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Sums of cubes of primes in short intervals

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Abstract In this paper, we are able to sharpen Hua's result by proving that almost all integers satisfying some necessary congruence conditions can be represented as

$$N = p_1^3 + \cdots + p_s^3 \quad \text{with} \left| p_j - \sqrt[3]{\frac{N}{s}} \right| \leqslant U,\ j = 1, \ldots, s,$$

where p_j are primes and $U = N^{\frac{1}{3}-\delta_s+\varepsilon}$ with $\delta_s = \frac{s-4}{6s+72}$, where $s = 5, 6, 7, 8$.

Keywords The additive theory of prime numbers · Short intervals · Circle method

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1 Introduction

It is conjectured that all sufficiently large integers satisfying some necessary congruence conditions are sums of four cubes of primes. Such a result seems out of reach at present. The best record in this direction is due to Hua [2] who proved in 1938 that:

- All sufficiently large odd integers are sums of nine cubes of primes.
- Almost all integers satisfying some necessary congruence conditions are sums of s cubes of primes, where $s = 5, 6, 7, 8$.

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More precisely, define the subsets $\mathcal{N}_s$ of $\mathbb{N}$ by

$$\begin{aligned}
\mathcal{N}_5 &= \{n \in \mathbb{N} : n \equiv 1 (\bmod 2), n \not\equiv 0, \pm 2 (\bmod 9), n \not\equiv 0 (\bmod 7)\},\\
\mathcal{N}_6 &= \{n \in \mathbb{N} : n \equiv 0 (\bmod 2), n \not\equiv \pm 1 (\bmod 9)\},\\
\mathcal{N}_7 &= \{n \in \mathbb{N} : n \equiv 1 (\bmod 2), n \not\equiv 0 (\bmod 9)\},\\
\mathcal{N}_s &= \{n \in \mathbb{N} : n \equiv 0 (\bmod 2)\} \quad (s \geqslant 8).
\end{aligned}$$

Let $E_s(N)$ denote the number of integers $n \in \mathcal{N}_s$, not exceeding N that cannot be written as sums of s cubes of primes. Then Hua's second result actually states that $E_s(N) \ll NL^{-A}$ for some positive A, where $L = \log N$, $s = 5, 6, 7, 8$.

In this paper, we shall consider the above problem in short intervals,

$$\begin{cases} n = p_1^3 + \cdots + p_s^3, \\ |p_j - \sqrt[3]{\frac{N}{s}}| \leqslant U, \quad j = 1, \ldots, s, \end{cases} \tag{1.1}$$

where $s = 5, 6, 7, 8$ and p_j are primes. Let $E_s(N, U)$ denote the number of integers $n \in \mathcal{N}_s$, $N \leqslant n \leqslant N + N^{\frac{2}{3}} U$, which cannot be represented as (1.1). Our results are the following.

Theorem 1 *For* $U = N^{\frac{1}{3} - \delta_s + \varepsilon}$ *with* $\delta_s = \frac{s-4}{6s+72}$, *we have* $E_s(N, U) \ll N^{\frac{2}{3}} U^{1-\varepsilon}$, $s = 5, 6, 7, 8$.

We will prove Theorem 1 by circle method and others. Similar approach was used in [9] to prove that almost all large integers can be represented as sums of four almost equal squares of primes. The proof depends on the iterative method in Liu [6], the new estimates for Dirichlet polynomials in Choi and Kumchev [1], and the new exponential sums estimates in Liu, Lü and Zhan [7].

Theorem 2 *For* $U = N^{\frac{1}{3} - \delta_7' + \varepsilon}$ *with* $\delta_7' = \frac{1}{150}$, *we have* $E_7(N, U) \ll N^{\frac{1}{3}} U^{1-\varepsilon}$. *Furthermore, for* $U = N^{\frac{1}{3} - \delta_8' + \varepsilon}$ *with* $\delta_8' = \frac{1}{198}$, *we have* $E_8(N, U) \ll U^{1-\varepsilon}$.

For the absence of short intervals, the exceptional sets for sums of seven and eight cubes of primes are much smaller than those for five and six such cubes. Obviously in Theorem 1, we pay our main attention to the size of U, namely to finding how small U that we can take on the premise that almost all such integers can be written as (1.1). While in Theorem 2, we are not only interested in the size of U, but also concerned with the cardinality of $E_s(N, U)$.

2 Outline of the method and proof of Theorem 1

Let N be a sufficiently large integer and $n \in \mathcal{N}_s$ satisfying $N \leqslant n \leqslant N + N^{\frac{2}{3}} U$. Let

$$R(n, U) = \sum_{\substack{n = p_1^3 + \cdots + p_s^3 \\ |p_j - \sqrt[3]{\frac{N}{s}}| \leqslant U}} (\log p_1) \cdots (\log p_s),$$

where $U = N^{\frac{s+28}{6s+72}+\varepsilon}$ and p_j are primes. For

$$N_1 = \sqrt[3]{\frac{N}{s}} - U, \qquad N_2 = \sqrt[3]{\frac{N}{s}} + U,$$

we define

$$S(\alpha) = \sum_{N_1 < p \leqslant N_2} (\log p) e\big(p^3 \alpha\big), \quad e(z) = e^{2\pi i z}, \alpha \in [0, 1].$$

