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A generalization of Clausen's identity

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Abstract The paper gives an extension of Clausen's identity to the square of any Gauss hypergeometric function. Accordingly, solutions of the related third-order linear differential equation are found in terms of certain bivariate series that can reduce to ${}_3F_2$ series similar to those in Clausen's identity. The general contiguous variation of Clausen's identity is found as well. The related Chaundy's identity is generalized without any restriction on the parameters of the Gauss hypergeometric function. The special case of dihedral Gauss hypergeometric functions is underscored.

Keywords Gauss hypergeometric function · Bivariate hypergeometric series

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1 Introduction

Clausen's identity [4] is

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix} \middle| \, z \right)^2 = {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix} \middle| \, z \right). \tag{1} $$

A related identity is [3, (5)]

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix} \middle| \, z \right) {}_2F_1\left(\begin{matrix} \frac{1}{2}-a,\ \frac{1}{2}-b \\ \frac{3}{2}-a-b \end{matrix} \middle| \, z \right) = {}_3F_2\left(\begin{matrix} \frac{1}{2},\ a-b+\frac{1}{2},\ b-a+\frac{1}{2} \\ a+b+\frac{1}{2},\ \frac{3}{2}-a-b \end{matrix} \middle| \, z \right). \tag{2} $$

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Up to a power factor, the two ${}_2\mathrm{F}_1$ functions in the second identity are solutions of the same Euler's hypergeometric differential equation. The two identities demonstrate a case where the *symmetric tensor square* of Euler's hypergeometric equation coincides with the hypergeometric differential equation for ${}_3\mathrm{F}_2$ functions. Accordingly, formulas (1) and (2) relate quadratic forms in ${}_2\mathrm{F}_1$ functions and ${}_3\mathrm{F}_2$ functions as solutions of the same third-order linear differential equation; see Exercise 13 in [1, p. 116].

This paper aims to generalize Clausen's identity to the square of a general ${}_2\mathrm{F}_1$ function, without any restriction on the three parameters. Accordingly, we try to find other attractive solutions of the symmetric tensor square equation for general Euler's hypergeometric equation. We present the symmetric tensor square equation in Sect. 3; see formula (31).

We find that the general symmetric square equation has solutions expressible as specializations of the following double hypergeometric series:

$$F^{2:1;1}_{1:1;1}\left(\begin{matrix} a;\, b;\, p_1,\, p_2 \\ c;\, q_1,\, q_2 \end{matrix}\,\middle|\, x,\, y\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{i+j}\,(b)_{i+j}\,(p_1)_i\,(p_2)_j}{(c)_{i+j}\,(q_1)_i\,(q_2)_j\; i!\; j!}\, x^i\, y^j, \tag{3}$$

$$F^{1:2;2}_{2:0;0}\left(\begin{matrix} a;\, p_1,\, p_2;\, q_1,\, q_2 \\ b;\; c \end{matrix}\,\middle|\, x,\, y\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{i+j}\,(p_1)_i\,(p_2)_j\,(q_1)_i\,(q_2)_j}{(b)_{i+j}\;(c)_{i+j}\; i!\; j!}\, x^i\, y^j. \tag{4}$$

If $a = c$, these two series reduce to the bivariate Appell's F_2 or F_3 hypergeometric series, respectively. These functions are special cases of *Kampé de Fériet series*.

Surely, the left-hand sides of (1)–(2) are trivially double hypergeometric series as well. The significance of new solutions is that they reduce to ${}_3\mathrm{F}_2$ functions of similar shape as in (1)–(2) in special cases and that they represent elementary solutions (that is, a polynomial times a power function) if the symmetric square equation has such solutions.

Our main results are summarized in the following two theorems. In Theorem 1, by a univariate specialization of $F^{2:1;1}_{1:1;1}$ or $F^{2:1;1}_{1:1;1}$ function we understand the restriction of any branch (under analytic continuation) of the bivariate $F^{2:1;1}_{1:1;1}$ or $F^{2:1;1}_{1:1;1}$ function to the curve parameterized by z. In Theorem 2, we refer to the $F^{2:1;1}_{1:1;1}$ or $F^{2:1;1}_{1:1;1}$ series as defined in (3)–(4). A direct generalization of Clausen's identity is formula (10). It applies to the Gauss hypergeometric functions contiguous to Clausen's instance. We prove these theorems in Sects. 4 and 5, respectively.

Theorem 1 *The univariate functions*

$$F^{2:1;1}_{1:1;1}\left(\begin{matrix} 2a;\, 2b;\; c-\frac{1}{2},\, a+b-c+\frac{1}{2} \\ a+b+\frac{1}{2};\; 2c-1,\, 2a+2b-2c+1 \end{matrix}\,\middle|\, z,\, 1-z\right), \tag{5}$$

$$z^{1-c}(1-z)^{c-a-b-\frac{1}{2}} \times F^{1:2;2}_{2:0;0}\left(\begin{matrix} \frac{1}{2};\, a-b+\frac{1}{2},\, a+b-c+\frac{1}{2};\, b-a+\frac{1}{2},\, c-a-b+\frac{1}{2} \\ c;\; 2-c \end{matrix}\,\middle|\, z,\, \frac{z}{z-1}\right), \tag{6}$$

$$z^{\frac{1}{2}-c}(1-z)^{c-a-b} \times F^{1:2;2}_{2:0;0}\left(\left.\begin{matrix} \frac{1}{2};\; a-b+\frac{1}{2}, c-\frac{1}{2};\; b-a+\frac{1}{2}, \frac{3}{2}-c \\ a+b-c+1;\; c-a-b+1 \end{matrix}\,\right|\, 1-z, 1-\frac{1}{z}\right), \tag{7}$$

$$z^{\frac{1}{2}-c}(1-z)^{c-a-b-\frac{1}{2}} \times F^{1:2;2}_{2:0;0}\left(\left.\begin{matrix} \frac{1}{2};\; c-\frac{1}{2}, a+b-c+\frac{1}{2};\; \frac{3}{2}-c, c-a-b+\frac{1}{2} \\ a-b+1;\; b-a+1 \end{matrix}\,\right|\, \frac{1}{z}, \frac{1}{1-z}\right) \tag{8}$$

satisfy the symmetric tensor square equation for ${}_2\mathrm{F}_1\left(\left.{a,\, b \atop c}\right| z\right)^2$.

