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Poly-Cauchy numbers with a q parameter

Takao Komatsu

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Abstract The concept of poly-Cauchy numbers was recently introduced by the author. The poly-Cauchy number is a generalization of the Cauchy number just as the poly-Bernoulli number is a generalization of the classical Bernoulli number. In this paper we give some more generalizations of poly-Cauchy numbers and show some arithmetical properties.

Keywords Poly-Cauchy numbers · Cauchy numbers · Poly-Bernoulli numbers

Mathematics Subject Classification 05A15 · 11B75

1 Introduction

The Cauchy numbers (of the first kind) c_n are given by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\,dx = n!\int_0^1 \binom{x}{n}\,dx$$

[7, Chap. VII]. The numbers $c_n/n!$ are sometimes called the Bernoulli numbers of the second kind (see, e.g., [2, 24]). Such numbers have been studied by several authors [6, 20, 21, 23, 25] because they are related to various special combinatorial numbers, including Stirling numbers of both kinds, Bernoulli numbers and harmonic numbers. Remarkably, the Cauchy numbers of the first kind c_n and the Bernoulli numbers B_n

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T. Komatsu (✉)
Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan
e-mail: komatsu@cc.hirosaki-u.ac.jp

have several symmetric properties. The generating function of the Cauchy numbers of the first kind c_n is given by

$$\frac{x}{\ln(1+x)}=\sum_{n=0}^\infty c_n\frac{x^n}{n!}$$

[7, 21], and the generating function of Bernoulli numbers B_n is given by

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}$$

[7] or

$$\frac{x}{1-e^{-x}}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}$$

[16]. In this paper we use the latter definition of B_n. In addition, Cauchy numbers of the first kind c_n can be written explicitly as

$$c_n=(-1)^n\sum_{m=0}^n\left[{n\atop m}\right]\frac{(-1)^m}{m+1}$$

([7, Chap. VII], [21, p. 1908]), where $\left[{n\atop m}\right]$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1)=\sum_{m=0}^n\left[{n\atop m}\right]x^m$$

(see, e.g., [9]). Bernoulli numbers B_n (in the latter definition) can be also written explicitly as

$$B_n=(-1)^n\sum_{m=0}^n\left\{{n\atop m}\right\}\frac{(-1)^m m!}{m+1},$$

where $\left\{{n\atop m}\right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{{n\atop m}\right\}=\frac{1}{m!}\sum_{j=0}^m(-1)^j\binom{m}{j}(m-j)^n$$

(see, e.g., [9]). Recently, Liu, Qi and Ding [20] established some recurrence relations about Cauchy numbers of the first kind as analogues of results for Bernoulli numbers by Agoh and Dilcher [1].

In 1997 Kaneko [16] introduced the poly-Bernoulli numbers $B_n^{(k)}$ ($n\ge 0$, $k\ge 1$) by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^\infty B_n^{(k)}\frac{x^n}{n!},$$

where

$$\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k}$$

is the kth polylogarithm function. When $k=1$, $B_n^{(1)}=B_n$ is the classical Bernoulli number with $B_1^{(1)}=1/2$.

Recently, the author [17] introduced the poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ as a generalization of the Cauchy numbers and an analogue of the poly-Bernoulli numbers by

$$c_n^{(k)}=\underbrace{\int_0^1\cdots\int_0^1}_{k}(x_1x_2\cdots x_k)(x_1x_2\cdots x_k-1)$$
$$\cdots(x_1x_2\cdots x_k-n+1)\,dx_1\,dx_2\cdots dx_k.$$

In addition, the generating function of poly-Cauchy numbers is given by

$$\mathrm{Lif}_k\bigl(\ln(1+x)\bigr)=\sum_{n=0}^{\infty}c_n^{(k)}\frac{x^n}{n!},$$

where

$$\mathrm{Lif}_k(z):=\sum_{m=0}^{\infty}\frac{z^m}{m!(m+1)^k}$$

is the kth *polylogarithm factorial* function, which is also introduced by the author [17, 18]. If $k=1$, then $c_n^{(1)}=c_n$ is the classical Cauchy number. One different extension of Cauchy numbers is on hypergeometric Cauchy numbers [19], as that of hypergeometric Bernoulli numbers is a different extension of Bernoulli numbers (e.g., [12, 13]).

The concept of the poly-Bernoulli numbers have been extended by several authors, including Bayad and Hamahata [3, 4], Hamahata and Masubuchi [10, 11], Sasaki [22] and Jolany [14]. Some applications of the poly-Bernoulli numbers have been studied (e.g., [5, 15]).

In this paper, we give a generalization of the poly-Cauchy numbers and show several combinatorial properties. The poly-Cauchy numbers are special ones with $q=1$ in the poly-Cauchy numbers with q parameter.