Then we have

$$R(n, U) = \int_0^1 S^s(\alpha) e(-n\alpha)\, \mathrm{d}\alpha. \tag{2.1}$$

In order to apply the circle method, we set

$$P_0 = \big(N^{\frac{1}{3}} U^{-1}\big)^{\frac{16}{s-4}} N^{11\varepsilon}, \qquad Q_0 = U^{5-11\varepsilon} N^{-\frac{2}{3}} \big(N^{\frac{1}{3}} U^{-1}\big)^{-\frac{16}{s-4}}.$$

By Dirichlet's Lemma on rational approximations for each $\alpha \in [\frac{1}{Q_0}, 1 + \frac{1}{Q_0}]$ there are coprime integers a, q satisfying $1 \leqslant a \leqslant q \leqslant Q_0$ and

$$\alpha = \frac{a}{q} + \lambda, \qquad |\lambda| \leqslant \frac{1}{q Q_0}. \tag{2.2}$$

We denote by $\mathfrak{m}(a, q)$ the set of all α satisfying (2.2), and define the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{m} = \bigcup_{P_0 \leqslant q \leqslant Q_0} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{m}(a, q).$$

To define the major arcs, we set

$$P = \big(N^{\frac{1}{3}} U^{-1}\big)^{\frac{2}{s-4}} N^{\varepsilon}, \qquad Q = N^{\frac{31}{36}+2\varepsilon}. \tag{2.3}$$

Then the major arcs $\mathfrak{M}$ are defined as the union of all intervals $[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}]$ with $1 \leqslant a \leqslant q \leqslant P$. Let the intermediate arcs $\mathfrak{l}$ be the complement of $\mathfrak{M}$ and $\mathfrak{m}$ in $[\frac{1}{Q_0}, 1 + \frac{1}{Q_0}]$, so that $[\frac{1}{Q_0}, 1 + \frac{1}{Q_0}] = \mathfrak{M} \cup \mathfrak{l} \cup \mathfrak{m}$, and consequently (2.1) becomes

$$R(n, U) = \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m} \cup \mathfrak{l}} \right\} S^s(\alpha) e(-n\alpha)\, \mathrm{d}\alpha. \tag{2.4}$$

We shall establish the following asymptotic formula on the major arcs $\mathfrak{M}$ in the next section.

Lemma 2.1 *Let $n \in \mathcal{N}_s$ satisfying $N \leqslant n \leqslant N + N^{\frac{2}{3}} U$. Let $Q \geq N^{\frac{31}{36}+\varepsilon}$ and $PQ \leq U N^{\frac{2}{3}} L^{-A}$, and the major arcs $\mathfrak{M}$ be defined as above. Then for any $A > 0$,*

$$\int_{\mathfrak{M}} S^s(\alpha) e(-n\alpha)\, \mathrm{d}\alpha = \frac{1}{3^s} \mathfrak{S}(n) \mathfrak{J}(n) + O\big(U^{s-1} N^{-\frac{2}{3}} L^{-A}\big),$$

where

$$\mathfrak{J}(n) := \sum_{\substack{m_1+\cdots+m_s=n \\ N_1^3<m_j\leqslant N_2^3}} (m_1\cdots m_s)^{-\frac{2}{3}} \asymp U^{s-1}N^{-\frac{2}{3}},$$

and $\mathfrak{S}(n)$ *is the singular series defined in* (3.3) *which converges and satisfies* $\mathfrak{S}(n) \gg 1$ *for* $n \in \mathcal{N}_s$.

Next we first estimate $S(\alpha)$ on $\mathfrak{m}\cup\mathfrak{l}$. By Lemma 2.1 in [8] we know that

$$\sup_{\alpha\in\mathfrak{m}} \left|S(\alpha)\right| \ll U^{1+\varepsilon}\left(P_0^{-\frac{1}{16}} + \frac{N^{\frac{1}{96}}}{U^{\frac{1}{16}}} + \frac{N^{\frac{1}{15}}}{U^{\frac{1}{4}}} + \frac{Q_0^{\frac{1}{16}}N^{\frac{1}{24}}}{U^{\frac{5}{16}}}\right) \ll U^{1-\varepsilon}\left(UN^{-\frac{1}{3}}\right)^{\frac{1}{s-4}}. \tag{2.5}$$

To estimate $S(\alpha)$ on the intermediate arcs $\mathfrak{l}$, we quote the following results in [7].

Lemma 2.2 *Let* $k\geqslant 1, 2\leqslant y\leqslant x$ *and* $\alpha = a/q+\lambda$ *subject to* $1\leqslant a\leqslant q,\ (a,q)=1$. *We have*

$$\sum_{x<p\leqslant x+y} (\log p)e\left(p^k\alpha\right) \ll (qx)^{\varepsilon}\left\{\frac{q^{\frac{1}{2}}y\Xi^{\frac{1}{2}}}{x^{\frac{1}{2}}} + q^{\frac{1}{2}}x^{\frac{1}{2}}\Xi^{\frac{1}{6}} + y^{\frac{1}{2}}x^{\frac{3}{10}} + \frac{x^{\frac{4}{5}}}{\Xi^{\frac{1}{6}}} + \frac{x}{q^{\frac{1}{2}}\Xi^{\frac{1}{2}}}\right\},$$

where $\Xi = |\lambda|x^k + x^2y^{-2}$.

