Ramanujan J (2013) 30:211–222
DOI 10.1007/s11139-011-9360-8

A note on Ramanujan's identities involving the hypergeometric function ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$

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Received: 29 September 2010 / Accepted: 1 December 2011 / Published online: 5 December 2012

Abstract We study a class of elliptic functions associated with the hypergeometric function ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$. From the perspective of the properties of conformal mappings and differential equations, we provide new insight into a set of identities of Ramanujan associated with the above hypergeometric function.

Keywords Eisenstein series · Hypergeometric function · Legendre function · Weierstrass elliptic function

Mathematics Subject Classification (2000) Primary 33C75

1 Introduction

We begin by stating a set of identities of Ramanujan concerning the Eisenstein series and the hypergeometric function ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$ [1, pp. 161–164]. The definitions of the Eisenstein series $E_4(\tau)$, $E_6(\tau)$ and the theta functions $\theta_i(\tau)$ will be given in the next section.

Let $x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$. If x_2 satisfies the equation

$$4x_2(1-x_2) = \frac{27}{4}\frac{x^2(1-x)^2}{(1-x+x^2)^3}, \tag{1.1}$$

then

$$\tau = i\frac{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; 1-x_2)}{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x_2)} \tag{1.2}$$

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and

$$E_4(\tau) = {}_2F_1^4\left(\frac{1}{6}, \frac{5}{6}; 1; x_2\right), \tag{1.3}$$

$$E_6(\tau) = (1-2x_2){}_2F_1^6\left(\frac{1}{6}, \frac{5}{6}; 1; x_2\right). \tag{1.4}$$

Moreover,

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; x_2\right) = \left(1-x+x^2\right)^{1/4} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right). \tag{1.5}$$

Although these identities can be verified directly by manipulating some existing hypergeometric identities, we will provide an alternative approach by studying a differential equation naturally associated with a class of elliptic function related to the hypergeometric series ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$; in the process of proving these identities we shall uncover and bring out interesting connections between this class of elliptic function and the conformal mapping that arises from the above hypergeometric function.

The paper is organized as follows. In Sect. 2, we summarize pertinent properties of the Weierstrass elliptic function. The crucial differential equation, its solution, and its properties are studied in Sect. 3. In Sect. 4, we study the quantity x_2 mentioned in (1.1) with the help of the theory of conformal mappings and derive its geometric significance.

We assume the reader is reasonably familiar with the theories of the Weierstrass elliptic function and the theta functions contained in [8, Chaps. 20 and 21]. Finally, we would like to refer the reader to an important pioneering work of the subject from a different perspective [2].

2 Brief summary of some basic properties of the Weierstrass elliptic function

We will use the same notation ${}_2F_1(a, b; c; z)$ to denote the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n$$

and the hypergeometric function derived from the analytic continuation of the above series which typically have branch points at $z = 1$ and ∞.

Throughout the paper, we will always assume that ω_1 and ω_2 are a pair of complex numbers such that $\Im \frac{\omega_2}{\omega_1} > 0$. Define

$$E_4(\omega_1, \omega_2) = 45 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4},$$

$$E_6(\omega_1, \omega_2) = \frac{945}{2} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6}.$$

Let τ be a complex number such that $\Im\tau > 0$. The Eisenstein series $E_4(\tau)$ and $E_6(\tau)$ are defined as

$$E_4(\tau) = 45 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\pi + n\pi\tau)^4},$$
$$E_6(\tau) = \frac{945}{2} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\pi + n\pi\tau)^6}.$$

Let $\tau = \frac{\omega_2}{\omega_1}$. Then

$$E_4(\omega_1, \omega_2) = \frac{\pi^4}{\omega_1^4} E_4(\tau)$$

and

$$E_6(\omega_1, \omega_2) = \frac{\pi^6}{\omega_1^6} E_6(\tau).$$

The theta functions of Jacobi are

$$\theta_2(\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}, \qquad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2};$$

where $q = e^{i\pi\tau}$. We remind the reader of the following well-known identity:

$$\theta_3^4(\tau) = \theta_2^4(\tau) + \theta_4^4(\tau). \tag{2.1}$$

We first review the pertinent facts about the Weierstrass elliptic function. Let $\wp(z; \omega_1, \omega_2)$ be the Weierstrass elliptic function of periods ω_1 and ω_2. We recall its definition [8, p. 434]:

