Ramanujan J (2016) 39:201–224
DOI 10.1007/s11139-014-9666-4

# Counting corners in partitions

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Received: 17 June 2014 / Accepted: 8 December 2014 / Published online: 22 January 2015


**Abstract** A *partition* of a positive integer $n$ is a non-increasing sequence of positive integers whose sum is $n$. It may be represented by a Ferrers diagram. These diagrams contain corners which are points of degree two. We define corners of types $(a, b)$, $(a+, b)$ and $(a+, b+)$, and also define the size of a corner. Via a generating function, we count corners of each type and corners of size $m$. We also find asymptotics for the number of corners as $n$ tends to infinity.

**Keywords** Partitions · Generating functions · Corners · Asymptotics

**Mathematics Subject Classification** Primary 05A15 · 05A16

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Charlotte Brennan and Arnold Knopfmacher were supported by the National Research Foundation under Grant Numbers 86329 and 81021, respectively.

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## 1 Introduction

A *partition* of a positive integer $n$ is a non-increasing sequence of positive integers whose sum is $n$, and the number of such partitions is denoted by $P(n)$ (we define $P(0) = 1$). A standard result is

$$\sum_{j\geq 0} P(j)x^j = \frac{1}{\prod_{j\geq 1}(1-x^j)}.$$

Many researchers, for example George Andrews, have focused on the number of partitions satisfying certain conditions (for example, see [1–5,8,10,11] and references therein). For instance, if $Q(n)$ is the number of partitions of $n$ with distinct parts and $Q(0) := 1$, then we have the generating function

$$\sum_{j\geq 0} Q(j)x^j = \prod_{j\geq 0}(1+x^j).$$

In this paper, we define a new statistic, namely a corner for such partitions. This is related both to descents and the number of occurrences of a part (of fixed size). It is simple to describe corners in terms of the associated Ferrers diagrams.

The *Ferrers diagram* of an integer partition proves to be a useful tool for visualizing partitions. It is constructed by stacking left-justified rows of cells, where the number of cells in the $i$th row corresponds to the size of the $i$th part.

A *corner* of a partition $\pi$ is a point of degree two in the corresponding Ferrers diagram. We denote the number of corners of $\pi$ by $cor(\pi)$. For example, if $\pi = 4422111$, then $cor(\pi) = 6$, see Fig. 1. Moreover, we define $cor_k(\pi)$ to be the number of corners at line $y = k$ in the Ferrers diagram of $\pi$, where the topmost horizontal line of the Ferrers diagram is the line $y = 0$. For example, for the partition illustrated below, $cor_0(\pi) = 2$ (corners $E$ and $F$), $cor_2(\pi) = 1$ (corner $C$) and so on. More generally, we are interested in several types of corners. We say that a *corner is of type* $(a, b)$ if it is at the bottom right of a specific maximal $a \times b$ rectangle (where $b$ is its length and $a$ its height). For such a maximal rectangle, there are no cells below it and no cells to its right. Thus, corner $B$ is at the bottom right of the 2 by 1 rectangle, with cells marked by X. So that $B$ is a (2, 1) corner. Here, we shall only consider corners which are at the bottom-right extremities of such $b$ by $a$ rectangles. So for

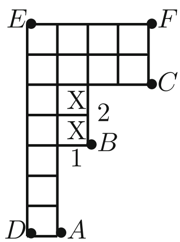



**Fig. 1** The 6 corners of $\pi = 4422111$, with $A$, $B$ and $C$ of type $(a, b)$

**Table 1** Summary of main results

| Types of corners | Generating function for the total number of such corners | Main term asymptotics for the average number of such corners |
|---|---|---|
| All corners | $-3+\dfrac{3-2x}{(1-x)\prod_{j\geq 1}(1-x^j)}$ | $\dfrac{\sqrt{6n}}{\pi}$ |
| $(a, b)$ | $\dfrac{x^{ab}\prod_{i=1}^{a}(1-x^i)}{\prod_{i\geq 1}(1-x^i)\prod_{i=b+1}^{b+a+1}(1-x^i)}$ | $\dfrac{a!b!\sqrt{6n}}{\pi(a+b+1)!}$ |
| $(a+, b)$ | $\dfrac{x^{ab}\prod_{i=1}^{a}(1-x^i)}{\prod_{i\geq 1}(1-x^i)\prod_{i=b}^{a+b}(1-x^i)}$ | $\dfrac{a!(b-1)!\sqrt{6n}}{\pi(a+b)!}$ |
| $(a+, b+)$ | $\dfrac{x^{ab}\prod_{i=1}^{a-1}(1-x^i)}{\prod_{i\geq 1}(1-x^i)\prod_{i=b}^{a+b-1}(1-x^i)}$ | $\dfrac{(a-1)!(b-1)!\sqrt{6n}}{\pi\left(a+b-1\right)!}$ |
| size $m$ | $\dfrac{1}{\prod_{l=1}^{\infty}(1-x^l)}\sum_{p=1}^{m-1}\dfrac{x^{p(m-p)}\prod_{i=1}^{p}(1-x^i)}{\prod_{i=m-p+1}^{m+1}(1-x^i)}$ | $\dfrac{1}{m+1}\sum_{a=1}^{m-1}\binom{m}{a}^{-1}\dfrac{\sqrt{6n}}{\pi}$ |

convenience, we ignore the 3 corners $D$, $E$ and $F$ at levels $x=0$ and $y=0$. Thus, the partition in Fig. 1 has corners $C$ of type $(2, 2)$, $B$ of type $(2, 1)$ and $A$ of type $(3, 1)$.

We observe that a corner of type $(a, b)$ in the Ferrers diagram of $\pi$ corresponds to consecutive parts $c$ , $c-b$ in the partition $\pi$ such that the multiplicity of $c$ is exactly $a$ and the last occurrence of $c$ is followed by a part of size $c-b$. We also consider corners of type $(a+, b)$; these are corners of type $(j, b)$ for any $j \geq a$. Similarly, we define corners of type $(a+, b+)$. Finally, we define the size of corners of type $(a, b)$ to be $a+b$.

We summarize some of the main results obtained in this paper in Table 1.

## 2 Corners at level *k*

Let $P_m = P_m(x, t_0, t_1, \ldots)$ be the generating function for the number of partitions of $n$ where the first part is $m$ and $t_k$ marks the number of corners at level $k$ ($y=k$, measured downwards with $y=0$ at the top). Each partition $\pi$ with first part $m$ can be written as $\pi = m\pi'$, where $\pi'$ is either empty or its largest part is less than or equal to $m$. Therefore we obtain

$$P_m(x, t_0, t_1, \ldots) = x^m\left(\frac{t_0^2}{t_1^2} P_m(x, t_1, t_2, \ldots) + \frac{t_0^2}{t_1}\sum_{j=1}^{m-1} P_j(x, t_1, t_2, \ldots) + t_0^2 t_1^2\right).$$

Define $P(z, x, t_0, t_1, \ldots) := \sum_{m\geq 1} P_m(x, t_0, t_1, \ldots)z^m$. Multiplying the above equation by $z^m$ and summing over $m \geq 1$, we derive the following result.