To bound $S(\alpha)$ on $\mathfrak{l}$, we further write $\mathfrak{l} = \mathfrak{l}_1\cup\mathfrak{l}_2$, where

$$\mathfrak{l}_1 = \left\{\alpha : 1\leqslant q\leqslant P, \frac{1}{qQ} < |\lambda| \leqslant \frac{1}{qQ_0}\right\},$$
$$\mathfrak{l}_2 \subset \left\{\alpha : P<q\leqslant P_0, \qquad |\lambda| \leqslant \frac{1}{qQ_0}\right\}.$$

For $\alpha\in\mathfrak{l}_1$, by Lemma 2.2, we have

$$\begin{aligned}\sup_{\alpha\in\mathfrak{l}_1}\left|S(\alpha)\right| &\ll N^{\varepsilon}\left\{\frac{U\sqrt{q|\lambda|N}}{N^{\frac{1}{6}}} + N^{\frac{1}{6}}q^{\frac{1}{2}}(|\lambda|N)^{\frac{1}{6}} + N^{\frac{1}{10}}U^{\frac{1}{2}}\right.\\ &\quad\left. + \frac{N^{\frac{4}{15}}}{(|\lambda|N)^{\frac{1}{6}}} + \frac{N^{\frac{1}{3}}}{\sqrt{q|\lambda|N}}\right\} + N^{\frac{1}{6}}\\ &\ll U^{1-\varepsilon}\left(UN^{-\frac{1}{3}}\right)^{\frac{1}{s-4}}.\end{aligned}$$

For $\alpha\in\mathfrak{l}_2$, we have

$$\Xi \asymp |\lambda|N + N^{\frac{2}{3}}U^{-2} \gg N^{\frac{2}{3}}U^{-2}, \qquad q\Xi \ll Q_0^{-1}N + P_0N^{\frac{2}{3}}U^{-2} \ll Q_0^{-1}N.$$

Thus by Lemma 2.2,

$$\begin{aligned}\sup_{\alpha\in\mathfrak{l}_2}\left|S(\alpha)\right| &\ll N^{\varepsilon}\left\{\frac{U(q\Xi)^{\frac{1}{2}}}{N^{\frac{1}{6}}} + N^{\frac{1}{6}}q^{\frac{1}{3}}(q\Xi)^{\frac{1}{6}} + N^{\frac{1}{10}}U^{\frac{1}{2}} + \frac{N^{\frac{4}{15}}}{\Xi^{\frac{1}{6}}} + \frac{N^{\frac{1}{3}}}{(P\Xi)^{\frac{1}{2}}}\right\} + N^{\frac{1}{6}}\\ &\ll U^{1-\varepsilon}\left(UN^{-\frac{1}{3}}\right)^{\frac{1}{s-4}}.\end{aligned}$$

These estimates combined with (2.5) give us

$$\sup_{\alpha\in\mathfrak{m}\cup\mathfrak{l}} |S(\alpha)| \ll U^{1-\varepsilon}\big(UN^{-\frac{1}{3}}\big)^{\frac{1}{s-4}}. \tag{2.6}$$

Now we can establish Theorem 1.

Proof of Theorem 1 By Bessel's inequality, we have

$$\begin{aligned}
\sum_{N\leqslant n\leqslant N+N^{\frac{2}{3}}U} \left|\int_{\mathfrak{m}\cup\mathfrak{l}} S^s(\alpha)e(-n\alpha)\,\mathrm{d}\alpha\right|^2 &\ll \int_{\mathfrak{m}\cup\mathfrak{l}} |S(\alpha)|^{2s}\,\mathrm{d}\alpha \\
&\ll \Big(\sup_{\alpha\in\mathfrak{m}\cup\mathfrak{l}} |S(\alpha)|\Big)^{2s-8}\int_0^1 |S(\alpha)|^8\,\mathrm{d}\alpha \\
&\ll \Big(U^{1-\varepsilon}\big(UN^{-\frac{1}{3}}\big)^{\frac{1}{s-4}}\Big)^{2s-8}U^{5+\frac{\varepsilon}{2}} \\
&\ll U^{2s-1-\frac{3\varepsilon}{2}}N^{-\frac{2}{3}},
\end{aligned}$$

where we have used (2.6) and

$$\int_0^1 |S(\alpha)|^8\,\mathrm{d}\alpha \ll U^{5+\frac{\varepsilon}{2}}.$$

This can be established by a very similar argument with Hua's estimate (see Lemma 4.1 in [5] for example).

Therefore, for all sufficiently large integers $n\in\mathcal{N}_s$ satisfying $N\leqslant n\leqslant N+N^{\frac{2}{3}}U$, with at most $O(N^{\frac{2}{3}}U^{1-\varepsilon})$ exceptions,

$$\int_{\mathfrak{m}\cup\mathfrak{l}} S^s(\alpha)e(-n\alpha)\,\mathrm{d}\alpha \ll U^{s-1-\frac{\varepsilon}{4}}N^{-\frac{2}{3}}. \tag{2.7}$$

Consequently, Theorem 1 follows from (2.4), (2.7) and Lemma 2.1. □

3 Proof of Lemma 2.1

In this section, we apply the iterative idea in [6] to establish Lemma 2.1.

