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More parity results for broken 8-diamond partitions

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Abstract Broken k-diamond partitions were introduced in 2007 by Andrews and Paule. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. In 2010, Radu and Sellers provided many beautiful congruences for $\Delta_k(n)$ modulo 2 when $k = 2, 3, 5, 6, 8, 9, 11$. Among them when $k = 8$, they showed that $\Delta_8(34n + r) \equiv 0 \pmod 2$ when $r \in \{11, 15, 17, 19, 25, 27, 29, 33\}$. In this article, by using properties of modular forms, we extend this result for $\Delta_8(n)$. We have completely determined the behavior of $\Delta_8(2n + 1)$ modulo 2. As a consequence, we obtain many more congruences for $\Delta_8(n)$ modulo 2.

Keywords Broken k-diamond partitions · Congruences · Modular forms

Mathematics Subject Classification 11P83 · 11F11

1 Introduction

In 2007, Andrews and Paule [1] introduced a new class of combinatorial objects called broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n, and they proved that

$$\sum_{n=0}^{\infty} \Delta_k(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}. \tag{1}$$

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In [1], they proved a congruence for $\Delta_1(n)$ that for all $n \geq 0$, $\Delta_1(2n+1) \equiv 0 \pmod 3$. They also conjectured several other congruences modulo 2 satisfied by certain $\Delta_k(n)$. Since then, mathematicians have provided numerous additional congruences satisfied by $\Delta_k(n)$ for small integers k. For example, Hirschhorn and Sellers [3] proved some parity results for $\Delta_1(n)$ and $\Delta_2(n)$:

$$\begin{aligned}
\Delta_1(4n+2) &\equiv 0 \pmod 2\\
\Delta_1(4n+3) &\equiv 0 \pmod 2\\
\Delta_2(10n+2) &\equiv 0 \pmod 2\\
\Delta_2(10n+6) &\equiv 0 \pmod 2.
\end{aligned}$$

After that, Chan [2] provided a different proof of the above results and obtained some new congruences for $\Delta_2(n)$ modulo 5. Paule and Radu [6] extended the results of $\Delta_2(n)$ modulo 5, and made four conjectures about $\Delta_3(n)$ modulo 7 and $\Delta_5(n)$ modulo 11. Two of those conjectures were proved by Xiong [10] in 2011, and the other two were proved by Jameson [4] recently. See more results of congruences for $\Delta_k(n)$ in Radu and Sellers [7–9], Yao [11], etc.

In 2010, Radu and Sellers [7] provided many beautiful congruences for $\Delta_k(n)$ modulo 2 when $k = 2, 3, 5, 6, 8, 9, 11$. Among them when $k = 8$, they proved that for all $n \geq 0$,

$$\Delta_8(34n+r) \equiv 0 \pmod 2 \tag{2}$$

when $r \in \{11, 15, 17, 19, 25, 27, 29, 33\}$. In our article, by using properties of modular forms, we have obtained many more congruences for $\Delta_8(n)$ modulo 2. In fact, we have completely determined the behavior of $\Delta_8(2n+1)$ modulo 2. It can be characterized in the following theorem:

Theorem 1 *The broken 8-diamond partition function is defined by*

$$\sum_{n=0}^{\infty} \Delta_8(n)q^n := \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{17n})}{(1-q^n)^3(1-q^{34n})}, \tag{3}$$

and then we have

$$\sum_{n=0}^{\infty} \Delta_8(2n+1)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)^3 + q^2\prod_{n=1}^{\infty}(1-q^{17n})^3 \pmod 2. \tag{4}$$

With the help of the well-known identity [5, thm 1.60]

$$q \cdot \prod_{n=1}^{\infty}(1-q^{8n})^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}, \tag{5}$$

equation (4) can be changed to

$$\sum_{n=0}^{\infty} \Delta_8(2n+1)q^{8n+1} \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} + \sum_{n=0}^{\infty} q^{17(2n+1)^2} \pmod{2}. \tag{6}$$

