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On Riemann's posthumous fragment II on the limit values of elliptic modular functions

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Abstract We shall resurrect the instinctive direction of B. Riemann on his posthumous fragment on the limit values of elliptic modular functions à la C.G.J. Jacobi, Fundamenta Nova. In the spirit of Riemann who considered the odd part, we shall realize the situation where there is no singularity occurring in taking the radial limits, thus streamlining and elucidating the recent investigation by Arias de Reyna. By the new Dirichlet–Abel theorem (which should be within reach of Riemann), we may directly sum the series in question, which allows us to condense Arias de Reyna's paper into a few pages.

Keywords Elliptic modular function · Dedekind eta function · Dirichlet–Abel theorem · Riemann's posthumous fragment II

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1 Introduction

Riemann's posthumous Fragment, based on Jacobi's Fundamenta Nova, §40 [2], consists of two parts, Fragment I and Fragment II, the latter of which contains only formulas and almost no text. Dedekind [6] succeeded in elucidating the genesis of all the formulas in Fragment II by introducing the most celebrated Dedekind eta-function. All the results in Fragment II deal with the asymptotic behavior of those modular functions from Jacobi's Fundamenta Nova, for which the variable tends to

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rational points on the unit circle. After Dedekind, several authors including Smith [3], Hardy [4], and Rademacher [5, 8] made some more incorporations of Fragment II.

Recently Arias de Reyna [1] has analyzed all the formulas in Fragment II again by painstaking efforts of applying his Theorem 3, which will turn out to be a weaker disguised form of the Dirichlet–Abel theorem, and applying ad hoc methods to each of the formulas. He criticizes Dedekind and claims his own view on Riemann's fragment. A serious defect is that he overlooked the more informative paper of Wintner [7] published in 1941, which already gave a close analysis of Fragment I and some far-reaching comments on Fragment II. Thus, as soon as [1] has been published in 2004, it has got immediately out of date regarding those comments as to why Riemann was interested in the imaginary parts only (denoting the real part by A, leaving untouched) of those elliptic modular functions in question.

The main purpose of the present paper is to make clear the thought (or maybe instinctive direction) of Riemann and show that although he might not have the proofs given below at hand, he could have conceived of them in his mind. What he has done is to eliminate the singular part, which turns out to be the Clausen function, by taking the odd part. Our objective is well summarized in the passage by Wintner [7, p. 634, ll. 8–11] to the effect that all expansions considered by Jacobi in his §40 must be identical consequences of the corresponding expansion of the (principal) logarithm of the fundamental invariant:

$$\varDelta = \varDelta(z) = z\prod_{n=1}^{\infty}\left(1 - z^n\right)^{24}, \quad |z| < 1.$$

Following these lines, we shall first enhance Theorem 3 of [1] into our new Theorem 1 which is a disguised form of the Dirichlet–Abel theorem and could have been within reach of Riemann, as well as unifying Theorems 4–7 of [1] into one concise theorem (Theorem 2 below). At least in his mind Riemann must have felt the truth of this type of Dirichlet–Abel theorem.

In [1], Arias de Reyna needs to remove the singular part $\frac{R_n(x)}{n}$, $n \equiv Q \bmod 2Q$ (for instance, cf. [1, Proof of Theorem 10]), from the series first, then evaluate the series, and then to give back the removed part into the series. This process of summing the series with some terms removed and giving those terms back to the series makes the argument unnecessarily complicated. At a first glance, it looks like Riemann would have thought in this complicated fashion.

However, in the present paper, with this enhanced theorem at hand, we may simply sum the series $\sum_{n=1}^{\infty} \frac{R_n(1)}{n}$ directly, whereby using, as our great advantage, the universal expression (31) for $R_n(1)$ in terms of a finite weighted exponential sum.

Accordingly, as soon as we substitute that closed formula, we almost immediately get the expression in terms of the (differences of) polylogarithm function of order 1, without singularity, and we confirm that this is what Riemann intended to do but could not do because of lack of time. He simply noted those formulas for record and may have not argued in the order of events as presented in [1].

In conclusion, we may describe our contribution to be a realization of Riemann's (subconscious) idea with the help of the Dirichlet–Abel theorem and the Dedekind eta function.

