt = \Gamma(s)[\zeta(s, kz) - \zeta(s, z)],$$

giving

$$\begin{aligned} & \int_0^\infty t^{s-1} e^{-(z-1)t} \frac{[e^{-(k-1)tz} - 1]}{e^t - 1} \ln t \, dt \\ &= \Gamma(s)\psi(s)[\zeta(s, kz) - \zeta(s, z)] + \Gamma(s) \left[\frac{\partial \zeta}{\partial s}(s, kz) - \frac{\partial \zeta}{\partial s}(s, z) \right]. \end{aligned}$$

Since by (1.1)

$$\begin{aligned} \frac{\partial \zeta}{\partial s}(s, a) &= -\frac{1}{(s-1)^2} + \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n!} \gamma_{n+1}(a)(s-1)^n \\ &= -\frac{1}{(s-1)^2} - \gamma_1(a)(s-1)^0 + \frac{\gamma_2(a)}{2}(s-1)^1 + O[(s-1)^2], \end{aligned}$$

we find

$$\int_0^\infty e^{-(z-1)t} \frac{[e^{-(k-1)tz} - 1]}{e^t - 1} \ln t \, dt = \gamma[\psi(kz) - \psi(z)] + \gamma_1(kz) - \gamma_1(z),$$

and the Proposition again follows. □

Remarks Obviously we may evaluate the integrals

$$\int_0^\infty e^{-(z-1)t} \frac{[e^{-(k-1)tz} - 1]}{e^t - 1} \ln^j t \, dt$$

in terms of the difference $\gamma_j(kz) - \gamma_j(z)$.

The harmonic numbers $H_n = \sum_{k=1}^n 1/k = \psi(n) + \gamma$ and generalized harmonic numbers

$$\begin{aligned} H_n^{(r)} &= \sum_{k=1}^n \frac{1}{k^r} = \frac{(-1)^{r-1}}{(r-1)!} \left[\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1) \right] \\ &= \frac{(-1)^{r-1}}{(r-1)!} \int_0^1 \frac{(t^n - 1)}{t - 1} \ln^{r-1} t \, dt \end{aligned}$$

enter the representations of Proposition 3 and for the higher Stieltjes constants. This is part of the elaboration of the following discussion section.

Proposition 4. We first write, for $0 < p < q$,

$$I_{pq}^k \equiv q \int_0^1 \left(\frac{x^{q-1} - x^{p-1}}{1 - x^q} \right) \ln^k(-\ln x) dx = \sum_{k=1}^{q-1} \int_0^1 \left(\frac{\omega_k^p - 1}{x - \omega_k} \right) \ln^k(-\ln x) dx.$$

By performing logarithmic differentiation on the integral

$$\int_0^1 \frac{(-\ln x)^a}{x - \omega} dx = -\Gamma(a) \text{Li}_a(\omega^{-1}),$$

one then has the following expressions:

$$I_{pq}^1 = - \sum_{k=1}^{q-1} (\omega_k^p - 1) \left[\gamma \ln \left(\frac{\omega_k - 1}{\omega_k} \right) + \frac{\partial \text{Li}_s}{\partial s} (\omega_k^{-1}) \Big|_{s=1} \right]$$

and

$$\begin{aligned} I_{pq}^2 &= \sum_{k=1}^{q-1} (\omega_k^p - 1) \left[\left(\gamma^2 + \frac{\pi^2}{6} \right) \ln \left(\frac{\omega_k - 1}{\omega_k} \right) \right. \\ &\quad \left. + 2\gamma \frac{\partial \text{Li}_s}{\partial s} (\omega_k^{-1}) \Big|_{s=1} - \frac{\partial^2 \text{Li}_s}{\partial s^2} (\omega_k^{-1}) \Big|_{s=1} \right]. \end{aligned}$$

