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Colored partition identities conjectured by Sandon and Zanello

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Abstract Recently, Sandon and Zanello (Ramanujan J 33: 83–120, 2014) conjectured 29 highly non-trivial colored partition identities. In this paper, we establish 17 of them and prove analogous colored partition identities of the remaining 12 conjectural identities by using the theory of Ramanujan's theta functions. We also present some new colored partition identities of the same type.

Keywords Colored partitions · Theta function identities · Modular equations · Continued fractions

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1 Introduction

Recently, Sandon and Zanello [12] determined a unified combinatorial framework to look at a large number of colored partition identities, and studied combinatorially the five identities, proved by Berndt [10], corresponding to modular equations of prime degrees 3, 5, 7,11, and 23 of the Schröter, Russell, and Ramanujan type. In [13], they further found several new and highly non-trivial colored partition identities by using their master bijection, i.e., Theorem 2.1 in [12], and conjectured 29 more identities (In fact, they conjectured 30 identities, but analytic proof of one of the identities was already given by Baruah and Berndt in [3, Theorem 8.1]). Their conjectures are

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formulated in terms of certain sets of integers satisfying the conditions of their master bijection and the partition identities are stated as corollaries. As mentioned by Berndt and Zhou [6], *these conjectures and corollaries are different formulations of the same phenomena; their corollaries are not less general than the corresponding conjectures*. Three of their conjectured identities are proved analytically by Berndt and Zhou [6] with the help of Ramanujan's formulas for multipliers. In [7], they proved all the remaining conjectures of Sandon and Zanello [13].

Following Sandon and Zanello [13], for given integers $C \geq 1$, $0 \leq A_i \leq C/2$ and $0 \leq B_i \leq C/2$, let S be the set containing one copy of all positive integers congruent to $\pm A_i$ modulo C for each i, and let T be the set containing one copy of all positive integers congruent to $\pm B_i$ modulo C for each i. Let $D_S(N)$ (respectively, $D_T(N)$) be the number of partitions of N into distinct elements of S (respectively, T), where *such partitions require to have an odd number of parts if no A_i (respectively, no B_i) is equal to zero*. Then the theorems and conjectures on colored partitions of Sandon and Zanello in [12] and [13] are identities connecting $D_S(N)$ and $D_T(N)$. For example, Corollary to Conjecture 3.24 in [13] can be stated as follows.

Conjecture 1.1 *Let S be the set containing one copy of the even positive integers that are not multiples of 25, and T be the set containing one copy of the odd positive integers that are not multiples of 25. Then, for any $N \geq 4$,*

$$D_S(N) = D_T(N-3).$$

Obviously, the above is incorrect for odd N as S contains only even elements. The corrected form, which has been proved by Berndt and Zhou [6] by using a 25th degree modular equation of Ramanujan, can be stated as follows.

Theorem 1.2 *If S and T are as defined in Corollary 1.1, then, for any $N \geq 2$, the number of partitions of $2N$ into an odd number of distinct elements of S is equinumerous to the number of partitions of $2N-3$ into distinct elements of S.*

Berndt and Zhou [6] also proved two similar conjectures of Sandon and Zanello [13] as well as several new partition identities arising from Ramanujan's formulas for multipliers.

Now, refer to the italicized text in the second paragraph of this introduction. If we do not restrict the parity of the number of partitions into distinct elements of S (or, T), then some of the partition identities conjectured by Sandon and Zanello [13] take different forms. The aim of the paper is to present new partition identities without restricting the parity of number of partitions into distinct elements of S (or, T). We also prove 17 conjectures of Sandon and Zanello [13] that do not require to restrict the parity of number of partitions into distinct elements of S (or, T). These correspond to Corollaries to Conjectures 3.26, 3.32, 3.34–3.38, 3.40, 3.41, 3.43–3.46, and 3.48–3.51 in [13].

In each of our partition identities, the number of partitions of a positive integer n into distinct elements of a particular set A will be denoted by $P_A(n)$. For example, the identity analogous to the previous theorem is

$$P_S(2N) = 2P_T(2N-3) + a(N),$$

where $a(N)$ is the difference of the number of partitions of N into an even number of distinct nonmultiples of 25 and the number of partitions of N into an odd number of distinct nonmultiples of 25. By Euler's famous pentagonal number theorem, it is clear that

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty},$$

where, here and in the sequel, for any complex number a and $|q| < 1$, we define

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

We extensively employ Ramanujan's theta function identities to arrive at the partition identities. In the next few paragraphs, we introduce Ramanujan's theta functions and some preliminary results.

Ramanujan's general theta function $f(a,b)$ for $|ab| < 1$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}. \tag{1.1}$$

If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i \tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta functions in its standard notation. Jacobi's famous triple product identity can be written as

$$f(a,b) = (-a,ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2}$$

Three special cases of $f(a,b)$ are given in the following lemma.

Lemma 1.3 *[8, p. 36, Entry 22]. If $|q| < 1$, then*

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \frac{1}{2}f(1,q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and $$f(-q) = f(-q,-q^2) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty,$$

where the product representations in the above arise from (1.2).

After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty,$$

which is the generating function for the number of partitions of a positive integer into distinct odd parts.

Some more results are given in the next two lemmas.

Lemma 1.4 *[8, p. 45, Entry 29]. If $ab = cd$, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{1.3}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(b/c, ac^2d\right) f\left(b/d, acd^2\right). \tag{1.4}$$

Lemma 1.5 *[8, p. 40, Entry 25]. We have*

$$\begin{aligned} \varphi(q) + \varphi(-q) &= 2\varphi(q^4), \\ \varphi(q) - \varphi(-q) &= 4q\psi(q^8), \\ \varphi^2(q) - \varphi^2(-q) &= 8q\psi^2(q^4), \end{aligned}$$

and

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2).$$

Next, we give the definition of a modular equation as understood by Ramanujan. The complete elliptic integral of first kind $K(k)$ is defined as

$$\begin{aligned} K(k) :&= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} k^{2n} \\ &= \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad (0 < k < 1) \end{aligned} \tag{1.5}$$

where the series representation is found by expanding the integrand in a binomial series and integrating termwise, and ${}_2F_1(a, b; c; z)$, $|z| < 1$ denotes the ordinary hypergeometric function. The number k is called the modulus, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote complete elliptic integrals of first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ, which is implied by (1.6). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = \ell^2$. We then say that β has degree n over α. The corresponding multiplier m is defined by

$$m = \frac{K}{L}.$$

If $q = \exp(-\pi K'/K)$, then one of the fundamental results in the theory of elliptic functions [8, Entry 6, p. 101] is given by

$$\varphi^2(q) = \frac{2}{\pi} K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where φ is as defined in Lemma 1.3.

The above identity enables one to derive formulas for φ, ψ, f, and χ at different arguments in terms of α , q, and $z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$. In particular, Ramanujan recorded the following identities.

Lemma 1.6 *[8, pp. 122 – 124, Entries 10–12]. We have*

$$\begin{aligned}
\varphi(q) &= \sqrt{z}, \\
\varphi(-q) &= \sqrt{z}(1-\alpha)^{1/4}, \\
\varphi(-q^2) &= \sqrt{z}(1-\alpha)^{1/8}, \\
\psi(q) &= \frac{\sqrt{z}\alpha^{1/8}}{\sqrt{2}q^{1/8}}, \\
\psi(-q) &= \frac{\sqrt{z}\,(\alpha(1-\alpha))^{1/8}}{\sqrt{2}q^{1/8}}, \\
\psi(q^2) &= \frac{\sqrt{z}\alpha^{1/4}}{2q^{1/4}}, \\
f(q) &= \frac{\sqrt{z}\,(\alpha(1-\alpha))^{1/24}}{2^{1/6}q^{1/24}}, \\
f(-q^2) &= \frac{\sqrt{z}\,(\alpha(1-\alpha))^{1/12}}{2^{1/3}q^{1/12}}.
\end{aligned}$$

Suppose that β has degree n over α. If we replace q by q^n above, then the same evaluations hold with α replaced by β and z replaced by $z_n := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

In the next six sections, we prove 17 of the conjectures in [13] and find analogous partition identities for the remaining 12 conjectures. It would be clear from our proofs of the partition identities that more such colored partition identities could be found. In the last section of this paper, we present some new colored partition identities of the same type.

2 Partition identities analogous to conjectures 3.24, 3.25, and 3.27 of [13]

As mentioned in the previous section, each of the modified versions of Conjectures 3.24, 3.25, and 3.27 of [13] have been proved by Berndt and Zhou [7] by employing a certain kind of Ramanujan modular equation involving multipliers. In this section, we present three analogous partition identities without restricting the parity of the number of distinct elements of S (and/or, T). It is worthwhile to mention that the same kind of partition identities may be obtained from other analogous modular equations of Ramanujan involving multipliers.

Theorem 2.1 *(*Analogs to Corollary to Conjecture 3.24 of [13] and to Theorem 3.3 of [6]*) Let S be the set containing one copy of the even positive integers that are not multiples of* 25*, and T be the set containing one copy of the odd positive integers that are not multiples of* 25*. Let $a(N)$ be the difference of the number of partitions of N into an even number of distinct non multiples of* 25 *and the number of partitions of N into an odd number of distinct non multiples of* 25*. Then $P_S(2) = 2 + a(1)$ and for $N > 1$, we have*

$$P_S(2N) = 2P_T(2N-3) + a(N). \tag{2.1}$$

Proof From [8, p. 291, Entry 15(i)], we recall a modular equation of degree 25 as

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = \sqrt{m},$$

where β has degree 25 over α, and m is the multiplier connecting α and β. Transcribing this modular equation, with the aid of Lemma 1.6, we have

$$q^3\left\{\frac{\psi(q^{25})}{\psi(q)} - \frac{\psi(-q^{25})}{\psi(-q)}\right\} = 1 + 2q^2\frac{f(-q^{50})}{f(-q^2)} - \frac{\varphi(-q^{50})}{\varphi(-q^2)},$$

which can be transformed into the q-product identity

$$\frac{(-q^2;q^2)_\infty}{(-q^{50};q^{50})_\infty} = q^3\left\{\frac{(-q;q^2)_\infty}{(-q^{25};q^{50})_\infty} - \frac{(q;q^2)_\infty}{(q^{25};q^{50})_\infty}\right\} + 2q^2 + \frac{f(-q^2)}{f(-q^{50})}.$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^3\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + 2q^2 + \frac{f(-q^2)}{f(-q^{50})}.$$

Equating the coefficients of q^{2N} from both sides of the above, and noting that

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f(-q)}{f(-q^{25})},$$

we easily arrive at (2.1).

Corollary 2.2 *For* $N \geq 0$,

$$P_S(10N+6) = 2P_T(10N+3), \tag{2.2}$$
$$P_S(10N+8) = 2P_T(10N+5), \tag{2.3}$$

and, for $N \geq 1$,

$$P_S(10N+2) = 2P_T(10N-1). \tag{2.4}$$

Furthermore,

$$a(10) = a(17) = a(20) = a(43) = a(45) = a(67) = a(117) = 0;$$

i.e.,

$$P_S(20)=2P_T(17),\ P_S(34) = 2P_T(31),\ P_S(40) = 2P_T(37),\ P_S(86) = 2P_T(83),$$
$$P_S(90) = 2P_T(87),\ P_S(134) = 2P_T(131),\ P_S(234) = 2P_T(231),$$

and

$$a(25n) > 0, a(25n+5) > 0, a(25n+7) > 0, a(25n+17) > 0, a(25n+22 > 0;$$
$$a(25n+2) < 0, a(25n+10) < 0, a(25n+12) < 0, a(25n+15) < 0,$$
$$a(25n+20) < 0.$$

Proof We recall from [8, p. 82] that

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f(-q)}{f(-q^{25})} = \frac{f(-q^{10}, -q^{15})}{f(-q^5, -q^{20})} - q - q^2 \frac{f(-q^5, -q^{20})}{f(-q^{10}, -q^{15})}.$$

Extracting various terms from both sides of the above, we find that

$$\begin{aligned} &a(1) = -1,\ a(5n+1) = 0,\ \text{for } n \geq 1, \\ &a(5n+3) = 0 = a(5n+4),\ \text{for } n \geq 0, \\ &\sum_{n=0}^{\infty} a(5n)q^n = \frac{f(-q^2, -q^3)}{f(-q, -q^4)}, \end{aligned} \tag{2.5}$$

and

$$\sum_{n=0}^{\infty} a(5n+2)q^n = -\frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

From (2.5) and (2.1), we arrive at (2.2)–(2.4).