Theorem 2 *The following identities hold in a neighborhood of* $z=0$:

$$\begin{aligned} &{}_2\mathrm{F}_1\left(\left.\begin{matrix} a,\, b \\ c \end{matrix}\,\right|\, z\right) {}_2\mathrm{F}_1\left(\left.\begin{matrix} 1+a-c,\, 1+b-c \\ 2-c \end{matrix}\,\right|\, z\right) \\ &\quad = (1-z)^{c-a-b-\frac{1}{2}} \\ &\qquad \times F^{1:2;2}_{2:0;0}\left(\left.\begin{matrix} \frac{1}{2}; a-b+\frac{1}{2}, a+b-c+\frac{1}{2}; b-a+\frac{1}{2}, c-a-b+\frac{1}{2} \\ c;\; 2-c \end{matrix}\,\right|\, z, \frac{z}{z-1}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} &{}_2\mathrm{F}_1\left(\left.\begin{matrix} a,\, b \\ a+b+n+\frac{1}{2} \end{matrix}\,\right|\, z\right)^2 \\ &\quad = \frac{(\frac{1}{2})_n\, (a+b+\frac{1}{2})_n}{(a+\frac{1}{2})_n\, (b+\frac{1}{2})_n}\, F^{2:1;1}_{1:1;1}\left(\left.\begin{matrix} 2a;\; 2b;\; a+b+n,\, -n \\ a+b+\frac{1}{2};\; 2a+2b+2n, -2n \end{matrix}\,\right|\, z, 1-z\right), \end{aligned} \tag{10}$$

$$\begin{aligned} &\frac{(a+\frac{1}{2})_n (a+m+\frac{1}{2})_n}{(\frac{1}{2})_n (m+\frac{1}{2})_n}\, {}_2\mathrm{F}_1\left(\left.\begin{matrix} a,\, -a-m-n \\ \frac{1}{2}-m \end{matrix}\,\right|\, z\right)^2 \\ &\quad + \frac{(a+\frac{1}{2})_m (a+n+\frac{1}{2})_m (a)^2_{m+n+1}}{(\frac{1}{2})_m (\frac{1}{2})^2_{m+1} (\frac{1}{2})_n (\frac{1}{2})_{m+n}}\, z^{2m+1}\, {}_2\mathrm{F}_1\left(\left.\begin{matrix} a+m+\frac{1}{2}, \frac{1}{2}-a-n \\ m+\frac{3}{2} \end{matrix}\,\right|\, z\right)^2 \\ &\quad = F^{2:1;1}_{1:1;1}\left(\left.\begin{matrix} 2a;\; -2a-2m-2n;\; -m,\, -n \\ \frac{1}{2}-m-n;\; -2m, -2n \end{matrix}\,\right|\, z, 1-z\right). \end{aligned} \tag{11}$$

Here a, b, c *can be any complex numbers if only the involved lower parameters* c, $2-c$, $a+b+\frac{1}{2}$ *are not zero or negative integers. The numbers* m, n *are assumed to be nonnegative integers.*

The $F^{2:1;1}_{1:1;1}$ series in (10) is understood to terminate in the second argument at the power $(1-z)^n$, but it is never a terminating series in the first argument even if $a+b+n$ is a nonpositive integer $-m$ (when the term-wise limit $b \to -a-n-m$ should be applied). The $F^{2:1;1}_{1:1;1}$ series in (11) is a finite rectangular sum with $(m+1)(n+1)$ terms.

If $n=0$, formula (10) reduces to Clausen's identity (1). If $c=a+b+\frac{1}{2}$, formula (9) reduces to (2). We should observe that Clausen's identity is wrong if $a+b$ is a nonpositive integer and the ${}_3\mathrm{F}_2$ series is interpreted as terminating. A correct identity with the terminating ${}_3\mathrm{F}_2$ series is then the special case $n=0$ of (11):

$$
{}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,\ -a-m \\ \frac{1}{2}-m \end{array}\right| z\right)^2 + \frac{(a+\frac{1}{2})_m^2\,(a)_{m+1}^2}{(\frac{1}{2})_m^2\,(\frac{1}{2})_{m+1}^2}\, z^{2m+1}\, {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a+m+\frac{1}{2},\ \frac{1}{2}-a \\ m+\frac{3}{2} \end{array}\right| z\right)^2
$$
$$
= {}_3\mathrm{F}_2\left(\left.\begin{array}{c} 2a,\ -2a-2m,\ -m \\ \frac{1}{2}-m,\ -2m \end{array}\right| z\right). \tag{12}
$$

On the other hand, if $2a$ is an integer such that $-2m \ge 2a \ge 0$, Clausen's formula is evidently correct again, as used in [5, (8.34)].