For $q\leqslant P$ and $N_1<p\leqslant N_2$, we write

$$S\Big(\frac{a}{q}+\lambda\Big) = \frac{C(q,a)}{\varphi(q)}V(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C(\chi,a)W(\chi,\lambda),$$

where

$$C(\chi,a) = \sum_{h=1}^{q}\overline{\chi}(h)e\Big(\frac{ah^3}{q}\Big), \qquad C(q,a) = C\big(\chi^0,a\big),$$

$$V(\lambda) = \sum_{N_1<m\leqslant N_2} e\big(m^3\lambda\big), \qquad W(\chi,\lambda) = \sum_{N_1<p\leqslant N_2} (\log p)\chi(p)e\big(p^3\lambda\big) - \delta_\chi V(\lambda),$$

where $\delta_\chi = 1$ or 0 according as χ is the principal character or not. Thus,

$$\int_{\mathfrak{M}} S^s(\alpha)e(-n\alpha)\,\mathrm{d}\alpha = I_0 + sI_1 + \frac{s(s-1)}{2}I_2 + \cdots + \frac{s(s-1)}{2}I_{s-2} + sI_{s-1} + I_s, \tag{3.1}$$

where

$$I_j = \sum_{q\leqslant P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{C^{s-j}(q,a)}{\varphi^s(q)}e\left(-\frac{an}{q}\right) \times \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} V^{s-j}(\lambda)\left\{\sum_{\chi \bmod q} C(\chi,a)W(\chi,\lambda)\right\}^j e(-n\lambda)\,\mathrm{d}\lambda. \tag{3.2}$$

We shall prove that I_0 gives the main term and $I_1, \ldots, I_s$ contribute to the error term.

We first compute the main term I_0. Define

$$B(n,q,\chi_1,\ldots,\chi_s) = \sum_{\substack{a=1\\(a,q)=1}}^{q} C(\chi_1,a)\cdots C(\chi_s,a)e\left(-\frac{an}{q}\right),$$
$$B(n,q) = B\left(n,q,\chi^0,\ldots,\chi^0\right),$$

and

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty}\frac{B(n,q)}{\varphi^s(q)}. \tag{3.3}$$

Then $\mathfrak{S}(n)$ converges and satisfies $\mathfrak{S}(n) \gg 1$ for $n \in \mathcal{N}_s$. Note that

$$I_0 = \sum_{q\leqslant P}\frac{B(n,q)}{\varphi^s(q)}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} V^s(\lambda)e(-n\lambda)\,\mathrm{d}\lambda. \tag{3.4}$$

Applying Lemma 8.8 in [3] to $V(\lambda)$ we get

$$\begin{aligned} V(\lambda) &= \int_{N_1}^{N_2} e\left(\lambda u^3\right)\mathrm{d}u + O(1) \\ &= \frac{1}{3}\int_{N_1^3}^{N_2^3} v^{-\frac{2}{3}}e(\lambda v)\,\mathrm{d}v + O(1) \\ &= \frac{1}{3}\sum_{N_1^3\leqslant m\leqslant N_2^3} m^{-\frac{2}{3}}e(\lambda m) + O(1). \end{aligned}$$

Substituting this into I_0 and extending the integral to $[-\frac{1}{2}, \frac{1}{2}]$, we have

$$\begin{aligned} I_0 &= \frac{1}{3^s}\sum_{q\leqslant P}\frac{B(n,q)}{\varphi^s(q)}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left(\sum_{N_1^3\leqslant m\leqslant N_2^3} m^{-\frac{2}{3}}e(\lambda m)\right)^s e(-n\lambda)\,\mathrm{d}\lambda \\ &\quad + O\left(\sum_{q\leqslant P}\frac{B(n,q)|}{\varphi^s(q)}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left|\sum_{N_1^3\leqslant m\leqslant N_2^3} m^{-\frac{2}{3}}e(\lambda m)\right|^{s-1}\mathrm{d}\lambda\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3^s} \sum_{q \leqslant P} \frac{B(n,q)}{\varphi^s(q)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Bigg(\sum_{N_1^3 \leqslant m \leqslant N_2^3} m^{-\frac{2}{3}} e(\lambda m) \Bigg)^s e(-n\lambda)\, \mathrm{d}\lambda \\
&\quad + O\Bigg(\sum_{q \leqslant P} \frac{|B(n,q)|}{\varphi^s(q)} \int_{\frac{1}{qQ}}^{\frac{1}{2}} \Bigg| \sum_{N_1^3 \leqslant m \leqslant N_2^3} m^{-\frac{2}{3}} e(\lambda m) \Bigg|^s \mathrm{d}\lambda \Bigg) \\
&\quad + O\Bigg(\sum_{q \leqslant P} \frac{|B(n,q)|}{\varphi^s(q)} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \Bigg| \sum_{N_1^3 \leqslant m \leqslant N_2^3} m^{-\frac{2}{3}} e(\lambda m) \Bigg|^{s-1} \mathrm{d}\lambda \Bigg).
\end{aligned}
$$