$$\wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}.$$

It satisfies the differential equation

$$y'^2 = 4y^3 - g_2 y - g_3, \qquad y(0) = \infty, \tag{2.2}$$

where

$$g_2 = \frac{4}{3}E_4(\omega_1, \omega_2), \qquad g_3 = \frac{8}{27}E_6(\omega_1, \omega_2) \tag{2.3}$$

and

$$g_2^3 - 27g_3^2 \neq 0.$$

Conversely, suppose $g_2^3 - 27g_3^2 \neq 0$. Then there exists a pair of complex numbers ω_1 and ω_2 such that the elliptic function $\wp(z;\omega_1,\omega_2)$ satisfies (2.2) and (2.3). Define

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{E_4^3}{E_4^3 - E_6^2}.$$

Let

$$e_1 = \wp\left(\frac{\omega_1}{2};\omega_1,\omega_2\right), \qquad e_2 = \wp\left(\frac{\omega_1+\omega_2}{2};\omega_1,\omega_2\right), \qquad e_3 = \wp\left(\frac{\omega_2}{2};\omega_1,\omega_2\right). \tag{2.4}$$

Then

$$4y^3 - g_2 y - g_3 = 4(y-e_1)(y-e_2)(y-e_3).$$

The notation $\wp(z;\omega_1,\omega_2)$ and $\wp(z;g_2,g_3)$ will be used interchangeably.

For simplicity, let $\wp(z|\tau) = \wp(z;\pi,\pi\tau)$.The values of the Weierstrass elliptic function at the points of half periods can be expressed in terms of the Jacobi theta functions [6, p. 133]:

$$\wp\left(\frac{\pi}{2}\middle|\tau\right) = \frac{\theta_3^4(\tau)+\theta_4^4(\tau)}{3},$$
$$\wp\left(\frac{\pi+\pi\tau}{2}\middle|\tau\right) = \frac{\theta_2^4(\tau)-\theta_4^4(\tau)}{3}$$

and

$$\wp\left(\frac{\pi\tau}{2}\middle|\tau\right) = -\frac{\theta_2^4(\tau)+\theta_3^4(\tau)}{3}.$$

We recall the homogeneous relation resulting from scaling the fundamental periods ω_1 and ω_2 with a factor of α:

$$\wp(\alpha z;\alpha\omega_1,\alpha\omega_2) = \alpha^{-2}\wp(z;\omega_1,\omega_2) \tag{2.5}$$

or equivalently

$$\wp\left(\alpha z;\alpha^{-4}g_2,\alpha^{-6}g_3\right) = \alpha^{-2}\wp(z;g_2,g_3). \tag{2.6}$$

From (2.4) and (2.5), we have

Lemma 2.1

$$e_1 = \frac{\pi^2}{\omega_1^2}\frac{\theta_3^4(\tau)+\theta_4^4(\tau)}{3},$$
$$e_2 = \frac{\pi^2}{\omega_1^2}\frac{\theta_2^4(\tau)-\theta_4^4(\tau)}{3}$$

and

$$e_3 = -\frac{\pi^2}{\omega_1^2}\frac{\theta_2^4(\tau)+\theta_3^4(\tau)}{3}.$$

From (2.3), we have

Lemma 2.2

$$g_2 = \frac{4}{3}\frac{\pi^4}{\omega_1^4}E_4(\tau)$$

and

$$g_3 = \frac{8}{27}\frac{\pi^6}{\omega_1^6}E_6(\tau).$$

Before we proceed to the main part of the paper, it is instructive to recall the well-known classical identities between the Eisenstein series and the hypergeometric function ${}_2F_1(\frac{1}{2},\frac{1}{2};1;z)$ associated with the theory of the Jacobi elliptic functions (for details, see Chap. 21 of [8]). Consider the differential equation

$$y'^2 = \left(1-y^2\right)\left(1-k^2y^2\right), \qquad y(0)=0.$$

The solution of this equation is the Jacobian sine function $\mathrm{sn}(z|k)$, and its fundamental periods $4K$ and $2iK'$ are given by