2 Poly-Cauchy numbers with q parameter

Let n and k be integers with $n \ge 0$ and $k \ge 1$. Let q be a real number with $q \ne 0$. Define the *poly-Cauchy numbers with q parameter* of the first kind $c_{n,q}^{(k)}$ by

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \cdots x_k)(x_1 x_2 \cdots x_k - q) \cdots \bigl(x_1 x_2 \cdots x_k - (n-1)q\bigr)\, dx_1\, dx_2 \cdots dx_k.$$

Hence, if $q = 1$, then $c_{n,1}^{(k)} = c_n^{(k)}$ are the poly-Cauchy numbers, defined in [17]. We may define the *Cauchy numbers with q parameter* of the first kind $c_{n,q}^{(1)} = c_{n,q}$ by

$$c_{n,q} = \int_0^1 x(x-q)\cdots\bigl(x-(n-1)q\bigr)\, dx.$$

We record the first several Cauchy numbers with q parameter of the first kind:

$$\begin{aligned}
c_{1,q} &= \frac{1}{2},\\
c_{2,q} &= \frac{1}{3} - \frac{1}{2}q,\\
c_{3,q} &= \frac{1}{4} - q + q^2,\\
c_{4,q} &= \frac{1}{5} - \frac{3}{2}q + \frac{11}{3}q^2 - 3q^3,\\
c_{5,q} &= \frac{1}{6} - 2q + \frac{35}{4}q^2 - \frac{50}{3}q^3 + 12q^4,\\
c_{6,q} &= \frac{1}{7} - \frac{5}{2}q + 17q^2 - \frac{225}{4}q^3 + \frac{274}{3}q^4 - 60q^5,\\
c_{7,q} &= \frac{1}{8} - 3q + \frac{175}{6}q^2 - 147q^3 + 406q^4 - 588q^5 + 360q^6.
\end{aligned}$$

As a general case of the poly-Cauchy numbers and the Cauchy numbers, the poly-Cauchy numbers with q parameter $c_{n,q}^{(k)}$ can be expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$.

Theorem 1 *For a real number $q \ne 0$,*

$$c_{n,q}^{(k)} = \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{(-q)^{n-m}}{(m+1)^k} \quad (n \ge 0,\ k \ge 1).$$

Proof By the identity

$$x(x-1)\cdots(x-n+1)=\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-1)^{n-m}x^m$$

(see, e.g., [9, Chap. 6]), we get

$$\begin{aligned}\underbrace{X(X-q)\cdots\bigl(X-(n-1)q\bigr)}_{n}&=q^n\cdot\frac{X}{q}\left(\frac{X}{q}-1\right)\cdots\left(\frac{X}{q}-n+1\right)\\&=q^n\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-1)^{n-m}\left(\frac{X}{q}\right)^m\\&=\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-q)^{n-m}X^m.\end{aligned}$$

Hence, putting $X=x_1x_2\cdots x_k$, we have

$$\begin{aligned}c_{n,q}^{(k)}&=\underbrace{\int_0^1\cdots\int_0^1}_{k}(x_1x_2\cdots x_k)(x_1x_2\cdots x_k-q)\\&\quad\cdots\bigl(x_1x_2\cdots x_k-(n-1)q\bigr)\,dx_1\,dx_2\,\cdots\,dx_k\\&=\underbrace{\int_0^1\cdots\int_0^1}_{k}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}(-q)^{n-m}(x_1x_2\cdots x_k)^m\,dx_1\,dx_2\,\cdots\,dx_k\\&=\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\frac{(-q)^{n-m}}{(m+1)^k}.\end{aligned}$$

□

We also obtain the generating function of the poly-Cauchy numbers with q parameter by using the *polylogarithm factorial* function $\mathrm{Lif}_k(z)$ [17, 18] defined by

$$\mathrm{Lif}_k(z):=\sum_{m=0}^{\infty}\frac{z^m}{m!(m+1)^k}.$$

We may define the poly-Cauchy numbers with q parameter by the generating function. If $q=1$, the result is reduced to that of poly-Cauchy numbers.

Theorem 2 *The generating function of the poly-Cauchy numbers with q parameter $c_{n,q}^{(k)}$ is given by*

$$\mathrm{Lif}_k\left(\frac{\ln(1+qx)}{q}\right)=\sum_{n=0}^{\infty}c_{n,q}^{(k)}\frac{x^n}{n!}\quad(q\neq 0).$$

Proof Since

$$\frac{(\ln(1+x))^m}{m!} = (-1)^m \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!},$$

by Theorem 1 we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m}}{(m+1)^k} \frac{x^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{(-q)^{-m}}{(m+1)^k} \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-qx)^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!(m+1)^k} \left(\frac{\ln(1+qx)}{q}\right)^m = \mathrm{Lif}_k\left(\frac{\ln(1+qx)}{q}\right). \qquad \square\end{aligned}$$

The generating function of the poly-Cauchy numbers with q parameter in Theorem 2 can be also written in the form of iterated integrals as that of the poly-Cauchy numbers.

Corollary 1 *For $k = 1$, we have*

$$f_q(x) := \frac{q((1+qx)^{1/q} - 1)}{\ln(1+qx)} = \sum_{n=0}^{\infty} c_{n,q} \frac{x^n}{n!}.$$

For $k \ge 2$, we have

$$\underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k-1} \frac{f_q(x)}{1+qx} \underbrace{dx\,dx \cdots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{x^n}{n!}.$$

Proof Since

$$\mathrm{Lif}_k(z) = \frac{1}{z} \int_0^z \mathrm{Lif}_{k-1}(t)\,dt \quad (k \ge 2)$$

with $\mathrm{Lif}_1(z) = (e^z - 1)/z$, for $k \ge 2$, we have

$$\mathrm{Lif}_k(z) = \underbrace{\frac{1}{z} \int_0^z \frac{1}{z} \int_0^z \cdots \frac{1}{z} \int_0^z}_{k-1} \frac{e^z - 1}{z} \underbrace{dz\,dz \cdots dz}_{k-1}.$$

Putting $z = \ln(1+qx)/q$, we get the result for $k \ge 2$.