**Proposition 2.1** *The generating function $P(z, x, t_0, t_1, \ldots)$ satisfies the following functional equation:*

$$P(z, x, t_0, t_1, \ldots) = \frac{t_0^2}{t_1^2} P(xz, x, t_1, t_2, \ldots) + \frac{t_0^2 t_1^2 xz}{1 - xz} + \frac{t_0^2 xz}{t_1(1 - xz)} P(xz, x, t_1, t_2, \ldots).$$

Now, we are ready to study the total number of corners in a partition. Let $P(z, x, t) := P(z, x, t, t, \ldots)$. Proposition 2.1 with $t_j = t$ for all $j \geq 0$ gives

$$P(z, x, t) = P(xz, x, t) + \frac{t^4 xz}{1 - xz} + \frac{txz}{(1 - xz)} P(xz, x, t),$$

which is equivalent to

$$P(z, x, t) = \frac{t^4 xz}{1 - xz} + \left(1 + \frac{xzt}{1 - xz}\right) P(xz, x, t).$$

By iterating this equation for $z = 1, x, x^2, \ldots$, and solving for $P(1, x, t)$, we obtain the following result.

**Theorem 2.2** *The generating function for the number of partitions of $n$ according to the number of corners counted by $t$ is given by*

$$P(1, x, t) = t^4 \sum_{j\geq 1} \frac{x^j}{1 - x^j} \prod_{i=1}^{j-1} \frac{1 - (1 - t)x^i}{1 - x^i}.$$

We shall need the lemma below for the work that follows:

**Lemma 2.3** *For all $a, b \geq 1$,*

$$(a) \quad \sum_{j\geq 0} \frac{x^{jb}}{\prod_{i=1}^{j}(1 - x^i)} = \frac{1}{\prod_{i\geq b}(1 - x^i)}$$

*and*

$$(b)\quad \sum_{j\geq 0} x^{jb}\prod_{i=j+1}^{j+a}\left(1-x^i\right)=\frac{\prod_{i=1}^{a}\left(1-x^i\right)}{\prod_{i=b}^{b+a}\left(1-x^i\right)}.$$

*Proof* (a) We have

$$\begin{aligned}
f(b) := \sum_{j\geq 0}\frac{x^{jb}}{\prod_{i=1}^{j}\left(1-x^i\right)} &= 1+\sum_{j\geq 0}\frac{x^{(j+1)b}}{\prod_{i=1}^{j+1}\left(1-x^i\right)}\\
&= 1+\sum_{j\geq 0}\frac{x^{(j+1)b}-x^{(j+1)b+j+1}+x^{(j+1)b+j+1}}{\prod_{i=1}^{j+1}\left(1-x^i\right)}\\
&= 1+\sum_{j\geq 0}\frac{x^{(j+1)b}}{\prod_{i=1}^{j}\left(1-x^i\right)}+\sum_{j\geq 1}\frac{x^{j(b+1)}}{\prod_{i=1}^{j}\left(1-x^i\right)} = x^b f(b)+f(b+1),
\end{aligned}$$

which implies that $f(b)=\frac{1}{1-x^b}f(b+1)$. We use the fact that $f(1)=\frac{1}{\prod_{i\geq 1}(1-x^i)}$ to complete the proof of the first identity.

(b) Next we prove the second identity using induction on $a$. For $a=1$, we have

$$\sum_{j\geq 0} x^{jb}\prod_{i=j+1}^{j+1}\left(1-x^i\right)=\frac{1-x}{\left(1-x^b\right)\left(1-x^{b+1}\right)}.$$

Now, assume that the claim holds for all $1,2,\ldots,a$, and let us prove it for $a+1$. Firstly, we have

$$\begin{aligned}
g(a,b) &:= \sum_{j\geq 0} x^{jb}\prod_{i=j+1}^{j+a}(1-x^i)\\
&= \sum_{j\geq 0}\left(x^{jb}-x^{jb+j+a+1}+x^{jb+j+a+1}\right)\prod_{i=j+1}^{j+a}\left(1-x^i\right)\\
&= \sum_{j\geq 0} x^{jb}\prod_{i=j+1}^{j+a+1}(1-x^i)+x^{a+1}\sum_{j\geq 0}x^{j(b+1)}\prod_{i=j+1}^{j+a}(1-x^i)\\
&= g(a+1,b)+x^{a+1}g(a,b+1).
\end{aligned}$$

Thus, by the induction hypothesis, we have

$$\begin{aligned} g(a+1,b) &= \frac{\prod_{i=1}^{a}\left(1-x^i\right)}{\prod_{i=b}^{b+a}\left(1-x^i\right)} - x^{a+1}\frac{\prod_{i=1}^{a}\left(1-x^i\right)}{\prod_{i=b+1}^{b+a+1}\left(1-x^i\right)} \\ &= \frac{\prod_{i=1}^{a+1}\left(1-x^i\right)}{\prod_{i=b}^{b+a+1}\left(1-x^i\right)}\left(\frac{1-x^{a+b+1}}{1-x^{a+1}} - \frac{x^{a+1}(1-x^b)}{1-x^{a+1}}\right) \\ &= \frac{\prod_{i=1}^{a+1}\left(1-x^i\right)}{\prod_{i=b}^{b+a+1}\left(1-x^i\right)}, \end{aligned}$$

which completes the induction. □

We now find the derivative of the generating function $t^{-4}P(1,x,t)$ with respect to $t$ and thereafter substitute $t=1$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(t^{-4}P(1,x,t))\Big|_{t=1} &= \sum_{j\geq 1}\frac{x^j}{1-x^j}\sum_{i=1}^{j-1}\frac{x^i}{\prod_{m=1}^{j-1}\left(1-x^m\right)} \\ &= \frac{x}{1-x}\sum_{j\geq 0}\frac{x^j}{\prod_{m=1}^{j-1}\left(1-x^m\right)} \\ &\quad - \frac{1}{1-x}\sum_{j\geq 0}\frac{x^{2j}}{\prod_{m=1}^{j-1}\left(1-x^m\right)} - 1 \quad \text{by Lemma 2.3} \\ &= \frac{x}{1-x}\frac{1}{\prod_{i\geq 1}\left(1-x^i\right)} - \frac{1}{1-x}\frac{1}{\prod_{i\geq 2}\left(1-x^i\right)} - 1 \\ &= -1 + \frac{2x-1}{(1-x)\prod_{j\geq 1}(1-x^j)}. \end{aligned}$$

**Theorem 2.4** *Thus, the generating function for the total number of corners is*

$$-3 + \frac{3-2x}{(1-x)\prod_{j\geq 1}\left(1-x^j\right)}.$$

*This implies that the total number of corners in all partitions of $n$ is given by*

$$3P(n) + \sum_{j=1}^{n} P(n-j).$$

*Remark 2.5* For any given partition $\pi$, the total number of corners is equal to the number of distinct part sizes in $\pi$ plus 3. Distinct part sizes in partitions have previously been studied in [7,10].