Formula (11) has the following significance. The ${}_2\mathrm{F}_1$ functions on the left-hand side are *dihedral Gauss hypergeometric functions*, as the monodromy group of their Euler's hypergeometric equation is a dihedral group. Then the symmetric tensor square equation is reducible, and its monodromy representation has an invariant one-dimensional subspace. The $F^{2:1;1}_{1:1;1}$ function on the right-hand side is a polynomial, so it is obviously an invariant of the monodromy group. Formula (11) identifies a generator for the invariant space as a linear combination of ${}_2\mathrm{F}_1(z)^2$ solutions and as an explicit terminating $F^{2:1;1}_{1:1;1}$ expression [8, Sect. 3]. In more plain terms, the $F^{2:1;1}_{1:1;1}$ expression gives an expected elementary solution of the symmetric square equation. Similarly, the $F^{1:2;2}_{2:0;0}$ series in (9) is terminating in both summation directions in the following specialization with dihedral ${}_2\mathrm{F}_1$ functions:

$$
{}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,\ a+m+\frac{1}{2} \\ 2a+m+n+1 \end{array}\right| z\right) {}_2\mathrm{F}_1\left(\left.\begin{array}{c} -a-m-n,\ \frac{1}{2}-a-n \\ 1-2a-m-n \end{array}\right| z\right)
$$
$$
= (1-z)^n\, F^{1:2;2}_{2:0;0}\left(\left.\begin{array}{c} \frac{1}{2};\ -m,-n;\ m+1,n+1 \\ 2a+m+n+1;\ 1-2a-m-n \end{array}\right| z,\ \frac{z}{z-1}\right). \tag{13}
$$

The terminating $F^{2:1;1}_{1:1;1}$ and $F^{1:2;2}_{2:0;0}$ sums are closely related. Up to a constant multiple, one can rewrite the terminating $F^{2:1;1}_{1:1;1}$ sum as a terminating $F^{1:2;2}_{2:0;0}$ sum, and vice versa, simply by reversing the order of summation in both directions.

In addition to traditional applications of Clausen's formula to show the positivity of hypergeometric sums, Clausen formula is used in [5] to obtain a single series expression for the probability of transition between quantum states of a parametric oscillator.

2 Variation of formulas

A product of two single hypergeometric series is trivially a double hypergeometric series. In particular, we can write

$$
{}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,\ b \\ c \end{array}\right| z\right)^2 = F^{0:2;2}_{0:1;1}\left(\left.\begin{array}{c} a,\ a;\ b,\ b \\ c,\ c \end{array}\right| z,\ z\right). \tag{14}
$$

Chaundy [3, (6)] gives the following double series expansion:

$$
{}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right)^2 = \sum_{k=0}^{\infty} \frac{(2a)_k (2b)_k (c-\frac{1}{2})_k}{(c)_k\,(2c-1)_k\,k!} \times {}_4\mathrm{F}_3\left(\begin{matrix} -\frac{k}{2}, -\frac{k-1}{2}, \frac{1}{2}, a+b-c+\frac{1}{2} \\ a+\frac{1}{2},\ b+\frac{1}{2},\ \frac{3}{2}-c-k \end{matrix}\,\middle|\, 1\right) z^k. \tag{15}
$$

But our formulas (10) and (9) are particularly interesting as direct generalizations of the classical formulas (1)–(2).

Application of Euler's transformation [1, (2.2.7)] to the second ${}_2\mathrm{F}_1$ factor on the left-hand side of (9) gives

$$
{}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right) {}_2\mathrm{F}_1\left(\begin{matrix} 1-a,\ 1-b \\ 2-c \end{matrix}\,\middle|\, z\right) = \frac{1}{\sqrt{1-z}} \times F^{1:2;2}_{2:0;0}\left(\begin{matrix} \frac{1}{2}; a-b+\frac{1}{2}, a+b-c+\frac{1}{2}; b-a+\frac{1}{2}, c-a-b+\frac{1}{2} \\ c;\ 2-c \end{matrix}\,\middle|\, z, \frac{z}{z-1}\right). \tag{16}
$$

The substitution $z \mapsto z/(z-1)$ and application of Pfaff's transformation [1, (2.2.6)] to the ${}_2\mathrm{F}_1$ functions in (11) and (12) give, respectively:

$$
\begin{aligned}
&\frac{(a+\frac{1}{2})_n (a+m+\frac{1}{2})_n}{(\frac{1}{2})_n (m+\frac{1}{2})_n}\, {}_2\mathrm{F}_1\left(\begin{matrix} a,\ a+n+\frac{1}{2} \\ \frac{1}{2}-m \end{matrix}\,\middle|\, z\right)^2 \\
&\quad - \frac{(a+\frac{1}{2})_m (a+n+\frac{1}{2})_m (a)^2_{m+n+1}}{(\frac{1}{2})_m (\frac{1}{2})^2_{m+1} (\frac{1}{2})_n (\frac{1}{2})_{m+n}}\, z^{2m+1} \\
&\quad \times {}_2\mathrm{F}_1\left(\begin{matrix} a+m+\frac{1}{2}, a+n+m+1 \\ m+\frac{3}{2} \end{matrix}\,\middle|\, z\right)^2 \\
&= (1-z)^{-2a}\, F^{2:1;1}_{1:1;1}\left(\begin{matrix} 2a;\ -2a-2m-2n;\ -m,\ -n \\ \frac{1}{2}-m-n;\ -2m, -2n \end{matrix}\,\middle|\, \frac{z}{z-1}, \frac{1}{1-z}\right),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&{}_2\mathrm{F}_1\left(\begin{matrix} a,\ a+\frac{1}{2} \\ \frac{1}{2}-m \end{matrix}\,\middle|\, z\right)^2 - \frac{(a+\frac{1}{2})^2_m (a)^2_{m+1}}{(\frac{1}{2})^2_m (\frac{1}{2})^2_{m+1}}\, z^{2m+1}\, {}_2\mathrm{F}_1\left(\begin{matrix} a+m+\frac{1}{2}, a+m+1 \\ m+\frac{3}{2} \end{matrix}\,\middle|\, z\right)^2 \\
&= (1-z)^{-2a}\, {}_3\mathrm{F}_2\left(\begin{matrix} 2a,\ -2a-2m,\ -m \\ \frac{1}{2}-m,\ -2m \end{matrix}\,\middle|\, \frac{z}{z-1}\right).
\end{aligned} \tag{18}
$$

Note that

$$
\frac{(a+\frac{1}{2})_m (a)_{m+1}}{(\frac{1}{2})_m (\frac{1}{2})_{m+1}} = \frac{2^{2m}\,(m!)^2\,(2a)_{2m+1}}{(2m)!\,(2m+1)!}. \tag{19}
$$