Next, let $\gamma(n)$ and $\delta(n)$ be defined by

$$\sum_{n=0}^{\infty}\gamma(n)q^n := \sum_{n=0}^{\infty}a(5n)q^n = \frac{f(-q^2,-q^3)}{f(-q,-q^4)} = \frac{1}{q^{-1/5}R(q)}$$

and

$$\sum_{n=0}^{\infty}\delta(n)q^n := \sum_{n=0}^{\infty}a(5n+2)q^n = -\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = -q^{-1/5}R(q),$$

where $R(q)$ is the famous Rogers-Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \ldots, \quad |q| < 1. \tag{2.6}$$

The coefficients $\gamma(n)$ and $\delta(n)$ have been extensively studied by various authors. We refer to Chapt. 4 of [1] for many references. In particular, from [1, pp. 111–113, Corollary 4.2.1 and Corollary 4.2.2], we have

$$\begin{aligned}
&\gamma(2)=\gamma(4)=\gamma(9)=0, \delta(3)=\delta(8)=\delta(13)=\delta(23)=0,\\
&\gamma(5n)>0,\ \gamma(5n+1)>0,\ \gamma(5n+2)<0,\ \gamma(5n+3)<0,\ \gamma(5n+4)<0,\\
&\delta(5n)<0,\ \delta(5n+2)<0,\ \delta(5n+1)>0,\ \delta(5n+3)>0,\ \delta(5n+4)>0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&a(10)=a(20)=a(17)=a(42)=a(45)=a(67)=a(117)=0,\\
&a(25n)>0,\ a(25n+5)>0,\ a(25n+7)>0,\ a(25n+17)>0,\ a(25n+22)>0,\\
&a(25n+2)<0,\ a(25n+10)<0,\ a(25n+12)<0,\ a(25n+15)<0,\\
&\quad a(25n+20)<0,
\end{aligned}$$

which completes the proof. □

Theorem 2.3 (Analogs to Corollary to Conjecture 3.25 of [13] and to Theorem 2.7 of [6]) *Let S be the set containing 2 copies of the even positive integers that are not multiples of 13, and T be the set containing 2 copies of the odd positive integers that are not multiples of 13. Let $a(N)$ be the difference of the number of 2-colored partitions of N into an even number of distinct non multiples of 13 and the number of 2-colored partitions of N into an odd number of distinct non multiples of 13. Then $P_S(2) = 4 + a(1)$ and for any $N > 1$,*

$$P_S(2N) = 2P_T(2N-3) + a(N). \tag{2.7}$$

Proof If β has degree 13 over α, and m is the multiplier connecting α and β, then from [8, p. 376, Entry 8(iii)], we have

$$\left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}-4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}=m,$$

which can be transcribed, with the help of the identities in Lemma 1.6, into

$$\frac{\varphi^2(-q^{26})}{\varphi^2(-q^2)}=q^3\frac{\psi(q^{26})}{\psi(q^2)}\left\{\frac{\varphi(-q^{13})}{\varphi(-q)}-\frac{\varphi(q^{13})}{\varphi(q)}\right\}+4q^2\frac{f^4(q^{13})\varphi^2(q)}{f^4(q)\varphi^2(q^{13})}+1.$$

The above can be further transformed into

$$\frac{(-q^2;q^2)^2_\infty}{(-q^{26};q^{26})^2_\infty}=q^3\left\{\frac{(-q;q^2)^2_\infty}{(-q^{13};q^{26})^2_\infty}-\frac{(q;q^2)^2_\infty}{(q^{13};q^{26})^2_\infty}\right\}+4q^2+\frac{f^2(-q^2)}{f^2(-q^{26})}.$$

Thus,

$$\sum_{n=0}^{\infty}P_S(n)q^n=q^3\left\{\sum_{n=0}^{\infty}P_T(n)q^n-\sum_{n=0}^{\infty}P_T(n)(-q)^n\right\}+4q^2+\frac{f^2(-q^2)}{f^2(-q^{26})}. \quad (2.8)$$

Now, the generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty}a(n)q^n=\frac{f^2(-q)}{f^2(-q^{13})}.$$

Therefore, equating the coefficients of q^{2N} from both sides of (2.8), we easily arrive at the desired identity to complete the proof.

Theorem 2.4 (Analogs to Corollary to Conjecture 3.27 of [13] and to Theorem 2.5 of [6]) *Let S be the set containing 4 copies of the even positive integers that are not multiples of 7, and T the set containing 4 copies of the odd positive integers that are not multiples of 7. Furthermore, let $a(N)$ be the difference of the number of 4-colored partitions of N into an even number of distinct non multiples of 7 and the number of 4-colored partitions of N into an odd number of distinct non multiples of 7. Then*

$$P_S(2)=8+a(1)\quad and\quad for\ N>1,\quad P_S(2N)=2P_T(2N-3)+a(N). \quad (2.9)$$

Proof First of all, we note that

$$\sum_{n=0}^{\infty}a(n)q^n=\frac{f^4(-q)}{f^4(-q^7)}.$$

Next, if β has degree 7 over α, and m is the multiplier connecting α and β, then, from [8, p. 314, Entry 19(v)]

$$\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}-8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}=m^2,$$

which can be transcribed, with the help of Lemma 1.6, into

$$\frac{\varphi^4(-q^{14})}{\varphi^4(-q^2)}=1+q^3\left\{\frac{\psi^4(-q^7)}{\psi^4(-q)}-\frac{\psi^4(q^7)}{\psi^4(q)}\right\}+8q^2\frac{f^4(-q^{14})}{f^4(-q^2)}.$$

Transforming the theta functions into q-products, with the aid of Lemma 1.3, we find that

$$\frac{(-q^2;q^2)_\infty^4}{(-q^{14};q^{14})_\infty^4}=q^3\left\{\frac{(-q;q^2)_\infty^4}{(-q^7;q^{14})_\infty^4}-\frac{(q;q^2)_\infty^4}{(q^7;q^{14})_\infty^4}\right\}+8q^2+\frac{f^4(-q^2)}{f^4(-q^{14})},$$

which can be written as

$$\sum_{n=0}^{\infty}P_S(n)q^n=q^3\left\{\sum_{n=0}^{\infty}P_T(n)q^n-\sum_{n=0}^{\infty}P_T(n)(-q)^n\right\}+8q^2+\sum_{n=0}^{\infty}a(n)q^{2n}.$$

Equating the coefficients of q^{2N} from both sides of the above, we easily arrived at (2.9) to complete the proof. □

3 Conjectures 3.51 and 3.26 of [13]

Theorem 3.1 (Corollary to Conjecture 3.51 of [13]) *Let S be the set containing one copy of the even positive integers,* 2 *copies of the odd positive integers, one more copy of the positive multiples of* 14*, and* 2 *more copies of the odd positive multiples of* 7*; let T be the set containing* 2 *copies of the even positive integers, one copy of the odd positive integers,* 2 *more copies of the positive multiples of* 14*, and one more copy of the odd positive multiples of* 7*. Then, for any $N\geq 1$,*

$$D_S(N)=2D_T(N-1),$$

or equivalently,

$$P_S(N)=2P_T(N-1).$$

Proof We recall from Berndt's book [8, p. 304] that

$$\varphi(q)\varphi(q^7)-\varphi(-q^2)\varphi(-q^{14})=2q\psi(q)\psi(q^7). \tag{3.1}$$

Transforming this into q-products with the aid of Lemma 1.3 and canceling $(q^2;q^2)_\infty(q^{14};q^{14})_\infty$ from both sides, we find that

$$(-q;q^2)_\infty^2(-q^7;q^{14})_\infty^2-(q^2;q^4)_\infty(q^{14};q^{28})_\infty=\frac{2q}{(q;q^2)_\infty(q^7;q^{14})_\infty}.$$

Multiplying both sides by $(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty$ and then using Euler's identity $(-q;q)_\infty=(q;q^2)_\infty^{-1}$, we obtain

$$(-q;q^2)_\infty^2(-q^7;q^{14})_\infty^2(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty=$$
$$2q(-q;q^2)_\infty(-q^7;q^{14})_\infty(-q^2;q^2)_\infty^2(-q^{14};q^{14})_\infty^2+1,$$

which is equivalent to

$$\sum_{n=0}^{\infty}D_S(n)q^n=2q\sum_{n=0}^{\infty}D_T(n)q^n+1.$$

Equating the coefficients of q^N, we arrive at the desired result. □

Conjecture 3.2 (Corollary to Conjecture 3.26 of [13]) *Let S be the set containing 3 copies of the odd positive integers and 3 more copies of the odd positive multiples of 7, and T the set containing 3 copies of the even positive integers and 3 more copies of the positive multiples of 14. Then, for any $N\geq 3$,*

$$D_S(N)=4D_T(N-3).$$

Obviously, the above conjecture is not true when N is even as T contains only even elements. We prove the following modified version of the conjecture.

Theorem 3.3 *Let S and T be as defined in Conjecture 3.2. Then, for any $N\geq 1$,*

$$D_S(2N+1)=4D_T(2N-2), \tag{3.2}$$

or equivalently,

$$P_S(2N+1)=4P_T(2N-2). \tag{3.3}$$

Proof Cubing (3.1), we find that

$$\varphi^3(q)\varphi^3(q^7)-\varphi^3(-q^2)\varphi^3(-q^{14})=8q^3\psi^3(q)\psi^3(q^7)+6q\psi(q)\psi(q^7)\varphi(q)\varphi(q^7)$$
$$\times\varphi(-q^2)\varphi(-q^{14}),$$

which can be transformed, with the aid of Lemma 1.3, into

$$\{(-q;q^2)_\infty^6(-q^7;q^{14})_\infty^6-(q^2;q^4)_\infty^3(q^{14};q^{28})_\infty^3\}(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3$$
$$=8q^3\frac{(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3}{(q;q^2)_\infty^3(q^7;q^{14})_\infty^3}+6q(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3.$$

Dividing both sides of the above identity by $(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3(-q;q^2)_\infty^3$ $(-q^7;q^{14})_\infty^3$ and then using the trivial identity $(q^2;q^4)_\infty=(q;q^2)_\infty(-q;q^2)_\infty$, we arrive at

$$(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3-(q;q^2)_\infty^3(q^7;q^{14})_\infty^3=8q^3\frac{1}{(q^2;q^4)_\infty^3(q^{14};q^{28})_\infty^3}+6q,$$

which by Euler's identity $(-q;q)_\infty=(q,q^2)_\infty^{-1}$ reduces to

$$(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3-(q;q^2)_\infty^3(q^7;q^{14})_\infty^3=8q^3(-q^2;q^2)_\infty^3(-q^{14};q^{14})_\infty^3+6q.$$

Thus,

$$\sum_{n=0}^{\infty}D_S(n)q^n-\sum_{n=0}^{\infty}D_S(n)(-q)^n=8q^3\sum_{n=0}^{\infty}D_T(n)q^n+6q$$

or

$$\sum_{n=0}^{\infty}P_S(n)q^n-\sum_{n=0}^{\infty}P_S(n)(-q)^n=8q^3\sum_{n=0}^{\infty}P_T(n)q^n+6q.$$

Comparing the coefficients of q^{2N+1} from both sides of the above identities, we readily arrive at (3.2) and (3.3) to complete the proof.

Remark 3.4 The above two theorems have also been proved by Berndt and Zhou [7] by using Ramanujan's modular equations.

4 Partition identities in conjectures 3.38, 3.28, 3.30, and 3.42 of [13]

Conjecture 4.1 (Corollary to Conjecture 3.38 of [13]) *Let S be the set containing 3 copies of the odd positive integers that are not multiples of 3, one copy of the odd positive multiples of 3 that are not multiples of 9, and 4 copies of the odd positive multiples of 9; let T be the set containing 3 copies of the even positive integers that are not multiples of 3, one copy of the positive multiples of 6 that are not multiples of 18, and 4 copies of the positive multiples of 18. Then, for any $N \geq 3$,*

$$D_S(N)=2D_T(N-3).$$

Obviously, Conjecture 4.1 is incorrect for even N as T contains only even elements. We find the following modified result.

Theorem 4.2 *If S and T are as defined in Conjecture 4.1, then $D_S(1) = 3$ and for $N > 1$,*

$$D_S(2N+1) = 2D_T(2N-2), \tag{4.1}$$

or equivalently,

$$P_S(2N+1) = 2P_T(2N-2). \tag{4.2}$$

Proof First we recall from [8, p. 49, Corollary (ii)] that

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{4.3}$$

Now, by Jacobi triple product identity, (1.2), and the definition of φ and $\chi(q)$, we have

$$f(q, q^2) = (-q; q^3)_\infty(-q^2; q^3)_\infty(q^3; q^3)_\infty = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{4.4}$$

With the help of the above, we rewrite (4.3) as

$$\psi(q) = \frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9). \tag{4.5}$$

Replacing q by $-q$ and then employing the trivial identity $\chi^3(q) = \varphi(q)/\psi(-q)$, we have

$$\frac{\chi^3(q^9)}{\chi(q^3)} = q + \frac{\psi(-q)}{\psi(-q^9)}. \tag{4.6}$$

Again, from [9, p. 202, Entry 50(i)], we recall that

$$\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q\frac{\psi(-q^9)}{\psi(-q)}. \tag{4.7}$$

Multiplying the previous two identities, we have

$$\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)} = 4q + 3q^2\frac{\psi(-q^9)}{\psi(-q)} + \frac{\psi(-q)}{\psi(-q^9)}. \tag{4.8}$$

Next, by [8, p. 358, Entries 4(i) and (ii)],

$$\frac{\varphi(-q^{18})}{\varphi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1$$

and

$$\frac{\varphi(-q^2)}{\varphi(-q^{18})}+\frac{1}{q}\left(\frac{\psi(q)}{\psi(q^9)}-\frac{\psi(-q)}{\psi(-q^9)}\right)=3.$$

Replacing q by $-q$ in (4.8) and then subtracting from (4.8) and using the above two identities, we obtain

$$\begin{aligned}\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)}-\frac{\chi^3(-q)\chi^3(-q^9)}{\chi^2(-q^3)}&=2q+3q\frac{\varphi(-q^{18})}{\varphi(-q^2)}+q\frac{\varphi(-q^2)}{\varphi(-q^{18})},\\&=2q+\frac{q}{\varphi(-q^2)\varphi(-q^{18})}\\&\quad\times\left\{\varphi^2(-q^2)+3\varphi^2(-q^{18})\right\}.\end{aligned}\tag{4.9}$$

Now, by [8, p. 49, Corollary (i)],

$$\varphi(q)=\varphi(q^9)+2qf(q^3,q^{15}).$$

Noting by (1.2), that

$$f(q,q^5)=(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty=\psi(-q^3)\chi(q),$$

we rewrite the previous identity as

$$\varphi(q)=\varphi(q^9)+2q\psi(-q^9)\chi(q^3),\tag{4.10}$$

i.e.,

$$\varphi(q)-\varphi(q^9)=2q\psi(-q^9)\chi(q^3).$$

Again, by [2, Eq. (3.37)],

$$3\varphi(q^9)-\varphi(q)=2\psi(-q)\chi(q^3).$$

Multiplying the above two identities and then replacing q by $-q^2$, we find that

$$\varphi^2(-q^2)+3\varphi^2(-q^{18})=4\varphi(-q^2)\varphi(-q^{18})+4q^2\psi(q^2)\psi(q^{18})\chi^2(-q^6).\tag{4.11}$$

Employing (4.11) in (4.9), we obtain

$$\begin{aligned}\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)}-\frac{\chi^3(-q)\chi^3(-q^9)}{\chi^2(-q^3)}&=6q+4q^3\frac{\psi(q^2)\psi(q^{18})}{\varphi(-q^2)\varphi(-q^{18})}\chi^2(-q^6),\\&=6q+4q^3\frac{\chi^2(-q^6)}{\chi^3(-q^2)\chi^3(-q^{18})}.\end{aligned}\tag{4.12}$$

The above identity can be rewritten as

$$\sum_{n=0}^{\infty} D_S(n)q^n - \sum_{n=0}^{\infty} D_S(n)(-q)^n = 6q + 4q^3 \sum_{n=0}^{\infty} D_T(n)q^n$$

or

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 6q + 4q^3 \sum_{n=0}^{\infty} P_T(n)q^n.$$

Equating the coefficients of q^{2N+1} from both sides of the above identities, we readily arrive at (4.1) and (4.2) to finish the proof.

Remark 4.3 The identity (4.1) has also been established by Berndt and Zhou [7] by using Ramanujan's modular equations.