We next present the partial derivatives of the polylogarithm function. These result from expansion of the following expression in powers of $s - 1$ [14]:

$$\text{Li}_s(z) = \Gamma(1 - s) \ln^{s-1} \left(\frac{1}{z} \right) + \sum_{n=0}^{\infty} \zeta(s - n) \frac{\ln^n z}{n!}, \quad |\ln z| < 2\pi,$$

wherein the polar part of the first term on the right is cancelled by the pole $1/(s - 1)$ of the $n = 0$ term of the sum. We obtain

$$\begin{aligned} \left. \frac{\partial \text{Li}_s}{\partial s}(z) \right|_{s=1} &= -\gamma_1 - \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \gamma \ln(-\ln z) - \frac{1}{2} \ln^2(-\ln z) \\ &\quad + \sum_{n=1}^{\infty} \zeta'(1 - n) \frac{\ln^n z}{n!} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \left. \frac{\partial^2 \text{Li}_s}{\partial s^2}(z) \right|_{s=1} &= \frac{1}{6} \left[-2\gamma^3 - \gamma\pi^2 - 6\gamma^2 \ln(-\ln z) - \pi^2 \ln(-\ln z) - 6\gamma \ln^2(-\ln z) \right. \\ &\quad \left. - 2\ln^3(-\ln z) - 4\zeta(3) \right] + \gamma_2 + \sum_{n=1}^{\infty} \zeta''(1 - n) \frac{\ln^n z}{n!}. \end{aligned} \tag{2.7}$$

Part (a) then makes use of the first derivative and the sum $\sum_{k=1}^{q-1} (\omega_k^p - 1) = -q$.

For the second evaluation of I_{pq}^k , we use

$$\begin{aligned} I_{pq}^k &= q \sum_{m=0}^{\infty} \int_0^1 (x^{q-1} - x^{p-1}) x^{qm} \ln^k(-\ln x) dx \\ &= q \sum_{m=0}^{\infty} \int_0^{\infty} \left(e^{-(q-1)u} - e^{-(p-1)u} \right) e^{-(qm+1)u} \ln^k u \, du. \end{aligned}$$

By using logarithmic differentiation of the Gamma function integral,

$$\begin{aligned} I_{pq}^1 &= q \sum_{m=0}^{\infty} \left[\frac{\gamma + \ln(p + mq)}{p + mq} - \frac{\gamma + \ln(m + 1)q}{(m + 1)q} \right] \\ &= -(\gamma + \ln q) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] + \sum_{m=0}^{\infty} \left[\frac{\ln(m + p/q)}{m + p/q} - \frac{\ln(m + 1)}{m + 1} \right] \\ &= -(\gamma + \ln q) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] + \gamma_1 \left(\frac{p}{q} \right) - \gamma_1, \end{aligned}$$

wherein we applied (1.2) and the well-known summation (e.g., [16, p. 943])

$$\psi(x) = -\gamma - \sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k+1} \right).$$

The other evaluation of (b) goes similarly, with

$$\begin{aligned} I_{pq}^2 &= q \sum_{m=0}^{\infty} \int_0^{\infty} \left(e^{-(q-1)u} - e^{-(p-1)u} \right) e^{-(qm+1)u} \ln^2 u \, du \\ &= \sum_{m=0}^{\infty} \left\{ \frac{1}{6(m+1)} \left[6\gamma^2 + \pi^2 + 12\gamma \ln(m+1)q + 6 \ln^2(m+1)q \right] \right. \\ &\quad \left. - \frac{1}{6(m+p/q)} \left[6\gamma^2 + \pi^2 + 12\gamma \ln(mq+p)q + 6 \ln^2(mq+p) \right] \right\} \\ &= \left(\gamma^2 + \frac{\pi^2}{6} \right) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] + 2\gamma \sum_{m=0}^{\infty} \left[\frac{\ln(m+1)q}{m+1} - \frac{\ln(mq+p)}{m+p/q} \right] \\ &\quad + \sum_{m=0}^{\infty} \left[\frac{\ln^2(m+1)q}{m+1} - \frac{\ln^2(mq+p)}{m+p/q} \right] \\ &= \left(\gamma^2 + \frac{\pi^2}{6} + 2\gamma \ln q \right) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] - 2\gamma \left[\gamma_1 \left(\frac{p}{q} \right) - \gamma_1 \right] \\ &\quad + \sum_{m=0}^{\infty} \left[\frac{\ln^2(m+1)}{m+1} - \frac{\ln^2(m+p/q)}{m+p/q} + 2 \ln q \left(\frac{\ln(m+1)}{m+1} - \frac{\ln(m+p/q)}{m+p/q} \right) \right. \\ &\quad \left. + \ln^2 q \left(\frac{1}{m+1} - \frac{1}{m+p/q} \right) \right] \\ &= \left(\gamma^2 + \frac{\pi^2}{6} + 2\gamma \ln q + \ln^2 q \right) \left[\gamma + \psi \left(\frac{p}{q} \right) \right] \\ &\quad - 2(\gamma + \ln q) \left[\gamma_1 \left(\frac{p}{q} \right) - \gamma_1 \right] + \gamma_2 - \gamma_2 \left(\frac{p}{q} \right). \end{aligned}$$