For $k = 1$, we have

$$\mathrm{Lif}_1\left(\frac{\ln(1+qx)}{q}\right) = \frac{q((1+qx)^{1/q} - 1)}{\ln(1+qx)}. \qquad \square$$

For $q = 1$, we have

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} c_{m,1}^{(k)} = \frac{1}{(n+1)^k}$$

[17, Theorem 3]. However, we have not had a simple form of $\sum_{m=0}^{n} \left\{ {n \atop m} \right\} c_{m,q}^{(k)}$ for general q.

3 Poly-Cauchy numbers with q parameter of the second kind

The Cauchy numbers of the second kind $\hat{c}_n$ are defined by

$$\hat{c}_n = \int_0^1 (-x)(-x-1)\cdots(-x-n+1)\,dx$$

[7, Chap. VII]. Poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1x_2\cdots x_k)(-x_1x_2\cdots x_k - 1)$$
$$\cdots(-x_1x_2\cdots x_k - n + 1)\,dx_1\,dx_2\,\cdots\,dx_k$$

[17]. Now, we define the poly-Cauchy numbers with q parameter of the second kind $c_{n,q}^{(k)}$ by

$$\hat{c}_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1x_2\cdots x_k)(-x_1x_2\cdots x_k - q)$$
$$\cdots\big(-x_1x_2\cdots x_k - (n-1)q\big)\,dx_1\,dx_2\,\cdots\,dx_k.$$

Therefore, if $q = 1$, then $\hat{c}_{n,1}^{(k)} = \hat{c}_n^{(k)}$ are the poly-Cauchy numbers of the second kind. If $q = k = 1$, then $\hat{c}_{n,1}^{(1)} = \hat{c}_n$ are the Cauchy numbers of the second kind. In addition, we shall call $\hat{c}_{n,q}^{(1)} = \hat{c}_{n,q}$ as the Cauchy numbers with q parameter of the second kind. We record the first several Cauchy numbers with q parameter of the second kind:

$$\hat{c}_{1,q} = -\frac{1}{2},$$

$$\hat{c}_{2,q} = \frac{1}{3} + \frac{1}{2}q,$$

$$\hat{c}_{3,q} = -\frac{1}{4} - q - q^2,$$
$$\hat{c}_{4,q} = \frac{1}{5} + \frac{3}{2}q + \frac{11}{3}q^2 + 3q^3,$$
$$\hat{c}_{5,q} = -\frac{1}{6} - 2q - \frac{35}{4}q^2 - \frac{50}{3}q^3 - 12q^4,$$
$$\hat{c}_{6,q} = \frac{1}{7} + \frac{5}{2}q + 17q^2 + \frac{225}{4}q^3 + \frac{274}{3}q^4 + 60q^5,$$
$$\hat{c}_{7,q} = -\frac{1}{8} - 3q - \frac{175}{6}q^2 - 147q^3 - 406q^4 - 588q^5 - 360q^6.$$

Similarly to the Poly-Cauchy numbers with q parameter of the first kind, the poly-Cauchy numbers with q parameter of the second kind can be expressed in terms of the Stirling numbers of the first kind. This is a general case of the results by Merlini et al. [21] for $k = q = 1$ and by the author [17] for $q = 1$. The proof is similar to that of Theorem 1 and omitted.

Theorem 3 *For a real number $q \neq 0$,*

$$\hat{c}_{n,q}^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{q^{n-m}}{(m+1)^k} \quad (n \geq 0,\ k \geq 1).$$

The generating function of the poly-Cauchy numbers with q parameter of the second kind $\hat{c}_{n,q}^{(k)}$ can be also expressed by using the polylogarithm factorial function $\mathrm{Lif}_k(z)$. If $q = 1$, then this generating function is reduced to that of poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)} = \hat{c}_{n,q}^{(k)}$.

Theorem 4 *The generating function of the poly-Cauchy numbers with q parameter of the second kind $\hat{c}_{n,q}^{(k)}$ is given by*

$$\mathrm{Lif}_k\left(-\frac{\ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)} \frac{x^n}{n!} \quad (q \neq 0).$$

The generating function of the poly-Cauchy numbers of the second kind can be also written in the form of iterated integrals by putting $z = -\ln(1+qx)/q$ in

$$\mathrm{Lif}_k(z) = \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \frac{e^z - 1}{z} \underbrace{dz\,dz \cdots dz}_{k-1}.$$

Corollary 2 *For $k = 1$, we have*

$$g_q(x) := \frac{q(1-(1+qx)^{-1/q})}{\ln(1+qx)} = \sum_{n=0}^{\infty} \hat{c}_{n,q} \frac{x^n}{n!}.$$

For $k \ge 2$, *we have*

$$\underbrace{\frac{q}{\ln(1+qx)}\int_0^x \frac{q}{(1+qx)\ln(1+qx)}\int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)}\int_0^x}_{k-1} \frac{g_q(x)}{1+qx}\underbrace{dx\,dx\,\cdots\,dx}_{k-1}$$

$$=\sum_{n=0}^{\infty}\hat{c}_{n,q}^{(k)}\frac{x^n}{n!}.$$

For $q=1$, we have

$$\sum_{m=0}^{n}\left\{ {n \atop m} \right\}\hat{c}_{m,1}^{(k)}=\frac{(-1)^n}{(n+1)^k}$$

[17, Theorem 6]. However, we have not had a simple form of $\sum_{m=0}^{n}\left\{ {n \atop m} \right\}\hat{c}_{m,q}^{(k)}$ for general q.