In order to remove singularities, Riemann used a well-known device of taking the odd part or an alternate sum described respectively by

$$\sum_{2\nmid n} a_n = \sum_{n} a_n - \sum_{2|n} a_n \tag{1}$$

or

$$\sum_{n} (-1)^n a_n = \sum_{2|n} a_n - \sum_{2\nmid n} a_n = 2\sum_{2|n} a_n - \sum_{n} a_n, \tag{2}$$

by (1), where n runs over a finite range, or the series are absolutely convergent.

We shall reveal that all the important series of the form $\sum_{n=1}^{\infty} \frac{1}{n} R_n(1)$ can be easily evaluated using the polylogarithm function $l_1(x)$ of order 1 defined by (6) with $s = 1$ for $0 < x < 1$ and what remains is the imaginary part of $l_1(x)$, which is the periodic Bernoulli polynomial $\bar{B}_1(x) = x - [x] - \frac{1}{2}$ having the Fourier expansion in [7]. This is because by taking the alternate sum, the real part, which is the Clausen function and which is the main source of the singularities at rational points on the unit circle, has been eliminated.

Then incorporating the Bernoulli formula

$$\bar{B}_1(2x) - \bar{B}_1(x) = \bar{B}_1\left(x + \frac{1}{2}\right)$$

whose right-hand side is the Fourier series

$$-\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi n\left(x + \frac{1}{2}\right),$$

which is $\bar{B}_1(x + \frac{1}{2})$ for $x \notin \mathbb{Z}$ and is 0 for $x \in \mathbb{Z} + \frac{1}{2}$ and is denoted by $\varphi(x)$ in [1], we may rewrite the results in the form as presented there.

2 Dirichlet–Abel theorem and the Dedekind eta function

Theorem 1 (Dirichlet–Abel) *Let q be a fixed modulus >1. For each $n, k \in \mathbb{Z}$, let $R_n(x)$ denote a complex-valued function defined on $\mathrm{I} = [0, 1]$ such that $R_n(x) = R_k(x)$ if and only if $n \equiv k \pmod q$. Further, assume the functions $R_n(x)$ to fulfil the Lipschitz condition for some α, $\alpha \geq 1$, $R_k(x) \in \mathrm{Lip}\,\alpha$, and the vanishing condition*

$$\sum_{k=1}^{q} R_k(x) = 0 \tag{3}$$

for each $x \in \mathrm{I}$. Then the Dirichlet series

$$F(s) = F(s, x) = \sum_{n=1}^{\infty} \frac{R_n(x^n)}{n^s} \tag{4}$$

is uniformly convergent in $\mathrm{Re}\, s = \sigma > 0$ and $x \in \mathrm{I}$.

If, further, all the R_k *are continuous on* I, *then* $F(s)$ *is also continuous on* I, *and*

$$F(1,1)=\sum_{n=1}^{\infty}\frac{R_n(1)}{n}=-\frac{1}{q}\sum_{k=1}^{q}R_k(1)\psi\left(\frac{k}{q}\right)=\sum_{k=1}^{q}\hat{R}_k(1)l_1\left(\frac{k}{q}\right), \tag{5}$$

where $\psi=\frac{\Gamma'}{\Gamma}$ *is the Euler digamma function, and*

$$l_s(x)=\sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s}\quad \textit{for } s\in\mathbb{Z},\ \mathrm{Re}\,s=\sigma>0,\ \mathrm{Im}\,x\geq 0 \tag{6}$$

with $s=1$ *the first polylogarithm function.*

Since for any fixed positive integer $m>1$, $\sum_{j=1}^{m}e^{2\pi i\frac{j}{m}}=\frac{e^{2\pi i\frac{1}{m}}(1-e^{2\pi i})}{1-e^{2\pi i\frac{1}{m}}}=0$, by applying Theorem 1, we conclude that $l_s(z)$ is uniformly convergent for $\mathrm{Re}\,s>0$ if $z=\frac{k}{m}\in\mathbb{Q},\ 0<k<m$.

Proof We apply a generalization of Dirichlet's test: the series $\sum_{n=1}^{\infty}a_n(x)b_n(s)$ is uniformly convergent in $\sigma>0$ and $x\in$ I, we may check that the partial sums of $a_n(x)$ are bounded uniformly in x, $\lim_{n\to\infty}b_n(s)=0$ uniformly in $\sigma>0$, and $|b_n(s)-b_{n+1}(s)|\leq c_n$, where $\sum_{n=1}^{\infty}c_n<\infty$.