□

Remark For applications or computation with Proposition 4, it is important that the values of $\ln(\pm\omega_k^{\pm 1})$ are kept to the principal branch, e.g., with $-\pi < \text{Im} \ln z \leq \pi$. This requirement maintains a real-valued result for I_{pq}^k .

Elaboration of the partial derivative (2.6).

The partial derivative (2.6) may also be represented as

$$-\frac{\partial \text{Li}_s(z)}{\partial s} \Big|_{s=1} = \int_1^{\infty} \frac{z^x}{x} \ln x \, dx + \int_1^{\infty} z^x \left(\frac{(\ln z) \ln x}{x} + \frac{1}{x^2} - \frac{\ln x}{x^2} \right) P_1(x) dx, \tag{2.8}$$

following from [8]

$$-\frac{\partial \text{Li}_s(z)}{\partial s} = \int_1^\infty \frac{z^x}{x^s} \ln x \, dx + \int_1^\infty z^x \left(\frac{(\ln z) \ln x}{x^s} + \frac{1}{x^{s+1}} - s \frac{\ln x}{x^{s+1}} \right) P_1(x) dx,$$

wherein $P_1(x) \equiv B_1(x - [x])$. We relate (2.6) and (2.8). By using the expansion $z^x = \sum_{j=0}^\infty \ln^j z \frac{x^j}{j!}$, we have

$$\begin{aligned} \int_1^a \frac{z^x}{x} \ln x \, dx &= \sum_{j=0}^\infty \frac{\ln^j z}{j! j^2} [1 + a^j (j \ln a - 1)] \\ &= -\gamma \ln a - \Gamma(0, -a \ln z) \ln a + \frac{\ln^2 a}{2} \\ &\quad + \ln z {}_3F_3(1, 1, 1; 2, 2, 2; \ln z) \\ &\quad - a \ln z {}_3F_3(1, 1, 1; 2, 2, 2; a \ln z) - \ln a \ln(-a \ln z), \end{aligned}$$

where ${}_pF_q$ is the generalized hypergeometric function and the incomplete Gamma function $\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^\infty \frac{(-1)^n x^{n+\alpha}}{n!(n+\alpha)}$. The asymptotic form of the ${}_3F_3$ function as $a \rightarrow \infty$ may be considered as in [11] and the result is

$$\begin{aligned} \int_1^\infty \frac{z^x}{x} \ln x \, dx &= \frac{\gamma^2}{2} + \frac{\pi^2}{12} + \ln z {}_3F_3(1, 1, 1; 2, 2, 2; \ln z) \\ &\quad + \gamma \ln(-\ln z) + \frac{1}{2} \ln^2(-\ln z), \quad |z| < 1. \end{aligned} \tag{2.9}$$

This result (2.9) may also be obtained as a reduction of a Meijer- G function. However, we omit details of this evaluation.

By using respectively the partial derivative of (1.1) with respect to s and then (2.8) and (2.9), we have these additional expressions for the partial derivative (2.6):

$$\begin{aligned} \left. \frac{\partial \text{Li}_s(z)}{\partial s} \right|_{s=1} &= -\gamma_1 - \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \gamma \ln(-\ln z) - \frac{1}{2} \ln^2(-\ln z) \\ &\quad - \ln z {}_3F_3(1, 1, 1; 2, 2, 2; \ln z) - \sum_{j=0}^\infty \frac{\gamma_{j+1}}{j!} \sum_{n=1}^\infty \frac{\ln^n z}{n!} n^j \\ &= -\frac{\gamma^2}{2} - \frac{\pi^2}{12} - \gamma \ln(-\ln z) - \frac{1}{2} \ln^2(-\ln z) \\ &\quad - \ln z {}_3F_3(1, 1, 1; 2, 2, 2; \ln z) \\ &\quad - \int_1^\infty z^x \left(\frac{(\ln z) \ln x}{x} + \frac{1}{x^2} - \frac{\ln x}{x^2} \right) P_1(x) dx \\ &= -\frac{\gamma^2}{2} - \frac{\pi^2}{12} - \gamma \ln(-\ln z) - \frac{1}{2} \ln^2(-\ln z) \\ &\quad - \ln z {}_3F_3(1, 1, 1; 2, 2, 2; \ln z) \end{aligned}$$