4 Poly-Cauchy polynomials with q parameter

Define the poly-Cauchy polynomials with q parameter of the first kind $c_{n,q}^{(k)}(z)$ and of the second kind $\hat{c}_{n,q}^{(k)}(z)$ by

$$c_{n,q}^{(k)}(z)=\underbrace{\int_0^1\cdots\int_0^1}_{k}(x_1x_2\cdots x_k-z)(x_1x_2\cdots x_k-q-z)$$
$$\cdots\big(x_1x_2\cdots x_k-(n-1)q-z\big)\,dx_1d\,x_2\,\cdots\,dx_k$$

and

$$\hat{c}_{n,q}^{(k)}(z)=\underbrace{\int_0^1\cdots\int_0^1}_{k}(-x_1x_2\cdots x_k+z)(-x_1x_2\cdots x_k-q+z)$$
$$\cdots\big(-x_1x_2\cdots x_k-(n-1)q+z\big)\,dx_1\,dx_2\,\cdots\,dx_k,$$

respectively. If $q=1$, then $c_{n,1}^{(k)}(z)=c_n^{(k)}(z)$ and $\hat{c}_{n,1}^{(k)}(z)=\hat{c}_n^{(k)}(z)$ are poly-Cauchy polynomials of the first kind and of the second kind, respectively [18]. Note that we also have a different definition where z and $-z$ are interchanged [18]. If $z=0$, then $c_{n,q}^{(k)}(0)=c_{n,q}^{(k)}$ and $\hat{c}_{n,q}^{(k)}(0)=\hat{c}_{n,q}^{(k)}$ are poly-Cauchy numbers with q parameter of the first kind and of the second kind, respectively.

Poly-Cauchy polynomials with q parameter of the first kind $c_{n,q}^{(k)}(z)$ and of the second kind $\hat{c}_{n,q}^{(k)}(z)$ are expressed by using the (unsigned) Stirling numbers of the first kind $\left[{n \atop m} \right]$.

Theorem 5 *For integers n and k with $n \ge 0$ and $k \ge 1$ and a real number $q \ne 0$, we have*

$$c_{n,q}^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-q)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k},$$

$$\hat{c}_{n,q}^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n q^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}.$$

Proof Similarly to the proof of Theorem 1, putting $X = x_1 x_2 \cdots x_k - z$, we have

$$c_{n,q}^{(k)}(z) = \underbrace{\int_0^1 \cdots \int_0^1}_{k} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-q)^{n-m} (x_1 x_2 \cdots x_k - z)^m \, dx_1 \, dx_2 \cdots dx_k$$

$$= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-q)^{n-m}$$

$$\times \sum_{i=0}^{m} \binom{m}{i} (x_1 x_2 \cdots x_k)^{m-i} (-z)^i \, dx_1 \, dx_2 \cdots dx_k$$

$$= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-q)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}.$$

The second identity is proven similarly and omitted. □

Theorem 6 *Let n and k be integers with $n \ge 0$ and $k \ge 1$, and q be a real number with $q \ne 0$. Then the generating functions of the poly-Cauchy polynomials with q parameter of the first kind $c_{n,q}^{(k)}(z)$ and of the second kind $\hat{c}_{n,q}^{(k)}(z)$ are given by*

$$(1+qx)^{-z/q} \mathrm{Lif}_k\left(\frac{\ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} c_{n,q}^{(k)}(z) \frac{x^n}{n!}$$

and

$$(1+qx)^{z/q} \mathrm{Lif}_k\left(-\frac{\ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)}(z) \frac{x^n}{n!},$$

respectively.

Proof Similarly to the proof of Theorem 2, by the first identity of Theorem 5 we have

$$\sum_{n=0}^{\infty} c_{n,q}^{(k)}(z) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-q)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k} \frac{x^n}{n!}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} (-q)^m \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k} \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-qx)^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\ln(1+qx)}{q} \right)^m \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k} \\
&= \sum_{i=0}^{\infty} \frac{(-z)^i}{i!} \sum_{m=i}^{\infty} \frac{1}{(m-i)!(m-i+1)^k} \left(\frac{\ln(1+qx)}{q} \right)^m \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{-z\ln(1+qx)}{q} \right)^i \sum_{\nu=0}^{\infty} \frac{1}{\nu!(\nu+1)^k} \left(\frac{\ln(1+qx)}{q} \right)^\nu \\
&= (1+qx)^{-z/q} \mathrm{Lif}_k \left(\frac{\ln(1+qx)}{q} \right).
\end{aligned}
$$

The second identity is proven similarly and omitted. □

Therefore, by Corollaries 1 and 2 with Theorem 6 we obtain the generating function of poly-Cauchy polynomials with q parameter in the form of iterated integrals. Let $f_q(x)$ and $g_q(x)$ be as in Corollaries 1 and 2, respectively.