The specializations $m=0$ of (11) and (17) are interesting as well:

$$\left(a+\frac{1}{2}\right)_n^2 {}_2\mathrm{F}_1\left(\begin{matrix} a,\ -a-n \\ 1/2 \end{matrix}\,\middle|\, z\right)^2 + 2\,z\,(a)_{n+1}^2\,{}_2\mathrm{F}_1\left(\begin{matrix} a+\frac{1}{2}, \frac{1}{2}-a-n \\ 3/2 \end{matrix}\,\middle|\, z\right)^2$$
$$= \left(\frac{1}{2}\right)_n^2 {}_3\mathrm{F}_2\left(\begin{matrix} 2a,\ -2a-2n,\ -n \\ \frac{1}{2}-n,\ -2n \end{matrix}\,\middle|\, 1-z\right), \tag{20}$$

$$\left(a+\frac{1}{2}\right)_n^2 {}_2\mathrm{F}_1\left(\begin{matrix} a,\ a+n+\frac{1}{2} \\ 1/2 \end{matrix}\,\middle|\, z\right)^2 - 2\,z\,(a)_{n+1}^2\,{}_2\mathrm{F}_1\left(\begin{matrix} a+\frac{1}{2}, a+n+1 \\ 3/2 \end{matrix}\,\middle|\, z\right)^2$$
$$= \left(\frac{1}{2}\right)_n^2 (1-z)^{-2a}\,{}_3\mathrm{F}_2\left(\begin{matrix} 2a,\ -2a-2n,\ -n \\ \frac{1}{2}-n,\ -2n \end{matrix}\,\middle|\, \frac{1}{1-z}\right). \tag{21}$$

It is tempting to equate ${}_2\mathrm{F}_1\left(\begin{smallmatrix} a,\, b \\ c \end{smallmatrix}\middle| z\right)^2$ to the $F^{2:1;1}_{1:1;1}$ function in (5) up to a constant multiple, since the latter appears to have the power series expansion at $z=0$ as well. The kth derivative of the $F^{2:1;1}_{1:1;1}$ function at $z=0$ would evaluate to a linear combination of the values

$$ {}_3\mathrm{F}_2\left(\begin{matrix} 2a+k,\ 2b+k,\ a+b-c+j+\frac{1}{2} \\ a+b+k+\frac{1}{2},\ 2a+2b-2c+j+1 \end{matrix}\,\middle|\, 1\right), \qquad j=0,1,\ldots,k. \tag{22}$$

However, the convergence condition of these ${}_3\mathrm{F}_2(1)$ sums is $\mathrm{Re}(1-c-k)>0$, so the ${}_3\mathrm{F}_2(1)$ values are undefined for large enough k. The $F^{2:1;1}_{1:1;1}$ function can have branching behavior at $z=0$ in general, as the local exponents z^{1-c}, z^{2-2c} of the symmetric square equation can come into play. See [7] for explicit details.

The $F^{2:1;1}_{1:1;1}$ function in (5) can be evaluated at $z=0$ and $z=1$ using the ${}_3\mathrm{F}_2(1)$ evaluation in [1, Theorem 3.5.5(i)]:

$$F^{2:1;1}_{1:1;1}\left(\begin{matrix} 2a;\ 2b;\ c-\frac{1}{2},\ a+b-c+\frac{1}{2} \\ a+b+\frac{1}{2};\ 2c-1, 2a+2b-2c+1 \end{matrix}\,\middle|\, 0, 1\right)$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(a+b+\frac{1}{2})\Gamma(1-c)\,\Gamma(1+a+b-c)}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(1+a-c)\,\Gamma(1+b-c)}, \tag{23}$$

$$F^{2:1;1}_{1:1;1}\left(\begin{matrix} 2a;\ 2b;\ c-\frac{1}{2},\ a+b-c+\frac{1}{2} \\ a+b+\frac{1}{2};\ 2c-1, 2a+2b-2c+1 \end{matrix}\,\middle|\, 1, 0\right)$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(a+b+\frac{1}{2})\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(c-a)\,\Gamma(c-b)}. \tag{24}$$

The convergence conditions are $\mathrm{Re}(1-c)>0$ and $\mathrm{Re}(c-a-b)>0$, respectively.

It is worth mentioning that Bailey's identity [2]

$$F_4\left(\begin{matrix} a;\ b \\ c, a+b-c+1 \end{matrix}\,\middle|\, x(1-y), y(1-x)\right) = {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, x\right) {}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ a+b-c+1 \end{matrix}\,\middle|\, y\right) \tag{25}$$

specializes to

$$F_4\left(\begin{matrix} a;\ b \\ c, a+b-c+1 \end{matrix}\,\middle|\, x^2, (1-x)^2\right) = {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, x\right) {}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ a+b-c+1 \end{matrix}\,\middle|\, 1-x\right). \tag{26}$$

The $F_4(x^2, (1-x)^2)$ function is another solution of the symmetric tensor square equation. In particular, we have the following identity for $x \in [0, 1]$ if $\mathrm{Re}\, c < 1$ and $\mathrm{Re}(c-a-b) > 0$, due to the connection formula in [1, (2.3.13)]:

$$\begin{aligned} {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, z\right)^2 &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F_4\left(\begin{matrix} a;\ b \\ c, a+b-c+1 \end{matrix}\,\middle|\, x^2, (1-x)^2\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)} (1-t)^{2c-2a-2b} \\ &\quad \times F_4\left(\begin{matrix} c-a;\ c-b \\ c, c-a-b+1 \end{matrix}\,\middle|\, x^2, (1-x)^2\right). \end{aligned} \tag{27}$$