Conjecture 4.4 (Corollary to Conjecture 3.28 of [13]) *Let S be the set containing 3 copies of the even positive integers that are not multiples of 9, and T be the set containing 3 copies of the odd positive integers that are not multiples of 9. Then, for any $N \geq 4$,*

$$D_S(N) = D_T(N-3).$$

Obviously, Conjecture 4.4 is incorrect for odd N as S contains only even elements. For even N, the conjecture is proved by Berndt and Zhou [7]. Here we find the following analogous result.

Theorem 4.5 *Let S and T be as defined in Conjecture 4.4 and let $a(N)$ be the difference of the number of 3-colored partitions of N into an even number of distinct nonmultiples of 9 and the number of 3-colored partitions of N into an odd number of distinct nonmultiples of 9. Then, $P_S(2) = 3 + a(1)$ and for $N > 1$, we have*

$$P_S(2N) = 2P_T(2N-3) + a(N). \tag{4.13}$$

Proof Note that

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f^3(-q)}{f^3(-q^9)}.$$

Multiplying (4.5) by $\varphi(q^9)$ and (4.10) by $q\psi(q^9)$ and then subtracting the second from the first, we have

$$\psi(q)\varphi(q^9) - q\varphi(q)\psi(q^9) = \frac{\varphi^2(-q^{18}) - 2q^2\psi(q^{18})\varphi(-q^{18})\chi(-q^6)}{\chi(-q^3)},$$

where we have also used the trivial identities $\chi(q)\chi(-q) = \chi(-q^2)$, $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$, and $\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2)$. Replacing q by $-q^2$ in (4.10) and then using it in the above identity, we obtain

$$\psi(q)\varphi(q^9) - q\varphi(q)\psi(q^9) = \frac{\varphi(-q^2)\varphi(-q^{18})}{\chi(-q^3)}. \tag{4.14}$$

Cubing, we have

$$\psi^3(q)\varphi^3(q^9) - q^3\varphi^3(q)\psi^3(q^9) = \frac{\varphi^3(-q^2)\varphi^3(-q^{18})}{\chi^3(-q^3)} + 3q\psi(q)\varphi(q^9)\varphi(q)\psi(q^9)\frac{\varphi(-q^2)\varphi(-q^{18})}{\chi(-q^3)}.$$

Dividing both sides of the above by $\psi^3(q)\psi^3(q^9)$ and using $\dfrac{\varphi(q)}{\psi(q)} = \chi(q)\chi(-q^2)$ and then simplifying further, we find that

$$\frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = q^3\frac{\chi^3(q)}{\chi^3(q^9)} + \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + 3q\frac{\chi^3(-q^9)}{\chi(-q^3)}. \tag{4.15}$$

Replacing q by $-q$ in (4.15) and then adding the resulting identity with (4.15), we have

$$2\frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = q^3\left\{\frac{\chi^3(q)}{\chi^3(q^9)} - \frac{\chi^3(-q)}{\chi^3(-q^9)}\right\} + \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + \frac{\chi^3(q)\chi^6(q^9)}{\chi^3(q^3)} + 3q\left\{\frac{\chi^3(-q^9)}{\chi(-q^3)} - \frac{\chi^3(q^9)}{\chi(q^3)}\right\}. \tag{4.16}$$

Now, Ramanujan's third degree modular equation

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} = 1$$

can be transformed into (see [3], Theorem 4.1)

$$\frac{f^3(-q^3)}{f(-q)} - \frac{f^3(q^3)}{f(q)} = 2q\frac{f^3(-q^{12})}{f(-q^4)}.$$

Multiplying both sides of the above by $\dfrac{f(-q^2)}{f^3(-q^6)}$ and noting that $f(-q) = \chi(-q)f(-q^2)$, we find that

$$\frac{\chi^3(-q^3)}{\chi(-q)} - \frac{\chi^3(q^3)}{\chi(q)} = 2q\frac{\chi(-q^2)}{\chi^3(-q^6)}. \tag{4.17}$$

Employing the above, with q replaced by q^3, in (4.16), we obtain

$$2\frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = q^3\left\{\frac{\chi^3(q)}{\chi^3(q^9)} - \frac{\chi^3(-q)}{\chi^3(-q^9)}\right\} + \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + \frac{\chi^3(q)\chi^6(q^9)}{\chi^3(q^3)} + 6q^4\frac{\chi(-q^6)}{\chi^3(-q^{18})},$$

which can be recast as

$$2L = q^3R + A + 6q^4\frac{\chi(-q^6)}{\chi^3(-q^{18})}, \tag{4.18}$$

where

$$L = \frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = \sum_{n=0}^{\infty} P_S(n)q^n,$$

$$R = \left\{\frac{\chi^3(q)}{\chi^3(q^9)} - \frac{\chi^3(-q)}{\chi^3(-q^9)}\right\} = \sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n,$$

and

$$A = \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + \frac{\chi^3(q)\chi^6(q^9)}{\chi^3(q^3)}.$$

From (4.12) and (4.17), we have

$$\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)} - \frac{\chi^3(-q)\chi^3(-q^9)}{\chi^2(-q^3)} = 6q + 4q^3\frac{\chi^2(-q^6)}{\chi^3(-q^2)\chi^3(-q^{18})} \tag{4.19}$$

and

$$\frac{\chi^3(q^9)}{\chi(q^3)} - \frac{\chi^3(-q^9)}{\chi(-q^3)} = -2q^3\frac{\chi(-q^6)}{\chi^3(-q^{18})}. \tag{4.20}$$

Multiplying (4.19) and (4.20), we find that

$$2L = A + 12q^4\frac{\chi(-q^6)}{\chi^3(-q^{18})} + 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})}. \tag{4.21}$$

Multiplying (4.18) by 2 and then subtracting (4.21), we obtain

$$2L = 2q^3R + A - 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})}. \tag{4.22}$$

We want a simplified expression for $A - 8q^6 \dfrac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})}$. We do this by employing some results involving Ramanujan's cubic continued fraction [8, p. 345], $G(q)$, defined by

$$G(q) := \frac{q^{1/3}\chi(-q)}{\chi^3(-q^3)} = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q| < 1. \tag{4.23}$$

We note from [1, pp. 95–96, Theorem 3.3.1] that

$$G(q)G(-q) + G(q^2) = 0, \tag{4.24}$$

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0, \tag{4.25}$$

and

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0. \tag{4.26}$$

Now, by (4.6) and (4.7), we have

$$\frac{\psi(-q)}{\psi(-q^9)} = -q + \frac{\chi^3(q^9)}{\chi(q^3)}$$

and

$$3q\frac{\psi(-q^9)}{\psi(-q)} = -1 + \frac{\chi^3(q)}{\chi(q^3)}.$$

Multiplying the above two identities, we obtain

$$\frac{\chi^3(q)}{\chi(q^3)} = \frac{1-2G(-q^3)}{1+G(-q^3)}.$$

Thus,

$$\begin{aligned} A - 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})} &= \frac{q^2}{G^2(q^3)}\left(\frac{1-2G(q^3)}{1+G(q^3)}\right) + \frac{q^2}{G^2(-q^3)}\left(\frac{1-2G(-q^3)}{1+G(-q^3)}\right) \\ &\quad - 8q^2G^2(q^6)\left(\frac{1+G(q^6)}{1-2G(q^6)}\right) \\ &= 2q^2\left(\frac{1}{G(q^6)} + 3 + 4G^2(q^6)\right), \end{aligned} \tag{4.27}$$

where (4.24)–(4.26) have also been utilized to arrive at the last expression.

But, by [8, p. 95, Entry 1(iv)],

$$4G^2(q) - 3 + \frac{1}{G(q)} = \frac{f^3(-q^{1/3})}{q^{1/3} f^3(-q^3)}.$$

Employing the above, with q replaced by q^6, in (4.27), we have

$$A - 8q^6 \frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})} = 12q^2 + 2\frac{f^3(-q^2)}{f^3(-q^{18})},$$

and hence, (4.22) reduces to

$$L = q^3 R + 6q^2 + \frac{f^3(-q^2)}{f^3(-q^{18})}.$$

In terms of $P_S(n)$, $P_T(n)$, and $a(n)$, the above can be recast as

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^3 \left\{ \sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n \right\} + 6q^2 + \sum_{n=0}^{\infty} a(n)q^{2n}.$$

Equating the coefficient of q^{2N} from both sides of the above, we arrived at the desired result. □

Theorem 4.6 (Corollary to Conjecture 3.30 of [13]) *Let S be the set containing one copy of the odd positive integers that are not multiples of 9 and 2 copies of the even positive integers that are not multiples of 9, and T be the set containing 2 copies of the odd positive integers that are not multiples of 9 and one copy of the even positive integers that are not multiples of 9. Then, for any $N \geq 2$,*

$$D_S(N) = D_T(N-1).$$

Berndt and Zhou [7] proved the above theorem. Here we present an analogous theorem involving $P_S(n)$ and $P_T(n)$.

Theorem 4.7 *If S and T are as defined in Conjecture 4.6, then*

$$P_S(3N+1) = P_T(3N), \tag{4.28}$$

$$P_S(3N+2) = P_T(3N+1), \tag{4.29}$$

$$P_S(3N) = P_T(3N-1) + a(N), \tag{4.30}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^3(-q^3)}{\chi(-q)} = \frac{1}{C(q)},$$

with $C(q) = q^{-1/3}G(q)$, where $G(q)$ is Ramanujan's cubic continued fraction as defined by (4.23).

Furthermore, $a(n)$ is nonzero except if $n = 5$ and 8.

Proof Transforming the theta functions in (4.14) into q-products by using Lemma 1.3, we find that

$$\frac{(-q;q^2)_\infty(-q^2;q^2)_\infty^2}{(-q^9;q^{18})_\infty(-q^{18};q^{18})_\infty^2} = q\frac{(-q;q^2)_\infty^2(-q^2;q^2)_\infty}{(-q^9;q^{18})_\infty^2(-q^{18};q^{18})_\infty} + \frac{\chi^3(-q^9)}{\chi(-q^3)}.$$

In terms of $P_S(n)$ and $P_T(n)$, the above can be written as

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + \frac{\chi^3(-q^9)}{\chi(-q^3)}.$$

Equating the coefficients of q^{3N+1}, q^{3N+2}, and q^{3N}, from both sides of the above, we readily arrive at (4.28)–(4.30), respectively.

Now, Hirschhorn and Roselin [11, Theorem 1.5] proved that $a_{6n} > 0$, $a_{6n+1} > 0$, $a_{6n+2} > 0$, $a_{6n+3} < 0$, $a_{6n+4} < 0$, $a_{6n+5} < 0$ except $a_5 = a_8 = 0$. Thus, we finish the proof. □

Theorem 4.8 (Analog to Corollary to Conjecture 3.42 of [13]) *Let S be the set containing 2 copies of the positive integers that are not multiples of 4, and 2 more copies of the positive multiples of 3 that are not multiples of 4; let T be the set containing 2 copies of the positive integers that are not congruent to 2 modulo 4, and 2 more copies of the positive multiples of 3 that are not congruent to 2 modulo 4. Then $P_S(1) = 2$ and for $N \geq 1$,*

$$P_S(2N+1) = 4P_T(2N-1).$$

Proof We recall from [8, p. 232] that

$$\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3) = 4q\psi(q^2)\psi(q^6),$$

which can be transformed into

$$(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2 - (q;q^2)_\infty^2(q^3;q^6)_\infty^2 = \frac{4q}{(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2}. \tag{4.31}$$

Replacing q by q^2 in the above, multiplying both sides of the resulting identity by $(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2$ and then using Euler's identity and the trivial identity $(-q;q^2)_\infty(q;q^2)_\infty = (q^2;q^4)_\infty$, we find that

$$\frac{(-q;q)_\infty^2(-q^3;q^3)_\infty^2}{(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2} - (-q;q^2)_\infty^2(-q^3;q^6)_\infty^2(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2$$
$$= \frac{4q^2(-q;q)_\infty^2(-q^3;q^3)_\infty^2}{(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2}.$$

We rewrite the above as

$$\sum_{n=0}^{\infty} P_S(n)q^n - (-q;q^2)_\infty^2(-q^3;q^6)_\infty^2(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2 = 4q^2\sum_{n=0}^{\infty} P_T(n)q^n. \tag{4.32}$$

Replacing q by $-q$ in (4.32) and then subtracting the resulting identity from (4.32), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n - (q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2\\ &\quad\times\left\{(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2 - (-q;q^2)_\infty^2(-q^3;q^6)_\infty^2\right\}\\ &\quad= 4q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\}.\end{aligned} \tag{4.33}$$

Employing (4.31) in (4.33), we arrive at

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 4q + 4q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\}.$$

Equating the coefficients of q^{2N+1} from both sides, we complete the proof. □

5 Conjectures 3.32, 3.33, 3.31, 3.35–3.37 and 3.52 of [13]

Theorem 5.1 (Corollary to Conjecture 3.32 of [13]) *Let S be the set containing 2 copies of the positive integers that are either odd or multiples of 8, and 7 copies of the positive integer that are congruent to 2 modulo 4; let T be the set containing 4 copies of the positive integers that are either odd or multiples of 8, and 2 copies of the positive integers that are congruent to 2 modulo 4. Then, for any $N \geq 1$,*

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof By Lemma 1.5,

$$\frac{\varphi(q)-\varphi(-q)}{\varphi(q)+\varphi(-q)} = 2q\frac{\psi(q^8)}{\phi(q^4)}.$$

Thus,

$$\frac{\varphi^2(q)-\varphi(q)\varphi(-q)}{\varphi(q)+\varphi(-q)}=2q\frac{\varphi(q)\psi(q^8)}{\phi(q^4)}.$$

Adding $\varphi(-q)$ to both sides of the above and then using Lemma 1.5 again, we have

$$\frac{\varphi^2(q^2)}{\varphi(q^4)}=2q\frac{\varphi(q)\psi(q^8)}{\phi(q^4)}+\varphi(-q).$$

Dividing both sides by $\varphi(-q)$,

$$\frac{\varphi^2(q^2)}{\varphi(-q)\varphi(q^4)}=2q\frac{\varphi(q)\psi(q^8)}{\varphi(-q)\phi(q^4)}+1,$$

which can be transformed into

$$(-q;q^2)_\infty^2(-q^8;q^8)_\infty^2(-q^2;q^4)_\infty^7=2q(-q;q^2)_\infty^4(-q^8;q^8)_\infty^4(-q^2;q^4)_\infty^2+1, \tag{5.1}$$

where we also applied Euler's identity $(-q;q)_\infty=(q;q^2)_\infty^{-1}$ and the trivial identity $(q;q^2)_\infty(-q;q^2)_\infty=(q^2;q^4)_\infty$.