Since with $b_n(s)=n^{-s}$ for $\sigma>0$, $\lim_{n\to\infty}b_n(s)=0$ and $|b_n(s)-b_{n+1}(s)|=|\frac{1}{s}\int_n^{n+1}t^{-\sigma-1}\,\mathrm{d}t|=O(n^{-\sigma-1})$, we have $\sum_{n=1}^{\infty}c_n=\zeta(\sigma+1)<\infty$ for $\sigma>0$.

The boundedness of the partial sum of $a_n(x)=R_n(x)$ follows from the following remark.

Writing $n=jm+k,\ 1\leq k\leq m,\ 0\leq j\leq[\frac{n}{m}]$, we may express $F(s,x)$ as

$$F(s,x)=\sum_{k=1}^{m}\sum_{\substack{n=1\\ n\equiv k \pmod m}}^{\infty}\frac{R_n(x^n)}{n^s}=m^{-s}\sum_{k=1}^{m}\sum_{j=0}^{\infty}\frac{R_k(x^{mj+k})}{(j+\frac{k}{m})^s}. \tag{7}$$

Hence it suffices to consider the partial zeta function

$$F_k(s,x)=\sum_{\substack{n=1\\ n\equiv k \pmod m}}^{\infty}\frac{R_k(x^n)}{n^s}.$$

By assumption,

$$\left|R_k\left(x^{jm+k}\right)-R_k\left(x^{jm}\right)\right|=O\left(x^{jm\alpha}\left|x^k-1\right|^{\alpha}\right).$$

Hence,

$$\sum_{n=1}^{N} R_n\big(x^n\big) = \sum_{j=0}^{[N/m]} \sum_{k=1}^{m} R_k\big(x^{jm+k}\big)$$
$$= \sum_{j=0}^{[N/m]} \sum_{k=1}^{m} R_k\big(x^{jm}\big) + O\left(\sum_{j=0}^{[N/m]} x^{\alpha jm} \sum_{k=1}^{m} \big(1-x^k\big)^{\alpha}\right).$$

Now

$$\sum_{j=0}^{[N/m]} \big(x^{\alpha m}\big)^j \sum_{k=1}^{m} \big(1-x^k\big)^{\alpha}$$
$$\leq \begin{cases} 0, & x=1 \\ \frac{1}{1-x^{\alpha m}}(1-x)^{\alpha} \sum_{k=1}^{m} (1+x+\cdots+x^{k-1})^{\alpha}, & 0 \leq x < 1 \end{cases}$$
$$= \begin{cases} 0, & x=1 \\ O\big(\big(\frac{1-x}{(1-x^{\alpha m})^{\frac{1}{\alpha}}}\big)^{\alpha}\big), & 0 \leq x < 1 \end{cases} \tag{8}$$

(the last line being valid because $\alpha > 0$).

Now note that

$$\lim_{x\to 1^-} \frac{1-x}{(1-x^{\alpha m})^{\frac{1}{\alpha}}} = \lim_{x\to 1^-} \frac{(1-x^{\alpha m})^{1-\frac{1}{\alpha}}}{m x^{\alpha m-1}} = 0$$

for $1-\frac{1}{\alpha} > 0$, i.e., $\alpha > 1$, and that for $\alpha = 1$, the above sum is $O(1)$. Hence altogether it follows that

$$\sum_{n=1}^{N} R_n\big(x^n\big) = \sum_{k=1}^{m} \sum_{j=0}^{[N/m]} R_k\big(x^{jm}\big) + O(1) = 0 + O(1) = O(1)$$

by assumption. Hence the series for $F(s,x)$ is uniformly convergent for $\sigma > 0,\ x \in \mathrm{I}$.

If the $R_k(x)$ are continuous, then the uniformly convergent series of continuous functions being continuous, we conclude that $F(s,x)$ is continuous in $x \in \mathrm{I}$ for each s in $\{s|\sigma > 0\}$. Hence, in particular,

$$F(1,x) = \sum_{n=1}^{\infty} \frac{R_n(x^n)}{n}$$

being continuous in $x \in \mathrm{I}$, we conclude (5). □

We observe that Theorems 4–7 [1] all depend on the same principle and can be readily unified in the following Theorem 2. We look at [1, Theorem 6] which in a sense gives all that is necessary to uplift Theorems 4–7 to our Theorem 2.