Corollary 3 *For $k = 1$, we have*

$$
\begin{aligned}
(1+qx)^{-z/q} f_q(x) &= \sum_{n=0}^{\infty} c_{n,q}(z) \frac{x^n}{n!}, \\
(1+qx)^{z/q} g_q(x) \ln(1+qx) &= \sum_{n=0}^{\infty} \hat{c}_{n,q}(z) \frac{x^n}{n!}.
\end{aligned}
$$

For $k \geq 2$, we have

$$
\begin{aligned}
&(1+qx)^{-z/q} \underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k-1} \\
&\quad \frac{f_q(x)}{1+qx} \underbrace{dx\, dx \cdots dx}_{k-1} \\
&= \sum_{n=0}^{\infty} c_{n,q}^{(k)}(z) \frac{x^n}{n!},
\end{aligned}
$$

$$(1+qx)^{z/q}\underbrace{\frac{q}{\ln(1+qx)}\int_0^x\frac{q}{(1+qx)\ln(1+qx)}\int_0^x\cdots\frac{q}{(1+qx)\ln(1+qx)}\int_0^x}_{k-1}$$

$$\frac{g_q(x)}{1+qx)}\underbrace{dx\,dx\,\cdots\,dx}_{k-1}$$

$$=\sum_{n=0}^{\infty}\hat{c}_{n,q}^{(k)}(z)\frac{x^n}{n!}.$$

5 Some properties of poly-Cauchy numbers and polynomials with q parameter

It is known that poly-Bernoulli numbers satisfy the duality theorem $B_n^{(-k)}=B_k^{(-n)}$ for $n,k\ge 0$ [16, Theorem 2] because of the symmetric formula

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^{x+y}}{e^x+e^y-e^{x+y}}.$$

However, poly-Cauchy numbers with q parameter do not satisfy the duality theorem for any $q\ne 0$, by the following results.

Proposition 1 *For nonnegative integers n and k and a real number $q\ne 0$, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}c_{n,q}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=e^y(1+qx)^{e^y/q},$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\hat{c}_{n,q}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^y}{(1+qx)^{e^y/q}}.$$

Proof We shall prove the first identity. The second identity is proven similarly. By Theorem 2 we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}c_{n,q}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}&=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}c_{n,q}^{(-k)}\frac{x^n}{n!}\right)\frac{y^k}{k!}\\&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(m+1)^k}{m!}\left(\frac{\ln(1+qx)}{q}\right)^m\frac{y^k}{k!}\\&=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\ln(1+qx)}{q}\right)^m\sum_{k=0}^{\infty}\frac{((m+1)y)^k}{k!}\\&=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\ln(1+qx)}{q}\right)^m e^{(m+1)y}\end{aligned}$$

$$= e^y \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{e^y \ln(1+qx)}{q} \right)^m$$

$$= e^y (1+qx)^{e^y/q}. \qquad \square$$

Poly-Bernoulli polynomials $B_n^{(k)}(z)$ are defined as

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} e^{xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z) \frac{x^n}{n!}$$

[3]. Note that $B_n^{(k)}(z)$ are defined in [8] by replacing e^{xz} by e^{-xz}. Concerning the poly-Bernoulli polynomials, for an integer k and a positive integer n, we have

$$\frac{d}{dz} B_n^{(k)}(z) = n B_{n-1}^{(k)}(z)$$

[3, Theorem 1.4]. However, poly-Cauchy polynomials with q parameter are not such sequences for any $q \neq 0$. By differentiating $c_{n,q}^{(k)}(z)$ or $\hat{c}_{n,q}^{(k)}(z)$, we have the following:

Proposition 2 *For nonnegative integers n and k and a real number $q \neq 0$, we have*

$$\frac{d}{dz} c_{n,q}^{(k)}(z) = -n! \sum_{l=0}^{n-1} \frac{(-q)^{n-l-1}}{(n-l)l!} c_{l,q}^{(k)}(z),$$

$$\frac{d}{dz} \hat{c}_{n,q}^{(k)}(z) = n! \sum_{l=0}^{n-1} \frac{(-q)^{n-l-1}}{(n-l)l!} \hat{c}_{l,q}^{(k)}(z).$$

Proof Differentiating both sides of the first identity in Theorem 6 with respect to z, we have

$$-\frac{\ln(1+qx)}{q} (1+qx)^{-z/q} \mathrm{Lif}_k \left(\frac{\ln(1+qx)}{q} \right) = \sum_{n=0}^{\infty} \frac{d}{dz} c_{n,q}^{(k)}(z) \frac{x^n}{n!}.$$