3 The differential equations

Let Θ_z, Θ_x, Θ_y denote the differential operators

$$\Theta_z = z\,\frac{d}{dz}, \qquad \Theta_x = x\,\frac{\partial}{\partial x}, \qquad \Theta_y = y\,\frac{\partial}{\partial y}. \tag{28}$$

Euler's hypergeometric differential equation [1, (2.3.5)] for a general Gauss hypergeometric function ${}_2\mathrm{F}_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix}\,\middle|\, z\right)$ can be compactly written as follows:

$$z(\Theta_z + a)(\Theta_z + b)y(z) - \Theta_z(\Theta_z + c - 1)y(z) = 0. \tag{29}$$

This is a Fuchsian equation with three regular singular points $z = 0,\ 1$, and ∞. The local exponents at the singular points are:

$$0,\, 1-c \quad \text{at } z=0; \qquad 0,\, c-a-b \quad \text{at } z=1; \quad \text{and} \quad a, b \quad \text{at } z=\infty.$$

In general, a basis of solutions is

$${}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right), \qquad z^{1-c}\,{}_2\mathrm{F}_1\left(\begin{matrix} 1+a-c,\ 1+b-c \\ 2-c \end{matrix}\,\middle|\, z\right). \tag{30}$$

Notice that the left-hand sides of formulas (2) and (9) are products of these two solutions of the same Euler's equation, with the power factor z^{1-c} ignored. The classical formulas (1)–(2) apply when the difference of local exponents at $z = 1$ is equal to $1/2$.

If y_1, y_2 are two independent solutions of a second-order homogeneous linear differential equation like (29), the functions y_1^2, $y_1 y_2$, y_2^2 satisfy a third-order linear differential equation called the *symmetric tensor square* of the second-order equation. The symmetric tensor square equation for Euler's equation (29) can be written as follows [3, (2)]:

$$z(\Theta_z + 2a)(\Theta_z + 2b)(\Theta_z + a + b)y(z) - \Theta_z(\Theta_z + c - 1)(\Theta_z + 2c - 2)y(z) + \frac{(2a + 2b - 2c + 1)z}{z - 1}\big((a + b - c + 1)\Theta_z + 2ab\big)y(z) = 0. \tag{31}$$

If $c = a + b + \frac{1}{2}$, this is a differential equation for the ${}_3F_2$ function in (1).

Partial differential equations for the $F^{2:1;1}_{1:1;1}(x, y)$ function in (3) can be obtained by considering the first-order recurrence relations between its coefficients. Writing the $F^{2:1;1}_{1:1;1}$ sum as $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}$, we have the relations

$$\frac{c_{i+1,j}}{c_{i,j}} = \frac{(a + i + j)(b + i + j)(p_1 + i)x}{(c + i + j)(q_1 + i)(1 + i)},$$
$$\frac{c_{i,j+1}}{c_{i,j}} = \frac{(a + i + j)(b + i + j)(p_2 + j)y}{(c + i + j)(q_2 + i)(1 + j)},$$

and also $c_{i+1,j}/c_{i,j+1} = (p_1 + i)(q_2 + j)(1 + j)\,x/(p_2 + j)(q_1 + i)(1 + i)\,y$. These relations translate to the following partial differential equations for the $F^{2:1;1}_{1:1;1}(x, y)$ function:

$$P_1 = x(\Theta_x + \Theta_y + a)(\Theta_x + \Theta_y + b)(\Theta_x + p_1) - \Theta_x(\Theta_x + \Theta_y + c - 1)(\Theta_x + q_1 - 1), \tag{32}$$
$$P_2 = y(\Theta_x + \Theta_y + a)(\Theta_x + \Theta_y + b)(\Theta_y + p_2) - \Theta_y(\Theta_x + \Theta_y + c - 1)(\Theta_y + q_2 - 1), \tag{33}$$
$$P_3 = x\,\Theta_y\,(\Theta_y + q_2 - 1)(\Theta_x + p_1) - y\,\Theta_x(\Theta_x + q_1 - 1)(\Theta_y + p_2). \tag{34}$$

In general, two of these operators generate the ideal J in $\mathbf{C}(x, y)\langle\Theta_x, \Theta_y\rangle$ annihilating the $F^{2:1;1}_{1:1;1}(x, y)$ series. In particular, we have the following obvious syzygies:

$$-y(\Theta_y + p_2)P_1 + x(\Theta_x + p_1)P_2 + (\Theta_x + \Theta_y + c - 2)P_3 = 0,$$
$$\Theta_y(\Theta_y + q_2 - 1)P_1 - \Theta_x(\Theta_x + q_1 - 1)P_2 + (\Theta_x + \Theta_y + a - 1)(\Theta_x + \Theta_y + b - 1)P_3 = 0.$$

If the coefficient $\Theta_x + \Theta_y + c - 2$ does not divide $(\Theta_x + \Theta_y + a - 1)(\Theta_x + \Theta_y + b - 1)$, we can express P_3 in terms of P_1, P_2, etc.

Note the following commutation relations in $\mathbf{C}(x, y)\langle\Theta_x, \Theta_y\rangle$:

$$\Theta_x x = x\Theta_x + x = x\,(\Theta_x + 1), \qquad \Theta_y y = y\,(\Theta_y + 1). \tag{35}$$

The variables x, Θ_x commute with y, Θ_y. Up to the multiplication order, these relations are the same as shift operator relations, such as $S_n n = (n+1)S_n$ in the Ore algebra $\mathbf{C}\langle n, S_n\rangle$. We will keep working with Euler-type differentiation operators in (28) in our transformations of differential equations.