To bound the O-terms, we use the elementary estimates

$$
\sum_{N_1^3 \leqslant m \leqslant N_2^3} m^{-\frac{2}{3}} e(\lambda m) \ll \min\Bigg(U, \frac{1}{N^{\frac{2}{3}} \|\lambda\|} \Bigg),
$$

thus

$$
\begin{aligned}
I_0 &= \frac{1}{3^s} \mathfrak{J}(n) \sum_{q \leqslant P} \frac{B(n,q)}{\varphi^s(q)} + O\big(N^{-\frac{2s}{3}} (PQ)^{s-1}\big) + O\big(U^{s-2} N^{-\frac{2}{3}}\big) \\
&= \frac{1}{3^s} \mathfrak{J}(n) \mathfrak{S}(n) + O\big(U^{s-1} N^{-\frac{2}{3}} L^{-A}\big)
\end{aligned}
$$

holds for any $A > 0$, where

$$
\mathfrak{J}(n) := \sum_{\substack{m_1 + \cdots + m_s = n \\ N_1^3 < m_j \leqslant N_2^3}} (m_1 \cdots m_s)^{-\frac{2}{3}} \asymp U^{s-1} N^{-\frac{2}{3}}, \tag{3.5}
$$

and $\mathfrak{S}(n)$ is defined by (3.3) and satisfies $\mathfrak{S}(n) \gg 1$ for $n \in \mathcal{N}_s$. By (3.1), (3.4) and (3.5), to prove Lemma 2.1, we only need to prove

$$
I_j \ll U^{s-1} N^{-\frac{2}{3}} L^{-A}, \quad j = 1, \ldots, s. \tag{3.6}
$$

For simplicity, we only prove (3.6) for I_s, the most complicated one. Define

$$
\begin{aligned}
J(g) &= \sum_{r \leqslant P} [g, r]^{-\frac{3}{2}+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda| \leqslant \frac{1}{rQ}} \big| W(\chi, \lambda) \big|, \\
K(g) &= \sum_{r \leqslant P} [g, r]^{-\frac{3}{2}+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \Bigg(\int_{-\frac{1}{rQ}}^{\frac{1}{rQ}} \big| W(\chi, \lambda) \big|^2 \mathrm{d}\lambda \Bigg)^{\frac{1}{2}},
\end{aligned}
$$

where $\sum_{\chi \bmod r}^*$ denotes the sum over all primitive characters modulo r.

The proof of Lemma 2.1 depends on the following four lemmas.

Lemma 3.1 *Let χ_j mod r_j with $j = 1, \ldots, s$ be primitive characters, χ^0 mod q the principal character and $r_0 = [r_1, \ldots, r_s]$. Then we have*

$$
\sum_{\substack{q \leqslant x \\ r_0 | q}} \frac{|B(n, q, \chi_1 \chi^0, \ldots, \chi_s \chi^0)|}{\varphi^5(q)} \ll r_0^{-\frac{3}{2}+\varepsilon} \log^c x.
$$

Lemma 3.2 *Let $Q \geq N^{\frac{31}{36}+\varepsilon}$ and $PQ \leq UN^{\frac{2}{3}}L^{-A}$. We have*

$$J(g) \ll g^{-\frac{3}{2}+\varepsilon}UL^c.$$

Lemma 3.3 *Let $Q \geq N^{\frac{31}{36}+\varepsilon}$ and $PQ \leq UN^{\frac{2}{3}}L^{-A}$. For $g = 1$ we further have for any $B > 0$*

$$J(1) \ll UL^{-B}.$$

Lemma 3.4 *Let $Q \geq N^{\frac{31}{36}+\varepsilon}$ and $PQ \leq UN^{\frac{2}{3}}L^{-A}$. We have*

$$K(g) \ll g^{-\frac{3}{2}+\varepsilon}U^{\frac{1}{2}}N^{-\frac{1}{3}}L^c.$$

The proofs of these lemmas are standard now (for example, Lemma 3.1 see [4], and others see [8]). However, for completeness we will give a proof of Lemma 3.2 in Sect. 4.

Now we can prove (3.6) for I_s. Reducing the characters in I_s into primitive characters and then applying Lemma 3.1, we have

$$\begin{aligned}
I_s &\ll \sum_{\substack{r_j \leqslant P \\ j=1,\ldots,s}} \sum_{\substack{\chi_j \bmod r_j \\ j=1,\ldots,s}}{}^{*} \sum_{\substack{q \leqslant P \\ r_0 | q}} \frac{|B(n,q,\chi_1\chi^0,\ldots,\chi_s\chi^0)|}{\varphi^s(q)} \\
&\quad \times \int_{-1/(r_0Q)}^{1/(r_0Q)} \big|W(\chi_1,\lambda)\big| \cdots \big|W(\chi_s,\lambda)\big| \,\mathrm{d}\lambda \\
&\ll L^c \sum_{\substack{r_j \leqslant P \\ j=1,\ldots,s}} r_0^{-\frac{3}{2}+\varepsilon} \sum_{\substack{\chi_j \bmod r_j \\ j=1,\ldots,s}}{}^{*} \int_{-1/(r_0Q)}^{1/(r_0Q)} \big|W(\chi_1,\lambda)\big| \cdots \big|W(\chi_s,\lambda)\big| \,\mathrm{d}\lambda.
\end{aligned}$$