We consider the limit as $z = e^{\pi i\tau} \to 1$ (we write here z in place of q in [1]) i.e., $\tau \to 0$ in the upper half-plane, which we refer to as $\tau \to 0$ (in $\mathcal{H}$). The following is an example for $\tau \to 0$ only philosophically speaking $\tau' := -\frac{1}{\tau}$; introducing the function

$$h(y) = 2\log\eta\left(-\frac{1}{\tau}\right) - 4\log\eta\left(-\frac{1}{2}\frac{1}{\tau}\right), \tag{9}$$

we have $-\frac{1}{\tau} \to 0$ (in $\mathcal{H}$), where $-\frac{1}{\tau} = \frac{i}{\pi}\log 1/y$, $y = e^{-\pi i/\tau}$, and $y \to 1^-$.

By definition,

$$\eta(\tau) = z^{1/12}\prod_{n=1}^{\infty}\left(1 - z^{2n}\right), \quad z = e^{\pi i\tau} \ (\tau \in \mathcal{H}), \tag{10}$$

whence as $z \to 0$ ($\tau \in \mathcal{H}$), we get

$$\lim_{z\to 0}\log\eta(\tau) = \frac{1}{12}\pi i\tau + \tilde{\omega}(\tau), \tag{11}$$

where $\tilde{\omega}(\tau)$ is a continuous function, and $\tilde{\omega}(0) = 0$. This function is rather immaterial and can be regarded as something like $o(1)$, a negligible term.

By using the eta transformation formula

$$\log\eta\left(-\frac{1}{\tau}\right) = \log\eta(\tau) + \frac{1}{4}\log\left(-\tau^2\right), \tag{12}$$

we rewrite (9) as

$$h(y) = 2\log\eta(\tau) - 4\log\eta(2\tau) - \frac{1}{2}\log\left(-\frac{1}{\tau^2}\right) + \log\left(-\frac{1}{(2\tau)^2}\right), \tag{13}$$

which is [1, p. 71, lines 2–3] with $\tau = \pi i(\log\frac{1}{y})^{-1}$, which is one half of τ in [1].

Applying (11), we obtain

$$h(y) = \frac{\pi^2}{2\log\frac{1}{y}} + \log\left(\log\left(\frac{1}{y}\right)\right) - \log 4\pi + \omega(y), \tag{14}$$

where we denote *once and for all* by $\omega(y)$ a continuous function in $y \in (0, 1]$ such that $\omega(1) = 0$. By using

$$\left(\log\frac{1}{y}\right)^{-1} = \frac{1}{1-y} - \frac{1}{2} + \omega(y) \tag{15}$$

and

$$\log\left(\log\frac{1}{y}\right)^{-1} = -\log(1-y) + \omega(y), \tag{16}$$

as $y \to 1^-$, $0 < y < 1$, we have

$$h(y) = \frac{\pi^2}{2(1-y)} - \frac{\pi^2}{4} + \log(1-y) - \log 4\pi + \omega(y),$$

which is [1, Theorem 6].

We slightly modify the above procedure and formulate it in the general setting and state it as the following:

Theorem 2 *Let*

$$F(\tau) = a \log \eta(\tau) + b \log \eta(2\tau) + c \log \eta\left(\frac{\tau}{2}\right), \tag{17}$$

where $a, b, c \in \mathbb{Q}$ *are arbitrary coefficients. Then*

$$F(\tau) = \frac{1}{12}\left(a + \frac{1}{2}b + 2c\right)\pi i\left(-\frac{1}{\tau}\right) - \frac{1}{4}\log\left(4^{b-c}\left(-\tau^2\right)^{a+b+c}\right) + \tilde{\omega}(\tau) \tag{18}$$

as $\tau \to i\infty$ *or* $-\frac{1}{\tau} \to 0$ (in $\mathcal{H}$).

Proof By (12),

$$\begin{aligned} F(\tau) = {} & a \log \eta\left(-\frac{1}{\tau}\right) + b \log \eta\left(-\frac{1}{2}\frac{1}{\tau}\right) + c \log \eta\left(-2\frac{1}{\tau}\right) \\ & - \frac{1}{4}\left(a \log\left(-\tau^2\right) + b \log\left(-4\tau^2\right) + c \log\left(-\frac{1}{4}\tau^2\right)\right), \end{aligned} \tag{19}$$

which by (11), becomes (18), completing the proof. □

In [1, Theorem 6] is formula (17) above without the last term and $a = 2$, $b = 0$, $c = -4$, $\tau = 1 + \frac{i}{\pi}\log\frac{1}{y}$, i.e., the value of $\sum_{p=1}^{\infty} \frac{4y^p}{p(1-y^p)}$ $(0 < y < 1)$.