That means $\Delta_8(2n+1) \equiv 1 \pmod{2}$ if and only if $8n+1$ is a square or 17 times a square. So we obtain a corollary to judge whether a congruence for $\Delta_8(n)$ holds or not:

Corollary 1 *Let A, B be two nonnegative integers, $B < A$, then the congruence*

$$\Delta_8(2An + 2B + 1) \equiv 0 \pmod{2} \tag{7}$$

holds for all $n \geq 0$ if and only if $8B+1$ is a quadratic nonresidue mod $8A$ and $136B+17$ is a quadratic nonresidue mod $136A$.

Based on the above corollary, we can derive many new congruences for $\Delta_8(n)$ modulo 2. Following the notation in [7], we write

$$f(tn + r_1, r_2, \ldots, r_m) \equiv 0 \pmod{2}$$

to mean that, for each $i \in \{1, 2, \ldots, m\}$,

$$f(tn + r_i) \equiv 0 \pmod{2}.$$

Example 1 The following congruences hold for all $n \geq 0$:

$$\begin{aligned}
&\Delta_8(18n + 9, 15) &&\equiv 0 \pmod{2}\\
&\Delta_8(26n + 9, 11, 15, 19, 23, 25) &&\equiv 0 \pmod{2}\\
&\Delta_8(34n + 11, 15, 17, 19, 25, 27, 29, 33) &&\equiv 0 \pmod{2}\\
&\Delta_8(36n + 9, 15, 27, 33) &&\equiv 0 \pmod{2}\\
&\Delta_8(38n + 9, 11, 17, 23, 25, 27, 29, 33, 37) &&\equiv 0 \pmod{2}\\
&\Delta_8(42n + 17, 19, 25, 29, 35, 37) &&\equiv 0 \pmod{2}\\
&\Delta_8(50n + 17, 27, 37, 47) &&\equiv 0 \pmod{2}.
\end{aligned}$$

We can also directly obtain infinite families of congruences for $\Delta_8(n)$ modulo 2.

Corollary 2 *Let p be a odd prime, $p \neq 17$, and $t > 0$ and $\alpha \geq 0$ are integers with $(p, t) = 1$, and $t \cdot p^{2\alpha+1} \equiv 1 \pmod{8}$. Then the following congruence holds for all $n \geq 0$:*

$$\Delta_8\left(2p^{2\alpha+2}n + \frac{1}{4}(p^{2\alpha+1}t - 1) + 1\right) \equiv 0 \pmod{2}. \tag{8}$$

We will prove these theorems and corollaries in Sects. 3 and 4. Note that after examining small congruences for $\Delta_8(n)$ modulo 2 by computer, we find that Corollary 1 has covered all congruences in the form $\Delta_8(An+B) \equiv 0 \pmod{2}$ for $A \leq 100$.

2 Preliminaries

In this section, we will introduce some notations of modular forms. The Dedekind's eta function $\eta(z)$ is defined as

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = 2\pi i z$, $z \in \mathcal{H}$, $\mathcal{H}$ is the upper half complex plane. A function $f(z)$ is called an eta-quotient if it can be written as

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where δ and N are positive integers and r_δ is an integer corresponding to δ. Let $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) denote the set of all holomorphic modular forms (resp. cusp forms) with respect to $\Gamma_0(N)$ with weight k and character χ; the following theorem helps us to determine when an eta-quotient is of a modular form:

Theorem 2 [5, thm 1.64] *If* $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ *is an eta-quotient with* $k = \frac{1}{2} \sum_{\delta|N} r_\delta$, *with the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *Here the character* χ *is defined by* $\chi(d) := \left(\frac{(-1)^k \cdot s}{d}\right)$, *where* $s := \prod_{\delta|N} \delta^{r_\delta}$. *Moreover, if* $f(z)$ *is holomorphic (resp. vanishes) at all of the cusps of* $\Gamma_0(N)$, *then* $f(z) \in M_k(\Gamma_0(N), \chi)$ *(resp.* $S_k(\Gamma_0(N), \chi)$*).*