$$-\sum_{j=1}^{\infty} \int_0^1 z^{y+j} \left[\frac{(\ln z) \ln(y+j)}{y+j} + \frac{1}{(y+j)^2} - \frac{\ln(y+j)}{(y+j)^2} \right] \left(y - \frac{1}{2} \right) dy.$$

By comparing (2.6) with the second expression above, we conclude that

$$\begin{aligned} \gamma_1 - \sum_{n=1}^{\infty} \zeta'(1-n) \frac{\ln^n z}{n!} \\ = \ln z {}_3F_3(1, 1, 1; 2, 2, 2; \ln z) + \int_1^{\infty} z^x \left(\frac{(\ln z) \ln x}{x} + \frac{1}{x^2} - \frac{\ln x}{x^2} \right) P_1(x) dx. \end{aligned}$$

Proposition 5. Let $C_k(a) \equiv \gamma_k(a) - (\ln^k a)/a$. With $B_n(x)$ being the Bernoulli polynomials, their periodic extension is denoted as $P_n(x) \equiv B_n(x - [x])$ and we have the representation [32]

$$C_n(a) = (-1)^{n-1} n! \sum_{k=0}^{n+1} \frac{s(n+1, n+1-k)}{k!} \int_1^{\infty} P_n(x-a) \frac{\ln^k x}{x^{n+1}} dx, \quad n \geq 1, \tag{2.10}$$

with $s(n, k)$ as the Stirling numbers of the first kind. We recall the Fourier expansions of $P_n(x)$ [1, p. 805],

$$P_{2n}(x-a) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2\pi k(x-a)}{k^{2n}},$$

and

$$P_{2n-1}(x-a) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2\pi k(x-a)}{k^{2n-1}}.$$

We therefore obtain

$$P_{2n}(x-a) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} [\cos 2\pi kx + 2\pi ka \sin 2\pi kx + O(a^2)]$$

and

$$P_{2n-1}(x-a) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{1}{k^{2n-1}} [\sin 2\pi kx - 2\pi ka \cos 2\pi kx + O(a^2)].$$

These forms are then inserted into (2.10). Noting that $C_k(1) = \gamma_k$, the a^0 term produces γ_k , $C_k(a) \rightarrow \gamma_k$ as $a \rightarrow 0$ and hence the result. □

3 Discussion

Here we first discuss the equivalence of Proposition 3 as a case of an addition formula which we have previously presented [12] (Proposition 1). We then show applications of differences of Stieltjes constants to some classic integrals of analytic number theory. We exhibit a new proof technique for certain log–log integrals.

As regards the Truesdell representation of $\zeta(s, kz)$, we note

$$\begin{aligned} \binom{s+n-1}{n} &= \frac{(-1)^n}{n!} (1-s-n)_n = \frac{(-1)^n}{n!} \frac{\Gamma(1-s)}{\Gamma(1-s-n)} \\ &= \frac{1}{n!} \frac{\Gamma(s+n)}{\Gamma(s)} = \frac{1}{n!} (s)_n, \end{aligned}$$

so that

$$\zeta(s, kz) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} (1-k)^n z^n \zeta(s+n, z).$$

On the other hand, an old formula of Wilton [31] may be written as

$$\zeta(s, a+b) = \sum_{j=0}^{\infty} \frac{(-b)^j}{j!} (s)_j \zeta(s+j, a), \quad |b| < |a|, \quad \text{Re } a > 0.$$

Thus the two formulas correspond with $b = -(1-k)z$ and $a = z$. Lemma 1 of [12] provides the derivative values

$$\left(\frac{d}{ds}\right)^\ell (s)_j \Big|_{s=1} = (-1)^{j+\ell} \ell! s(j+1, \ell+1).$$