By using Theorem 2.2 and the $q$-binomial theorem, we obtain

**Theorem 2.6** *The generating function for the number of partitions with exactly $m+4$ corners is given by*

$$\sum_{j\geq 1}\frac{x^j}{\prod_{i=1}^{j}\left(1-x^i\right)}\sum_{i=0}^{j-1}(-1)^{i-m}x^{\binom{i+1}{2}}\begin{bmatrix} j-1 \\ i \end{bmatrix}_x\binom{i}{m}.$$

Another application of our general result in Proposition 2.1 is to study the total number of corners at line $y=2k$ for any $k\geq 0$. In order to do that, we define

$$Q(z,x,t)=P(z,x,t,1,t,1,\ldots)\quad\text{and}\quad Q'(z,x,t)=P(z,x,1,t,1,t,\ldots).$$

Proposition 2.1 shows

$$Q(z,x,t)=\frac{t^2xz}{1-xz}+\frac{t^2}{1-xz}Q'(xz,x,t)$$

and

$$Q'(z,x,t)=\frac{1}{t^2}Q(xz,x,t)+\frac{t^2xz}{1-xz}+\frac{xz}{t(1-xz)}Q(xz,x,t).$$

Therefore,

$$Q(z,x,t)=\frac{t^2xz(1+t^2x-x^2z)}{(1-xz)(1-x^2z)}+\frac{1+(t-1)x^2z}{(1-xz)(1-x^2z)}Q(x^2z,x,t).$$

Iterating infinitely many times, we obtain

$$Q(z,x,t)=t^2xz\sum_{j\geq 0}\left(x^{2j}(1+t^2x-x^{2j+2}z)\frac{\prod_{i=1}^{j}(1+(t-1)x^{2i}z)}{\prod_{i=1}^{2j+2}(1-x^iz)}\right).$$

Thus, we may state the following result.

**Theorem 2.7** *The generating function $Q(1,x,t)$, where $t$ marks the number of corners at even levels, is given by*

$$Q(1,x,t)=t^2x\sum_{j\geq 0}\frac{x^{2j}\prod_{i=1}^{j}\left(1+(t-1)x^{2i}\right)}{\prod_{i=1}^{2j+1}\left(1-x^i\right)}+t^4x^2\sum_{j\geq 0}\frac{x^{2j}\prod_{i=1}^{j}\left(1+(t-1)x^{2i}\right)}{\prod_{i=1}^{2j+2}\left(1-x^i\right)}.$$

For the total number of such corners, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(1,x,t)\Big|_{t=1} = 2\sum_{j\ge 0}\frac{x^{2j+1}}{\prod_{i=1}^{2j+1}\left(1-x^i\right)} + x\sum_{j\ge 0}\frac{x^{2j}\sum_{i=1}^{j}x^{2i}}{\prod_{i=1}^{2j+1}\left(1-x^i\right)} + 4\sum_{j\ge 0}\frac{x^{2j+2}}{\prod_{i=1}^{2j+2}\left(1-x^i\right)} + x^2\sum_{j\ge 0}\frac{x^{2j}\sum_{i=1}^{j}x^{2i}}{\prod_{i=1}^{2j+2}\left(1-x^i\right)},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(1,x,t)\Big|_{t=1} = \frac{2-x^2}{1-x^2}\sum_{j\ge 0}\frac{x^{2j+1}}{\prod_{i=1}^{2j+1}\left(1-x^i\right)} + \frac{3-2x^2}{1-x^2}\sum_{j\ge 0}\frac{x^{2j}}{\prod_{i=1}^{2j}\left(1-x^i\right)} - 3. \tag{1}$$

Note that $\sum_{j\ge 0}\frac{x^j y^j}{\prod_{i=1}^{j}\left(1-x^i\right)} = \frac{1}{\prod_{j\ge 1}\left(1-yx^j\right)}$. By splitting the respective sums above into those with the largest part odd (and generating function $\prod_{j\ge 1}\frac{1}{1-x^j} - \prod_{j\ge 1}\frac{1}{1+x^j}$), or the largest part even (with generating function $\prod_{j\ge 1}\frac{1}{1-x^j} + \prod_{j\ge 1}\frac{1}{1+x^j}$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(1,x,t)\ |_{t=1} = \frac{5-3x^2}{2(1-x^2)\prod_{j\ge 1}\left(1-x^j\right)} + \frac{1}{2\prod_{j\ge 1}\left(1+x^j\right)} - 3. \tag{2}$$

Using the facts that $\frac{1}{\prod_{j\ge 0}\left(1-x^j\right)} = \sum_{j\ge 0} P(j)x^j$ and $\frac{1}{\prod_{j\ge 0}\left(1+x^j\right)} = \sum_{j\ge 0} P'(j)x^j$ (see sequence A081362 in [9]), we find

**Theorem 2.8** *Let $n \ge 1$, then the total number of corners at even level $2k$, where $k \ge 0$, in all partitions of $n$ is given by*

$$\frac{1}{2}(5P(n) + P'(n)) + \sum_{j\ge 1} P(n-2j).$$

## 3 Corners of type $(a, b)$

In this section, we study partitions according to the number of corners of type $(a, b)$. Let

$$P_{k;a,b}(x) := P_{k;a,b}(x, q)$$

be the generating function for the number of partitions of $n$ according to the number of corners of type $(a, b)$ with first part of size $k$, where $x$ marks the size of the partition and $q$ the number of such corners. Then

$$P_{k;a,b}(x) = \frac{x^k}{1-x^k}\sum_{j=0}^{k-1} P_{j;a,b}(x) + (q-1)x^{ka}P_{k-b;a,b}(x),$$

where $P_{0;a,b}(x) := 1$. This is equivalent to

$$P_{k;a,b}(x) = x^k \sum_{j=0}^{k} P_{j;a,b}(x) + (q-1)x^{ka}(1-x^k)P_{k-b;a,b}(x)\delta_{k\geq b} \quad \text{for } k \geq 0.$$

Define $P_{a,b}(x, y, q) := \sum_{k\geq 0} P_{k;a,b}(x)y^k$. By multiplying the last recurrence by $y^k$ and summing over $k \geq 0$, we get the following result.

**Theorem 3.1** *The generating function $P_{a,b}(x, y, q)$ satisfies*

$$\begin{aligned} P_{a,b}(x, y, q) =& \frac{1}{1-xy} P_{a,b}(x, xy, q) \\ &+ (q-1)y^b x^{ab}\big(P_{a,b}\big(x, x^a y, q\big) - x^b P_{a,b}\big(x, x^{a+1} y, q\big)\big). \end{aligned}$$

Now we will find an explicit formula for the generating function $P_{1,b}(x, y, q)$. To do that we need the following lemma.

**Lemma 3.2** *Let $\mathcal{A}_0 = \{1\}$ and $\mathcal{A}_1 = \{a_0\}$. For $n \geq 2$, we define $\mathcal{A}_n$ to be the set*

$$\mathcal{A}_{n-1} \cdot a_{n-1} \cup \mathcal{A}_{n-2} \cdot b_{n-2},$$

*where $B \cdot x = \{\pi \cdot x \mid \pi \in B\}$. Assume that $\alpha_i = a_i\alpha_{i+1} + b_i\alpha_{i+2}$ for all $i \geq 0$. Then*

$$\alpha_0 = \left(\sum_{\pi\in\mathcal{A}_{n+1}} \pi\right)\alpha_{n+1} + \left(\sum_{\pi\in\mathcal{A}_n} \pi\right) b_n\alpha_{n+2},$$