And the orders of an eta-quotient at cusps are determined by

Theorem 3 [5, thm 1.65] *Let* c, d, *and* N *be the positive integers with* $d|N$ *and* $\gcd(c, d) = 1$. *If* $f(z)$ *is an eta-quotient satisfying the conditions of Theorem 2.1 for* N, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is*

$$\frac{N}{24}\sum_{\delta|N}\frac{gcd(d,\delta)^2 r_\delta}{gcd(d,\frac{N}{d})d\delta}.$$

If d is a positive integer and $f(q)=\sum_{n=0}^{\infty}a(n)q^n$ is a formal power series, we define the operator $U(d)$ by

$$f(q)|U(d):=\sum_{n=0}^{\infty}a(dn)q^n.$$

Proposition 2.22 in [5] shows that if $d|N$, $f(z)\in M_k(\Gamma_0(N),\chi)$, then $f(z)|U(d)\in M_k(\Gamma_0(N),\chi)$. Also note that the $U(d)$ operator has the property

$$\left[\left(\sum_{n=0}^{\infty}a(n)q^n\right)\left(\sum_{n=0}^{\infty}b(n)q^{dn}\right)\right]|U(d)=\left(\sum_{n=0}^{\infty}a(dn)q^n\right)\left(\sum_{n=0}^{\infty}b(n)q^n\right). \quad (9)$$

For the purposes of studying congruences, we introduce the Sturm's Theorem, to show that every holomorphic modular form modulo M is determined by its "first few" coefficients. Let $f(q)=\sum_{n=0}^{\infty}a(n)q^n$ be a formal power series with $a(n)\in\mathbb{Z}$ and M be a positive integer, and define the order of f modulo M by

$$\mathrm{ord}_M(f):=\min\{n:a(n)\not\equiv 0 \pmod{M}\},$$

then Sturm's Theorem can be stated as

Theorem 4 [5, thm 2.58] *Suppose that $f(z), g(z)\in M_k(\Gamma_0(N),\chi)\bigcap\mathbb{Z}[[q]]$ and M is prime. If*

$$\mathrm{ord}_M(f(z)-g(z))\geq 1+\frac{kN}{12}\prod_{p|N}\left(1+\frac{1}{p}\right),$$

where the product is over all prime divisors p of N, then $f(z)\equiv g(z) \pmod{M}$.

With the above theorems, we can now start our proof of Theorem 1. We will frequently use the following congruence during the proof:

$$\prod_{n=1}^{\infty}(1-q^n)^2\equiv\prod_{n=1}^{\infty}(1-q^{2n}) \pmod{2}. \quad (10)$$

3 Proof of Theorem 1

Proof In this section, we will prove Eq. (4). Define eta-quotients $F(z)$ and $G(z)$ as

$$\begin{aligned}F(z) &:= \frac{\eta(z)^3\eta(2z)^{43}\eta(17z)}{\eta(34z)}\\ &= q^3\prod_{n=1}^{\infty}\frac{(1-q^n)^3(1-q^{2n})^{43}(1-q^{17n})}{(1-q^{34n})}\end{aligned}$$

and

$$\begin{aligned}G(z) &:= \eta(z)^{44}\eta(2z)^2+\eta(z)^{43}\eta(2z)\eta(17z)\eta(34z)\\ &= q^2\prod_{n=1}^{\infty}(1-q^n)^{44}(1-q^{2n})^2\\ &\quad + q^4\prod_{n=1}^{\infty}(1-q^n)^{43}(1-q^{2n})(1-q^{17n})(1-q^{34n}).\end{aligned}$$