Therefore, from Proposition 1 of [12], we know the general form of the multiplication formula for the Stieltjes constants,

$$\gamma_\ell(kz) = \gamma_\ell(z) + (-1)^\ell \sum_{j=2}^{\infty} \frac{(k-1)^{j-1} z^{j-1}}{(j-1)!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} s(j, k+1) k! \zeta^{(\ell-k)}(j, z).$$

The Stirling numbers of the first kind may indeed be written with the generalized harmonic numbers, and the first few are given by $s(n+1, 1) = (-1)^n n!$, $s(n+1, 2) = (-1)^{n+1} n! H_n$, $s(n+1, 3) = (-1)^n \frac{n!}{2} [H_n^2 - H_n^{(2)}]$, and $s(n+1, 4) = (-1)^{n+1} \frac{n!}{6} [H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}]$.

We demonstrate how differences of Stieltjes constants may be used to efficiently evaluate some example log–log integrals, including

$$I_{\pm} \equiv \int_0^1 \frac{\ln(-\ln x)}{1 \pm x + x^2} dx.$$

For I_- ,

$$\begin{aligned}
 I_- &= \int_0^1 \left(\frac{1+x}{1+x^3} \right) \ln(-\ln x) dx \\
 &= \sum_{m=0}^{\infty} (-1)^m \int_0^1 (1+x)x^{3m} \ln(-\ln x) dx \\
 &= \sum_{m=0}^{\infty} (-1)^m \int_0^{\infty} (1+e^{-u})e^{-(3m+1)u} \ln u \, du \\
 &= - \sum_{m=0}^{\infty} (-1)^m \left[\gamma \left(\frac{1}{3m+2} + \frac{1}{3m+1} \right) + \frac{\ln(3m+2)}{3m+2} + \frac{\ln(3m+1)}{3m+1} \right] \\
 &= -\gamma \frac{2\pi}{3\sqrt{3}} - \frac{1}{6} \sum_{m=0}^{\infty} \left[\ln 6 \left(\frac{1}{m+1/3} + \frac{1}{m+1/6} - \frac{1}{m+5/6} - \frac{1}{m+2/3} \right) \right. \\
 &\quad \left. + \frac{\ln(m+1/3)}{m+1/3} + \frac{\ln(m+1/6)}{m+1/6} - \frac{\ln(m+5/6)}{m+5/6} - \frac{\ln(m+2/3)}{m+2/3} \right] \\
 &= -(\gamma + \ln 6) \frac{2\pi}{3\sqrt{3}} + \frac{1}{6} \left[\gamma_1 \left(\frac{2}{3} \right) - \gamma_1 \left(\frac{1}{3} \right) + \gamma_1 \left(\frac{5}{6} \right) - \gamma_1 \left(\frac{1}{6} \right) \right].
 \end{aligned}$$

We used polygamma function values and (1.2).

For I_+ ,

$$\begin{aligned}
 I_+ &= \int_0^1 \left(\frac{1-x}{1-x^3} \right) \ln(-\ln x) dx \\
 &= \sum_{m=0}^{\infty} \int_0^1 (1-x)x^{3m} \ln(-\ln x) dx \\
 &= \sum_{m=0}^{\infty} \int_0^{\infty} (1-e^{-u})e^{-(3m+1)u} \ln u \, du \\
 &= \sum_{m=0}^{\infty} \left[\gamma \left(\frac{1}{3m+2} - \frac{1}{3m+1} \right) + \frac{\ln(3m+2)}{3m+2} - \frac{\ln(3m+1)}{3m+1} \right] \\
 &= -\frac{\gamma\pi}{3\sqrt{3}} + \frac{1}{3} \sum_{m=0}^{\infty} \left[\frac{\ln 3(m+2/3)}{m+2/3} - \frac{\ln 3(m+1/3)}{m+1/3} \right] \\
 &= -(\gamma + \ln 3) \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \left[\gamma_1 \left(\frac{2}{3} \right) - \gamma_1 \left(\frac{1}{3} \right) \right].
 \end{aligned}$$