Since the above is equivalent to

$$\sum_{n=0}^{\infty}D_S(n)q^n=2q\sum_{n=0}^{\infty}D_T(n)q^n+1$$

or equivalently,

$$\sum_{n=0}^{\infty}P_S(n)q^n=2q\sum_{n=0}^{\infty}P_T(n)q^n+1,$$

equating the coefficients of q^N from both sides, we readily arrive at the desired result. □

Theorem 5.2 (Corollary to Conjecture 3.33 of [13]) *Let S be the set containing 4 copies of the positive integers that are either odd or congruent to 4 modulo 8, and 2 copies of the positive integers that are congruent to 2 modulo 4; let T be the set containing 2 copies of the positive integers that are either odd or multiples of 8, and 7 copies of the positive integers that are congruent to 2 modulo 4. Then, for any $N\geq 2$,*

$$D_S(N)=D_T(N-1).$$

The above theorem has been proved by Berndt and Zhou [7]. In the following theorem we prove an analogous result.

Theorem 5.3 *If S and T are as defined in Theorem 5.2, then $P_S(1) = 4$, $P_T(0) = 1$, and for $N \geq 1$,*

$$P_S(2N+1) = 2P_T(2N) \tag{5.2}$$

and

$$P_S(2N) = 2P_T(2N-1) + a(N), \tag{5.3}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^8(-q^2)}{\chi^4(-q)}. \tag{5.4}$$

Proof It is easy to see, or by Lemma 1.5,

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{5.5}$$

Multiplying both sides by $\varphi^2(q^2)$ and again using the identities of Lemma 1.5, we have

$$\varphi(q)\left(2\varphi^2(q^4) - \varphi^2(-q^2)\right) = 2q\psi(q^8)\varphi^2(q^2) + \varphi^2(q^2)\left(\varphi(-q) + 2q\psi(q^8)\right),$$

i.e.,

$$\begin{aligned}\varphi(q)\varphi^2(q^4) &= 2q\psi(q^8)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\varphi^2(q)\\ &= 2q\psi(q^8)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\left\{\varphi^2(q^2) + 4q\psi^2(q^4)\right\}\\ &= 2q\psi(q^8)\varphi^2(q^2) + \varphi(-q)\varphi^2(q^2) + 2q\varphi(-q)\psi^2(q^4).\end{aligned}$$

Dividing both sides by $\varphi(-q)\psi^2(q^4)$ and simplifying by using $\varphi(q)\psi(q^2) = \psi^2(q)$, we find that

$$\frac{\varphi(q)\varphi(q^4)}{\varphi(-q)\psi(q^8)} = 2q\frac{\varphi^2(q^2)}{\varphi(-q)\varphi(q^4)} + 2q + \frac{\varphi^2(q^2)}{\psi^2(q^4)}.$$

Expressing the above in q-products, we have

$$\begin{aligned}(-q;q^2)_\infty^4(-q^4:q^8)_\infty^4(-q^2;q^4)_\infty^2 &= 2q(-q;q^2)_\infty^2(-q^8:q^8)_\infty^2(-q^2;q^4)_\infty^7\\ &\quad +2q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)},\end{aligned} \tag{5.6}$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 2q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}.$$

Equating the coefficients of q^{2N+1} and q^{2N} on both sides, we easily arrive at (5.2) and (5.3), respectively. □

Theorem 5.4 (Corollary to Conjecture 3.31 of [13]) *Let S be the set containing 4 copies of the positive integers that are either odd or congruent to 4 modulo 8, and 2 copies of the positive integers that are congruent to 2 modulo 4; let T be the set containing 4 copies of the positive integers that are either odd or multiples of 8, and 2 copies of the positive integers that are congruent to 2 modulo 4. Then, for any $N \geq 2$,*

$$D_S(N) = 2D_T(N-2).$$

A proof of the above theorem can be found in Berndt and Zhou [7]. We find the following result.

Theorem 5.5 *If S and T are as defined in Theorem 5.4, then $P_S(1) = 4$ and for $N \geq 1$,*

$$P_S(2N+1) = 4P_T(2N-1), \tag{5.7}$$

$$P_S(2N) = 4P_T(2N-2) + a(N), \tag{5.8}$$

where $a(n)$ is as defined in (5.4).

Proof From (5.1) and (5.6), we have

$$(-q;q^2)_\infty^2(-q^8;q^8)_\infty^2(-q^2;q^4)_\infty^7 = 2q(-q;q^2)_\infty^4(-q^8;q^8)_\infty^4(-q^2;q^4)_\infty^2 + 1$$

and

$$(-q;q^2)_\infty^4(-q^4;q^8)_\infty^4(-q^2;q^4)_\infty^2 = 2q(-q;q^2)_\infty^2(-q^8;q^8)_\infty^2(-q^2;q^4)_\infty^7 + 2q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}.$$

Thus,

$$(-q;q^2)_\infty^4(-q^4;q^8)_\infty^4(-q^2;q^4)_\infty^2 = 4q^2(-q;q^2)_\infty^4(-q^4;q^8)_\infty^4(-q^2;q^4)_\infty^2 + 4q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)},$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q^2\sum_{n=0}^{\infty} P_T(n)q^n + 4q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}.$$

Equating the coefficient of q^{2N+1} and q^{2N} from both sides, we readily deduce (5.7) and (5.8). □

Theorem 5.6 (Corollary to Conjecture 3.35 of [13]) *Let S be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 6 copies of the positive integers that are congruent to 4 modulo 8, and 4 copies of the positive multiples of 8; let T be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 4 copies of the positive integers that are congruent to 4 modulo 8, and 6 copies of the positive multiples of 8. Then, for any $N \geq 1$,*

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof Replacing q by $-q$ in (5.5) and then dividing both side by $\varphi(-q)$, we have

$$\frac{\varphi(q^4)}{\varphi(-q)} = 2q\frac{\psi(q^8)}{\varphi(-q)} + 1,$$

which can be transformed into

$$\begin{aligned}(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^6(-q^8;q^8)_\infty^4 \\ = 2q(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^4(-q^8;q^8)_\infty^6 + 1.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} D_S(n)q^n = 2q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides, we arrive at the desired result. □

Theorem 5.7 (Corollary to Conjecture 3.36 of [13]) *Let S be the set containing 2 copies of the positive integers and 2 more copies of the odd positive integers; let T be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 4 copies of the positive integers that are congruent to 4 modulo 8, and 6 copies of the positive multiples of 8. Then, for any $N \geq 1$,*

$$D_S(N) = 4D_T(N-1)$$

or equivalently,

$$P_S(N) = 4P_T(N-1).$$

Proof From Lemma 1.5,

$$\varphi(q) = 4q\psi(q^8) + \varphi(-q).$$

Dividing both sides by $\varphi(-q)$ and then transforming into q-products, we find that

$$(-q;q^2)_\infty^2(-q;q)_\infty^2 = 4q(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^4(-q^8;q^8)_\infty^6 + 1.$$

Since the above is equivalent to

$$\sum_{n=0}^{\infty} D_S(n)q^n = 4q \textstyle\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or, equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q \sum_{n=0}^{\infty} P_T(n)q^n + 1,$$

we complete the proof by comparing the coefficients of q^N from both sides. □

Theorem 5.8 (Corollary to Conjecture 3.37 of [13]) *Let S be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 6 copies of the positive integers that are congruent to 4 modulo 8, and 4 copies of the positive multiples of 8; let T be the set containing 2 copies of the positive integers and 2 more copies of the odd positive integers. Then, for any $N \geq 1$,*

$$D_S(N) = \frac{1}{2}D_T(N)$$

or equivalently,

$$P_S(N) = \frac{1}{2}P_T(N).$$

Proof It is easy to see, or by Lemma 1.5,

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

which can be transformed into

$$\begin{aligned}(-q;q^2)_\infty^2 + (q;q^2)_\infty^2 &= 2\frac{(-q^4;q^8)_\infty^2(q^8;q^8)_\infty}{(q^2;q^2)_\infty}\\ &= 2\frac{(-q^4;q^8)_\infty^2}{(q^2;q^4)_\infty(q^4;q^8)_\infty}.\end{aligned}$$

Dividing both sides by $2(q;q^2)_\infty^2$, and then employing Euler's theorem $(q;q^2)_\infty = (-q;q)_\infty^{-1}$, we find that

$$\begin{aligned}\frac{1}{2}(-q;q)_\infty^2(-q;q^2)_\infty^2+\frac{1}{2} &= \frac{(-q^4;q^8)_\infty^2}{(q;q^2)_\infty^2(q^2;q^4)_\infty(q^4;q^8)_\infty}\\ &= \frac{(-q;q^2)_\infty^2(-q^4;q^8)_\infty^2}{(q^2;q^4)_\infty^3(q^4;q^8)_\infty}\\ &= \frac{(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^2}{(q^4;q^8)_\infty^4}\\ &= \frac{(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^6}{(q^8;q^{16})_\infty^4}\\ &= (-q^4;q^8)_\infty^6(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^8;q^8)_\infty^4,\end{aligned}$$

which can be rewritten in either of the forms

$$\frac{1}{2}\sum_{n=0}^{\infty} D_T(n)q^n + \frac{1}{2} = \sum_{n=0}^{\infty} D_S(n)q^n$$

and

$$\frac{1}{2}\sum_{n=0}^{\infty} P_T(n)q^n + \frac{1}{2} = \sum_{n=0}^{\infty} P_S(n)q^n.$$

Comparing the coefficients of q^N from both sides, we finish the proof. □

Theorem 5.9 (Corollary to Conjecture 3.52 of [13]) *Let S be the set containing 2 copies of the odd positive integers, one copy of the even positive integers that are not multiples of 16, and one more copy of the positive odd multiples of 8; let T be the set containing 2 copies of the odd positive integers, one copy of the even positive integers that are not odd multiples of 8, and one more copy of the positive multiples of 16. Then, for any $N \geq 2$,*

$$D_S(N) = D_T(N-2).$$

Theorem 5.9 has been proved by Berndt and Zhou [7]. We present the following result involving $P_S(n)$ and $P_T(n)$.

Theorem 5.10 *If S and T are as defined in Theorem 5.9, then $P_S(1) = 2$, and for $N \geq 1$,*

$$P_S(4N+1) = 2P_T(4N-1), \tag{5.9}$$

$$P_S(4N+2) = 2P_T(4N), \tag{5.10}$$

$$P_S(4N+3) = 2P_T(4N+1), \tag{5.11}$$

$$P_S(4N) = 2P_T(4N-2) + a(N), \tag{5.12}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^4(-q^2)}{\chi^2(-q)}.$$

Proof With repeated applications of the identities in Lemma 1.5, we have

$$\begin{aligned}
\varphi(q)\left(\varphi^2(q^8) - 2q^2\psi^2(q^8)\right) &= \varphi(q)\left(\left(\frac{\varphi(q^2)+\varphi(-q^2)}{2}\right)^2 \right. \\
&\quad \left. -2q^2\left(\frac{\varphi^2(q^2)-\varphi^2(-q^2)}{8}\right)\right) \\
&= \frac{1}{2}\varphi(q)\left(\varphi^2(-q^2)+\varphi^2(-q^4)\right) \\
&= \frac{1}{2}\varphi(q)\varphi(-q^2)\left(\varphi(-q^2)+\varphi(q^2)\right) \\
&= \varphi(q)\varphi(-q^2)\varphi(q^8) \\
&= \varphi(-q^2)\varphi(q^8)\left(2q\psi(q^8)+\varphi(q^4)\right) \\
&= 2q\varphi(-q^2)\varphi(q^8)\psi(q^8)+\varphi(-q^2)\varphi(q^8)\varphi(q^4).
\end{aligned}$$

Hence,

$$\varphi(q)\varphi^2(q^8) = 2q^2\varphi(q)\psi^2(q^8) + 2q\varphi(-q^2)\varphi(q^8)\psi(q^8) + \varphi(-q^2)\varphi(q^8)\varphi(q^4).$$

Dividing both sides by $\varphi(-q^2)\varphi(q^8)\psi(q^8)$, we have

$$\frac{\varphi(q)\varphi(q^8)}{\varphi(-q^2)\psi(q^8)} = 2q^2\frac{\varphi(q)\psi(q^8)}{\varphi(-q^2)\varphi(q^8)} + 2q + \frac{\varphi(q^4)}{\psi(q^8)},$$

which can be transformed into

$$\begin{aligned}
\frac{(-q;q^2)_\infty^2(-q^2;q^2)_\infty(-q^8;q^{16})_\infty}{(-q^{16};q^{16})_\infty} &= 2q^2\frac{(-q;q^2)_\infty^2(-q^2;q^2)_\infty(-q^{16};q^{16})_\infty}{(-q^8;q^{16})_\infty} \\
&\quad + 2q + \frac{\chi^4(-q^8)}{\chi^2(-q^4)}.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q^2\sum_{n=0}^{\infty} P_T(n)q^n + 2q + \frac{\chi^4(-q^8)}{\chi^2(-q^4)}.$$

Equating the coefficients of q^{4N+1}, q^{4N+2}, q^{4N+3}, and q^{4N} from both sides of the above, we finish the proof. □

6 Conjectures 3.39 and 3.40 of [13]

Theorem 6.1 (Corollary to Conjecture 3.39 of [13]) *Let S be the set containing 2 copies of the positive integers that are not multiples of 10, one more copy of the odd positive integers, and one more copy of the odd positive multiples of 5; let T be the set containing 2 copies of the positive integers that are not odd multiples of 5, one more copy of the even positive integers, and one more copy of the positive multiples of 10. Then, for any $N \geq 2$,*

$$D_S(N) = 2D_T(N-2).$$

Berndt and Zhou [7] have proved Theorem 6.1. Here we give an analogous result.