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

and

$$iK' = \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

We can express K and K' in terms of the hypergeometric function:

$$K = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$$

and

$$K' = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-k^2\right).$$

Let

$$\tau = i\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}.$$

Then, k^2 can be expressed as a function of τ:

$$k^2(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)};$$

moreover, we have the following relationship between the Eisenstein series and the hypergeometric function:

$$E_4(\tau) = \left(1 - k^2 + k^4\right) {}_2F_1^4\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

and

$$E_6(\tau) = \left(1 + k^2\right)\left(1 - 2k^2\right)\left(1 - \frac{k^2}{2}\right) {}_2F_1^6\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

The relation between $k^2(\tau)$ and $J(\tau)$ is given by

$$J = \frac{4}{27} \frac{(1 - k^2 + k^4)^3}{k^4(1 - k^2)^2}. \tag{2.7}$$

For future reference, we record the following identities:

$$k^2(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)},$$

$$\left(\theta_3^2\right)^2(\tau) = \left(\theta_4^2\right)^2(\tau) + \left(\theta_2^2\right)^2(\tau),$$

and

$$\theta_3^2(\tau) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2(\tau)\right). \tag{2.8}$$

3 A differential equation associated with Ramanujan's identities involving ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$

We now present the differential equation associated with the set of identities of Ramanujan stated in the Introduction.

Consider the differential equation

$$y'^2 = 4y^3 - 3y - \left(1 - 2\mu^2\right), \qquad y(0) = \infty. \tag{3.1}$$

We note that (3.1) is a special case of (2.2) with the invariant

$$J = \frac{1}{4\mu^2(1 - \mu^2)}. \tag{3.2}$$

We will derive the solution of the differential equation and its associated identities by first restricting the parameter μ in the interval $[-1, 1]$; the restriction is then removed by a standard argument of analytic continuation. Since the solution of this differential equation is completely determined by a pair of fundamental periods ω_1 and ω_2, we consider the solution found once they are known.

Lemma 3.1 *Suppose* $\mu = \sin\beta$. *Then*

$$4x^3 - 3x - \left(1 - 2\mu^2\right) = 4(x - e_1)(x - e_2)(x - e_3),$$

where

$$e_1 = \cos\frac{2\beta}{3}, \qquad e_2 = \cos\frac{2\pi - 2\beta}{3}, \qquad e_3 = \cos\frac{2\beta + 2\pi}{3}.$$

If, in addition $0 \le \beta < \frac{\pi}{2}$, *then*

$$e_3 < 0 < e_2 < e_1 < 1. \tag{3.3}$$

Proof We note that $1 - 2\mu^2 = \cos 2\beta$ and

$$4\cos^3\theta - 3\cos\theta - \left(1 - 2\mu^2\right) = \cos 3\theta - \cos 2\beta.$$

The conclusion follows readily by observing that the right-hand side of the trigonometric identity equals zero, for $0 \le \theta \le 2\pi$, when $\theta = \pm\frac{2\beta}{3}, \pm\frac{2\pi - 2\beta}{3}, \pm\frac{2\beta + 2\pi}{3}$. □

Theorem 3.2 *Suppose* $0 \le \beta < \frac{\pi}{2}$. *Let a pair of fundamental periods* ω_1 *and* ω_2 *be defined by* [6, p. 88]

$$\frac{\omega_1}{2} = \int_{e_3}^{e_2} \frac{dx}{\sqrt{4x^3 - 3x - (1 - 2\mu^2)}}$$

and

$$\frac{\omega_2}{2} = \int_{e_2}^{e_1} \frac{dx}{\sqrt{4x^3 - 3x - (1 - 2\mu^2)}}.$$

Then

$$\omega_1 = \pi\sqrt{\frac{2}{3}}\,{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \mu^2\right) = \pi\sqrt{\frac{2}{3}}\,{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \sin^2\beta\right)$$

and

$$\omega_2 = i\pi\sqrt{\frac{2}{3}}\,{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1 - \mu^2\right) = i\pi\sqrt{\frac{2}{3}}\,{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \cos^2\beta\right).$$