More generally,

$$\begin{aligned}
 I_{+n} &= \int_0^1 \frac{\ln(-\ln x)}{x^{n-1} + x^{n-2} + \dots + x + 1} dx \\
 &= \frac{(\gamma + \ln n)}{n} \left[\psi \left(\frac{1}{n} \right) - \psi \left(\frac{2}{n} \right) \right] + \frac{1}{n} \left[\gamma_1 \left(\frac{2}{n} \right) - \gamma_1 \left(\frac{1}{n} \right) \right], \\
 I_{+n}^q &= \int_0^1 \frac{x^q \ln(-\ln x)}{x^{n-1} + x^{n-2} + \dots + x + 1} dx \\
 &= \frac{(\gamma + \ln n)}{n} \left[\psi \left(\frac{q+1}{n} \right) - \psi \left(\frac{q+2}{n} \right) \right] \\
 &\quad + \frac{1}{n} \left[\gamma_1 \left(\frac{q+2}{n} \right) - \gamma_1 \left(\frac{q+1}{n} \right) \right], \quad \text{Re } q > -1,
 \end{aligned}$$

and for n odd,

$$\begin{aligned}
 I_{-n}^q &= \int_0^1 \frac{x^q \ln(-\ln x)}{x^{n-1} - x^{n-2} + \dots - x + 1} dx \\
 &= \frac{(\gamma + \ln n)}{2n} \left[\psi \left(\frac{q+2}{2n} \right) + \psi \left(\frac{q+1}{2n} \right) \right. \\
 &\quad \left. - \psi \left(\frac{n+q+2}{2n} \right) - \psi \left(\frac{n+q+1}{2n} \right) \right] \\
 &\quad + \frac{1}{2n} \left[\gamma_1 \left(\frac{n+q+2}{2n} \right) + \gamma_1 \left(\frac{n+q+1}{2n} \right) \right. \\
 &\quad \left. - \gamma_1 \left(\frac{q+2}{2n} \right) - \gamma_1 \left(\frac{q+1}{2n} \right) \right], \quad \text{Re } q > -1.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \int_0^1 \frac{\ln(-\ln x)}{1+x^2} dx &= -(\gamma + \ln 4) \frac{\pi}{4} + \frac{1}{4} \left[\gamma_1 \left(\frac{3}{4} \right) - \gamma_1 \left(\frac{1}{4} \right) \right], \\
 J_p &\equiv \int_0^1 \frac{\ln(-\ln x)}{1+x^p} dx = \frac{1}{2} \int_0^\infty \frac{e^{-(1-p/2)u} \ln u}{\cosh(pu/2)} du \\
 &= \frac{(\gamma + \ln 2p)}{2p} \left[\psi \left(\frac{1}{2p} \right) - \psi \left(\frac{p+1}{2p} \right) \right] \\
 &\quad + \frac{1}{2p} \left[\gamma_1 \left(\frac{p+1}{2p} \right) - \gamma_1 \left(\frac{1}{2p} \right) \right], \quad \text{Re } p > 0,
 \end{aligned}$$

and

$$\begin{aligned}
 J_p^2 &\equiv \int_0^1 \frac{\ln^2(-\ln x)}{1+x^p} dx \\
 &= \frac{1}{2p} \left[\gamma^2 + \frac{\pi^2}{6} + 2\gamma \ln 2p + \ln^2 2p \right] \left[\psi \left(\frac{p+1}{2p} \right) - \psi \left(\frac{1}{2p} \right) \right] \\
 &\quad + \frac{1}{p} (\gamma + \ln 2p) \left[\gamma_1 \left(\frac{1}{2p} \right) - \gamma_1 \left(\frac{p+1}{2p} \right) \right] \\
 &\quad + \frac{1}{2p} \left[\gamma_2 \left(\frac{1}{2p} \right) - \gamma_2 \left(\frac{p+1}{2p} \right) \right],
 \end{aligned}$$

with the limits

$$\lim_{p \rightarrow \infty} J_p = -\gamma, \quad \lim_{p \rightarrow \infty} J_p^2 = \gamma^2 + \zeta(2).$$

As they should be, these limits are consistent with Corollary 3 and the more general Proposition 5.