Example 1 For $a = 12,\ b = -4,\ c = -8$, we have

$$f(y) = F\left(\frac{i}{\pi}\log\frac{1}{y}\right) = \frac{\pi^2}{2}\left(\log\frac{1}{y}\right)^{-1} - \log 4 + \omega(y),$$

which reduces, by (15), to

$$f(y) = \frac{\pi^2}{2(1-y)} - \frac{\pi^2}{4} - \log 4 + \omega(y),$$

which is [1, Theorem 4], i.e., the value of $\sum_{p=1}^{\infty} \frac{8y^p}{p(1-y^{2p})}$ $(0 < y < 1)$, and is the singular part of $-\log k'$.

Example 2 For $a = 10,\ b = -4,\ c = -4$, we have

$$g(y) = F\left(\frac{i}{\pi}\log\frac{1}{y}\right) = \log\frac{2K}{\pi} = -\frac{1}{4}\log\tau^4 + \omega(\tau)$$
$$= -\log\left(\log\frac{1}{y}\right) + \log\pi + \omega(y),$$

which reduces, by (16), to

$$g(y) = -\log(1-y) + \log\pi + \omega(y),$$

which is [1, Theorem 5], i.e., the value of $\sum_{p=1}^{\infty}\frac{4y^p}{p(1+y^p)}$ $(0 < y < 1)$.

Example 3 For $a = -3,\ b = \frac{4}{3},\ c = \frac{2}{3}$, we have

$$u(y) = \log\frac{2K}{\pi} = \frac{1}{6}\left(\frac{\pi^2}{2\log\frac{1}{y}} + \log\left(\log\frac{1}{y}\right) - \log 4\pi\right.$$
$$\left. + 2\log\left(\log\frac{1}{y}\right) - 2\log\pi\right) + \omega(y),$$

which reduces, by (15) and (16), to

$$u(y) = \frac{\pi^2}{12}\frac{1}{1-y} + \frac{1}{2}\log(1-y) - \frac{\pi^2}{24} - \frac{1}{2}\log\pi + \omega(y),$$

which is [1, Theorem 7], i.e., the value of $\sum_{n=1}^{\infty}\frac{y^{2n}}{n(1-y^{2n})}$ $(0 < y < 1)$.

3 Riemann's posthumous fragment II revisited

All the subsequent theorems are rephrases of the results of Jacobi, and we state them as the following:

Definition The elliptic modular functions $k = k(z),\ K = K(z)$, and $k' = k'(z)$ are defined respectively by

$$\log k - \log 4\sqrt{z} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{4z^n}{1+z^n}, \tag{20}$$

$$\log\frac{2K}{\pi} = \sum_{p=1}^{\infty}\frac{4z^p}{p(1+z^p)}, \tag{21}$$

and

$$-\log k' = \sum_{p=1}^{\infty}\frac{8z^p}{p(1-z^{2p})}, \tag{22}$$

where in the last two sums, we follow Riemann and let p run through odd integers, i.e., these sums are odd parts.

The procedure remains the same for all cases, and we fix the following descriptions once and for all in what follows.

Given a rational number $\xi = \frac{M'}{Q'}$ in its lowest terms, i.e., $(M', Q') = 1$, there are three cases possible:

(i) M' is even, Q' is odd, in which case we think of $M = M'$ and $Q = Q'$.
(ii) M', Q' are both odd, in which case we think of $M = 2M'$ even and $Q = 2Q'$ even.
(iii) M' is odd, and Q' is even, in which case we think of $M = 2M'$ and $Q = 2Q'$.

We may fix $\xi = \frac{M}{Q}$ with an even integer M, and $\alpha = e^{\pi i/Q}$, so that α is a primitive $2Q$th root of unity. It follows that $\alpha^{\frac{MQ}{2}} = (-1)^{\frac{M}{2}}$ and $(M, Q) = 1$ resp. 2 corresponding to Q being odd resp. even. Further, let $z = ye^{\xi\pi i} = y\alpha^M$, $0 < y < 1$, $y \to 1-$. Hereby we note that since the R-function is chosen essentially in the form

$$R_n(x) = (-1)^n \frac{4\alpha^{Mn}x}{1+\alpha^{Mn}x}, \quad x \in [0, 1], \tag{23}$$

it follows that the singularity can occur only when $n \equiv \frac{Q}{2} \pmod{Q}$, i.e., only in cases (ii) and (iii), where the denominator Q is even.