Proof Substituting $x = \cos\frac{2\theta}{3}$ and $1 - 2\mu^2 = \cos 2\beta$ in (2.2), we have

$$\frac{\omega_1}{2} = \frac{2}{3}\int_{\pi-\beta}^{\pi+\beta} \frac{\sin\frac{2\theta}{3}\,d\theta}{\sqrt{\cos 2\theta - \cos 2\beta}} = \frac{2}{3}\int_{-\beta}^{\beta} \frac{\sin(\frac{2\theta}{3} + \frac{2\pi}{3})\,d\theta}{\sqrt{\cos 2\theta - \cos 2\beta}}$$

$$= \frac{2}{3}\sin\frac{2\pi}{3}\int_{-\beta}^{\beta} \frac{\cos\frac{2\theta}{3}\,d\theta}{\sqrt{\cos 2\theta - \cos 2\beta}} + \frac{2}{3}\cos\frac{2\pi}{3}\int_{-\beta}^{\beta} \frac{\sin\frac{2\theta}{3}\,d\theta}{\sqrt{\cos 2\theta - \cos 2\beta}}$$

$$= \frac{2}{\sqrt{3}} \int_0^{\beta} \frac{\cos \frac{2\theta}{3}\, d\theta}{\sqrt{\cos 2\theta - \cos 2\beta}} = \frac{1}{\sqrt{3}} \int_0^{2\beta} \frac{\cos \frac{\theta}{3}\, d\theta}{\sqrt{\cos \theta - \cos 2\beta}}.$$

We now appeal to an integral representation of the Legendre functions [5, p. 174, (7.4.10)]:

$${}_2F_1\left(-\nu, \nu+1; 1; \sin^2 \frac{\beta}{2}\right) = \frac{\sqrt{2}}{\pi} \int_0^{\beta} \frac{\cos(\nu + \frac{1}{2})\theta\, d\theta}{\sqrt{\cos\theta - \cos\beta}}.$$

Now choose $\nu = -\frac{1}{6}$. We obtain the first identity. Since the proof of the second identity follows from an identical argument, we omit the details. □

We observe that, from (3.3), $\Im \frac{\omega_2}{\omega_1} > 0$.

We remarked earlier that (3.1) is a special case of (2.2); we can, however, scale the solution of one into the other using the homogeneous relation (2.6). Thus we have

$$\wp(z; g_2, g_3) = a\wp\left(\sqrt{a}z; 3, 1 - 2\mu^2\right) \tag{3.4}$$

where $a = \sqrt{\frac{g_2}{3}}$ and $\mu^2 = \frac{1}{2}(1 - \frac{g_3}{a^3})$. From (3.4), we derive the following corollary.

Corollary 3.3 *Suppose* $g_2^3 - 27g_3^2 \neq 0$. *For* $g_2 \neq 0$, *let*

$$\omega_1 = \pi \sqrt{\frac{2}{\sqrt{3g_2}}}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{1}{2}\left(1 - \sqrt{27g_3^2/g_2^3}\right)\right)$$

and

$$\omega_2 = i\pi \sqrt{\frac{2}{\sqrt{3g_2}}}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{1}{2}\left(1 + \sqrt{27g_3^2/g_2^3}\right)\right);$$

for $g_2 = 0$, *let*

$$\omega_1 = \sqrt{\pi}\, 2^{1/3} 3^{-1} g_3^{-1/6} \frac{\Gamma(1/6)}{\Gamma(2/3)}$$

and

$$\omega_2 = e^{i2\pi/3} \sqrt{\pi}\, 2^{1/3} 3^{-1} g_3^{-1/6} \frac{\Gamma(1/6)}{\Gamma(2/3)}.$$

Then ω_1, ω_2 *generate the period lattice for* $\wp(z; g_2, g_3)$.

We only need to prove the case for $g_2 = 0$.