Theorem 6.2 *If S and T are as defined in Theorem 6.1, then $P_S(1) = 2 + b(1)$ and for $N > 1$*

$$P_S(N) = 4P_T(N-2) + b(N),$$

where

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^5;q^{10})_\infty^5}{(q;q^2)_\infty} = \frac{\chi^5(-q^5)}{\chi(-q)}.$$

Proof Recall from [3, p. 1039, Eq. (7.16)] that

$$(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 - (q;q^2)_\infty(q^5;q^{10})_\infty^3 = 4q^2(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty^3 + 2q\frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}.$$

Employing Euler's identity $(-q;q)_\infty = (q;q^2)_\infty^{-1}$, the above can be written as

$$\frac{(-q;q)_\infty^2}{(-q^{10};q^{10})_\infty^2}(-q^5;q^{10})_\infty(-q;q^2)_\infty - \frac{(q^5;q^{10})_\infty^5}{(q;q^2)_\infty} = 4q^2\frac{(-q;q)_\infty^2}{(-q^5;q^{10})_\infty^2}(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty + 2q.$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q^2\sum_{n=0}^{\infty} P_T(n)q^n + 2q + \frac{(q^5;q^{10})_\infty^5}{(q;q^2)_\infty}.$$

Equating the coefficients of q^N from both sides, we finish the proof. □

Theorem 6.3 (Corollary to Conjecture 3.40 of [13]) *Let S be the set containing 3 copies of the even positive integers, one copy of the odd positive integers, 3 more copies of the odd positive multiples of 5, and one more copy of the positive multiples* *of* 10*; let* T *be the set containing* 3 *copies of the odd positive integers, one copy of the even positive integers, one more copy of the odd positive multiples of* 5*, and* 3 *more copies of the positive multiples of* 10*. Then, for any* $N \geq 1$*,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof From [1, p. 28, Entries 1.7.1 (i), (iv)], we have

$$\varphi(q) + \varphi(q^5) = 2q^{4/5} f(q, q^9) R^{-1}(q^4)$$

and

$$\psi(q^2) - q\psi(q^{10}) = q^{-1/5} f(q^4, q^6) R(q),$$

where $R(q)$ is the Rogers–Ramanujan continued fraction as defined in (2.6). Multiplying the above identities and simplifying by using the trivial identity $\varphi(q)\psi(q^2) = \psi^2(q)$, we find that

$$\varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) + \psi^2(q) - q\psi^2(q^5) = 2q^{3/5} f(q, q^9) f(q^4, q^6) \frac{R(q)}{R(q^4)}. \tag{6.1}$$

Since, by [8, p. 262, Entry 10(v)],

$$\psi^2(q) - q\psi^2(q^5) = f(q, q^4) f(q^2, q^2),$$

identity (6.1) reduces to

$$\varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) + f(q, q^4) f(q^2, q^3) = 2q^{3/5} f(q, q^9) f(q^4, q^6) \frac{R(q)}{R(q^4)}. \tag{6.2}$$

Now, setting $a = q$, $b = q^4$, $c = q^2$, and $d = q^3$ in (1.3), we have

$$f(q, q^4) f(q^2, q^3) + f(-q, -q^4) f(-q^2, -q^3) = 2f(q^3, q^7) f(q^4, q^6). \tag{6.3}$$

But, by Jacobi's triple product identity, (1.2),

$$\begin{aligned} f(-q, -q^4) f(-q^2, -q^3) &= (q; q^5)_\infty (q^2; q^5)_\infty (q^3; q^5)_\infty (q^4; q^5)_\infty (q^5; q^5)_\infty^2 \\ &= (q; q)_\infty (q^5; q^5)_\infty, \end{aligned}$$

and hence, from (6.3), we have

$$f(q,q^4)f(q^2,q^3)=2f(q^3,q^7)f(q^4,q^6)-(q;q)_\infty(q^5;q^5)_\infty. \tag{6.4}$$

Employing (6.4) in (6.2), we find that

$$\begin{aligned}\varphi(q^5)\psi(q^2)-q\varphi(q)\psi(q^{10})=&\left\{2q^{3/5}f(q,q^9)f(q^4,q^6)\frac{R(q)}{R(q^4)}\right.\\&\left.-2f(q^3,q^7)f(q^4,q^6)\right\}\\&+(q;q)_\infty(q^5;q^5)_\infty.\end{aligned} \tag{6.5}$$

Now, $R(q)$ has the product representation

$$R(q)=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty},$$

and therefore,

$$\frac{R(q)}{R(q^4)}=q^{-3/5}\frac{(q,q^6,q^9,q^{11},q^{14},q^{19};q^{20})_\infty}{(q^2,q^3,q^7,q^{13},q^{17},q^{18};q^{20})_\infty}. \tag{6.6}$$

On the other hand, by employing Jacobi's triple product identity, (1.2), and changing the base to q^{20}, we have

$$\frac{f(q^3,q^7)}{f(q,q^9)}=\frac{(q,q^6,q^9,q^{11},q^{14},q^{19};q^{20})_\infty}{(q^2,q^3,q^7,q^{13},q^{17},q^{18};q^{20})_\infty}. \tag{6.7}$$

From (6.6) and (6.7),

$$q^{3/5}\frac{R(q)}{R(q^4)}f(q,q^9)f(q^4,q^6)-f(q^3,q^7)f(q^4,q^6)=0,$$

and hence, from (6.5),

$$\varphi(q^5)\psi(q^2)-q\varphi(q)\psi(q^{10})=(q;q)_\infty(q^5;q^5)_\infty.$$

The above is equivalent to

$$\begin{aligned}&(-q^5;q^{10})^2_\infty(q^{10};q^{10})_\infty\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}-q(-q;q^2)^2_\infty(q^2;q^2)_\infty\frac{(q^{20};q^{20})_\infty}{(q^{10};q^{20})_\infty}\\&=(q;q)_\infty(q^5;q^5)_\infty.\end{aligned}$$

Dividing both sides by $(q;q)_\infty(q^5;q^5)_\infty$ and then employing $(q;q)_\infty=(q;q^2)_\infty$ $(q^2;q^2)_\infty$ and Euler's identity, we find that

$$(-q^2;q^2)_\infty^3(-q;q^2)_\infty(-q^5;q^{10})_\infty^3(-q^{10};q^{10})_\infty$$
$$= q(-q;q^2)_\infty^3(-q^2;q^2)_\infty(-q^5;q^{10})_\infty(-q^{10};q^{10})_\infty^3 + 1.$$

Since the above can put either of the forms

$$\sum_{n=0}^{\infty} D_S(n)q^n = q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

and

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + 1,$$

we complete the proof by equating the coefficients of q^N from both sides of the above two identities. □

7 Conjectures 3.34, 3.29, 3.41, 3.43–3.50 of [13]

Theorem 7.1 (Corollary to Conjecture 3.34 of [13]) *Let S be the set containing one copy of the positive integers congruent to* ± 1 *modulo* 6*,* 5 *copies of the positive integers congruent to* ± 2 *modulo* 6*, and* 6 *copies of the positive multiples of* 3*; let T be the set containing* 5 *copies of the positive integers congruent to* ± 1 *modulo* 6*, one copy of the positive integers congruent to* ± 2 *modulo* 6*, and* 6 *copies of the positive multiples of* 3*. Then, for any $N \geq 1$,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof Adding (1.3) and (1.4), we find that

$$f(a,b)f(c,d) = af(b/c, ac^2d)f(b/d, acd^2) + f(ac,bd)f(ad,bc). \tag{7.1}$$

Setting $a = q$, $b = q^5$, $c = q^3$, and $d = q^3$ in the above,

$$f(q,q^5)\varphi(q^3) = qf^2(q^2,q^{10}) + f^2(q^4,q^8). \tag{7.2}$$

Replacing q by $-q$, we have

$$f(-q,-q^5)\varphi(-q^3) = -qf^2(q^2,q^{10}) + f^2(q^4,q^8). \tag{7.3}$$

Multiplying the previous two identities, we find that

$$\begin{aligned} f^4(q^4,q^8)-q^2f^4(q^2,q^{10}) &= f(q,q^5)f(-q,-q^5)\varphi(q^3)\varphi(-q^3) \\ &= f(q,q^5)f(-q,-q^5)\varphi^2(-q^6). \end{aligned} \tag{7.4}$$

Now, setting $a=q$, $b=q^5$, $c=-q$, and $d=-q^5$ in (7.1) and noting that $f(-1,u)=0$, we have

$$f(q,q^5)f(-q,-q^5)=\varphi(-q^6)f(-q^2,-q^{10}). \tag{7.5}$$

Using the above in (7.4), we obtain

$$f^4(q^4,q^8)=q^2f^4(q^2,q^{10})+\varphi^3(-q^6)f(-q^2,-q^{10}). \tag{7.6}$$

Replacing q^2 by q in (7.6) and then noting, by (1.2), that

$$f(-q,-q^5)=(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty=\psi(q^3)\chi(-q),$$

we have

$$f^4(q^2,q^4)=qf^4(q,q^5)+\varphi^3(-q^3)\psi(q^3)\chi(-q).$$

Dividing both sides by the last expression, employing (1.2), and then simplifying, we deduce that

$$\begin{aligned} &(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)^5_\infty(-q^3;q^3)^6_\infty \\ &\quad =q(-q^{\pm1};q^6)^5_\infty(-q^{\pm2};q^6)_\infty(-q^3;q^3)^6_\infty+1, \end{aligned} \tag{7.7}$$

where, here and in the sequel,

$$(-q^{\pm r};q^s)_\infty:=(-q^r;q^s)_\infty(q^{s-r};q^s)_\infty.$$

Since (7.7) can be written

$$\sum_{n=0}^{\infty}D_S(n)q^n=q\sum_{n=0}^{\infty}D_T(n)q^n+1$$

or equivalently,

$$\sum_{n=0}^{\infty}P_S(n)q^n=q\sum_{n=0}^{\infty}P_T(n)q^n+1,$$

we complete the proof by equating the coefficients of q^N from both sides. □

Theorem 7.2 (Analog to Corollary to Conjecture 3.29 of [13]) *Let S be the set containing 4 copies of the positive integers that are either congruent to ± 1 modulo 6 or to ± 4 modulo 12, and T be the set containing 4 copies of the positive integers that are either congruent to ± 1 modulo 6 or to ± 2 modulo 12. Then, $P_S(1) = 4$ and for $N \geq 1$,*

$$P_S(2N+1) = P_T(2N-1). \tag{7.8}$$

Furthermore, let U be the set containing one copy of the even positive integers and one more copy of the even positive multiples of 3, V be the set containing two copies of the odd positive integers and two more copies of the odd positive multiples of 3, W be the set containing two copies of the even positive integers and two more copies of the even positive multiples of 3. If $S' = S \cup U$ and $T' = T \cup U$, then

$$P_{S'}(2N) = P_{T'}(2N-2) + P_V(N) + 4P_W(N-1). \tag{7.9}$$

Proof Dividing both sides of (7.6) by $f^4(-q^{12})$ and then transforming into q-products, we find that

$$(-q^{\pm 4}; q^{12})_\infty^4 = q^2(-q^{\pm 2}; q^{12})_\infty^4 + (q^2; q^4)_\infty (q^6; q^{12})_\infty^5.$$

Multiplying both sides of the above by $(-q^{\pm 1}; q^6)_\infty^4$ and then simplifying by using Euler's identity, we have

$$\begin{aligned}(-q^{\pm 1}; q^6)_\infty^4(-q^{\pm 4}; q^{12})_\infty^4 &= q^2(-q^{\pm 1}; q^6)_\infty^4(-q^{\pm 2}; q^{12})_\infty^4\\ &\quad + (-q; q^2)_\infty^4(q^2; q^4)_\infty(q^3; q^6)_\infty^4(q^6; q^{12})_\infty,\end{aligned}$$

which can be put in the form

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^2\sum_{n=0}^{\infty} P_T(n)q^n + (q^2; q^4)_\infty(q^6; q^{12})_\infty(-q; q^2)_\infty^4(q^3; q^6)_\infty^4. \tag{7.10}$$

Replacing q by $-q$ in (7.10) and then subtracting the identity from (7.10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n &= q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\}\\ &\quad + (q^2; q^4)_\infty(q^6; q^{12})_\infty\\ &\quad \times\left\{(-q; q^2)_\infty^4(q^3; q^6)_\infty^4 - (q; q^2)_\infty^4(-q^3; q^6)_\infty^4\right\}.\end{aligned} \tag{7.11}$$

Now, we note from [14, p. 84, Corollary 3.3] that

$$\varphi(q)\varphi(-q^3) = \varphi(-q^4)\varphi(-q^{12}) + 2q\psi(-q^2)\psi(-q^6).$$

Squaring, we have

$$\begin{aligned}\varphi^2(q)\varphi^2(-q^3) &= \varphi^2(-q^4)\varphi^2(-q^{12}) + 4q^2\psi^2(-q^2)\psi^2(-q^6)\\ &\quad + 4q\varphi(-q^4)\varphi(-q^{12})\psi(-q^2)\psi(-q^6). \qquad (7.12)\end{aligned}$$

Replacing q by $-q$ in (7.12) and then subtracting the resulting identity from (7.12), we find that

$$\varphi^2(q)\varphi^2(-q^3) - \varphi^2(-q)\varphi^2(q^3) = 8q\psi(-q^2)\varphi(-q^4)\psi(-q^6)\varphi(-q^{12}),$$

which can be transformed into

$$(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 - (q;q^2)_\infty^4(-q^3;q^6)_\infty^4 = \frac{8q}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}. \qquad (7.13)$$

Employing (7.13) in (7.11),

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + 8q,$$

from which, by equating the coefficients of q^{2N+1} from both sides, we arrive at (7.8).

Now we prove (7.9).

Replacing q by $-q$ in (7.10) and then adding the resulting identity with (7.10), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} P_S(n)q^n + \sum_{n=0}^{\infty} P_S(n)(-q)^n\\ &= q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + (q^2;q^4)_\infty(q^6;q^{12})_\infty\\ &\quad \times \left\{(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4\right\},\\ &= q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\}\\ &\quad + \frac{(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4}{(-q^2;q^2)_\infty(-q^6;q^6)_\infty},\end{aligned}$$

i.e.,

$$\begin{aligned}\sum_{n=0}^{\infty} P_{S'}(n)q^n + \sum_{n=0}^{\infty} P_{S'}(n)(-q)^n &= q^2\left\{\sum_{n=0}^{\infty} P_{T'}(n)q^n + \sum_{n=0}^{\infty} P_{T'}(n)(-q)^n\right\}\\ &\quad + (-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4. \qquad (7.14)\end{aligned}$$

Again, replacing q by $-q$ in (7.12) and then adding the resulting identity with (7.12), we have

$$\varphi^2(q)\varphi^2(-q^3)+\varphi^2(-q)\varphi^2(q^3)=2\varphi^2(-q^4)\varphi^2(-q^{12})+8q^2\psi^2(-q^2)\psi^2(-q^6),$$

which is equivalent to

$$(-q;q^2)_\infty^4(q^3;q^6)_\infty^4+(q;q^2)_\infty^4(-q^3;q^6)_\infty^4=2(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2 \\ +8q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2.$$

Employing the above identity in (7.14), we have

$$\sum_{n=0}^{\infty}P_{S'}(n)q^n+\sum_{n=0}^{\infty}P_{S'}(n)(-q)^n=q^2\left\{\sum_{n=0}^{\infty}P_{T'}(n)q^n+\sum_{n=0}^{\infty}P_{T'}(n)(-q)^n\right\} \\ +2(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2 \\ +8q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2.$$

Equating the coefficients of q^{2N} from both sides of the above and noting that

$$\sum_{n=0}^{\infty}P_V(n)q^n=(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2$$

and

$$\sum_{n=0}^{\infty}P_W(n)q^n=(-q^2;q^2)_\infty^2(-q^6;q^6)_\infty^2,$$

we readily arrive at (7.9) to finish the proof.