Then, by Cauchy's inequality, we get

$$\begin{aligned}
I_s &\ll L^c \sum_{r_1 \leqslant P} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda| \leqslant \frac{1}{r_1Q}} \big|W(\chi_1,\lambda)\big| \\
&\quad \times \cdots \times \sum_{r_{s-2} \leqslant P} \sum_{\chi_{s-2} \bmod r_{s-2}}{}^{*} \max_{|\lambda| \leqslant \frac{1}{r_{s-2}Q}} \big|W(\chi_{s-2},\lambda)\big| \\
&\quad \times \sum_{r_{s-1} \leqslant P} \sum_{\chi_{s-1} \bmod r_{s-1}}{}^{*} \left(\int_{-\frac{1}{r_{s-1}Q}}^{\frac{1}{r_{s-1}Q}} \big|W(\chi_{s-1},\lambda)\big|^2 \,\mathrm{d}\lambda \right)^{\frac{1}{2}} \\
&\quad \times \sum_{r_s \leqslant P} r_0^{-\frac{3}{2}+\varepsilon} \sum_{\chi_s \bmod r_s}{}^{*} \left(\int_{-\frac{1}{r_sQ}}^{\frac{1}{r_sQ}} \big|W(\chi_s,\lambda)\big|^2 \,\mathrm{d}\lambda \right)^{\frac{1}{2}}.
\end{aligned}$$

Following the iterative procedure in [6], we apply Lemma 3.4 to the sums over r_s, r_{s-1}, Lemma 3.2 to the sums over $r_{s-2},\ldots,r_2$, and Lemma 3.3 to the sum over r_1 consecutively, and finally obtain

$$I_s \ll U^{s-1}N^{-\frac{2}{3}}L^{-A},$$

for any $A > 0$. This proves (3.6) for I_s. □

4 Proof of Lemma 3.2

Let

$$\widehat{W}(\chi,\lambda)=\sum_{N_1<m\leqslant N_2}\big(\Lambda(m)\chi(m)-\delta_\chi\big)e\big(m^3\lambda\big).$$

Then

$$W(\chi,\lambda)-\widehat{W}(\chi,\lambda)\ll N^{\frac{1}{6}}.$$

Therefore we have

$$\begin{aligned}&\sum_{r\leqslant P}[g,r]^{-\frac{3}{2}+\varepsilon}\sum_{\chi \bmod r}{}^{*}\max_{|\lambda|\leqslant\frac{1}{rQ}}\big|W(\chi,\lambda)-\widehat{W}(\chi,\lambda)\big|\\&\quad\ll g^{-\frac{3}{2}+\varepsilon}N^{\frac{1}{6}}\sum_{r\leqslant P}\bigg(\frac{r}{(g,r)}\bigg)^{-\frac{3}{2}+\varepsilon}r\\&\quad\ll g^{-\frac{3}{2}+\varepsilon}N^{\frac{1}{6}}P\sum_{r\leqslant P}\bigg(\frac{r}{(g,r)}\bigg)^{-1+\varepsilon}\\&\quad\ll g^{-\frac{3}{2}+\varepsilon}N^{\frac{1}{6}}P\sum_{\substack{d\leqslant P\\ d|g}}d^{1-\varepsilon}\sum_{\substack{r\leqslant P\\ d|r}}r^{-1+\varepsilon}\\&\quad\ll g^{-\frac{3}{2}+\varepsilon}N^{\frac{1}{6}}P^{1+\varepsilon}\ll g^{-\frac{3}{2}+\varepsilon}UL^c.\end{aligned}$$

Thus to establish Lemma 3.2, it suffices to show that

$$\sum_{r\sim R}[g,r]^{-\frac{3}{2}+\varepsilon}\sum_{\chi \bmod r}{}^{*}\max_{|\lambda|\leqslant\frac{1}{rQ}}\big|\widehat{W}(\chi,\lambda)\big|\ll g^{-\frac{3}{2}+\varepsilon}UL^c,\qquad R\leqslant P.\tag{4.1}$$

By Perron's formula, we have

$$\sum_{N_1<m\leqslant u}\Lambda(m)\chi(m)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}F(s,\chi)\frac{u^s-N_1^s}{s}\,\mathrm{d}s+O\big(L^2\big),$$

where

$$F(s,\chi)=\sum_{N_1<m\leqslant u}\Lambda(m)\chi(m)m^{-s},\quad T=N,\quad 0<b\leqslant L^{-1}.$$