Then,

$$\mathrm{LHS} = \left(\sum_{m=1}^{\infty} \frac{(-1)^m q^{m-1} x^m}{m} \right) \left(\sum_{l=0}^{\infty} c_{l,q}^{(k)}(z) \frac{x^l}{l!} \right)$$

$$= \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \frac{(-1)^{n-l} q^{n-l-1} c_{l,q}^{(k)}(z)}{(n-l)l!} x^n$$

$$= \sum_{n=1}^{\infty} (-n!) \sum_{l=0}^{n-1} \frac{(-q)^{n-l-1} c_{l,q}^{(k)}(z)}{(n-l)l!} \frac{x^n}{n!}$$

and

$$\text{RHS} = \sum_{n=1}^{\infty} \frac{d}{dz} c_{n,q}^{(k)}(z) \frac{x^n}{n!}.$$

The second identity is proven similarly and omitted. □

Lastly, we show a recurrence formula for the poly-Cauchy polynomials $c_n^{(k)}(z) = c_{n,1}^{(k)}(z)$ in terms of the poly-Cauchy numbers $c_n^{(k)} = c_{n,1}^{(k)}$ and the Cauchy polynomials $c_n(z) = c_{n,1}^{(1)}(z)$.

Theorem 7 *For integers n and k with $n \geq 0$ and $k \geq 1$, we have*

$$c_n^{(k)}(z) = (-1)^n n! \sum_{m=0}^{n} \frac{(-1)^m c_m^{(k-1)}}{m!} \sum_{l=0}^{n-m} \frac{(-1)^l c_l(z)}{(n-l+1)l!},$$

$$\hat{c}_n^{(k)}(z) = (-1)^n n! \sum_{m=0}^{n} \frac{(-1)^m \hat{c}_m^{(k-1)}}{m!} \sum_{l=0}^{n-m} \frac{(-1)^l \hat{c}_l(z)}{(n-l+1)l!}$$
$$+ (-1)^n n! \sum_{m=0}^{n-1} \frac{(-1)^m \hat{c}_m^{(k-1)}}{m!} \sum_{l=0}^{n-m-1} \frac{(-1)^{l+1} \hat{c}_l(z)}{(n-l)l!}.$$

Proof

$$\frac{d}{dz}\bigl(z\text{Lif}_k(z)\bigr) = \text{Lif}_{k-1}(z),$$

so

$$\text{Lif}_k(z) = \frac{1}{z}\int_0^z \text{Lif}_{k-1}(t)\,dt. \tag{1}$$

If we put $z = \ln(1+x)$ and $t = \log(1+s)$ in the identity, then

$$\frac{\text{Lif}_k(\ln(1+x))}{(1+x)^z} = \frac{1}{(1+x)^z \ln(1+x)} \int_0^x \frac{\text{Lif}_{k-1}(\ln(1+s))}{1+s}\,ds.$$

By the generating function in Theorem (6),

$$\sum_{n=0}^{\infty} c_n^{(k)}(z)\frac{x^n}{n!} = \sum_{n=0}^{\infty} c_n(z)\frac{x^{n-1}}{n!} \int_0^x \left(\sum_{n=0}^{\infty}(-x)^n\right)\left(\sum_{n=0}^{\infty} c_n^{(k-1)}\frac{x^n}{n!}\right) dx$$
$$= \left(\sum_{l=0}^{\infty} c_l(z)\frac{x^{l-1}}{l!}\right) \int_0^1 \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n-m}}{m!} c_m^{(k-1)} x^n\right) dx$$
$$= \left(\sum_{l=0}^{\infty} c_l(z)\frac{x^{l-1}}{l!}\right) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n-m}}{m!} c_m^{(k-1)} \frac{x^{n+1}}{n+1}$$

$$
\begin{aligned}
&=\sum_{l=0}^{\infty}\frac{c_l(z)}{l!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n-m}}{m!}c_m^{(k-1)}\frac{x^{n+l}}{n+1}\\
&=\sum_{\nu=0}^{\infty}\sum_{n=0}^{\nu}\sum_{m=0}^{n}\frac{c_m^{(k-1)}}{m!}\frac{c_{\nu-n}(z)}{(\nu-n)!}\frac{(-1)^{n-m}}{n+1}x^{\nu}\\
&=\sum_{\nu=0}^{\infty}\left(\sum_{n=0}^{\nu}\sum_{m=0}^{n}\frac{c_m^{(k-1)}}{m!}\frac{(-1)^{n-m}}{n+1}\frac{\nu!}{(\nu-n)!}c_{\nu-n}(z)\right)\frac{x^{\nu}}{\nu!}\\
&=\sum_{\nu=0}^{\infty}\left(\nu!\sum_{m=0}^{\nu}\frac{c_m^{(k-1)}}{m!}(-1)^m\sum_{n=m}^{\nu}\frac{(-1)^n}{n+1}\frac{c_{\nu-n}(z)}{(\nu-n)!}\right)\frac{x^{\nu}}{\nu!}\\
&=\sum_{\nu=0}^{\infty}\left(\nu!\sum_{m=0}^{\nu}\frac{c_m^{(k-1)}}{m!}(-1)^m\sum_{n=0}^{\nu-m}\frac{(-1)^{\nu-n}}{\nu-n+1}\frac{c_n(z)}{n!}\right)\frac{x^{\nu}}{\nu!}.
\end{aligned}
$$

The second recurrence relation is obtained similarly. □

6 Some more extensions

We shall consider integrals of the definition of poly-Cauchy numbers with q parameter in the range $[0, l]$, where l is a real number with $l \neq 0$ instead of the range $[0, 1]$. Define $c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$, where $l_1, l_2, \dots, l_k$ are nonzero real numbers, by

$$
\begin{aligned}
&c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)\\
&\quad=\int_0^{l_1}\int_0^{l_2}\cdots\int_0^{l_k}(x_1x_2\cdots x_k)(x_1x_2\cdots x_k-q)\\
&\qquad\cdots\bigl(x_1x_2\cdots x_k-(n-1)q\bigr)\,dx_1\,dx_2\,\cdots\,dx_k.
\end{aligned}
$$

Then $c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$ can be also expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$.