*for all $n \geq 0$.*

*Proof* We proceed by induction on $n$. For $n = 0$, the statement reduces to $\alpha_0 = a_0\alpha_1 + b_0\alpha_2$, which holds. Assume that the claim holds for $n$, and let us prove it for $n + 1$. By the induction hypothesis, we have

$$\begin{aligned}\alpha_0 &= \left(\sum_{\pi\in\mathcal{A}_{n+1}}\pi\right)\alpha_{n+1}+\left(\sum_{\pi\in\mathcal{A}_n}\pi\right)b_n\alpha_{n+2}\\ &= \left(\sum_{\pi\in\mathcal{A}_{n+1}}\pi\right)\big(a_{n+1}\alpha_{n+2}+b_{n+1}\alpha_{n+3}\big)+\left(\sum_{\pi\in\mathcal{A}_n}\pi\right)b_n\alpha_{n+2}\\ &= \left(\sum_{\pi\in\mathcal{A}_{n+1}}\pi a_{n+1}+\sum_{\pi\in\mathcal{A}_n}\pi b_n\right)\alpha_{n+2}+\left(\sum_{\pi\in\mathcal{A}_{n+1}}\pi\right)b_{n+1}\alpha_{n+3},\end{aligned}$$

which, by the definition of the sets $\mathcal{A}_n$, implies

$$\alpha_0=\left(\sum_{\pi\in\mathcal{A}_{n+2}}\pi\right)\alpha_{n+2}+\left(\sum_{\pi\in\mathcal{A}_{n+1}}\pi\right)b_{n+1}\alpha_{n+3}.$$

This completes the induction step. □

Clearly, the set $\mathcal{A}_n$ in the statement of the above Lemma 3.2 can be described bijectively as all sequences $\pi_0\pi_1\cdots\pi_s$ such that $\pi_0=0$, $\pi_s\in\{n-1,n-2\}$ and $\pi_j-\pi_{j-1}=1,2$ for all $j=1,2,\ldots,s$. We denote all such sequences by $\mathcal{B}_n$. For each word $\pi=\pi_0\pi_1\cdots\pi_j\in\mathcal{B}_n$, we define

$$\begin{aligned}J(\pi)&=\{i|\pi_{i+1}-\pi_i=2 \text{ with } i=0,1,\ldots,j-1,\ \text{or } \pi_i=\pi_j=n-2\},\\ I(\pi)&=\{i|\pi_{i+1}-\pi_i=1 \text{ with } i=0,1,\ldots,j-1,\ \text{or } \pi_i=\pi_j=n-1\}.\end{aligned}$$

Therefore,

$$\sum_{\pi\in\mathcal{A}_n}\pi=\sum_{\pi\in\mathcal{B}_n}\prod_{i\in J(\pi)}b_{\pi_i}\prod_{i\in I(\pi)}a_{\pi_i}.$$

By Theorem 3.1 with $a=1$, the generating function $P_{a,b}(x,y,q)$ satisfies

$$\begin{aligned}P_{1,b}(x,y,q)=&\left(\frac{1}{1-xy}+(q-1)y^bx^b\right)P_{1,b}(x,xy,q)\\ &-(q-1)y^bx^{2b}P_{1,b}\big(x,x^2y,q\big).\end{aligned}$$

Define $\alpha_i=P_{1,b}\big(x,x^iy,q\big)$, $a_i=\frac{1}{1-x^{i+1}y}+(q-1)y^bx^{b(i+1)}$ and $b_i=(1-q)y^bx^{b(i+2)}$; so

$$\alpha_i=a_i\alpha_{i+1}+b_i\alpha_{i+2},$$

where $\lim_{j\to\infty}\alpha_j = 1$. Hence, by Lemma 3.2 (note that $b_i \to 0$ when $i \to \infty$), we have

$$\alpha_0 = \sum_{\pi\in\mathcal{B}_\infty} \prod_{i\in J(\pi)} b_{\pi_i} \prod_{i\in I(\pi)} a_{\pi_i},$$

where $\mathcal{B}_\infty = \lim_{n\to\infty}\mathcal{B}_n$ which is the set of all sequences $\pi_0\pi_1\pi_2\cdots$ such that $\pi_0 = 0$ and $\pi_{i+1} - \pi_i$ is 1 or 2. So we can state the following result.

**Theorem 3.3** *The generating function $P_{1,b}(x, y, q)$ is given by*

$$\frac{P_{1,b}(x,y,q)}{\prod_{i\ge1}\left(\frac{1}{1-x^iy} + (q-1)y^bx^{bi}\right)} = \sum_{\pi\in\mathcal{B}_\infty} \frac{(1-q)^{|J(\pi)|}y^{b|J(\pi)|}x^{2b|J(\pi)|+b\sum_{i\in J(\pi)}\pi_i}}{\prod_{i\in J(\pi),m=1,2}\left(\frac{1}{1-x^{\pi_i+m}y} + (q-1)y^bx^{b(\pi_i+m)}\right)}.$$

Note that $J(\pi) = \emptyset$ if and only if $\pi = 0123\cdots$. Thus the above theorem with $q = 1$ becomes

$$P_{1,b}(x,y,1) = \prod_{i\ge1}\frac{1}{1-x^iy},$$

as is well known.

Also, note that $|J(\pi)| = 1$ if and only if there exists $i \ge 0$ such that

$$\pi = 012\cdots i(i+2)(i+3)(i+4)\cdots,$$

and $|J(\pi)| = 2$ if and only if there exists $i' > i+1 \ge 1$ such that

$$\pi = 012\cdots i(i+2)(i+3)\cdots i'(i'+2)(i'+3)(i'+4)\cdots.$$

(Similarly, we can characterize all the infinite sequences with $J(\pi) = m$). Then the above theorem shows that

$$\begin{aligned}
&\frac{P_{1,b}(x,y,q)}{\prod_{i\ge1}\left(\frac{1}{1-x^iy} + (q-1)y^bx^{bi}\right)} \\
&\quad = 1 + (1-q)y^bx^b\sum_{s\ge1}\frac{x^{bs}}{\left(\frac{1}{1-x^sy} + (q-1)y^bx^{bs}\right)\left(\frac{1}{1-x^{s+1}y} + (q-1)y^bx^{bs+b}\right)} \\
&\qquad + \sum_{\pi\in\mathcal{B}_\infty,\,|J(\pi)|\ge2}\frac{(1-q)^{|J(\pi)|}y^{b|J(\pi)|}x^{2b|J(\pi)|+b\sum_{i\in J(\pi)}\pi_i}}{\prod_{i\in J(\pi),m=1,2}\left(\frac{1}{1-x^{\pi_i+m}y} + (q-1)y^bx^{b(\pi_i+m)}\right)}.
\end{aligned}$$

Another way to solve the recurrence $\alpha_i = a_i\alpha_{i+1} + b_i\alpha_{i+2}$ in Lemma 3.2 is by using continued fractions. Note that $\frac{\alpha_i}{\alpha_{i+1}} = a_i + \frac{b_i}{\frac{\alpha_{i+1}}{\alpha_{i+2}}}$, for all $i \geq 0$. Therefore,

$$\frac{\alpha_i}{\alpha_{i+1}} = a_i + \cfrac{b_i}{a_{i+1} + \cfrac{b_{i+1}}{a_{i+2} + \ddots}}.$$

So, multiplying over all $i \geq 0$, assuming that $\lim_{j\to\infty} \alpha_j = 1$, we obtain

$$\alpha_0 = \prod_{i\geq 0}\left( a_i + \cfrac{b_i}{a_{i+1} + \cfrac{b_{i+1}}{a_{i+2} + \ddots}} \right).$$

Thus, if we define $\alpha_i = P_{1,b}(x, x^i y, q)$, $a_i = \frac{1}{1-x^{i+1}y} + (q-1)y^b x^{b(i+1)}$ and $b_i = (1-q)y^b x^{(2+i)b}$, we have the following result.