Proof A pair of periods ω_1, ω_2 can be obtained from the following integrals:

$$\omega_1 = 2\int_{(g_3/4)^{1/3}}^{\infty} \frac{dx}{\sqrt{4x^3 - g_3}} = 2^{1/3} g_3^{-1/6} \int_1^{\infty} \frac{dx}{\sqrt{t^3 - 1}} = \sqrt{\pi}\, 2^{1/3} 3^{-1} g_3^{-1/6} \frac{\Gamma(1/6)}{\Gamma(2/3)}$$

and

$$\omega_2 = 2\int_{\rho(g_3/4)^{1/3}}^{\infty} \frac{dx}{\sqrt{4x^3 - g_3}} = \rho 2^{1/3} g_3^{-1/6} \int_1^{\infty\rho} \frac{dx}{\sqrt{t^3-1}}$$
$$= \rho\sqrt{\pi} 2^{1/3} 3^{-1} g_3^{-1/6} \frac{\Gamma(1/6)}{\Gamma(2/3)};$$

where $\rho = e^{i2\pi/3}$. We remind the reader of the beta function identity used in the calculation of both integrals:

$$\int_1^{\infty(\rho)} \frac{dx}{\sqrt{t^3-1}} = \frac{\sqrt{\pi}}{3} \frac{\Gamma(1/6)}{\Gamma(2/3)}. \qquad \square$$

It is interesting to note that the above identities allow us to compute a set of fundamental periods for $\wp(z; g_2, g_3)$ directly in terms of ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$ and the parameters g_2 and g_3. The standard method for computing fundamental periods ω_1, ω_2 of the Weierstrass elliptic function is to first find the roots $a < b < c$ of the cubic polynomial $4x^3 - g_2 x - g_3$, and then evaluate the periods by

$$\omega_1 = \frac{\pi}{M(\sqrt{c-a}, \sqrt{c-b})}$$

and

$$\omega_2 = \frac{i\pi}{M(\sqrt{c-a}, \sqrt{b-a})},$$

where $M(x, y)$ is the arithmetic–geometric mean of x and y (see Proposition 6.34 of [4]).

4 Representation of x_2 in terms of Schwarzian triangle function

In this section we will study the quantity x_2 with help from the properties of conformal mappings and derive its geometric significance. We first establish the crucial connection between the hypergeometric function ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)$ and a Schwarzian triangle function. One can find a very readable account of the basic facts used in this work in Sect. 7 of Chap. 5 and Sect. 5 of Chap. 7 of [7] and [3, pp. 96–99].

Let Δ denote the triangular region on the upper half plane $\tau : \Im\tau > 0$ bounded by three circular arcs with vertices located at $\infty, 0$ and $e^{i\pi/3}$. The angles of Δ at these points are 0, 0 and $\frac{2\pi}{3}$. Define

$$S(z) = i\frac{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; 1-z)}{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)}. \tag{4.1}$$

Then, $S(z)$ maps the upper half plane $\Im z > 0$ conformally to the triangle Δ. Let $\tau = S(z)$ and let $z = s(\tau)$ be its inverse. Applying the Schwarz reflection principle repeatedly, we can construct the domain of $s(\tau)$ which is a Riemann surface with branch points of order $-\frac{3}{2}$ at the points generated from $e^{i\pi/3}$ by the repeated applications of the reflection principle.

Let $\mathbb{C}$ denote the complex plane and let $\mathcal{F}$ be the region on the upper half plane obtained by reflecting the triangular region Δ across the y axis and identifying the two edges vertical to the real axis.

We state a consequence of the above discussion.

Theorem 4.1 *Let $s(\tau)$ be the inverse of $S(z)$. Then it maps $\mathcal{F}$ conformally to $\mathbb{C} - [1, \infty)$ such that $s(\infty) = 0, s(0) = 1$ and $s(e^{i\pi/3}) = \infty$. Moreover,*

$$s\left(\frac{-1}{\tau}\right) = 1 - s(\tau). \tag{4.2}$$

Proof The first part of the theorem follows from the above discussion; we need only to justify the identity (4.2). We observe that, from (4.1), we have $s(\tau) = z$ and

$$S(1 - z) = i \frac{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; z)}{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; 1 - z)} = \frac{-1}{\tau}.$$