Theorem 8 *For a real number $q \neq 0$,*

$$
c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)=\sum_{m=0}^{n}\left[{n \atop m}\right]\frac{(-q)^{n-m}(l_1l_2\cdots l_k)^{m+1}}{(m+1)^k}\quad(n\ge 0,\ k\ge 1).
$$

The numbers $l_1, l_2, \dots, l_k$ and q are not necessarily positive integers. For example, for $k = 2$, $n = 4$, $l_1 = \sqrt{2}$ and $l_2 = -1/3$, we have

$$
c_{4,q}^{(2)}\left(\sqrt{2}, -\frac{1}{3}\right)=\int_0^{\sqrt{2}}\int_0^{-\frac{1}{3}}(x_1x_2)(x_1x_2-q)(x_1x_2-2q)(x_1x_2-3q)\,dx_1\,dx_2
$$

$$= \sum_{m=0}^{n} \begin{bmatrix} 4 \\ m \end{bmatrix} \frac{(-q)^{4-m}(\sqrt{2}(-1/3))^{m+1}}{(m+1)^2}$$

$$= -\frac{1}{3}q^3 - \frac{22\sqrt{2}}{243}q^2 - \frac{1}{54}q - \frac{4\sqrt{2}}{6075}.$$

If $k = q = 1$ in Theorem 8, then

$$c_{n,1}^{(1)}(l) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^{n-m} l^{m+1}}{m+1},$$

which is the relation in [21, Corollary 3.3].

The generating function of $c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$ is also given by using the polylogarithm factorial function $\mathrm{Lif}_k(z)$.

Theorem 9 *For a real number $q \neq 0$,*

$$l_1 l_2 \cdots l_k \cdot \mathrm{Lif}_k\left(\frac{l_1 l_2 \dots l_k \ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} c_{n,q}^{(k)}(l_1, l_2, \dots, l_k) \frac{x^n}{n!} \quad (n \geq 0,\ k \geq 1).$$

If $k = q = 1$ in Theorem 9, then

$$\sum_{n=0}^{\infty} c_{n,1}^{(1)}(l) \frac{x^n}{n!} = l \cdot \mathrm{Lif}_1\big(l \ln(1+x)\big)$$

$$= \frac{(1+x)^l - 1}{\ln(1+x)},$$

which is the relation in [21, Theorem 3.2].

The generating function of $c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$ in Theorem 9 can be also written in the form of iterated integrals.

Corollary 4 *For $k = 1$, we have*

$$f_q(l_1, \dots, l_k; x) := \frac{q((1+qx)^{l_1 l_2 \cdots l_k / q} - 1)}{\ln(1+qx)} = \sum_{n=0}^{\infty} c_{n,q}^{(1)}(l_1, l_2, \dots, l_k) \frac{x^n}{n!}.$$

For $k \geq 2$, we have

$$\underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k-1}$$

$$\frac{f_q(l_1, \dots, l_k; x)}{1+qx} \underbrace{dx\, dx \cdots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} c_{n,q}^{(k)}(l_1, l_2, \dots, l_k) \frac{x^n}{n!}.$$

In similar manners, if we define $\hat{c}_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$ by

$$\begin{aligned}\hat{c}_{n,q}^{(k)}(l_1, l_2, \dots, l_k) &= \int_0^{l_1} \int_0^{l_2} \cdots \int_0^{l_k} (-x_1 x_2 \cdots x_k)(-x_1 x_2 \cdots x_k - q) \\ &\quad \cdots \bigl(-x_1 x_2 \cdots x_k - (n-1)q\bigr)\, dx_1\, dx_2 \cdots dx_k,\end{aligned}$$

then we have the following series of results.

Theorem 10 *For a real number $q \neq 0$,*

$$\hat{c}_{n,q}^{(k)}(l_1, l_2, \dots, l_k) = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{q^{n-m}(l_1 l_2 \cdots l_k)^{m+1}}{(m+1)^k} \quad (n \geq 0,\ k \geq 1).$$

Theorem 11 *The generating function of $\hat{c}_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$ $(q \neq 0)$ is given by*

$$l_1 l_2 \cdots l_k \cdot \mathrm{Lif}_k\left(-l_1 l_2 \cdots l_k \frac{\ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)}(l_1, l_2, \dots, l_k) \frac{x^n}{n!} \quad (n \geq 0,\ k \geq 1).$$

Corollary 5 *For $k = 1$, we have*

$$g_q(l_1, \dots, l_k; x) := \frac{q(1 - (1+qx)^{-l_1 l_2 \cdots l_k / q})}{\ln(1+qx)} = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(1)}(l_1, l_2, \dots, l_k) \frac{x^n}{n!}.$$