**Theorem 3.4** *The generating function* $P_{1,b}(x, y, q)$ *is given by*

$$\prod_{i\geq 0}\left( \frac{1}{1-x^{i+1}y} + (q-1)y^b x^{b(i+1)} + \cfrac{(1-q)y^b x^{(2+i)b}}{\frac{1}{1-x^{i+2}y} + (q-1)y^b x^{b(i+2)} + \cfrac{(1-q)y^b x^{(3+i)b}}{\frac{1}{1-x^{i+3}y} + (q-1)y^b x^{b(i+3)} + \ddots}} \right).$$

Note that by comparing Theorems 3.3 and 3.4, we get the following corollary.

**Corollary 3.5** *We have*

$$P_{1,b}(x, y, q) = \prod_{i\geq 1}\left( \frac{1}{1-x^i y} + (q-1)y^b x^{bi} \right) \times \sum_{\pi\in\mathcal{B}_\infty} \frac{(1-q)^{|J(\pi)|} y^{b|J(\pi)|} x^{2b|J(\pi)|+b\sum_{i\in J(\pi)}\pi_i}}{\prod_{i\in J(\pi),m=1,2}\left( \frac{1}{1-x^{\pi_i+m}y} + (q-1)y^b x^{b(\pi_i+m)} \right)}$$

$$=\prod_{i\geq 0}\left(\frac{1}{1-x^{i+1}y}+(q-1)y^b x^{b(i+1)}\right.$$
$$\left.+\frac{(1-q)y^b x^{(2+i)b}}{\frac{1}{1-x^{i+2}y}+(q-1)y^b x^{b(i+2)}+\frac{(1-q)y^b x^{(3+i)b}}{\frac{1}{1-x^{i+3}y}+(q-1)y^b x^{b(i+3)}+\ddots}}\right).$$

3.1 Total number of corners of type $(a,b)$

Using Theorem 3.1, we find the generating function for the total number of corners of type $(a,b)$ to be

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,y,q)\Big|_{q=1}=\frac{1}{1-xy}\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,xy,q)\Big|_{q=1}$$
$$+y^b x^{ab}\big(P_{a,b}\big(x,x^a y,1\big)-x^b P_{a,b}\big(x,x^{a+1}y,1\big)\big).$$

Using $P_{a,b}(x,y,1)=\frac{1}{\prod_{j\geq 1}(1-x^j y)}$, this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,y,q)\Big|_{q=1}=\frac{1}{1-xy}\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,xy,q)\Big|_{q=1}+y^b x^{ab}\frac{1-x^b+x^{a+b+1}y}{\prod_{j\geq a+1}(1-x^j y)}.$$

By iterating this an infinite number of times, and noting that $\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,x^s,q)\Big|_{q=1}\to 0$ as $s\to\infty$, we have

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,1,q)\Big|_{q=1}=\sum_{j\geq a}\frac{x^{jb}}{\prod_{i=1}^{j-a}(1-x^i)}\frac{1-x^b+x^{j+b+1}}{\prod_{i\geq j+1}(1-x^i)}$$
$$=\frac{x^{ab}}{\prod_{i\geq 1}(1-x^i)}\sum_{j\geq 0}x^{jb}\big(1-x^b+x^{j+a+b+1}\big)\prod_{i=j+1}^{j+a}\big(1-x^i\big). \tag{3}$$

Thus, by Lemma 2.3 and Eq. (3),

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x,y,q)\Big|_{q=1}=\frac{x^{ab}(1-x^b)}{\prod_{i\geq 1}(1-x^i)}\left(\frac{\prod_{i=1}^{a}(1-x^i)}{\prod_{i=b}^{b+a}(1-x^i)}+x^{a+b+1}\frac{\prod_{i=1}^{a}(1-x^i)}{\prod_{i=b}^{b+a+1}(1-x^i)}\right).$$

This is captured in the following theorem.

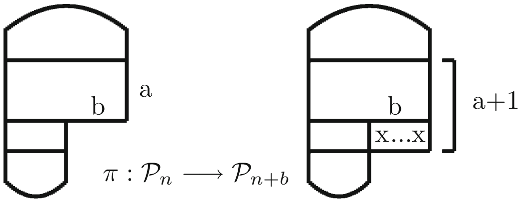



**Fig. 2** Bijection between number of $(a, b)$ corners in $\mathcal{P}_n$ and parts of multiplicity $a + 1$ in $\mathcal{P}_{n+b}$

**Theorem 3.6** *The generating function for the total number of corners of type* $(a, b)$ *in all partitions is given by*

$$\frac{\mathrm{d}}{\mathrm{d}q} P_{a,b}(x, 1, q)\Big|_{q=1} = \frac{x^{ab} \prod_{i=1}^{a} \left(1 - x^i\right)}{\prod_{i \geq 1} \left(1 - x^i\right) \prod_{i=b+1}^{b+a+1} \left(1 - x^i\right)}.$$

*Example 3.7* Theorem 3.6 for $b = 1$ leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}q} P_{a,1}(x, 1, q)\Big|_{q=1} &= \frac{x^a (1 - x)}{\left(1 - x^{a+1}\right)\left(1 - x^{a+2}\right) \prod_{i \geq 1} \left(1 - x^i\right)} \\ &= \frac{1}{x} \left( \frac{1}{1 - x^{a+1}} - \frac{1}{1 - x^{a+2}} \right) \frac{1}{\prod_{i \geq 1} \left(1 - x^i\right)}, \end{aligned}$$

which implies that the total number of corners of type $(a, 1)$ in all partitions of $n$ is given by

$$\sum_{j \geq 0} (P(n + 1 - j(a + 1)) - P(n + 1 - j(a + 2))).$$

Let $\mathcal{P}_n$ denote the set of all partitions of size $n$. There is a simple correspondence between the total number of corners of type $(a, b)$ in $\mathcal{P}_n$ and the number of parts $\geq b$ of multiplicity $a + 1$ in $\mathcal{P}_{n+b}$. We illustrate the bijection in Fig. 2.

Note, however, that the distributions of corners of type $(a, b)$ and of parts of multiplicity $a+1$ are quite different. For example, for the case where $a = 1$, the generating function for partitions with no parts of multiplicity 2 is $\prod_{i=1}^{\infty}(\frac{1}{1-x^i} - x^{2i})$, whereas by setting $q = 0$ in Theorem 3.4, the generating function for partitions with no corner of type $(1, b)$ is

$$P_{1,b}(x, y, 0) = \prod_{i \geq 0} \left( \frac{1}{1 - x^{i+1} y} - y^b x^{b(i+1)} + \frac{y^b x^{(2+i)b}}{\frac{1}{1-x^{i+2}y} - y^b x^{b(i+2)} + \frac{y^b x^{(3+i)b}}{\frac{1}{1-x^{i+3}y} - y^b x^{b(i+3)} + \ddots}} \right).$$

## 4 Corners of type $(a+, b)$

We define a corner of type $(a+, b)$ to be a corner of type $(j, b)$ for any $j \geq a$. Let

$$P_{k;a+,b}(x) := P_{k;a+,b}(x, q)$$

be the generating function for the number of partitions according to the number of corners of type $(a+, b)$ with first part $k$. Then for $k \geq b$,

$$P_{k;a+,b}(x) = \left(x^k + x^{2k} + \cdots + x^{k(a-1)}\right) \sum_{j=0}^{k-1} P_{j;a+,b}(x) + \frac{x^{ka}}{1-x^k}\left((q-1)P_{k-b;a+,b}(x) + \sum_{j=0}^{k-1} P_{j;a+,b}(x)\right)$$

with $P_{k;a+,b}(x) = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$ for $k = 0, 1, \ldots, b-1$. Thus,

$$P_{k;a+,b}(x) = x^k \sum_{j=0}^{k} P_{j;a+,b}(x) + x^{ka}(q-1)P_{k-b;a+,b}(x)\delta_{k\geq b}.$$

Define $P_{a+,b}(x, y, q) := \sum_{k\geq 0} P_{k;a+,b}(x, q)y^k$. By multiplying the last recurrence by $y^k$ and summing over $k \geq 0$, we get

$$P_{a+,b}(x, y, q) = \frac{1}{1-xy} P_{a+,b}(x, xy, q) + (q-1)x^{ab}y^b P_{a+,b}(x, x^a y, q). \quad (4)$$

We solve the recursion in the case where $a = 1$.