Thus,

$$\begin{aligned}\widehat{W}(\chi,\lambda)&=\sum_{N_1<m\leqslant N_2}\Lambda(m)\chi(m)e\big(m^3\lambda\big)\\&=\int_{N_1}^{N_2}e\big(\lambda u^3\big)\,\mathrm{d}\Big\{\sum_{N_1<m\leqslant u}\Lambda(m)\chi(m)\Big\}\\&=\int_{N_1}^{N_2}e\big(\lambda u^3\big)\,\mathrm{d}\Big\{\frac{1}{2\pi i}\int_{b-iT}^{b+iT}F(s,\chi)\frac{u^s-N_1^s}{s}\,\mathrm{d}s+O\big(L^2\big)\Big\}\end{aligned}$$

$$= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} F(s,\chi)\int_{N_1}^{N_2} e(\lambda u^3)u^{s-1}\,\mathrm{d}u\,\mathrm{d}s + O\big(L^2(1+|\lambda|N^{\frac{2}{3}}U)\big)$$

$$= \frac{1}{6\pi i}\int_{b-iT}^{b+iT} F(s,\chi)\int_{N_1^3}^{N_2^3} v^{\frac{b}{3}-1}e\bigg(\frac{t}{6\pi}\log v + \lambda v\bigg)\,\mathrm{d}v\,\mathrm{d}s$$

$$+ O\big(L^2(1+|\lambda|N^{\frac{2}{3}}U)\big),$$

where the error term is obviously admissible.

By Lemmas 4.3 and 4.4 in [10], we have

$$\int_{N_1^3}^{N_2^3} v^{\frac{b}{3}-1}e\bigg(\frac{t}{6\pi}\log v + \lambda v\bigg)\,\mathrm{d}v$$
$$\ll N^{\frac{b}{3}-1}\min\bigg\{N^{\frac{2}{3}}U, \frac{N}{\sqrt{|t|}}, \frac{N}{\min_{N_1^3<v\leqslant N_2^3}|t+6\pi\lambda v|}\bigg\}.$$

Let $T_0 = N^{\frac{2}{3}}U^{-2}$, $\widehat{T_0} = 12\pi N(RQ)^{-1}$ and $b \to 0$. Then

$$\widehat{W}(\chi,\lambda) \ll UN^{-\frac{1}{3}}\max_{T_1\leqslant T_0}\int_{-T_1}^{T_1}\big|F(it,\chi)\big|\,\mathrm{d}t + L\max_{T_0<T_2\leqslant \widehat{T_0}} T_2^{-\frac{1}{2}}\int_{-T_2}^{T_2}\big|F(it,\chi)\big|\,\mathrm{d}t$$
$$+ L\max_{\widehat{T_0}<T_3\leqslant T} T_3^{-1}\int_{-T_3}^{T_3}\big|F(it,\chi)\big|\,\mathrm{d}t.$$

Thus the left side of (4.1) is bounded by

$$\ll UN^{-\frac{1}{3}}\sum_{r\sim R}[g,r]^{-\frac{3}{2}+\varepsilon}\sum_{\chi \bmod r}{}^{*}\max_{T_1\leqslant T_0}\int_{-T_1}^{T_1}\big|F(it,\chi)\big|\,\mathrm{d}t \tag{4.2}$$

$$+ L\sum_{r\sim R}[g,r]^{-\frac{3}{2}+\varepsilon}\sum_{\chi \bmod r}{}^{*}\max_{T_0<T_2\leqslant \widehat{T_0}} T_2^{-\frac{1}{2}}\int_{-T_2}^{T_2}\big|F(it,\chi)\big|\,\mathrm{d}t \tag{4.3}$$

$$+ L\sum_{r\sim R}[g,r]^{-\frac{3}{2}+\varepsilon}\sum_{\chi \bmod r}{}^{*}\max_{\widehat{T_0}<T_3\leqslant T} T_3^{-1}\int_{-T_3}^{T_3}\big|F(it,\chi)\big|\,\mathrm{d}t. \tag{4.4}$$

Note that $[g,r](g,r) = gr$, we find that (4.2) is

$$\ll UN^{-\frac{1}{3}}g^{-\frac{3}{2}+\varepsilon}\max_{T_1\leqslant T_0}\sum_{\substack{d|g\\ d\leqslant R}}\bigg(\frac{R}{d}\bigg)^{-\frac{3}{2}+\varepsilon}\sum_{\substack{r\sim R\\ d|r}}\sum_{\chi \bmod r}{}^{*}\int_{-T_1}^{T_1}\big|F(it,\chi)\big|\,\mathrm{d}t.$$

By Theorem 1.1 in [1], we know that

$$\sum_{\substack{r\sim R\\ d|r}}\sum_{\chi \bmod r}{}^{*}\int_{-T_1}^{T_1}\big|F(it,\chi)\big|\,\mathrm{d}t \ll N^{\frac{1}{3}} + \frac{R^2T_1}{d}N^{\frac{11}{60}}.$$

Thus (4.2) is bounded by

$$\ll UN^{-\frac{1}{3}}g^{-\frac{3}{2}+\varepsilon}\max_{T_1\leqslant T_0}\max_{\substack{d|g\\ d\leqslant R}}\left(\frac{R}{d}\right)^{-\frac{3}{2}+\varepsilon}\left(N^{\frac{1}{3}}+\frac{R^2T_1}{d}N^{\frac{11}{60}}\right)$$
$$\ll g^{-\frac{3}{2}+\varepsilon}UL^c,$$

for $R\leqslant P$ and $U=N^{\frac{s+28}{6s+72}+\varepsilon}$.