This implies

$$s\left(\frac{-1}{\tau}\right) = 1 - z = 1 - s(\tau). \qquad \square$$

For a given $0 \le \mu^2 < 1$, let $x_2 = \mu^2$. According to Theorem 4.1, there is a unique τ on the imaginary axis such that $\mu^2 = s(\tau)$. From (2.7) and (3.2), we see that the identity (1.1) describes precisely the relation between $s(\tau)$ and $k^2(\tau)$:

$$4s(1 - s) = \frac{27}{4} \frac{k^4(1 - k^2)^2}{(1 - k^2 + k^4)^3}.$$

Solving s in terms of k^2, we obtain

$$s(\tau) = \frac{1}{2} - \frac{1}{2} \frac{(1 + k^2)(1 - 2k^2)(1 - \frac{k^2}{2})}{(1 - k^2 + k^4)^{3/2}}$$

and

$$s\left(\frac{-1}{\tau}\right) = \frac{1}{2} + \frac{1}{2} \frac{(1 + k^2)(1 - 2k^2)(1 - \frac{k^2}{2})}{(1 - k^2 + k^4)^{3/2}}.$$

From (3.2),

$$J^{-1}(\tau) = 4s(\tau)\big(1 - s(\tau)\big).$$

Thus

$$s(\tau) = \frac{1}{2}\big(1 - \sqrt{1 - J^{-1}(\tau)}\big)$$

and using the definition of J, we express s in terms of the Eisenstein series:

$$s(\tau)=\frac{1}{2}\left(1-\frac{E_6(\tau)}{E_4^{3/2}(\tau)}\right).$$

We now derive the Ramanujan identities stated in the Introduction. The identities (1.2) and (1.3) follow from Lemma 2.2 (with $g_2=3$, $g_3=1-2\mu^2$) and Theorem 3.2:

$$E_4(\tau)={}_2F_1^4\left(\frac{1}{6},\frac{5}{6};1;s(\tau)\right)$$

$$E_6(\tau)=\big(1-2s(\tau)\big){}_2F_1^6\left(\frac{1}{6},\frac{5}{6};1;s(\tau)\right).$$

Before proving (1.5), we first derive the following identity.

Lemma 4.2

$$\sin\frac{2\beta}{3}=\frac{1}{\sqrt{3}}\frac{\pi^2}{\omega_1^2}\theta_2^4(\tau).$$

Proof Equating the two different representations of e_2 in Lemma 2.1 and Lemma 3.1, we have

$$e_2=\frac{\pi^2}{\omega_1^2}\frac{\theta_2^4(\tau)-\theta_4^4(\tau)}{3}=\cos\frac{2\pi-2\beta}{3}=-\frac{1}{2}\cos\frac{2\beta}{3}+\frac{\sqrt{3}}{2}\sin\frac{2\beta}{3}.$$

Similarly,

$$e_3=-\frac{\pi^2}{\omega_1^2}\frac{\theta_2^4(\tau)+\theta_3^4(\tau)}{3}=\cos\frac{2\pi+\beta}{3}=-\frac{1}{2}\cos\frac{2\beta}{3}-\frac{\sqrt{3}}{2}\sin\frac{2\beta}{3}.$$

The difference of these two identities (along with (2.1)) gives the desired identity. □

We now prove (1.5).

Proof From Lemmas 2.1, 3.1, and 4.2,

$$1=\sin^2\frac{2\beta}{3}+\cos^2\frac{2\beta}{3}=\frac{4}{9}\frac{\pi^2}{\omega_1^2}\big(\theta_2^8-\theta_2^4\theta_3^4+\theta_3^8\big).$$

Thus

$$\frac{\omega_1^2}{\pi^2}=\frac{4}{9}\big(\theta_2^8-\theta_2^4\theta_3^4+\theta_3^8\big)=\frac{4}{9}\theta_3^8\big(1-k^2+k^4\big).$$

The identity (1.5) follows from this identity along with Theorem 3.2 and (2.8). □

We end the paper by observing the following parallel of (2.8): Let $a = E_4^{1/4}$, $b = [\frac{1}{2}(E_4^{3/2} + E_6)]^{1/6}$, and $c = [\frac{1}{2}(E_4^{3/2} - E_6)]^{1/6}$. Then

$$\mu^2(\tau) = \frac{c^6(\tau)}{a^6(\tau)}$$
$$a^6(\tau) = b^6(\tau) + c^6(\tau)$$

and

$$a(\tau) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \mu^2(\tau)\right).$$

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