Corollary 7.3 *If S' and T' are defined in Theorem 7.2, then*

$$P_{S'}(4N+2)=P_{T'}(4N)+3P_V(2N+1)$$

and

$$P_{S'}(4N)=P_{T'}(4N-2)+P_V(2N).$$

Proof It is known from Berndt's paper [10, Theorem 3.1] that $P_V(2N+1)=2P_W(2N)$. Therefore, from (7.9),

$$P_{S'}(4N+2)=P_{T'}(4N)+P_V(2N+1)+4P_W(2N)=P_{T'}(4N)+3P_V(2N+1)$$

and

$$P_{S'}(4N) = P_{T'}(4N-2) + P_V(2N) + 4P_W(2N-1) = P_{T'}(4N-2) \\ + P_V(2N+1),$$

since $P_W(2N-1) = 0$ as W contains only even elements. □

Theorem 7.4 (Corollary to Conjecture 3.41 of [13]) *Let S be the set containing 2 copies of the even positive integers, 2 more copies of the positive integers congruent to ± 2 modulo 12, and 4 copies of the odd multiples of 3; let T be the set containing 2 copies of the even positive integers, 4 more copies of the positive multiples of 12, one copy of the odd positive integers, and one more copy of the odd multiples of 3. Then, for any $N \geq 2$,*

$$D_S(N) = 4D_T(N-2)$$

or equivalently,

$$P_S(N) = 4P_T(N-2).$$

Proof Setting $a = q, b = q^5, c = -q^3$, and $d = -q^3$ in (1.4), we have

$$-f(-q,-q^5)\varphi(q^3) = -f(q,q^5)\varphi(-q^3) + 2qf^2(-q^2,-q^{10}). \tag{7.15}$$

Multiplying both sides of (7.15) by $\varphi(-q^3)$ and then adding $\varphi^2(q^3)f(q,q^5)$ to both sides, we find that

$$\varphi(q^3)\left\{\varphi(q^3)f(q,q^5) - \varphi(-q^3)f(-q,-q^5)\right\} = f(q,q^5)\left\{\varphi^2(q^3) - \varphi^2(-q^3)\right\} \\ + 2q\varphi(-q^3)f^2(-q^2,-q^{10}). \tag{7.16}$$

Again applying (7.15) and the third identity of Lemma 1.5, with q replaced by q^3, in (7.16), we find that

$$\varphi(q^3)f^2(q^2,q^{10}) = 4q^2f(q,q^5)\psi^2(q^{12}) + \varphi(-q^3)f^2(-q^2,-q^{10}).$$

Dividing both sides of the above by $\varphi(-q^3)f^2(-q^2,-q^{10})$ and then transforming to q-products, we obtain

$$(-q^2;q^2)_\infty^2(-q^{\pm 2};q^{12})_\infty^2(-q^3;q^6)_\infty^4 \\ = 4q^2(-q^2;q^2)_\infty^2(-q^{12};q^{12})_\infty^4(-q;q^2)_\infty(-q^3;q^6)_\infty + 1,$$

which is clearly

$$\sum_{n=0}^{\infty} D_S(n)q^n = 4q^2 \sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q^2 \sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides of the above, we find the desired result. □

Theorem 7.5 (Corollary to Conjecture 3.44 of [13]) *Let S be the set containing one copy of the positive integers that are not odd multiples of 6, one more copy of the positive multiples of 3 that are not odd multiples of 6, 2 more copies of the positive integers that are congruent to ± 2 modulo 12, and 3 more copies of the positive integers that are congruent to ± 4 modulo 12; let T be the set containing 2 copies of the positive integers that are not congruent to 6 or ± 4 modulo 12, one copy of the positive integers that are congruent to ± 4 modulo 12, and one more copy of the positive integers that are congruent to ± 1 modulo 6. Then, for any $N \geq 1$,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof Setting, in turn, $a = c = q$, $b = d = q^5$ and $a = c = q$, $b = d = q^2$, in Lemma 1.4, we have

$$f^2(q, q^5) + f^2(-q, -q^5) = 2f(q^2, q^{10})\varphi(q^6), \tag{7.17}$$

$$f^2(q, q^5) - f^2(-q, -q^5) = 4qf(q^4, q^8)\psi(q^{12}), \tag{7.18}$$

$$f^2(q, q^2) + f^2(-q, -q^2) = 2f(q^2, q^4)\varphi(q^3), \tag{7.19}$$

and

$$f^2(q, q^2) - f^2(-q, -q^2) = 4qf(q, q^5)\psi(q^6). \tag{7.20}$$

Multiplying (7.18) by $qf(q^2, q^{10})$ and then using (7.20) with q replaced by q^2, we find that

$$\begin{aligned} qf^2(q, q^5)f(q^2, q^{10}) = 4q^2 f(q^4, q^8)f^2(q^2, q^{10})\psi(q^{12}) \\ + qf(q^2, q^{10})f^2(-q, -q^5) \end{aligned}$$

$$\begin{aligned}
&= f(q^4,q^8)\left(f^2(q^2,q^4)-f^2(-q^2,-q^4)\right)\\
&\quad + qf(q^2,q^{10})f^2(-q,-q^5)\\
&= f^2(q^2,q^4)f(q^4,q^8)-\left(f^2(-q^2,-q^4)f(q^4,q^8)\right.\\
&\quad \left.-qf(q^2,q^{10})f^2(-q,-q^5)\right). \qquad (7.21)
\end{aligned}$$

Again, by (7.17)–(7.20),

$$\begin{aligned}
&f^2(-q^2,-q^4)f(q^4,q^8)-qf(q^2,q^{10})f^2(-q,-q^5)\\
&\quad = f^2(q^4,q^8)\varphi(q^6)-2q^2f(q^2,q^{10})f(q^4,q^8)\psi(q^{12})-qf^2(q^2,q^{10})\varphi(q^6)\\
&\qquad + 2q^2f(q^2,q^{10})f(q^4,q^8)\psi(q^{12})\\
&\quad = \varphi(q^6)\left(f^2(q^4,q^8)-qf^2(q^2,q^{10})\right),\\
&\quad = \varphi(q^6)\varphi(-q^3)f(-q,-q^5), \qquad (7.22)
\end{aligned}$$

where (7.3) is used to get the last equality.

Employing (7.22) in (7.21), we obtain

$$f^2(q^2,q^4)f(q^4,q^8)=qf^2(q,q^5)f(q^2,q^{10})+\varphi(-q^3)\varphi(q^6)f(-q,-q^5).$$

Multiplying the equation by $f(q^4,q^8)$ and then transforming the terms into q-products, and then simplifying further, we find that

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(-q^6;q^{12})_\infty}\frac{(-q^3;q^3)_\infty}{(-q^6;q^{12})_\infty}(-q^{\pm 2};q^{12})_\infty^2(-q^{\pm 4};q^{12})_\infty^3\\
&\quad = \frac{(-q;q)_\infty^2}{(-q^6;q^{12})_\infty^2(-q^{\pm 4};q^{12})_\infty^2}(-q^{\pm 4};q^{12})_\infty(-q^{\pm 1};q^6)_\infty+1. \qquad (7.23)
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}D_S(n)q^n=q\sum_{n=0}^{\infty}D_T(n)q^n+1$$

or equivalently,

$$\sum_{n=0}^{\infty}P_S(n)q^n=q\sum_{n=0}^{\infty}P_T(n)q^n+1.$$

Equating the coefficients of q^N from both sides, we complete the proof. □

Theorem 7.6 (Analog to Corollary to Conjecture 3.45 of [13]) *Let S be the set containing 2 copies of the positive integers that are not congruent to 0 or ±2 modulo*

12*, one copy of the positive integers that are congruent to* ± 2 *modulo* 12*, and one more copy of the positive integers that are congruent to* ± 1 *modulo* 6*; let* T *be the set containing one copy of the positive integers that are not odd multiples of* 6*, one more copy of the positive multiples of* 3 *that are not odd multiples of* 6*,* 2 *more copies of the positive integers that are congruent to* ± 2 *modulo* 12*, and* 3 *more copies of the positive integers that are congruent to* ± 4 *modulo* 12*. Then, for any* $N \geq 1$*,*

$$P_S(2N) = 2P_T(2N-1) + a(2N) \tag{7.24}$$

and

$$P_S(2N+1) = 2P_T(2N), \tag{7.25}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^3(-q^3)\chi^3(-q^6)}{\chi(-q)\chi(-q^2)}. \tag{7.26}$$

Proof From (7.17)–(7.20) and Lemma 1.5, we have

$$\begin{aligned}
&\varphi(q^6)f^2(q,q^5) - 2q\psi(q^{12})f^2(q^2,q^4)\\
&\quad = \left(f(q^2,q^{10})\varphi^2(q^6) + 2qf(q^4,q^6)\varphi(q^6)\psi(q^{12})\right)\\
&\qquad - 2q\left(f(q^4,q^6)\varphi(q^6)\psi(q^{12}) + 2q^2 f(q^2,q^{10})\psi^2(q^{12})\right)\\
&\quad = f(q^2,q^{10})\left(\varphi^2(q^6) - 4q^3\psi^2(q^{12})\right) = f(q^2,q^{10})\varphi^2(-q^3).
\end{aligned}$$

Transforming the above into q-products, multiplying both sides by $(-q;q)_\infty(-q^{\pm 4};q^{12})_\infty/(-q^6;q^{12})^2_\infty$ and then simplifying further, we deduce that

$$\begin{aligned}
&\frac{(-q;q)^2_\infty(-q^{\pm 2};q^{12})_\infty(-q^{\pm 1};q^6)_\infty}{(-q^{12};q^{12})^2_\infty(-q^{\pm 2};q^{12})^2_\infty}\\
&\quad = 2q\frac{(-q;q)_\infty(-q^3;q^3)_\infty(-q^{\pm 2};q^{12})^2_\infty(-q^{\pm 4};q^{12})^3_\infty}{(-q^6;q^{12})^2_\infty}\\
&\qquad + \frac{\chi^3(-q^3)\chi^3(-q^6)}{\chi(-q)\chi(-q^2)},
\end{aligned} \tag{7.27}$$

which also states that

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} a(n)q^n, \tag{7.28}$$

where $a(n)$ is as defined in (7.26). Equating the coefficients of q^{2N} and q^{2N+1} from both sides of (7.28), we deduce (7.24) and

$$P_S(2N+1) = 2P_T(2N) + a(2N+1), \tag{7.29}$$

respectively.

But, by (4.17), we have

$$\sum_{n=0}^{\infty} a(n)q^n - \sum_{n=0}^{\infty} a(n)(-q)^n = 2q.$$

Equating the coefficients of q^{2N+1} from both sides of the above, we have $a(1) = 1$ and for $n \geq 1$, $a(2n+1) = 0$, and therefore, (7.29) reduces to (7.25). □

Theorem 7.7 (Analog to Corollary to Conjecture 3.43 of [13]) *Let S be the set containing 2 copies of the positive integers that are not congruent to 0 or ± 2 modulo 12, one copy of the positive integers that are congruent to ± 2 modulo 12, and one more copy of the positive integers that are congruent to ± 1 modulo 6; let T be the set containing 2 copies of the positive integers that are not congruent to 6 or ± 4 modulo 12, one copy of the positive integers that are congruent to ± 4 modulo 12, and one more copy of the positive integers that are congruent to ± 1 modulo 6. Then, for any $N \geq 1$,*

$$P_S(2N) = 2P_T(2N-2) + a(2N) \tag{7.30}$$

and

$$P_S(2N+1) = 2P_T(2N-1), \tag{7.31}$$

where $a(n)$ is as defined in (7.26).

Proof Define

$$\begin{aligned}
A &:= \frac{(-q;q)_\infty}{(-q^6;q^{12})_\infty} \frac{(-q^3;q^3)_\infty}{(-q^6;q^{12})_\infty} (-q^{\pm 2};q^{12})_\infty^2 (-q^{\pm 4};q^{12})_\infty^3, \\
B &:= \frac{(-q;q)_\infty^2}{(-q^6;q^{12})_\infty^2 (-q^{\pm 4};q^{12})_\infty^2} (-q^{\pm 4};q^{12})_\infty (-q^{\pm 1};q^6)_\infty, \\
C &:= \frac{(-q;q)_\infty^2 (-q^{\pm 2};q^{12})_\infty (-q^{\pm 1};q^6)_\infty}{(-q^{12};q^{12})_\infty^2 (-q^{\pm 2};q^{12})_\infty^2}.
\end{aligned}$$

From (7.23) and (7.27), we have

$$A = qB + 1$$

and

$$C = 2qA + \sum_{n=0}^{\infty} a(n)q^n,$$

where $a(n)$ is defined by (7.26). It is easily seen from the above that

$$C = 2q^2 B + 2q + \sum_{n=0}^{\infty} a(n)q^n,$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q^2 \sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} a(n)q^n + 2q,$$

where S and T are as given in the statement of the theorem. Equating the coefficients of q^{2N} and q^{2N+1}, respectively, from both sides of the above, and also noting that $a(2N+1) = 0$, we readily arrive at (7.30) and (7.31) to complete the proof. □

Theorem 7.8 (Corollary to Conjecture 3.46 of [13]) *Let S be the set containing 2 copies of the positive integers that are not odd multiples of 3, one more copy of the positive integers that are congruent to ± 2 modulo 12, and 2 more copies of the positive odd multiples of 6; let T be the set containing 2 copies of the positive integers that are not odd multiples of 3, one more copy of the positive integers that are congruent to ± 4 modulo 12, and 2 more copies of the positive multiples of 12. Then, for any $N \geq 1$,*

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof Replacing q by q^2 in (7.2), we have

$$\begin{aligned} f(q^2, q^{10})\varphi(q^6) &= q^2 f^2(q^4, q^{20}) + f^2(q^8, q^{16}), \\ &= \Big(f(q^8, q^{16}) - qf(q^4, q^{20})\Big)^2 + 2qf(q^4, q^{20})f(q^8, q^{16}). \end{aligned} \tag{7.32}$$