For $k \geq 2$, we have

$$\begin{aligned}&\underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k-1} \\ &\quad \frac{g_q(l_1, \dots, l_k; x)}{1+qx} \underbrace{dx\, dx \cdots dx}_{k-1} \\ &= \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)}(l_1, l_2, \dots, l_k) \frac{x^n}{n!}.\end{aligned}$$

Polynomials $c_{n,q}^{(k)}(z; l_1, l_2, \dots, l_k)$ and $\hat{c}_{n,q}^{(k)}(z; l_1, l_2, \dots, l_k)$ are similarly defined, and their explicit formulae and generating functions are obtained by replacing the range of the integral [0, 1] by $[0, l]$.

7 Future work

There is the following relation between poly-Cauchy numbers and poly-Bernoulli numbers [17, Theorem 8]:

$$B_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)} \quad (n \geq 1).$$

However, any corresponding generalized poly-Bernoulli numbers to poly-Cauchy numbers with q parameter have not yet been studied, though one candidate may be

$$B_{n,q}^{(k)} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-q)^{n-m} m!}{(m+1)^k}.$$

On the other hand, though several generalized poly-Bernoulli numbers have been studied (see, e.g., [3, 4, 10, 11, 14, 22, 23]), the corresponding generalized poly-Cauchy numbers have not yet been studied, either. One of the reasons is that the method to generalize the poly-Cauchy numbers in this paper is based upon the definition of integrals and the methods to generalize the poly-Bernoulli numbers in other works are based upon the definition of generating functions.

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References

1. Agoh, T., Dilcher, K.: Shortened recurrence relations for Bernoulli numbers. Discrete Math. **309**, 887–898 (2009)
2. Agoh, T., Dilcher, K.: Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind. Fibonacci Q. **48**, 4–12 (2010)
3. Bayad, A., Hamahata, Y.: Polylogarithms and poly-Bernoulli polynomials. Kyushu J. Math. **65**, 15–24 (2011)
4. Bayad, A., Hamahata, Y.: Arakawa–Kaneko L-functions and generalized poly-Bernoulli polynomials. J. Number Theory **131**, 1020–1036 (2011)
5. Brewbaker, C.: A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues. Integers **8**, #A02 (2008)
6. Cheon, G.-S., Hwang, S.-G., Lee, S.-G.: Several polynomials associated with the harmonic numbers. Discrete Appl. Math. **155**, 2573–2584 (2007)
7. Comtet, L.: Advanced Combinatorics. Reidel, Dordrecht (1974)
8. Coppo, M.-A., Candelpergher, B.: The Arakawa–Kaneko zeta functions. Ramanujan J. **22**, 153–162 (2010)
9. Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics, 2nd edn. Addison-Wesley, Reading (1994)
10. Hamahata, Y., Masubuchi, H.: Special multi-poly-Bernoulli numbers. J. Integer Seq. **10**, Article 07.4.1 (2007)
11. Hamahata, Y., Masubuchi, H.: Recurrence formulae for multi-poly-Bernoulli numbers. Integers **7**, #A46 (2007)
12. Hassen, A., Nguyen, H.D.: Hypergeometric Bernoulli polynomials and Appell sequences. Int. J. Number Theory **4**, 767–774 (2008)
13. Hassen, A., Nguyen, H.D.: Hypergeometric zeta functions. Int. J. Number Theory **6**, 99–126 (2010)
14. Jolany, H.: Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a, b, c parameters. http://arxiv.org/pdf/1109.1387v1.pdf

15. Kamano, K.: Sums of products of Bernoulli numbers, including poly-Bernoulli numbers. J. Integer Seq. **13**, Article 10.5.2 (2010)
16. Kaneko, M.: Poly-Bernoulli numbers. J. Théor. Nr. Bordx. **9**, 199–206 (1997)
17. Komatsu, T.: Poly-Cauchy numbers. Kyushu J. Math. (to appear)
18. Kamano, K., Komatsu, T.: Poly-Cauchy polynomials. Preprint
19. Komatsu, T.: Hypergeometric Cauchy numbers. Int. J. Number Theory (to appear). doi:10.1142/S1793042112501473
20. Liu, H.-M., Qi, S.-H., Ding, S.-Y.: Some recurrence relations for Cauchy numbers of the first kind. J. Integer Seq. **13**, Article 10.3.8 (2010)
21. Merlini, D., Sprugnoli, R., Verri, M.C.: The Cauchy numbers. Discrete Math. **306**, 1906–1920 (2006)
22. Sasaki, Y.: On generalized poly-Bernoulli numbers and related L-functions. J. Number Theory **132**, 156–170 (2012)
23. Wang, W.: Generalized higher order Bernoulli number pairs and generalized Stirling number pairs. J. Math. Anal. Appl. **364**, 255–274 (2010)
24. Young, P.T.: A 2-adic formula for Bernoulli numbers of the second kind and for the Nörlund numbers. J. Number Theory **128**, 2951–2962 (2008)
25. Zhao, F.-Z.: Sums of products of Cauchy numbers. Discrete Math. **309**, 3830–3842 (2009)