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Another elementary proof that $p(11n+6) \equiv 0 \pmod{11}$

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Abstract In this paper we give an elementary proof of the partition congruence $p(11n+6) \equiv 0 \pmod{11}$, using only Euler's Pentagonal Number Theorem and Jacobi's Identity for $(q;q)_\infty^3$.

Keywords Partitions · Congruences · Congruential restrictions

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Let n be a positive integer, let $p(n)$, denotes the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.

In 1919, Ramanujan [1], [2, pp. 210–213] announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

He gave the proofs of the first two of the above congruences in [1] and later in a short one page note [2, p. 230] and [3] announced that he had also found a proof of the third identity above. Of the proofs given for the third identity above, the most elementary proof is due to L. Winquist [4] and uses Winquist's Identity. Another elementary approach of proving the third identity has been devised by Berndt, S.H. Chan, Z.-G. Liu, and H. Yesilyurt [5], who established a new identity for $(q;q)_\infty^{10}$. Hirschhorn [6] has devised a common approach to proving all three congruences.

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In an unpublished paper, the author proved the third congruence above in 2009. It was pointed out to us that this proof is similar to a proof by J.M. Rushforth which Berndt [8] has reproduced in 2007. In this paper, we prove the third congruence above along the same lines as the elementary proof given in [7, pp. 34–42] for the first two congruences. This is the first time such a proof has been given. First we review some q-series, and two corollaries whose proof can be found in [7].

Definition 1 Define

$$(a)_0 := (a;q)_0 := 1,$$

$$(a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad n \geq 1,$$

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad |q|<1.$$

We call q the base. The generating function for $p(n)$, due to Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty}\frac{1}{1-q^k} = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where, as usual, we define $p(0)=1$.

In the proof of the main result, we need Euler's Pentagonal Number Theorem and Jacobi's Identity.

Corollary 1 (Euler's Pentagonal Number Theorem) *We have*

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} = (q;q)_\infty. \tag{1.2}$$

Corollary 2 (Jacobi's Identity) *We have*

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = (q;q)_\infty^3. \tag{1.3}$$

This is all we need to prove our main theorem.

Theorem 1 *We have*

$$p(11n+6) \equiv 0 \pmod{11}.$$

Proof Using Corollary 1 and some elementary calculations on the exponents, it is easy to see that

$$\frac{(q;q)_\infty}{(q^{121};q^{121})_\infty} = a + bq + cq^2 + eq^4 + q^5 + fq^7, \tag{1.4}$$

where a, b, c, e, and f are power series in q^{11} with integer coefficients. Also it is easy to see using the Pentagonal Number Theorem that the contribution from $(q;q)_\infty$ when $n \equiv 2 \pmod{11}$ is $q^5(q^{121};q^{121})_\infty$, which counts for the term q^5 in (1.4). Let

$$L = a(q^{11}) + b(q^{11})q + c(q^{11})q^2 + e(q^{11})q^4 + q^5 + f(q^{11})q^7. \tag{1.5}$$

Using the fact that, for integers x and y, $(x+y)^{11} \equiv x^{11} + y^{11} \pmod{11}$, it is easy to see that

$$\frac{(q^{11};q^{11})_\infty^{12}}{(q^{121};q^{121})_\infty^{12}} \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} \equiv \frac{(q:q)_\infty^{10}}{(q^{121};q^{121})^{10}} \equiv L^{10} \pmod{11}, \tag{1.6}$$

where we have used (1.4) and (1.5).

Using (1.1) we have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{(q^{121};q^{121})^{11}}{(q^{11};q^{11})^{12}} \frac{(q^{11};q^{11})_\infty^{12}}{(q^{121};q^{121})_\infty^{12}} \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty}.$$

Using the latter and (1.6), we have

$$\sum_{n=0}^{\infty} p(n)q^n = J'(q^{11})L^{10} + 11J. \tag{1.7}$$

As in (1.4), but instead using Corollary 2, it is easy to show

$$\frac{(q;q)_\infty^3}{(q^{121};q^{121})_\infty^3} = M = A + Bq + Cq^3 - 11q^{15} + Eq^6 + Fq^{10}, \tag{1.8}$$

where A, B, C, E, and F are power series in q^{11} with integer coefficients. Using (1.4) and (1.8) it follows that $M = L^3$. So it follows that $L^{10} = LM^3$. Using the latter in (1.7) and doing some expansions and equating the coefficients of $11n+6$ powers on both sides and dividing both sides by q^6. We have

$$\sum_{n=0}^{\infty} p(11n+6)q^{11n} = J'(q^{11})H(q^{11}) + 11J, \tag{1.9}$$

where

$$\begin{aligned} H = {} & \big(3EF^2c + 3CF^2 + 3BF^2f\big)q^{22} \\ & + \big(6ACFe + 3C^2E + 3C^2Fb + 3BE^2e + 6BCEf + 6BEFa + 3AE^2 \\ & + 3B^2F + 6AEFb + 3CE^2c + 3A^2Ff\big)q^{11} \\ & + 3A^2B + 6ABCc + 3AB^2e + 3A^2Ea + 3B^2Cb + 3AC^2a. \end{aligned} \tag{1.10}$$