**Theorem 4.1** *The generating function $P_{1+,b}(x, y, q)$ is given by*

$$P_{1+,b}(x, y, q) = \prod_{i\geq 1}\left(\frac{1}{1-x^i y} + (q-1)(x^i y)^b\right).$$

*Moreover, the generating function for the number of partitions without corners of types $(1+, b)$ is given by*

$$P_{1+,b}(x, 1, 0) = \prod_{i\geq 1}\left(\frac{1}{1-x^i} - x^{ib}\right).$$

*Proof* By (4) with $a = 1$, we have

$$P_{1+,b}(x, y, q) = \left(\frac{1}{1-xy} + (q-1)(xy)^b\right) P_{1+,b}(x, xy, q),$$

which, by repeated iteration and using $\lim_{i\to\infty} P_{1+,b}(x, x^i y, q) = 1$, implies

$$P_{1+,b}(x, y, q) = \prod_{i\geq 1}\left(\frac{1}{1-x^i y} + (q-1)(x^i y)^b\right),$$

as required. □

### 4.1 Total number of corners of type $(a+, b)$

Differentiating Eq. (4) with respect to $q$, then substituting $q = 1$ and using $P_{a+,b}(x, y, 1) = \frac{1}{\prod_{j\geq 1}(1-x^j y)}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b}(x, y, q)\Big|_{q=1} = \frac{1}{1-xy}\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b}(x, xy, q)\Big|_{q=1} + x^{ab}y^b P_{a+,b}(x, x^a y, 1).$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b}(x, y, q)\Big|_{q=1} = \frac{x^{ab}y^b}{\prod_{j\geq a+1}(1-x^j y)} + \frac{1}{1-xy}\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b}(x, xy, q)\Big|_{q=1}.$$

Iterating infinitely many times, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b}(x, y, q)\Big|_{q=1} = \sum_{j\geq 0}\frac{x^{(a+j)b}y^b}{\prod_{i=1}^{j}\left(1-x^i y\right)\prod_{i\geq a+1+j}\left(1-x^i y\right)}.$$

By Lemma 2.3, this leads to the following result.

**Theorem 4.2** *The generating function for the total number of corners of type $(a+, b)$ in all partitions of $n$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b}(x, 1, q)\Big|_{q=1} = \frac{x^{ab}\prod_{i=1}^{a}\left(1-x^i\right)}{\prod_{i\geq 1}\left(1-x^i\right)\prod_{i=b}^{a+b}\left(1-x^i\right)}.$$

*Example 4.3* Theorem 4.2 for $a = 1$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{1+,b}(x, 1, q)\Big|_{q=1} = \frac{1}{\prod_{i\geq 1}\left(1-x^i\right)}\left(\frac{1}{1-x^b} - \frac{1}{1-x^{b+1}}\right).$$

Thus, the total number of corners of type $(1+, b)$ in all the partitions of $n$ is given by

$$\sum_{j\geq 1}(P(n-bj) - P(n-(b+1)j)).$$

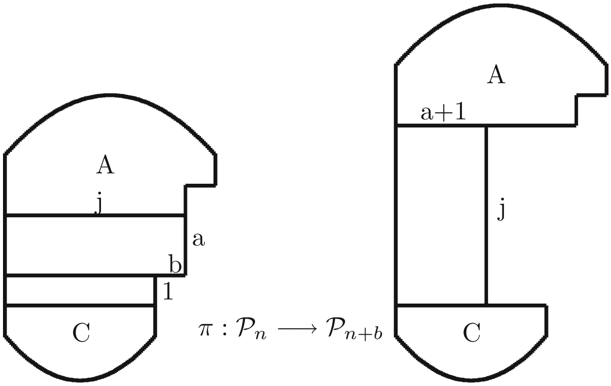



**Fig. 3** Bijection between number of $(a+, b)$ corners in $\mathcal{P}_n$ and parts in $\mathcal{P}_{n+b}$

We note that a simple bijection between a Ferrers diagram and its transpose implies that the above results also count corners of type $(a, b+)$.

We also note that the number of corners of type $(a+, b)$ in the partitions of $n$ equals the number $\geq b$ of parts of size $a+1$ in the partitions of $n+b$. We illustrate the bijection in Fig. 3.

First, we add $b$ cells to the first part of size $j-b$ and then conjugate the $a+1$ parts of size $j$; thereafter we rearrange all parts (in the right-hand side drawing in Fig. 3) in descending order.

*Example 4.4* For $a=2$, the corners of type (2+, 1) in the partition 2211 maps to the two different partitions: 322 and 331. Also, the corner of type (2+, 1) in the partition 111111 (resp. 21111, 3111, 411) maps to the partition 31111 (resp. 3211, 331, 43).

## 5 Corners of type $(a+, b+)$

Let $P_{k;a+,b+}(x) := P_{k;a+,b+}(x, q)$ be the generating function for the number of partitions of $n$ according to the number of corners of type $(a+, b+)$ with first part $k$. Then for $k \geq b$,

$$P_{k;a+,b+}(x) = \left(x^k + x^{2k} + \cdots + x^{k(a-1)}\right) \sum_{j=0}^{k-1} P_{j;a+,b+}(x) + \frac{x^{ka}}{1-x^k}\left(q \sum_{j=0}^{k-b} P_{j;a+,b+}(x) + \sum_{j=k-b+1}^{k-1} P_{j;a+,b+}(x)\right)$$

with $P_{k;a+,b+}(x) = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$ for $k = 0, 1, \ldots, b-1$. Thus,

$$P_{k;a+,b+}(x) = x^k \sum_{j=0}^{k} P_{j;a+,b+}(x) + (q-1)x^{ka} \sum_{j=0}^{k-b} P_{j;a+,b+}(x).$$

Define $P_{a+,b+}(x,y,q) := \sum_{k\geq 0} P_{k;a+,b+}(x,q)y^k$. Multiplying the last recurrence by $y^k$ and summing over $k \geq 0$, we get

$$P_{a+,b+}(x,y,q) = \frac{1}{1-xy}P_{a+,b+}(x,xy,q) + \frac{(q-1)x^{ab}y^b}{1-x^a y}P_{a+,b+}(x,x^a y,q). \tag{5}$$

For $a = 1$, we obtain the following result.