For $R\leqslant P$ and $U=N^{\frac{s+28}{6s+72}+\varepsilon}$, (4.3) and (4.4) can be estimated similarly. This finishes the proof of Lemma 3.2. □

5 The sketch of Proof of Theorem 2

In order to prove Theorem 2, we also use the circle method. We will give the proof of Theorem 2 for $E_8(N,U)$, and the proof for $E_7(N,U)$ is similar.

Take $U=N^{\frac{1}{3}-\frac{1}{198}+\varepsilon}$. For the major arcs, we put

$$P_0=N^{\frac{16}{99}+11\varepsilon},\qquad Q_0=N^{\frac{161}{198}-11\varepsilon},\qquad P=N^{\frac{2}{99}+\varepsilon},\qquad Q=N^{\frac{31}{36}+2\varepsilon},$$

and have the following asymptotic formula.

Lemma 5.1 *For $N\leq n\leq N+N^{\frac{2}{3}}U$ and any $A>0$, we have*

$$\int_{\mathfrak{M}}S^8(\alpha)e(-\alpha N)\,\mathrm{d}\alpha\sim C_8\mathfrak{S}_8(n)N^{-\frac{2}{3}}U^7. \tag{5.1}$$

Using the argument of preceding sections, we can show that the exponential sums $S(\alpha)$ over $\mathfrak{m}\cup\mathfrak{l}$ satisfies the following estimate.

Lemma 5.2 *We have*

$$\sup_{\alpha\in\mathfrak{m}\cup\mathfrak{l}}\left|S(\alpha)\right|\ll N^{-\frac{2}{3}}U^{3-\varepsilon}. \tag{5.2}$$

With Lemmas 5.1 and 5.2 known, we can use the method in [11] instead of Bessel's inequality to establish Theorem 2.

We assume $U=N^{\frac{1}{3}-\frac{1}{198}+\varepsilon}$. Denote by $E_8(N,U)$ the set of integers $n\in\mathcal{N}_8$, and yet the equation

$$\begin{cases} n=p_1^3+\cdots+p_8^3,\\ |p_j-\sqrt[3]{\frac{N}{8}}|\leqslant U,\quad j=1,\dots,8,\end{cases}$$

has no solutions in prime numbers $p_1,\dots,p_8$. Define the exponential sum

$$G(\alpha)=\sum_{n\in E_8(N,U)}e(n\alpha)$$

and write $Z = \mathrm{Card}(E_8(N, U))$. In view of the definition of $E_8(N, U)$, we have

$$\int_0^1 S^8(\alpha)G(-\alpha)\,\mathrm{d}\alpha = \sum_{n\in E_8(N,U)} \int_0^1 S^8(\alpha)e(-n\alpha)\,\mathrm{d}\alpha = 0.$$

By Lemma 5.1, one has

$$\int_{\mathfrak{M}} S^8(\alpha)G(-\alpha)\,\mathrm{d}\alpha = \sum_{n\in E_8(N,U)} \int_{\mathfrak{M}} S^8(\alpha)e(-n\alpha)\,\mathrm{d}\alpha \gg ZU^7N^{-\frac{2}{3}}$$

and thus we deduce that

$$\int_{\mathfrak{m}\cup\mathfrak{l}} S^8(\alpha)G(-\alpha)\,\mathrm{d}\alpha \gg ZU^7N^{-\frac{2}{3}}. \tag{5.3}$$

On the other hand, we have

$$\left|\int_{\mathfrak{m}\cup\mathfrak{l}} S^8(\alpha)G(-\alpha)\,\mathrm{d}\alpha\right| \le \sup_{\alpha\in\mathfrak{m}\cup\mathfrak{l}} \left|S(\alpha)\right| I_1^{\frac{1}{2}} I_2^{\frac{1}{2}}. \tag{5.4}$$

where $I_1 = \int_0^1 |S^6(\alpha)G(\alpha)|^2\,\mathrm{d}\alpha$ and $I_2 = \int_0^1 |S(\alpha)|^8\,\mathrm{d}\alpha$. One can easily find (see Sects. 6 and 7 in [11])

$$I_1 \ll N^{\frac{\varepsilon}{4}}\big(U^3Z^2 + U^4Z\big) \quad \text{and} \quad I_2 \ll N^{\frac{\varepsilon}{4}}U^5. \tag{5.5}$$

Collecting Lemma 3.2, (3.3–3.5), we obtain

$$ZU^7N^{-\frac{2}{3}} \ll N^{-\frac{2}{3}}U^{3-\varepsilon}N^{\frac{\varepsilon}{4}}\big(U^3Z^2 + U^4Z\big)^{\frac{1}{2}}U^{\frac{5}{2}}.$$

Noting that $N^{-\frac{2}{3}}U^{3-\varepsilon}N^{\frac{\varepsilon}{4}}U^{\frac{3}{2}}ZU^{\frac{5}{2}} = o(ZU^7N^{-\frac{2}{3}})$, we conclude that

$$ZU^7N^{-\frac{2}{3}} \ll N^{-\frac{2}{3}}U^{3-\varepsilon}N^{\frac{\varepsilon}{4}}\big(U^4Z\big)^{\frac{1}{2}}U^{\frac{5}{2}}$$

namely $Z \ll U^{1-\frac{\varepsilon}{5}}$. The proof for $E_7(N, U)$ is similar. This completes the proof of Theorem 2. □

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