Gröbner basis computations show that third-order partial differential operators in the ideal J are linearly generated by P_1, P_2, P_3. A residue basis over $\mathbf{C}(x, y)$ is formed by the 7 monomials $1, \Theta_x, \Theta_y, \Theta_x^2, \Theta_x\Theta_y, \Theta_y^2, \Theta_x^3$. Hence the rank of the differential system for $F^{2:1;1}_{1:1;1}(x, y)$ functions is equal to 7. The leading coefficients in various Gröbner bases suggest that the following lines are in the singular locus of the differential system:

$$x = 0, \qquad x = 1, \qquad y = 0, \qquad y = 1,$$
$$x + y = 0, \qquad x + y = 1, \qquad y + 2x = 1, \qquad 2y + x = 1.$$

It is easy to apply the transformations $F(x, y) \mapsto x^\alpha y^\beta F(x, y)$ or $x \mapsto 1/x$, $y \mapsto 1/y$ to the differential system generated by the operators P_1, P_2. In effect, we only have to replace additionally $\Theta_x \mapsto \Theta_x + \alpha$, $\Theta_y \mapsto \Theta_y + \beta$ or $\Theta_x \mapsto -\Theta_x$, $\Theta_y \mapsto -\Theta_y$, respectively. We may attempt to find transformations of the operators P_1, P_2 to a pair of similar hypergeometric operators. This gives us the following solutions of the same system of partial differential equations, besides the $F^{2:1;1}_{1:1;1}(x, y)$ function in (3):

$$x^{1-q_1}\, F^{2:1;1}_{1:1;1}\left(\begin{matrix} 1+a-q_1;\, 1+b-q_1;\, 1+p_1-q_1,\, p_2 \\ 1+c-q_1;\, 2-q_1,\, q_2 \end{matrix}\,\middle|\, x, y\right), \tag{36}$$

$$y^{1-q_2}\, F^{2:1;1}_{1:1;1}\left(\begin{matrix} 1+a-q_2;\, 1+b-q_2;\, p_1,\, 1+p_2-q_2 \\ 1+c-q_2;\, q_1,\, 2-q_2 \end{matrix}\,\middle|\, x, y\right), \tag{37}$$

$$x^{1-q_1}y^{1-q_2} \times F^{2:1;1}_{1:1;1}\left(\begin{matrix} 2+a-q_1-q_2;\, 2+b-q_1-q_2;\, 1+p_1-q_1,\, 1+p_2-q_2 \\ 2+c-q_1-q_2;\, 2-q_1,\, 2-q_2 \end{matrix}\,\middle|\, x, y\right), \tag{38}$$

$$x^{-p_1}\, y^{-p_2}\, F^{1:2;2}_{2:0;0}\left(\begin{matrix} 1+p_1+p_2-c;\, 1+p_1-q_1,\, 1+p_2-q_2;\, p_1,\, p_2 \\ 1+p_1+p_2-a;\, 1+p_1+p_2-b \end{matrix}\,\middle|\, \frac{1}{x}, \frac{1}{y}\right). \tag{39}$$

As we see, the systems of partial differential equations for the $F^{2:1;1}_{1:1;1}$ and $F^{1:2;2}_{2:0;0}$ functions are easily transformable to each other. In the other direction, here is an $F^{2:1;1}_{1:1;1}$ function satisfying the same partial differential equations as the $F^{1:2;2}_{2:0;0}(x, y)$ function in (4):

$$x^{-p_1}\, y^{-p_2}\, F^{2:1;1}_{1:1;1}\left(\begin{matrix} 1+p_1+p_2-a;\, 1+p_1+p_2-b;\, p_1,\, p_2 \\ 1+p_1+p_2-c;\, 1+p_1-q_1,\, 1+p_2-q_2 \end{matrix}\,\middle|\, \frac{1}{x}, \frac{1}{y}\right). \tag{40}$$

From this the other $F^{2:1;1}_{1:1;1}$ companion solutions for the $F^{1:2;2}_{2:0;0}(x, y)$ function in (4) can be obtained using the symmetries $p_1 \leftrightarrow q_1$ and $p_2 \leftrightarrow q_2$.

4 Proof of Theorem 1

First we prove that the $F^{2:1;1}_{1:1;1}$ function in (5) satisfies the symmetric square equation (31) for ${}_2\mathrm{F}_1\left(\begin{smallmatrix}a,\,b\\ c\end{smallmatrix}\middle|\,z\right)^2$. The ideal of partial differential operators annihilating

$$F^{2:1;1}_{1:1;1}\left(\begin{matrix}2a;\ 2b;\ c-\frac{1}{2},\ a+b-c+\frac{1}{2}\\ a+b+\frac{1}{2};\ 2c-1,\ 2a+2b-2c+1\end{matrix}\,\middle|\,x,y\right) \tag{41}$$

is generated by the polynomials P_1, P_2, P_3 in (32)–(34) with the parameters specialized to linear functions in a, b, c. Here are the specialized differential equations:

$$0 = x(\Theta_x+\Theta_y+2a)(\Theta_x+\Theta_y+2b)\left(\Theta_x+c-\frac{1}{2}\right) \\ -\Theta_x\left(\Theta_x+\Theta_y+a+b-\frac{1}{2}\right)(\Theta_x+2c-2), \tag{42}$$

$$0 = y(\Theta_x+\Theta_y+2a)(\Theta_x+\Theta_y+2b)\left(\Theta_y+a+b-c+\frac{1}{2}\right) \\ -\Theta_y\left(\Theta_x+\Theta_y+a+b-\frac{1}{2}\right)(\Theta_y+2a+2b-2c), \tag{43}$$

$$0 = x\Theta_y(\Theta_y+2a+2b-2c)\left(\Theta_x+c-\frac{1}{2}\right) \\ -y\Theta_x(\Theta_x+2c-2)\left(\Theta_y+a+b-c+\frac{1}{2}\right). \tag{44}$$

The variables and derivatives are related as follows under the univariate specialization under consideration:

$$x=z, \qquad y=1-z, \qquad \frac{d}{dz}=\frac{\partial}{\partial x}-\frac{\partial}{\partial y}, \qquad \Theta_z=\Theta_x+\frac{z}{z-1}\,\Theta_y. \tag{45}$$

Following [6, Definition 1.1], the *partial differential form* of an ordinary differential equation under a specialization like (45) is the expression where the univariate derivatives are replaced by respective linear combinations of partial derivatives. On the other hand, the *specialized form* of a partial differential equation under the same kind of specialization is the expression with coefficients to the partial derivatives specialized to univariate functions. In our setting, the two forms are "mixed" linear differential expressions in the partial derivatives Θ_x, Θ_y but with the coefficients univariate in z. Algebraically, we work in the $\mathbf{C}[z]\langle\Theta_z\rangle$ module generated by the partial derivatives of any order. The action of Θ_z on the partial derivatives is given by the identification in (45).