Using the fact that $L^3 = M$ and using the expressions for L and M, it follows by equating coefficients,

$$A = \left(3b + 6ce + 3c^2 f + 6aef\right)q^{11} + a^3, \tag{1.11}$$

$$B = \left(e^3 + 6bef + 3c + 6af\right)q^{11} + 3a^2 b, \tag{1.12}$$

$$C = \left(3af^2 + 6cf + 3e\right)q^{11} + 6abc + b^3, \tag{1.13}$$

$$E = 3fq^{11} + 6ab + 3b^2 e + c^3 + 6ace, \tag{1.14}$$

and

$$F = f^3 q^{11} + 3ce^2 + 6bcf + 6be + 3a. \tag{1.15}$$

Also we have the identities

$$(6cef + 3e^2 + 6bf)q^{11} + 3a^2 c + 3ab^2 = 0, \tag{1.16}$$

$$(12 + 3bf^2 + 3e^2 f)q^{11} + 3a^2 e + 3b^2 c + 3ac^2 = 0, \tag{1.17}$$

$$(2ef + cf^2)q^{11} + bc^2 + 2abe + a^2 = 0, \tag{1.18}$$

$$ef^2 q^{11} + \left(b^2 + 2ac + a^2 f + 2bce\right) = 0, \tag{1.19}$$

$$2ae + b^2 f + 2acf + c^2 + be^2 = 0, \tag{1.20}$$

and

$$f^2 q^{11} + \left(2abf + 2bc + c^2 e + ae^2\right) = 0. \tag{1.21}$$

Using Maple we substitute (1.11)–(1.15) in (1.10), and simplify to obtain an expression too big to display here. Then again using Maple and (1.16)–(1.21) in the latter expression to simplify, the following expression is obtained for H:

$$\begin{aligned}
H = {} & \left(-\frac{1197042}{13} f^{34} e^2 - \frac{1197042}{13} f^{33}\right) q^{110} \\
& + \left(-\frac{1197042}{13} f^{29} c - \frac{9576336}{13} f^{29} e^3 - \frac{3591126}{13} f^{28} e \right. \\
& \left. - \frac{2394084}{13} f^{30} e^5 - \frac{1197042}{13} f^{31} e^7\right) q^{99} \\
& + \left(\frac{35950530}{13} f^{22} - \frac{5985210}{13} e^8 f^{26} - \frac{11970420}{13} f^{24} e^4 \right. \\
& \left. - \frac{9576336}{13} f^{25} e^6 + \frac{31162362}{13} f^{23} e^2 - \frac{1197042}{13} e^{10} f^{27}\right) q^{88} \\
& + \left(\frac{263284560}{13} f^{18} e^3 + \frac{96293274}{13} f^{17} e - \frac{4788168}{13} e^9 f^{21} + \frac{34753488}{13} f^{18} c \right. \\
& \left. + \frac{26374194}{13} f^{20} e^7 + \frac{59139696}{13} f^{19} e^5 - \frac{1197042}{13} e^{11} f^{22}\right) q^{77}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{161418180}{13}e^8f^{15}+\frac{294848862}{13}f^{12}e^2+\frac{441732522}{13}f^{13}e^4\right.\\
&\left.+\frac{255013374}{13}f^{14}e^6+\frac{29174376}{13}e^{10}f^{16}+\frac{53420136}{13}f^{11}\right)q^{66}\\
&+\left(\frac{38837568}{13}e^{11}f^{11}+\frac{346896396}{13}e^3f^7+6468f^7c+\frac{243154758}{13}e^9f^{10}\right.\\
&\left.+\frac{71876574}{13}ef^6+\frac{471666888}{13}f^9e^7+\frac{577282398}{13}f^8e^5\right)q^{55}\\
&+\left(7177170e^4f^2+5281584e^{10}f^5+1428042e^2f\right.\\
&\left.+13870164e^6f^3+924+12825582f^4e^8\right)q^{44}\\
&+\left(-725802e^{13}f-1025640e^{15}f^2-7854c^5f^2-152922e^{19}f^4\right.\\
&\left.-645876e^{17}f^3-193116e^{11}\right)q^{33}+3696q^{22}c^8f-462c^{11}q^{11}.
\end{aligned}
\tag{1.22}
$$

It is easy using (1.22) to conclude that

$$
H=11J''\left(q^{11}\right). \tag{1.23}
$$

The result now follows from (1.9) and (1.23). □

It is also easy to use this method and give proofs of other congruences mod 5 and 7. It remains to be seen whether H can be simplified further.

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