We note the choice of R-function which should be made taking into account two aspects, removal of singularities and the vanishing condition. For even Q, this leads to the following modifications of (23):

– Considering $\log k$, we put

$$R_n(x) = \begin{cases} (-1)^n \frac{4\alpha^{Mn}x}{1+\alpha^{Mn}x}, & x \in [0, 1],\ n \not\equiv \frac{Q}{2} \pmod{Q}, \\ (-1)^n (\frac{-4x}{1-x} + \frac{4Qx^{Q/2}}{1-x^Q}), & x \in [0, 1),\ n \equiv \frac{Q}{2} \pmod{Q}, \\ 2(-1)^{\frac{Q}{2}}, & x = 1,\ n \equiv \frac{Q}{2} \pmod{Q}. \end{cases} \tag{24}$$

– For $\log \frac{2K}{\pi}$, we define

$$R_p(x) = \begin{cases} \frac{4\alpha^{Mp}x}{1+\alpha^{Mp}x} - 2x, & x \in [0, 1],\ p \not\equiv \frac{Q}{2} \pmod{Q}, \\ \frac{-4x}{1-x} - 2x + \frac{2Qx^{Q/2}}{1-x^{Q/2}} + Qx, & x \in [0, 1),\ p \equiv \frac{Q}{2} \pmod{Q}, \\ 0, & x = 1,\ p \equiv \frac{Q}{2} \pmod{Q}, \end{cases} \tag{25}$$

if $\frac{Q}{2}$ is odd and

$$R_p(x) = \frac{4\alpha^{Mp}x}{1+\alpha^{Mp}x} - \frac{4x^{Q/2}}{1+x^{Q/2}}, \quad x \in [0, 1], \tag{26}$$

if it is even.

– For $-\log k'$, we take

$$R_p(x) = \begin{cases} \frac{8\alpha^{Mp}x}{1-\alpha^{2Mp}x^2}, & x \in [0,1],\ p \not\equiv 0 \ (\mathrm{mod}\ \frac{Q}{2}), \\ -(\frac{8x}{1-x^2} - \frac{4Qx^{Q/2}}{1-x^Q}), & x \in [0,1),\ p \equiv 0 \ (\mathrm{mod}\ \frac{Q}{2}), \\ 0, & x = 1,\ p \equiv 0 \ (\mathrm{mod}\ \frac{Q}{2}), \end{cases} \tag{27}$$

if $\frac{Q}{2}$ is odd and

$$R_p(x) = \frac{8\alpha^{Mp}x}{1-\alpha^{2Mp}x^2}, \quad x \in [0,1], \tag{28}$$

if it is even.

Clearly, these modifications of (23), corresponding to cases (ii) and (iii), do not any longer have a singularity at $x = 1$ and satisfy the vanishing condition (3) with $2Q$ in place of q. Thus we are under the conditions to use (5) for even Q.

We shall state a bigger theorem by unifying all three cases, and it is easier to be compared with Arias de Reyna's results (see Remark 3 below). In the statement of our theorems we always assume these classifications and the notation.

The following two quite elementary lemmas are essentially due to [1] but are useful in unifying the proofs of our new theorem.

Lemma 1 *Let* $\alpha^M = e^{\pi i \xi}$ *be the first* $2Q$*th primitive root of unity,* $Q > 1$*. Then, for* $n \not\equiv \frac{Q}{2}$ (mod Q), *we have the identity*

$$\frac{4\alpha^{Mn}}{1+\alpha^{Mn}} = 2\sum_{r=1}^{Q-1}(-1)^r\left(\frac{r-Q}{Q} + (-1)^Q\frac{r}{Q}\right)\alpha^{Mnr} + 2, \tag{29}$$

resp.

$$\frac{4\alpha^{Mn}}{1+\alpha^{Mn}} = (-1)^Q 2\sum_{r=1}^{Q-1}(-1)^r\frac{r}{Q}\left(\alpha^{Mnr} - \alpha^{-Mnr}\right) + 2. \tag{29$'$}$$