To show that the univariate $F^{2:1;1}_{1:1;1}$ function in (10) satisfies the symmetric tensor square equation (31), we demonstrate that the partial differential form of the symmetric square equation coincides with the specialized form of a partial differential equation following from the partial differential equations (42)–(44). In the terminology of [6, Definition 1.1], we show that the symmetric square equation *follows fully* from the differential equations (42)–(44) under the specialization defined in (45).

To give a partial differential form of the symmetric square equation (31), we introduce a way to work with multiplicative expressions of (noncommutative) differential operators. Suppose that G, H are functions in z. Then

$$\begin{aligned}(\Theta_z+G)(\Theta_z+H) &= (\Theta_z+G)\left(\Theta_x+\frac{z}{z-1}\,\Theta_y+H\right)\\ &= \left(\Theta_x+\frac{z}{z-1}\,\Theta_y+A\right)\cdot\left(\Theta_x+\frac{z}{z-1}\,\Theta_y+B\right)\\ &\quad-\frac{z}{(z-1)^2}\,\Theta_y+z\,\frac{dH}{dz},\end{aligned}\tag{46}$$

where we use the dot between two differential factors to signify *commutative* formal multiplication. In other words, an expanded expression for $(\Theta_z+G)(\Theta_z+H)$ can be obtained by multiplying commutatively the two large factors in (46), collecting terms to the monomials in Θ_x, Θ_y, then interpreting those monomials as differential operators of suitable order, and writing the coefficients to those monomials as left-side factors.

If A, B, C are constants, the product $(\Theta_z+A)(\Theta_z+B)(\Theta_z+C)$ can be written as follows:

$$\begin{aligned}&\left(\Theta_x+\frac{z}{z-1}\Theta_y+A\right)\cdot\left(\left(\Theta_x+\frac{z}{z-1}\Theta_y+B\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+C\right)\right.\\ &\quad\left.-\frac{z}{(z-1)^2}\,\Theta_y\right)-\frac{z}{(z-1)^2}\,\Theta_y\cdot\left(2\Theta_x+\frac{2z}{z-1}\Theta_y+B+C\right)+\frac{z\,(z+1)}{(z-1)^3}\,\Theta_y,\end{aligned}$$

and finally, as

$$\begin{aligned}&\left(\Theta_x+\frac{z}{z-1}\Theta_y+A\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+B\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+C\right)\\ &\quad-\frac{z}{(z-1)^2}\,\Theta_y\cdot\left(3\Theta_x+\frac{3z}{z-1}\Theta_y+A+B+C\right)+\frac{z\,(z+1)}{(z-1)^3}\,\Theta_y.\end{aligned}\tag{47}$$

Following this "commutative" expression, the symmetric square equation (31) can be written as follows:

$$\begin{aligned}&z\left(\Theta_x+\frac{z}{z-1}\Theta_y+2a\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+2b\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+a+b\right)\\ &\quad-\left(\Theta_x+\frac{z}{z-1}\Theta_y\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+c-1\right)\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+2c-2\right)\\ &\quad-\frac{3z}{(z-1)^2}\,\Theta_y\cdot\left(\Theta_x+\frac{z}{z-1}\Theta_y+\frac{(a+b)\,z+1-c}{z-1}\right)+\frac{z(z+1)}{(z-1)^2}\,\Theta_y\end{aligned}$$

$$+\frac{(2a+2b-2c+1)z}{z-1}\left((a+b-c+1)\left(\Theta_x+\frac{z}{z-1}\Theta_y\right)+2ab\right)=0. \quad (48)$$

Using formal commutative computations, one can check that this equation coincides with the specialized (under $x=z,\ y=1-z$) form of

$$\big[\text{eq. (42)}\big]-\frac{z^2}{(z-1)^2}\big[\text{eq. (43)}\big]-\frac{1}{(z-1)^2}\big[\text{eq. (44)}\big]. \quad (49)$$

We proved that the $F^{2:1;1}_{1:1;1}$ function in (5) satisfies the symmetric square equation (31).

The relation between the functions in (4) and (40) implies that the function in (8) satisfies the same symmetric square equation. The symmetric square equation has the following two solutions as well:

$$z^{-2a}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}a,\ 1+a-c\\ 1+a-b\end{matrix}\,\right|\,\frac{1}{z}\right)^2,\qquad (1-z)^{-2a}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}a,\ c-b\\ 1+a-b\end{matrix}\,\right|\,\frac{1}{1-z}\right)^2. \quad (50)$$

The relation between the functions ${}_2\mathrm{F}_1\left(\left.{a,\,b \atop c}\right|z\right)^2$ and (8) translates into the claim that (6) and (7) are solutions of the same symmetric square equation as well. This completes the proof of Theorem 1.

Notice that the upper parameters in (6)–(8) contain the local exponent differences of ${}_2\mathrm{F}_1\left(\left.{a,\,b \atop c}\right|z\right)$ increased by $1/2$, while the lower parameters are the same local exponent differences increased by 1. A set of twelve $F^{2:1;1}_{1:1;1}$ solutions can be obtained using the relation between the functions in (4) and (40).