From the J_p evaluation follows the integral identity

$$\begin{aligned}
 &\int_0^1 \left\{ \frac{(\gamma + \ln 2p)}{2p} \left[\psi \left(\frac{1}{2p} \right) - \psi \left(\frac{p+1}{2p} \right) \right] + \frac{1}{2p} \left[\gamma_1 \left(\frac{p+1}{2p} \right) - \gamma_1 \left(\frac{1}{2p} \right) \right] \right\} dp \\
 &= \int_0^1 \frac{[\ln(2x) - \ln(x+1)]}{\ln x} \ln(-\ln x) dx.
 \end{aligned}$$

An analogous result applies for

$$\int_0^1 J_p^2 dp = \int_0^1 \frac{[\ln(2x) - \ln(x+1)]}{\ln x} \ln^2(-\ln x) dx.$$

As an extension of J_p , for $\text{Re } p > 0$ and $\text{Re } q > -1$, we have

$$\begin{aligned}
 J_p^q &\equiv \int_0^1 \frac{x^q}{1+x^p} \ln(-\ln x) dx \\
 &= \frac{(\gamma + \ln 2p)}{2p} \left[\psi \left(\frac{q+1}{2p} \right) - \psi \left(\frac{p+q+1}{2p} \right) \right] \\
 &\quad + \frac{1}{2p} \left[\gamma_1 \left(\frac{p+q+1}{2p} \right) - \gamma_1 \left(\frac{q+1}{2p} \right) \right],
 \end{aligned}$$

with

$$\lim_{p \rightarrow \infty} J_p^q = -\frac{\gamma + \ln(q+1)}{q+1}.$$

Propositions 1 and 4 apply to all of these integrals. As a brief example, one finds

$$I_2 \equiv \int_0^1 \frac{\ln(-\ln x)}{1+x^2} dx = \frac{\pi}{4} \left[\ln \frac{8\pi\Gamma^2(3/4)}{\Gamma^2(1/4)} - \ln 4 \right] = \frac{\pi}{2} \ln \frac{\sqrt{2\pi}\Gamma(3/4)}{\Gamma(1/4)}.$$

The value of I_2 has been known for a long time, and it may of course be written in many equivalent forms. However, the following method of evaluation may be new.

Demonstration 1

$$I_2 \equiv \int_0^1 \frac{\ln(-\ln x)}{1+x^2} dx = \int_{\pi/4}^{\pi/2} \ln(\ln(\tan x)) dx = \frac{\pi}{4} \left[\ln \left(\frac{\pi}{8} \right) + 2 \ln \frac{\Gamma(3/4)}{\Gamma(5/4)} \right].$$

The method below applies to a large variety of integrals, enabling another determination of differences of Stieltjes constants at rational arguments. A key feature of these integrals is integrands with polynomial denominators with zeros at roots of unity.

Proof Write

$$I_2 = \frac{1}{2i} \int_0^1 \left(\frac{1}{x-i} - \frac{1}{x+i} \right) \ln(-\ln x) dx,$$

and then apply

$$\int_0^1 \frac{\ln(-\ln x)}{x-a} dx = -\gamma \ln \left(\frac{a-1}{a} \right) - \left. \frac{\partial \text{Li}_s}{\partial s} \right|_{s=1} (a^{-1}), \tag{3.1}$$

and (2.6) for the partial derivative to find

$$\begin{aligned} I_2 &= \frac{\gamma\pi}{4} + \frac{\pi}{2} \ln \left(\frac{\pi}{2} \right) + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\zeta'(1-n)}{n!} \left(\frac{\pi}{2} \right)^n i^n [1 - (-1)^n] \\ &= \pi \left[\frac{\gamma}{4} + \frac{1}{2} \ln \left(\frac{\pi}{2} \right) \right] + \sum_{m=0}^{\infty} \frac{(-1)^m \zeta'(-2m)}{(2m+1)!} \left(\frac{\pi}{2} \right)^{2m+1}. \end{aligned}$$

Next separate the $m = 0$ term of the sum and use the functional equation of the zeta function, $\pi^{1-z}\zeta(z) = 2^z\Gamma(1-z)\zeta(1-z) \sin \frac{\pi z}{2}$, along with $\zeta(-2m) = 0$ for $m \geq 1$, to determine that for $m \geq 1$, $2(-1)^m \zeta'(-2m) = (2m)! \zeta(2m+1) / (2\pi)^{2m}$. There results

$$I_2 = \frac{\pi}{4} \left[\gamma + \ln \left(\frac{\pi}{8} \right) + \sum_{m=1}^{\infty} \frac{\zeta(2m+1)}{16^m(2m+1)} \right].$$