But, from [8, p. 46, Entries 30(ii) and 30(iii)],

$$f(-q, -q^5) = f(q^8, q^{16}) - qf(q^4, q^{20}). \tag{7.33}$$

Employing (7.33) in (7.32), we obtain

$$f(q^2, q^{10})\varphi(q^6) = 2qf(q^4, q^{20})f(q^8, q^{16}) + f^2(-q, -q^5).$$

Transforming into q-products and then simplifying, we find that

$$\frac{(-q;q)_\infty^2}{(-q^3;q^6)_\infty^2}(-q^{\pm 2};q^{12})_\infty(-q^6;q^{12})_\infty^2$$
$$= 2q\frac{(-q;q)_\infty^2}{(-q^3;q^6)_\infty^2}(-q^{\pm 4};q^{12})_\infty(-q^{12};q^{12})_\infty^2 + 1,$$

i.e.,

$$\sum_{n=0}^{\infty} D_S(n)q^n = 2q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

The proffered partition identity of the theorem follows immediately. □

Theorem 7.9 (Analog to Corollary to Conjecture 3.47 of [13]) *Let S be the set containing 2 copies of the positive integers that are not congruent to 2 modulo 4, one more copy of the positive integers that are congruent to ± 1 modulo 6, and one more copy of the positive integers that are congruent to ± 4 modulo 12; let T be the set containing 2 copies of the positive integers that are not multiples of 4, one more copy of the positive integers that are congruent to ± 1 modulo 6, and one more copy of the positive integers that are congruent to ± 2 modulo 12. Then, for any $N > 2$*

$$P_T(N) = P_S(N) + 3U(N-2), \tag{7.34}$$

where $U(N)$ is defined by

$$\sum_{n=0}^{\infty} U(n)q^n := \frac{(-q;q^2)_\infty^2(-q^{\pm 1};q^6)_\infty(-q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q^8;q^{24})_\infty^2(q^{16};q^{24})_\infty^2}. \tag{7.35}$$

Proof Baruah and Nath [5, Eq. 3.17] proved that

$$\varphi(q)f(q,q^5) = \psi(q^2)f(q^2,q^4) + 3q\frac{f^3(-q^{12})}{f(-q^4)}.$$

Replacing q by q^2 and then multiplying both sides by $f(q,q^5)$, we find that

$$\varphi(q^2)f(q^2,q^{10})f(q,q^5) = f(q,q^5)\psi(q^4)f(q^4,q^8) + 3q^2f(q,q^5)\frac{f^3(-q^{24})}{f(-q^8)}.$$

Transforming into q-products and simplifying, we have

$$\begin{aligned}&(-q^{\pm 1};q^6)_\infty(-q^{\pm 2};q^{12})_\infty(-q^2;q^4)_\infty^2\\&\quad=(-q^{\pm 1};q^6)_\infty(-q^{\pm 4};q^{12})_\infty\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty(q^4;q^4)_\infty}\\&\qquad+3q^2\frac{(q^{24};q^{24})_\infty^3(q^{\pm 1};q^6)_\infty}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q^8;q^8)_\infty}.\end{aligned}$$

Multiplying both sides of the above by $(-q;q^2)_\infty^2$ and simplifying by using Euler's identity, we find that

$$\begin{aligned}&(-q^{\pm 1};q^6)_\infty(-q^{\pm 2};q^{12})_\infty\frac{(-q;q)_\infty^2}{(-q^4;q^4)_\infty^2}\\&\quad=(-q^{\pm 1};q^6)_\infty(-q^{\pm 4};q^{12})_\infty\frac{(-q;q)_\infty^2}{(-q^2;q^4)_\infty^2}\\&\qquad+3q^2\frac{(-q;q^2)_\infty^2(q^{24};q^{24})_\infty^3(q^{\pm 1};q^6)_\infty}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q^8;q^8)_\infty}\\&\quad=(-q^{\pm 1};q^6)_\infty(-q^{\pm 4};q^{12})_\infty\frac{(-q;q)_\infty^2}{(-q^2;q^4)_\infty^2}\\&\qquad+3q^2\frac{(-q;q^2)_\infty^2(-q^{\pm 1};q^6)_\infty(-q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q^8;q^{24})_\infty^2(q^{16};q^{24})_\infty^2},\end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty}P_T(n)q^n+3q^2\sum_{n=0}^{\infty}U(n)q^n=\sum_{n=0}^{\infty}P_S(n)q^n. \tag{7.36}$$

Equating the coefficients of q^N from both sides, we easily arrive at the desired identity. □

Theorem 7.10 (Corollary to Conjecture 3.49 of [13]) *Let S be the set containing* 2 *copies of the positive multiples of* 6*,* 2 *copies of the positive integers that are congruent to* ± 1 *modulo* 6*, one copy of the positive integers that are congruent to* ± 2 *modulo* 6*, and* 4 *copies of the odd positive multiples of* 3*; let T be the set containing* 4 *copies of the positive multiples of* 6*, one copy of the positive integers that are congruent to* ± 1 *modulo* 6*, one copy of the positive integers that are congruent to* ± 2 *modulo* 6*,* 2 *more copies of the positive integers that are congruent to* ± 2 *modulo* 12*, and* 2 *copies of the odd positive multiples of* 3*. Then, for any $N \geq 1$,*

$$D_S(N)=2D_T(N-1)$$

or equivalently,

$$P_S(N)=2P_T(N-1).$$

Proof Setting $a = q$, $b = q^5$, $c = q^3$, and $d = q^3$ in (1.4), we have

$$f(q, q^5)\varphi(q^3) = 2qf^2(q^2, q^{10}) + f(-q, -q^5)\varphi(-q^3),$$

which can be rewritten, with the aid of the Jacobi triple product identity, (1.2), as

$$\begin{aligned}(-q^{\pm 1}; q^6)_\infty(-q^3; q^6)_\infty^2 &= 2q(-q^6; q^6)_\infty^2(-q^{\pm 2}; q^{12})_\infty^2 + (q^{\pm 1}; q^6)_\infty(q^3; q^6)_\infty^2 \\ &= 2q(-q^6; q^6)_\infty^2(-q^{\pm 2}; q^{12})_\infty^2 + (q; q^2)_\infty(q^3; q^6)_\infty \\ &= 2q(-q^6; q^6)_\infty^2(-q^{\pm 2}; q^{12})_\infty^2 + \frac{1}{(-q; q)_\infty(-q^3; q^3)_\infty},\end{aligned}$$

where Euler's identity is used in the last equality. The above can be put in the form

$$\begin{aligned}&(-q^6; q^6)_\infty^2(-q^{\pm 1}; q^6)_\infty^2(-q^{\pm 2}; q^6)_\infty(-q^3; q^6)_\infty^4 \\ &\quad = 2q(-q^6; q^6)_\infty^4(-q^{\pm 1}; q^6)_\infty(-q^{\pm 2}; q^6)_\infty(-q^{\pm 2}; q^{12})_\infty^2(-q^3; q^6)_\infty^2 + 1,\end{aligned} \tag{7.37}$$

which is

$$\sum_{n=0}^{\infty} D_S(N)q^n = 2q\sum_{n=0}^{\infty} D_T(N)q^n + 1,$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(N)q^n = 2q\sum_{n=0}^{\infty} P_T(N)q^n + 1.$$

Equating the coefficients of q^N from both sides of the above two identities, we complete the proof. □

Theorem 7.11 (Corollary to Conjecture 3.50 of [13]) *Let S be the set containing 4 copies of the positive multiples of 6, one copy of the positive integers that are congruent to ± 1 modulo 6, one copy of the positive integers that are congruent to ± 2 modulo 6, 2 more copies of the positive integers that are congruent to ± 4 modulo 12, and 2 copies of the odd positive multiples of 3; let T be the set containing 2 copies of the positive multiples of 6, 2 copies of the positive integers that are congruent to ± 1 modulo 6, one copy of the positive integers that are congruent to ± 2 modulo 6, and 4 copies of the odd positive multiples of 3. Then, for any $N \geq 1$,*

$$D_S(N) = \frac{1}{2}D_T(N)$$

or equivalently,

$$P_S(N) = \frac{1}{2}P_T(N).$$

Proof Setting $a = q, b = q^5, c = q^3$, and $d = q^3$ in (1.3), we have

$$2f^2(q^4, q^8) = f(q, q^5)\varphi(q^3) + f(-q, -q^5)\varphi(-q^3), \tag{7.38}$$

which can be rewritten, with the help of the Jacobi triple product identity and Euler's identity, as

$$2(-q^6; q^6)_\infty^2(-q^{\pm 4}; q^{12})_\infty^2 = (-q^{\pm 1}; q^6)_\infty(-q^3; q^6)_\infty^2 + \frac{1}{(-q; q)_\infty(-q^3; q^3)_\infty}.$$

After simplification, the above gives,

$$\begin{aligned}&(-q^6; q^6)_\infty^4(-q^{\pm 1}; q^6)_\infty(-q^{\pm 2}; q^6)_\infty(-q^{\pm 4}; q^{12})_\infty^2(-q^3; q^6)_\infty^2\\ &\quad = \frac{1}{2}(-q^{\pm 1}; q^6)_\infty^2(-q^3; q^6)_\infty^4(-q^{\pm 2}; q^6)_\infty(-q^6; q^6)_\infty^2 + \frac{1}{2},\end{aligned} \tag{7.39}$$

which is equivalent to

$$\sum_{n=0}^{\infty} D_S(n)q^n = \frac{1}{2}\sum_{n=0}^{\infty} D_T(n)q^n + \frac{1}{2}$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = \frac{1}{2}\sum_{n=0}^{\infty} P_T(n)q^n + \frac{1}{2}.$$

We complete the proof by equating the coefficients of q^N from both sides of the above two identities. □

Theorem 7.12 (Corollary to Conjecture 3.48 of [13]) *Let S be the set containing 4 copies of the positive multiples of 6, one copy of the positive integers that are congruent to ± 1 modulo 6, one copy of the positive integers that are congruent to ± 2 modulo 6, 2 more copies of the positive integers that are congruent to ± 4 modulo 12, and 2 copies of the odd positive multiples of 3; let T be the set containing 4 copies of the positive multiples of 6, one copy of the positive integers that are congruent to ± 1 modulo 6, one copy of the positive integers that are congruent to ± 4 modulo 12, 2 copies of the odd positive multiples of 3, and 3 copies of the positive integers that are congruent to ± 2 modulo 12. Then, for any $N \geq 1$,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof From (7.39) and (7.37), we find that

$$(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm4};q^{12})_\infty^2(-q^3;q^6)_\infty^2$$
$$=q(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm2};q^{12})_\infty^2(-q^3;q^6)_\infty^2+1.$$

With the help of the trivial identity $(-q^{\pm2};q^6)_\infty=(-q^{\pm4};q^{12})_\infty(-q^{\pm2};q^{12})_\infty$, the above reduces to

$$(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm4};q^{12})_\infty^2(-q^3;q^6)_\infty^2$$
$$=q(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm4};q^{12})_\infty(-q^{\pm2};q^{12})_\infty^3(-q^3;q^6)_\infty^2+1,$$

which can be rewritten as

$$\sum_{n=0}^{\infty}D_S(n)q^n=q\sum_{n=0}^{\infty}D_T(n)q^n+1$$

or equivalently,

$$\sum_{n=0}^{\infty}P_S(n)q^n=q\sum_{n=0}^{\infty}P_T(n)q^n+1.$$

Equating the coefficients of q^N, we complete the proof. □

To conclude this section, for completeness, we state the following theorem which is Corollary to Conjecture 3.53 of [13], and an analytic proof of this theorem has already been given by Baruah and Berndt [3, Theorem 8.1].

Theorem 7.13 (Corollary to Conjecture 3.53 of [13]) *Let S be the set containing one copy of the odd positive integers, one more copy of the odd positive multiples of* 3*, one more of the odd positive multiples of* 5*, and one more of the odd positive multiples of* 15*; let T be the set containing one copy of the even positive integers, one more copy of the positive multiples of* 6*, one more of the positive multiples of* 10*, and one more of the positive multiples of* 30*. Then, for any $N\geq 3$,*

$$D_S(N)=2D_T(N-3)$$

or equivalently,

$$P_S(N)=2P_T(N-3).$$

8 Some more colored partition identities

In this section, we present some more colored partition identities which are analogous to the partition identities discussed in the previous sections.

Theorem 8.1 *Let S be the set containing 2 copies of the even positive integers, 4 more copies of the odd positive multiples of 3, and 2 copies of the positive multiples of 4 that are not multiples of 12; let T be the set containing one copy of the odd positive integers, one more copy of the odd positive multiples of 3, two copies of the even positive integers, and 4 more copies of the odd multiples of 6. Then, for any $N \geq 1$,*

$$P_S(N) = P_T(N).$$

Proof Multiplying (7.15) by $\varphi(q^3)$ and then adding $\varphi^2(q^3)f(q,q^5)$ to both sides, we have

$$\varphi(q^3)\left\{\varphi(q^3)f(q,q^5)+\varphi(-q^3)f(-q,-q^5)\right\} = f(q,q^5)\left\{\varphi^2(q^3)+\varphi^2(-q^3)\right\} - 2q\varphi(-q^3)f^2(-q^2,-q^{10}),$$

which can be rewritten, with the aid of (7.38) and Lemma 1.5, as

$$\varphi(q^3)f^2(q^4,q^8) = \varphi^2(q^6)f(q,q^5) - q\varphi(-q^3)f^2(-q^2,-q^{10}).$$

Transforming the above into q-products and then simplifying, we obtain

$$(-q^2;q^2)_\infty^2(-q^3;q^6)_\infty^4\frac{(-q^4;q^4)_\infty^2}{(-q^{12};q^{12})_\infty^2} = (-q;q^2)_\infty(-q^3;q^6)_\infty(-q^2;q^2)_\infty^2(-q^6;q^{12})_\infty^4 - q,$$

which is equivalent to

$$\sum_{n=0}^{\infty}P_S(n)q^n = \sum_{n=0}^{\infty}P_T(n)q^n - q.$$

We complete the proof by equating the coefficients of q^N from both sides of the above. □

Theorem 8.2 *Let S be the set containing 4 copies of the positive integers that are congruent to ± 1 modulo 6 and 2 copies of the even positive integers that are not multiples of 6; let T be the set containing 2 copies of the even positive integers, 2 copies of the positive multiples of 12, two more copies of the positive integers that are congruent to ± 1 modulo 6 and one more copy of the positive integers that are congruent to ± 4 modulo 12. Then, for any $N \geq 1$,*

$$P_S(N) = 4P_T(N-1).$$

Proof With the help of (7.5), we can rewrite (7.18) as

$$f^4(q,q^5) = 4qf(q^4,q^8)f^2(q,q^5)\psi(q^{12}) + \varphi^2(-q^6)f^2(-q^2,-q^{10}).$$

Transcribing the above into q-products, we find that

$$(-q^{\pm 1}; q^6)_\infty^4 \frac{(-q^2; q^2)_\infty^2}{(-q^6; q^6)_\infty^2} = 4q(-q^2; q^2)_\infty^2 (-q^{12}; q^{12})_\infty^2 (-q^{\pm 1}; q^6)_\infty^2 (-q^{\pm 4}; q^{12})_\infty + 1,$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n) q^n = 4q \sum_{n=0}^{\infty} P_T(n) q^n + 1.$$

Now the proffered partition identity is apparent. □

The next two theorems easily follow from (7.38) and (7.15), respectively. We omit the proofs.