5 Proof of Theorem 2

To prove formula (9), we compare the function in (6) with the following solution of the symmetric square equation:

$$z^{1-c}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}a,\ b\\ c\end{matrix}\,\right|\,z\right)\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1+a-c,\ 1+b-c\\ 2-c\end{matrix}\,\right|\,z\right). \quad (51)$$

Both solutions have the same local exponent $1-c$ at $z=0$. For those values of c for which both ${}_2\mathrm{F}_1$ solutions are generally defined, the linear space of solutions with this local exponent is one-dimensional. After division by z^{1-c} both solutions (6) and (51) evaluate to 1 at $z=0$, and therefore formula (9) follows.

To prove formula (10), we observe that (for general a,b) the linear space of power series solutions at $z=0$ to the symmetric square equation is one-dimensional. The $F^{2:1;1}_{1:1;1}$ function in (10) and the square function in (14) are both power series solutions with the local exponent 0. Hence they must differ by a constant multiple. To compute the constant multiple, we evaluate both sides of $z=1$. Observe that the convergence condition $\mathrm{Re}(n+\frac{1}{2})>0$ for both sides is always satisfied. The left-hand side is evaluated by the Gauss formula [1, Theorem 2.2.2]; the right-hand side is evaluated by (24) with $c=a+b+n+\frac{1}{2}$. The Pochhammer-type factor follows.

To prove formula (11), we take the term-wise limit $b \to -a-m-n$ in (10). We use

$$\lim_{\varepsilon \to 0} \frac{(\varepsilon - k)_{2k+1}}{(2\varepsilon - 2k)_{2k+1}} = \frac{(-k)_k\, k!}{(-2k)_{2k} \cdot 2} = \frac{(-1)^k\,(k!)^2}{2 \cdot (2k)!} = \frac{(-1)^k\, k!}{2^{2k+1} (\frac{1}{2})_k} \tag{52}$$

to get the limiting formula

$$\begin{aligned} &\frac{(a+\frac{1}{2})_n (\frac{1}{2}-a-m-n)_n}{(\frac{1}{2})_n (\frac{1}{2}-m-n)_n}\, {}_2\mathrm{F}_1\!\left(\left. \begin{matrix} a,\ -a-m-n \\ \frac{1}{2}-m \end{matrix} \right| z \right)^2 \\ &= F^{2:1;1}_{1:1;1}\!\left(\left. \begin{matrix} 2a;\ -2a-2m-2n;\ -m,\ -n \\ \frac{1}{2}-m-n;\ -2m, -2n \end{matrix} \right| z, 1-z \right) \\ &\quad + \frac{(2a)_{2m+1}(-2a-2m-2n)_{2m+1}}{(\frac{1}{2}-m-n)_{2m+1}(2m+1)!}\, \frac{(-1)^m\, m!}{2^{2m+1}(\frac{1}{2})_m}\, z^{2m+1} \\ &\quad \times F^{2:1;1}_{1:1;1}\!\left(\left. \begin{matrix} 2a+2m+1;\ 1-2a-2n;\ m+1,\ -n \\ \frac{3}{2}+m-n;\ 2m+2, -2n \end{matrix} \right| z, 1-z \right). \end{aligned} \tag{53}$$

Due to (10), the latter $F^{2:1;1}_{1:1;1}$ function can be written as

$$\frac{(a+m+1)_n (1-a-n)_n}{(\frac{1}{2})_n (m-n+\frac{3}{2})_n}\, {}_2\mathrm{F}_1\!\left(\left. \begin{matrix} a+m+\frac{1}{2}, \frac{1}{2}-a-n \\ m+\frac{3}{2} \end{matrix} \right| z \right)^2. \tag{54}$$

We replace $(2m+1)! = 2^{2m+1} m!\,(\frac{1}{2})_{m+1}$ and rewrite identity (53) as follows:

$$\begin{aligned} &\frac{(a+\frac{1}{2})_n (a+m+\frac{1}{2})_n}{(\frac{1}{2})_n (m+\frac{1}{2})_n}\, {}_2\mathrm{F}_1\!\left(\left. \begin{matrix} a,\ -a-m-n \\ \frac{1}{2}-m \end{matrix} \right| z \right)^2 \\ &= F^{2:1;1}_{1:1;1}\!\left(\left. \begin{matrix} 2a;\ -2a-2m-2n;\ -m,\ -n \\ \frac{1}{2}-m-n;\ -2m, -2n \end{matrix} \right| z, 1-z \right) \\ &\quad + \frac{(-1)^{m+n+1}}{2^{4m+2} (\frac{1}{2})_m (\frac{1}{2})_{m+1}}\, \frac{(2a)_{2m+1}(2a+2n)_{2m+1}}{(\frac{1}{2}-m-n)_{2m+1}}\, z^{2m+1} \\ &\quad \times \frac{(a+m+1)_n (a)_n}{(\frac{1}{2})_n (m-n+\frac{3}{2})_n}\, {}_2\mathrm{F}_1\!\left(\left. \begin{matrix} a+m+\frac{1}{2}, \frac{1}{2}-a-n \\ m+\frac{3}{2} \end{matrix} \right| z \right)^2. \end{aligned} \tag{55}$$

After regrouping

$$\begin{aligned} &(1/2-m-n)_{2m+1} (m-n+3/2)_n \\ &\quad = (1/2-m-n)_{2m+n+1} = (-1)^{m+n} (1/2)_{m+n} (1/2)_{m+1}, \\ &(2a)_{2m+1} = 2^{2m+1}\,(a)_{m+1} (a+1/2)_m, \\ &(a)_{m+1}\,(a+m+1)_n = (a)_{m+n+1}, \qquad \text{etc.}, \end{aligned}$$

we obtain (11).

Alternatively, we may find the constant multiple in (10) by evaluating both sides at $z=0$. The evaluation of the right-hand side can be done by Zeilberger's algorithm. Formula (11) can be independently proved by observing that the linear space of power series solutions at $z=0$ to the symmetric square equation is two-dimensional and evaluating at the two points $z=0$ and $z=1$.

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