Using (e.g., [16, p. 939])

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} \zeta(2n+1) = -\gamma x + \frac{1}{2} [\ln \Gamma(1-x) - \ln \Gamma(x+1)], \quad |x| < 1, \tag{3.2}$$

completes the evaluation.¹ □

As a further indication of the applicability of this method, we mention

Lemma 1 For $-\pi < \delta \leq \pi$,

$$I_{\omega} \equiv \int_0^1 \frac{\ln(-\ln x)}{x^2 - 2x \cos \delta + 1} dx = -\frac{\pi}{2 \sin \delta} \left[\frac{\delta}{\pi} \ln(2\pi) - \ln \delta + \ln \frac{\Gamma\left(1 + \frac{\delta}{2p}\right)}{\Gamma\left(1 - \frac{\delta}{2p}\right)} \right].$$

We only sketch the proof as this is a known integral.

Proof We let $\omega = e^{i\delta}$ and use the factorization $(x - \omega)(x - \omega^*) = x^2 - (\omega + \omega^*)x + |\omega|^2 = x^2 - 2x \cos \delta + 1$, giving

$$I_{\omega} = \frac{1}{\omega - \omega^*} \int_0^1 \left(\frac{1}{x - \omega} - \frac{1}{x - \omega^*} \right) \ln(-\ln x) dx.$$

We employ the integral (3.1), the partial derivative (2.6), and finally the summation (3.2). Along the way we use elementary relations such as $1/\omega^* = \omega$ and

$$\ln \left[\frac{(\omega^* - 1)\omega}{\omega^*(\omega - 1)} \right] = \ln \left(\frac{1 - e^{i\delta}}{1 - e^{-i\delta}} \right) = \ln[e^{i(\delta+\pi)}] = i(\delta + \pi).$$

□

Similarly, integrals of the form

$$\int_0^1 \frac{\ln^2(-\ln x)}{p(x)} dx,$$

with p a polynomial having as zeros roots of unity, may be evaluated with the aid of

$$\begin{aligned} \int_0^1 \frac{\ln^2(-\ln x)}{x - a} dx &= \left(\gamma^2 + \frac{\pi^2}{6} \right) \ln \left(\frac{a - 1}{a} \right) \\ &+ 2\gamma \left. \frac{\partial \text{Li}_s}{\partial s} \right|_{s=1} (a^{-1}) - \left. \frac{\partial^2 \text{Li}_s}{\partial s^2} \right|_{s=1} (a^{-1}), \end{aligned}$$

¹ Details of the evaluation of the integrals I_{\pm} by this method are separately available from the author.

and the partial derivative (2.7). In summary, this method comprises the use of partial fractions, logarithmic differentiation of a polylogarithm integral, application of the partial derivatives of Li_s at $s = 1$, application of the functional equation of the Riemann zeta function, and summation to $\ln \Gamma$ constants where pertinent.

Many other integrals follow from the results of this paper. For instance, for $|a| = 1$ but $a \neq 1$ or $\text{Re } a > 1$, we have

$$\int_0^1 \frac{\ln(-\ln x)}{(x-a)^{m+1}} dx = \frac{\gamma}{m} (-1)^m \left[\frac{1}{(a-1)^m} - \frac{1}{a^m} \right] - \frac{1}{m!} \left(\frac{\partial}{\partial a} \right)^m \frac{\partial \text{Li}_s}{\partial s} \Big|_{s=1} \left(\frac{1}{a} \right).$$

This follows from (3.1) and (2.6). In particular,

$$\int_0^1 \frac{\ln(-\ln x)}{(x-a)^2} dx = \frac{1}{a} \left\{ \gamma \left[\frac{1}{\ln a} - \frac{1}{(a-1)} \right] + \frac{\ln(\ln a)}{\ln a} + \sum_{n=0}^{\infty} \frac{\zeta'(-n)}{n!} \ln^n(a^{-1}) \right\}$$

Such a result may be combined with the use of partial fractions to yield yet other integrals.

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