By Theorems 2 and 3, it is easy to verify that $F(z), G(z) \in M_k(\Gamma_0(N), \chi)$, where $k = 23$, $N = 34 * 24$, and $\chi(d) = \left(\frac{-1}{d}\right)$. Applying operator $U(2)$ to $F(z)$, and using Sturm's Theorem, after checking the first $23 * 8 * 18$ coefficients of $F(z)|U(2)$ and $G(z)$ by computer, we find that

$$F(z)|U(2) \equiv G(z) \pmod 2. \tag{11}$$

On the other hand, by the definition of $\Delta_8(n)$ in Eq. (3) (we define $\Delta_8(n) = 0$ if n is not a nonnegative integer), using Eq. (10), we can see that

$$\begin{aligned}F(z) &\equiv q^3\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{17n})}{(1-q^n)^3(1-q^{34n})}(1-q^{2n})^{45}\\ &\equiv \left(q^3\sum_{n=0}^{\infty}\Delta_8(n)q^n\right)\prod_{n=1}^{\infty}(1-q^{2n})^{45} \pmod 2,\end{aligned}$$

so we have

$$\begin{aligned}F(z)|U(2) &\equiv \left[\left(\sum_{n=0}^{\infty}\Delta_8(n-3)q^n\right)\left(\prod_{n=1}^{\infty}(1-q^{2n})^{45}\right)\right]|U(2)\\ &\equiv \left(\sum_{n=0}^{\infty}\Delta_8(2n-3)q^n\right)\left(\prod_{n=1}^{\infty}(1-q^n)^{45}\right) \quad \text{(by Eq. 9)}\\ &\equiv \left(q^2\sum_{n=0}^{\infty}\Delta_8(2n+1)q^n\right)\left(\prod_{n=1}^{\infty}(1-q^n)^{45}\right) \pmod 2. \end{aligned} \tag{12}$$

Also note that

$$G(z) \equiv q^2 \prod_{n=1}^{\infty}(1-q^n)^{48} + q^4 \prod_{n=1}^{\infty}(1-q^n)^{45}(1-q^{17n})^3 \pmod 2 \tag{13}$$

Combining (11), (12), and (13) together, we get Eq. (4). □

4 Proof of Corollaries 1 and 2

Since we have proved Theorem 1, those corollaries are quite simple. From Eq. (6), we know that if $\Delta_8(2(An+B)+1) \equiv 0 \pmod 2$ for all $n \geq 0$, then both $8(An+B)+1$ and $17 * (8(An+B)+1)$ cannot be squares, which is equivalent to say that $8B+1$ is a quadratic nonresidue mod $8A$ and $136B+17$ is a quadratic nonresidue mod $136A$. For example, let $A = 9$, then $8A = 72$, and $8B+1 \in S := \{1, 9, 17, 25, 33, 41, 49, 57, 65\}$. Among S, only 17, 33, 41, 57, 65 are quadratic nonresidues mod 72, which means $B \in \{2, 4, 5, 7, 8\}$. Similarly, $136B+17$ is a quadratic nonresidue mod $136A$ if and only if $B \in \{0, 3, 4, 6, 7\}$. Thus, only $B = 4$ and $B = 7$ suit both conditions. Substituting A and B in $\Delta_8(2(An+B)+1)$, we obtain the first result in Example 1.

To prove Corollary 2, write

$$2p^{2\alpha+2}n + \frac{1}{4}(p^{2\alpha+1}t-1) + 1 = 2\left(p^{2\alpha+2}n + \frac{1}{8}(p^{2\alpha+1}t-1)\right) + 1.$$

By Theorem 1, we need to show that

$$8\left(p^{2\alpha+2}n + \frac{1}{8}(p^{2\alpha+1}t-1)\right) + 1 = p^{2\alpha+1}(8pn+t)$$

is neither a square, nor 17 times a square. This is obvious because under our assumption, $\mathrm{ord}_p\left(p^{2\alpha+1}(8pn+t)\right) = 2\alpha+1$. Now we complete our proof.

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