**Theorem 5.1** *The generating function $P_{1+,b+}(x,y,q)$ is*

$$P_{1+,b+}(x,y,q) = \prod_{i\geq 1}\frac{1+(q-1)\left(x^i y\right)^b}{1-x^i y}.$$

Note that $P_{a+,b+}(x,y,1) = \frac{1}{\prod_{j\geq 1}(1-x^j y)}$. By differentiating Eq. (5) with respect to $q$ and then substituting $q = 1$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b+}(x,y,q)\Big|_{q=1} &= \tfrac{1}{1-xy}\tfrac{\mathrm{d}}{\mathrm{d}q}P_{a+,b+}(x,xy,q)\Big|_{q=1}\\ &\quad+\tfrac{x^{ab}y^b}{1-x^a y}P_{a+,b+}(x,x^a y,1),\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b+}(x,y,q)\Big|_{q=1} = \frac{1}{1-xy}\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b+}(x,xy,q)\Big|_{q=1} + \frac{x^{ab}y^b}{\prod_{j\geq a}\left(1-x^j y\right)}.$$

Iterating infinitely many times, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b+}(x,y,q)\Big|_{q=1} = \sum_{j\geq 0}\frac{x^{(a+j)b}y^b}{\prod_{i\geq a+j}\left(1-x^i y\right)\prod_{i=1}^{j}\left(1-x^i y\right)}.$$

By Lemma 2.3b, this implies the following result.

**Theorem 5.2** *The generating function for the total number of corners of type $(a+,b+)$ in all partitions is given by*

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{a+,b+}(x,1,q)\Big|_{q=1} = \frac{x^{ab}\prod_{i=1}^{a-1}\left(1-x^i\right)}{\prod_{i\geq 1}(1-x^i)\prod_{i=b}^{a+b-1}\left(1-x^i\right)}.$$

*Example 5.3* Theorem 5.2 for $a = 1$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}q}P_{1+,b+}(x,1,q)\Big|_{q=1} = \frac{x^b}{(1-x^b)\prod_{i\geq 1}\left(1-x^i\right)}.$$

Thus, the total number of corners of type $(1+, b+)$ in all partitions of $n$ is given by

$$\sum_{j\geq 1} P(n-bj).$$

## 6 Corners of size *m*

We shall define the *size* of a corner of type $(a, b)$ to be $m$ when $m = a + b$. In this section, we find the generating function that counts the total number of corners of size $m$ in partitions of $n$. For a corner of size $m$, where $m$ is fixed, there are $p$ parts of size $k$ followed by a part of size $k + p - m$, for $1 \leq p \leq m - 1$. We denote the generating function by $P_{k,m}(x) := P_{k,m}(x, q)$, where $q$ marks such corners, as illustrated in Fig. 4.

Clearly, the generating function satisfies the following equation:

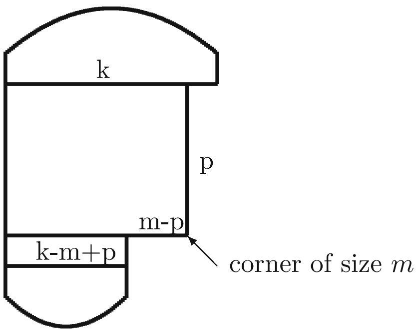
$$P_{k,m} = \frac{x^k}{1-x^k} \sum_{j=0}^{k-1} P_{j,m} + (q-1) \sum_{p=1}^{m-1} x^{kp} P_{k+p-m,m}.$$

Multiplying by $(1 - x^k)$ and simplifying,

$$P_{k,m} = x^k \sum_{j=0}^{k} P_{j,m} + (q-1)(1-x^k) \sum_{p=1}^{m-1} x^{kp} P_{k+p-m,m}. \tag{6}$$

We now define $P_m(x, y, q) := \sum_{k\geq 0} P_{k,m}(x, y, q) y^k$. Thus after multiplying (6) by $y^k$, summing over all $k$ and interchanging the order of summation, we have

**Fig. 4** Corner of size $m$



$$
\begin{aligned}
P_m(x,y,q) &= \sum_{j=0}^{\infty} P_{j,m}\frac{(xy)^j}{1-xy} + (q-1)\sum_{p=1}^{m-1}\left(x^p y\right)^{m-p}\sum_{r\geq 0}\left(x^p y\right)^{r-p+m} P_{r,m} \\
&\quad - (q-1)\sum_{p=1}^{m-1}\sum_{r\geq 0}\left(x^{1+p} y\right)^{r-p+m} P_{r,m} \qquad (\text{where } r = k+p-m) \\
&= \frac{P_m(x,xy,q)}{1-xy} + (q-1)\sum_{p=1}^{m-1} P_m\left(x, x^p y, q\right)\left(x^p y\right)^{m-p} \\
&\quad - (q-1)\sum_{p=1}^{m-1} P_m\left(x, x^{p+1} y, q\right)\left(x^{p+1} y\right)^{m-p}. \qquad (7)
\end{aligned}
$$

Following the previous procedure for finding the total number of corners, we compute

$$
\begin{aligned}
\left.\frac{\mathrm{d}P_m(x,y,q)}{\mathrm{d}q}\right|_{q=1} &= \frac{1}{1-xy}\left.\frac{\mathrm{d}P_m(x,xy,q)}{\mathrm{d}q}\right|_{q=1} + \sum_{p=1}^{m-1} P_m\left(x, x^p y, 1\right)\left(x^p y\right)^{m-p} \\
&\quad - \sum_{p=1}^{m-1} P_m\left(x, x^{p+1} y, 1\right)\left(x^{p+1} y\right)^{m-p}. \qquad (8)
\end{aligned}
$$

The generating function for all partitions is $P_m(x,y,1) = \frac{1}{\prod_{j\geq 1}(1-x^j y)}$. Therefore

$$
\begin{aligned}
\left.\frac{\mathrm{d}P_m(x,y,q)}{\mathrm{d}q}\right|_{q=1} &= \frac{1}{1-xy}\left.\frac{\mathrm{d}P_m(x,xy,q)}{\mathrm{d}q}\right|_{q=1} \\
&\quad + \sum_{p=1}^{m-1}\frac{1-x^{m-p}\left(1-x^{1+p}yv\right)}{\prod_{j\geq 1}\left(1-x^{j+p}y\right)}\left(x^p y\right)^{m-p}. \qquad (9)
\end{aligned}
$$

After infinitely many iterations and noticing that $\frac{\mathrm{d}P_m(x,x^s,q)}{\mathrm{d}q} \to 0$ as $s \to \infty$, we obtain our result

**Theorem 6.1** *The generating function for the total number of corners of size m is*

$$
\left.\frac{\mathrm{d}P_m(x,1,q)}{\mathrm{d}q}\right|_{q=1} = \sum_{i=0}^{\infty}\frac{1}{\prod_{l=1}^{i}\left(1-x^l\right)}\sum_{p=1}^{m-1}\frac{\left(x^{p+i}\right)^{m-p}\left(1-x^{m-p}+x^{m+1+i}\right)}{\prod_{j\geq 1}\left(1-x^{p+i+j}\right)} \qquad (10)
$$

$$
= \frac{1}{\prod_{l=1}^{\infty}\left(1-x^l\right)}\sum_{p=1}^{m-1}\frac{x^{p(m-p)}\prod_{i=1}^{p}\left(1-x^i\right)}{\prod_{i=m-p+1}^{m+1}\left(1-x^i\right)}. \qquad (11)
$$

Here, Eq. (11) follows from Eq. (10) by using Lemma 2.3.