Theorem 8.3 *Let S be the set containing one copy of the odd positive integers, one copy of the positive odd multiples of 3, one copy each of the positive integers, and the positive multiples of 3; let T be the set containing one copy of the positive integers, 2 copies of positive integers that are odd multiples of 6, one copy of the positive multiples of 3, and 2 more copies of the positive multiples of 4. Then, for any $N \geq 1$,*

$$P_S(N) = 2P_T(N).$$

Theorem 8.4 *Let S be the set containing one copy of the odd positive integers, one more copy of the positive odd multiples of 3, one copy each of the positive integers, and the positive multiples of 3; let T be the set containing one copy of the positive integers, 2 copies of the positive integers that are odd multiples of 2, one copy of the positive multiples of 3, and 2 more copies of the positive multiples of 12. Then, for any $N \geq 1$,*

$$P_S(N) = 2P_T(N-1).$$

Theorem 8.5 *Let S be the set containing 3 copies each of the positive integers, the odd positive integers, the positive multiples of 3, and the odd positive multiples of 3; let T be the set containing 6 copies each of the odd positive multiples of 2 and the positive multiples of 12 and 3 copies each of the positive integers and the positive multiples of 3, and let U be the set containing 2 copies each of the positive integers, the positive multiples of 3, the odd positive multiples of 2, and the positive multiples of 12, one copy each of the odd positive integers, and the odd positive multiples of 3. Then, for any $N \geq 1$,*

$$P_S(N) = 8P_T(N-3) + 6P_U(N-1).$$

Proof We recall from [9, p. 198, Entry 45] that

$$\psi(q)\psi(q^3)-\psi(-q)\psi(-q^3)=2q\varphi(q^2)\psi(q^{12}).$$

Cubing and then dividing both sides by $\psi^3(-q)\psi^3(-q^3)$, we have

$$\frac{\psi^3(q)\psi^3(q^3)}{\psi^3(-q)\psi^3(-q^3)}-1=8q^3\frac{\varphi^3(q^2)\psi^3(q^{12})}{\psi^3(-q)\psi^3(-q^3)}+6q\frac{\varphi(q^2)\psi(q^{12})\psi(q)\psi(q^3)}{\psi^2(-q)\psi^2(-q^3)},$$

which can be easily transformed, with the aid of Euler's identity, into

$$\begin{aligned}&(-q;q^2)_\infty^3(-q;q)_\infty^3(-q^3;q^6)_\infty^3(-q^3;q^3)_\infty^3\\&=8q^3(-q^2;q^4)_\infty^6(-q^{12};q^{12})_\infty^6(-q;q)_\infty^3(-q^3;q^3)_\infty^3\\&\quad+6q(-q^2;q^4)_\infty^2(-q;q)_\infty^2(-q^3;q^3)_\infty^2(-q^{12};q^{12})_\infty^2(-q;q^2)_\infty(-q^3;q^6)_\infty+1.\end{aligned}$$

Since the above is equivalent to

$$\sum_{n=0}^{\infty}P_S(n)q^n=8q^3\sum_{n=0}^{\infty}P_T(n)q^n+6q\sum_{n=0}^{\infty}P_U(n)q^n+1,$$

we complete the proof by equating the coefficients of q^N from both sides. □

Theorem 8.6 *Let S be the set containing* 6 *copies of the odd positive integers that are not multiple of* 5 *and one copy of the even positive integers; T be the set containing* 4 *copies of the even positive integers and* 9 *copies of the positive multiples of* 10*, and let U be the set containing* 5 *copies of the positive multiples of* 10*. Then, for any $N\geq 1$,*

$$P_S(2N+1)=32P_T(2N-2)+6P_U(2N).$$

Proof From [8, p. 278], we recall that

$$\varphi(q)\varphi(-q^5)-\varphi(-q)\varphi(q^5)=4qf(-q^4)f(-q^{20}).$$

Cubing the above, we obtain

$$\begin{aligned}\varphi^3(q)\varphi^3(-q^5)-\varphi^3(-q)\varphi^3(q^5)&=64q^3f^3(-q^4)f^3(-q^{20})\\&\quad+12q\varphi^2(-q^2)\varphi^2(-q^{10})f(-q^4)f(-q^{20}).\end{aligned}$$

Transforming into q-products, we have

$$\begin{aligned}&\frac{(-q;q^2)_\infty^6(q^5;q^{10})_\infty^6}{(q^2;q^4)_\infty(q^{10};q^{20})_\infty}-\frac{(q;q^2)_\infty^6(-q^5;q^{10})_\infty^6}{(q^2;q^4)_\infty(q^{10};q^{20})_\infty}\\&=64q^3(-q^2;q^2)_\infty^4(-q^{10};q^{10})_\infty^4+12q,\end{aligned}$$

which can further be reduced to

$$\frac{(-q;q^2)_\infty^6(-q^2;q^2)_\infty}{(-q^5;q^{10})_\infty^6}-\frac{(q;q^2)_\infty^6(-q^2;q^2)_\infty}{(q^5;q^{10})_\infty^6}=64q^3(-q^2;q^2)_\infty^4(-q^{10};q^{10})_\infty^9 \\ +12q(-q^{10};q^{10})_\infty^5.$$

Thus,

$$\sum_{n=0}^{\infty}P_S(n)q^n-\sum_{n=0}^{\infty}P_S(n)(-q)^n=64q^3\sum_{n=0}^{\infty}P_T(n)q^n+12q\sum_{n=0}^{\infty}P_U(n)q^n.$$

Equating the coefficients of q^{2N+1} from both sides, we easily arrive at the desired partition identity. □

Theorem 8.7 *Let S be the set containing one copy of the positive integers that are congruent to ±1 modulo 6 and 2 copies of the positive integers that are odd multiples of 3; T be the set containing 2 copies of the positive integers that are multiples of 6 and 2 more copies of the positive integers that are congruent to ±2 modulo 12, and let U be the set containing 2 copies of the positive multiples of 4 and 2 more copies of the odd multiple of 6. Then, for any $N\geq1$,*

$$P_S(N)=P_T(N-1)+P_U(N).$$

Proof We can easily transform (7.2) into

$$(-q^{\pm1};q^6)_\infty(-q^3;q^6)_\infty^2=q(-q^6;q^6)_\infty^2(-q^{\pm2};q^{12})_\infty^2+(-q^6;q^{12})_\infty^2(-q^4;q^4)_\infty^2,$$

which also states that

$$\sum_{n=0}^{\infty}P_S(n)q^n=q\sum_{n=0}^{\infty}P_T(n)q^n+\sum_{n=0}^{\infty}P_U(n)q^n.$$

Equating the coefficient of q^N from both sides, we complete the proof. □

Theorem 8.8 *Let S be the set containing one copy of the odd positive integers that are not multiples of 9 and 2 copies of the even positive integers that are not multiples of 18; let T be the set containing 2 copies of the even integers and one more copy of the even positive integers that are not multiples of 6. Then, for any $N\geq1$,*

$$P_S(2N+1)=P_T(2N).$$

Proof We recall from [4, Eq. (8.12)] that

$$\psi(q)\psi(-q^9)-\psi(-q)\psi(q^9)=2q\psi(q^{18})\psi(q^2)\chi(-q^6).$$

Dividing both sides by $\varphi(-q^2)\psi(q^{18})$, transforming into q-products, and then simplifying by using Euler's identity, we deduce that

$$\frac{(-q;q^2)_\infty(-q^2;q^2)_\infty^2}{(-q^9;q^{18})_\infty(-q^{18};q^{18})_\infty^2}-\frac{(q;q^2)_\infty(-q^2;q^2)_\infty^2}{(q^9;q^{18})_\infty(-q^{18};q^{18})_\infty^2}$$
$$=2q\,\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(-q^2;q^2)_\infty^2.$$

Thus,

$$\sum_{n=0}^\infty P_S(n)q^n-\sum_{n=0}^\infty P_S(n)(-q)^n=2q\sum_{n=0}^\infty P_T(n)q^n.$$

Equating the coefficients of q^{2N+1} from both sides, we finish the proof. □

Theorem 8.9 *Let S be the set containing one copy of the odd positive integers that are not multiples of* 9*, and let T be the set containing one copy of the even positive integers that are not multiple of* 6 *and* 2 *more copies of the positive integers that are multiples of* 18*. Then, for any $N \geq 1$,*

$$P_S(2N+1)=P_T(2N). \tag{8.1}$$

Proof From [8, p. 358, Entry 4(i)], we have

$$\frac{\varphi(-q^{18})}{\varphi(-q^2)}+q\left(\frac{\psi(q^9)}{\psi(q)}-\frac{\psi(-q^9)}{\psi(-q)}\right)=1. \tag{8.2}$$

Now, replacing q by $-q^2$ in (4.10), we have

$$\varphi(-q^2)=\varphi(-q^{18})-2q^2\psi(q^{18})\chi(-q^6).$$

Employing the above in (8.2), we find that

$$\frac{\psi(-q^9)}{\psi(-q)}-\frac{\psi(q^9)}{\psi(q)}=2q\,\frac{\psi(q^{18})\chi(-q^6)}{\varphi(-q^2)},$$

which can be transformed into

$$\frac{(-q;q^2)_\infty}{(-q^9;q^{18})_\infty}-\frac{(q;q^2)_\infty}{(q^9;q^{18})_\infty}=2q\,\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(-q^{18};q^{18})_\infty^2.$$

Since the above is equivalent to

$$\sum_{n=0}^\infty P_S(n)q^n-\sum_{n=0}^\infty P_S(n)(-q)^n=2q\sum_{n=0}^\infty P_T(n)q^n,$$

we complete the proof by equating the coefficients of q^{2N+1} from both sides. □

Theorem 8.10 *Let S be the set containing 2 copies of the odd positive integers that are not multiples of 5, and let T be the set containing one copy of the even positive integers and 3 more copies of the positive integers that are multiples of 10. Then, for any $N \geq 1$,*

$$P_S(2N+1) = 2P_T(2N).$$

Proof We recall from [8, p. 276] that

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) = 1. \tag{8.3}$$

But, from Entries 9(vii) and 10(iv) of [8, p. 258 and p. 262],

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7) = 4q\chi(q)f(-q^5)f(-q^{20}).$$

Replacing q by q^2 in the above and employing it in (8.3), we find that

$$\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)} = 4q\frac{\chi(-q^2)f(q^{10})f(-q^{40})}{\varphi^2(-q^2)},$$

which can be transformed into

$$\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2} = 4q(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty^3.$$

The above is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 4q\sum_{n=0}^{\infty} P_T(n)q^n,$$

and equating the coefficients of q^{2N+1} from both sides, we finish the proof. □

Theorem 8.11 *Let S be the set containing one copy of the positive integers that are not multiples of 3, one more copy of the odd positive integers that are not multiples of 3, 2 more copies of the positive integers, and 2 more copies of the odd positive integers; let T be the set containing one copy of the positive integers that are not multiples of 3, one more copy of the even positive integers that are not multiples of 6, 2 more copies of the positive integers, and 2 more copies of the even positive integers. Then, for any $N \geq 1$,*

$$P_S(N) = 2P_T(N).$$

Proof From [8, p. 359], we have

$$\frac{\varphi(-q)\psi(q^3)}{\psi(q)} + \frac{\varphi(q)\psi(-q^3)}{\psi(-q)} = 2\frac{\psi(q^2)\varphi(-q^6)}{\varphi(-q^2)}.$$

Dividing both sides by $\dfrac{\varphi(-q)\psi(q^3)}{\psi(q)}$ and then transforming into q-products, we find that

$$\begin{aligned}&\frac{(-q;q)_\infty(-q;q^2)_\infty}{(-q^3;q^3)_\infty(-q^3;q^6)_\infty}(-q;q^2)_\infty^2(-q;q)_\infty^2\\&\quad = 2\frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(-q^3;q^3)_\infty(-q^6;q^6)_\infty}(-q;q)_\infty^2(-q^2;q^2)_\infty^2 - 1,\end{aligned}$$

which is clearly

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2\sum_{n=0}^{\infty} P_T(n)q^n - 1.$$

Equating the coefficients of q^N from both sides, we readily arrive at the desired identity. □

Theorem 8.12 *Let S be the set containing two copies each of the positive integers, the odd positive integers, the positive multiples of* 3*, and the odd positive multiples of* 3*, and let T be the set containing two copies each of the positive integers, the even positive integers, the positive multiples of* 3*, and the positive multiples of* 6*. Then, for any $N \geq 1$,*

$$P_S(N) = 4P_T(N-1).$$

Proof We note from [9, p. 198, Entry 45] that

$$\psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\varphi(q^2)\psi(q^{12})$$

and

$$\psi(q)\psi(q^3) + \psi(-q)\psi(-q^3) = 2\varphi(q^6)\psi(q^4).$$

Multiplying together, we have

$$\psi^2(q)\psi^2(q^3) - \psi^2(-q)\psi^2(-q^3) = 4q\varphi(q^2)\psi(q^{12})\varphi(q^6)\psi(q^4).$$

Dividing both sides by $\psi^2(-q)\psi^2(-q^3)$ and simplifying by Euler's identity, we obtain

$$\begin{aligned}&(-q;q^2)_\infty^2(-q;q)_\infty^2(-q^3;q^3)_\infty^2(-q^3;q^6)_\infty^2\\&\quad = 4q(-q;q)_\infty^2(-q^2;q^2)_\infty^2(-q^3;q^3)_\infty^2(-q^6;q^6)_\infty^2 + 1,\end{aligned}$$

which is

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q \sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides, we finish the proof. □

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