## 7 Asymptotics

To find asymptotic estimates for the various types of corners studied above, we need to study generating functions of the form $P(x)F(x)$, where $P(x)$ is the generating function for the number of partitions. In the paper [6], the authors show how such asymptotic expansions can be obtained in a quasi-automatic way from expansions of $F(x)$ around $x = 1$. For the convenience of the reader, we state the relevant results from [6] that we need:

**Theorem 7.1** *Suppose that the function $F(z)$ satisfies*

$$|F(z)| = \mathcal{O}(e^{C/(1-|z|)^\eta}) \text{ as } |z| \to 1 \text{ for some } C > 0 \text{ and } \eta < 1, \tag{12}$$

*and $F(e^{-t}) = at^b + \mathcal{O}(f(|t|))$ as $t \to 0$, $\Re t > 0$, for real numbers $a, b$. Then one has*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \times \frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + \mathcal{O}\left(\exp\left(-n^{1/2-\epsilon}\right) + f\left(\frac{\pi}{\sqrt{6n}} + \mathcal{O}\left(n^{-1/2-\epsilon}\right)\right)\right)$$

*as $n \to \infty$ for any $0 < \epsilon < \frac{1-\eta}{2}$, where $I_\nu$ denotes a modified Bessel function of the first kind.*

It is also shown in [6] that the quotient of modified Bessel functions simplifies, with $h = |b + 3/2| - 1/2$ and $m = \sqrt{\frac{2\pi^2}{3}\left(n - \frac{1}{24}\right)}$, to

$$\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} = \frac{m}{m-1} \times \sum_{j=0}^{h} \frac{(h+j)!}{j!(h-j)!}\left(-\frac{1}{2m}\right)^j + \mathcal{O}(e^{-2m}). \tag{13}$$

In our applications, we apply (13) together with Theorem 7.1. Thereafter, we use computer algebra to obtain asymptotic formulae in terms of powers of $n$.

Firstly, in the case of the total number of corners of type $(a, b)$, we have $\frac{\mathrm{d}}{\mathrm{d}q} P_{a,b}(x, y, q)\Big|_{q=1} = P(x)F_{a,b}(x)$, where

$$F_{a,b}(x) = \frac{x^{ab}\prod_{i=1}^{a}(1-x^i)}{\prod_{i=b+1}^{b+a+1}(1-x^i)}.$$

To apply Theorem 7.1, we need to consider the Taylor expansion of $F_{a,b}(e^{-t})$:

$$\begin{aligned} F_{a,b}(e^{-t}) &= \frac{e^{-tab}\prod_{i=1}^{a}(1-e^{-it})}{\prod_{i=1}^{a+1}(1-e^{(i+b)t})} \\ &= \frac{1-abt}{t}\frac{\prod_{i=1}^{a}(i-i^2t/2)}{\prod_{i=1}^{a+1}((i+b)-(i+b)^2t/2)}+\mathcal{O}(t). \end{aligned} \tag{14}$$

Now

$$\prod_{i=1}^{a}(i-i^2t/2) = a!(1-ta(a+1)/4)+\mathcal{O}(t^2) \tag{15}$$

and

$$\prod_{i=1}^{a+1}((i+b)-(i+b)^2t/2) = \frac{(a+b+1)!}{b!}(1-t(a+1)(2b+a+2)/4)+\mathcal{O}(t^2). \tag{16}$$

Substituting these into (14) gives

$$F_{a,b}(e^{-t}) = \frac{a!b!}{(a+b+1)!}\left(\frac{1}{t}-\frac{ab-a-b-1}{2}\right)+\mathcal{O}(t). \tag{17}$$

Using this in Theorem 7.1 leads to

**Theorem 7.2** *The average number of corners of type* $(a, b)$ *in a random partition of* $n$ *is*

$$\frac{a!b!\sqrt{6n}}{\pi(a+b+1)!}-\frac{a!b!\left(\pi^2ab-\pi^2a-\pi^2b-\pi^2-6\right)}{2\pi^2(a+b+1)!}+\mathcal{O}(n^{-1/2}).$$

In the case of the total number of corners of type $(a+, b)$, we have $\frac{\mathrm{d}}{\mathrm{d}q}P_{a,b}(x, y, q)\Big|_{q=1} = P(x)F_{a,b}(x)$, where

$$F_{a,b}(x) = \frac{x^{ab}\prod_{i=1}^{a}(1-x^i)}{\prod_{i=b}^{b+a}(1-x^i)}.$$

From this we find

$$F_{a,b}(e^{-t}) = \frac{a!(b-1)!}{(a+b)!}\left(\frac{1}{t}-\frac{b(a-2)}{2}\right)+\mathcal{O}(t). \tag{18}$$

Using this in Theorem 7.1 leads to

**Theorem 7.3** *The average number of corners of type* $(a+, b)$ *in a random partition of* $n$ *is*

$$\frac{a!(b-1)!\sqrt{6n}}{\pi(a+b)!} - \frac{a!(b-1)!\left(\pi^2 ab - 2\pi^2 b - 6\right)}{2\pi^2(a+b)!} + \mathcal{O}(n^{-1/2}).$$

Next for the total number of corners of type $(a+, b+)$, we have

$$F_{a,b}(x) = \frac{x^{ab}\prod_{i=1}^{a-1}(1-x^i)}{\prod_{i=b}^{b+a-1}(1-x^i)}.$$

This yields

$$F_{a,b}(e^{-t}) = \frac{(a-1)!(b-1)!}{(a+b-1)!}\left(\frac{1}{t} - \frac{ab}{2}\right) + \mathcal{O}(t). \tag{19}$$

Using this in Theorem 7.1 leads to

**Theorem 7.4** *The average number of corners of type* $(a+, b+)$ *in a random partition of* $n$ *is*

$$\frac{(a-1)!(b-1)!\sqrt{6n}}{\pi(a+b-1)!} - \frac{(a-1)!(b-1)!\left(\pi^2 ab - 6\right)}{2\pi^2(a+b-1)!} + \mathcal{O}(n^{-1/2}).$$

For the total number of corners treated in Theorem 2.4, we use $F(x) = \frac{2x-1}{1-x} + 4$ in Theorem 7.1 to obtain

**Theorem 7.5** *The average number of corners in a random partition of* $n$ *is*

$$\frac{\sqrt{6n}}{\pi} + \frac{6 + 5\pi^2}{2\pi^2} + \mathcal{O}(n^{-1/2}).$$

For the total number of corners at even level, the contribution of $\frac{1}{2\prod_{j\geq 1}(1+x^j)}$ to (2) is asymptotically negligible; therefore, we may take $F(x) = \frac{5-3x^2}{2(1-x^2)}$ in Theorem 7.1 to obtain

**Theorem 7.6** *The average number of corners at even level in a random partition of* $n$ *is*

$$\frac{\sqrt{6n}}{2\pi} + \frac{3}{2\pi^2} + 2 + \mathcal{O}(n^{-1/2}).$$

Finally, for the total number of corners of size $m$ we take

$$F(x) = \sum_{a=1}^{m-1} \frac{x^{a(m-a)}\prod_{i=1}^{a}(1-x^i)}{\prod_{i=m-a+1}^{m+1}(1-x^i)},$$

in Theorem 7.1, with the aid of the expansions (17) applied to cases $(a, m-a)$ with $1 \leq a \leq m-1$, to obtain

**Theorem 7.7** *The average number of corners of size m in a random partition of n is*

$$\frac{1}{m+1}\sum_{a=1}^{m-1}\binom{m}{a}^{-1}\left(\frac{\sqrt{6n}}{\pi}+\frac{\pi^2\left(a^2-am+m+1\right)+6}{2\pi^2}\right)+\mathcal{O}(n^{-1/2}).$$

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