Proof of Lemma 1 In the elementary relation

$$\frac{1}{1+x} = \sum_{r=0}^{2Q-1}\frac{(-1)^r x^r}{1-x^{2Q}},$$

we put $x = t\alpha^{Mn}$ to obtain

$$\frac{1}{1+t\alpha^{Mn}} = \sum_{r=0}^{2Q-1}\frac{(-1)^r(t\alpha^{Mn})^r}{1-t^{2Q}}.$$

Noting that $\alpha^{Mn} \neq -1$ for $n \not\equiv \frac{Q}{2}$ (mod Q) and applying L'Hospital's rule to the right-hand side, we may take the limit of both sides as $t \to 1$ to obtain

$$\frac{2}{1+\alpha^{Mn}} = -\frac{1}{Q}\sum_{r=0}^{2Q-1}(-1)^r r\alpha^{Mnr}. \tag{30}$$

Multiplying (30) by $2\alpha^{Mn}$ and slightly rewriting, we obtain

$$\frac{4\alpha^{Mn}}{1+\alpha^{Mn}} = 2\sum_{r=1}^{2Q-1}(-1)^r\frac{r-Q}{Q}\alpha^{Mnr} + 2.$$

On the right-hand side, replacing r ($Q \le r \le 2Q-1$) by $Q+r$ ($0 \le r \le Q-1$), we conclude (29). Equation (29)$'$ follows by replacing r by $Q-r$, completing the proof. □

Remark 1 The RHS of (29) resp. (29)$'$ gives the modified value $R_p(1) = 0$ of (25) for $p \equiv 0$ (mod $\frac{Q}{2}$) and is a universal expression for $R_p(1)$ of (25), (26), and (47).

The RHS of (31) gives the modified value $R_n(1) = 2(-1)^{Q/2}$ of the R-function (24) for $n \equiv \frac{Q}{2}$ (mod Q) and $R_n(1) = -2$ of (23) for $n \equiv 0$ (mod Q); so that it is a universal expression for $R_n(1)$:

$$R_n(1) = 2(-1)^n\sum_{r=1}^{Q-1}(-1)^r\left(\frac{r-Q}{Q} + (-1)^Q\frac{r}{Q}\right)\alpha^{Mnr} + 2(-1)^n. \tag{31}$$

The identity

$$\frac{\alpha^{Mp}-1}{\alpha^{Mp}+1} = -\sum_{r=1}^{Q-1}(-1)^r\alpha^{Mpr} \quad \text{(if } Q \text{ odd)} \tag{32}$$

implies the formula for the odd part

$$2\sum_{p=1}^{\infty}\frac{1}{p}\frac{\alpha^{Mp}-1}{\alpha^{Mp}+1} = -\sum_{r=1}^{Q-1}(-1)^r\left(2l_1\left(\frac{Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right)\right), \tag{33}$$

which is only valid for Q odd. The RHS of (29)$'$ gives the general formula for the odd part

$$2\sum_{p=1}^{\infty}\frac{1}{p}\frac{\alpha^{Mp}-1}{\alpha^{Mp}+1} = (-1)^Q\sum_{r=1}^{Q-1}(-1)^r\frac{r}{Q} \times\left(2l_1\left(\frac{Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right). \tag{34}$$

The proof of (33) (or (34)) is almost trivial: Substituting (32), we have

$$LHS = -2\sum_{p=1}^{\infty}\frac{1}{p}\sum_{r=1}^{Q-1}(-1)^r\alpha^{Mpr}.$$

Since the sum over p is the odd part, we have, by (1),

$$LHS = -2\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{r=1}^{Q-1}(-1)^r\alpha^{Mnr} - \sum_{n=1}^{\infty}\frac{1}{2n}\sum_{r=1}^{Q-1}(-1)^r\alpha^{2Mnr}\right),$$

which amounts to (33).

Lemma 2 *Let p be an odd integer such that $p \not\equiv 0 \pmod{Q}$, and α, Q be defined as in Lemma 1. Then*

$$\frac{8\alpha^{Mp}}{1-\alpha^{2Mp}} = -4(-1)^Q\sum_{r=0}^{[\frac{Q}{2}]-1}\frac{2r+1}{Q}\bigl(\alpha^{Mp(2r+1)} - \alpha^{-Mp(2r+1)}\bigr). \tag{35}$$

This follows directly from (29)′.

Remark 2 The RHS of (35) gives the modified value $R_p(1) = 0$ of (48) for $p \equiv 0 \pmod{Q}$ (Q odd), it is a universal expression for $R_p(1)$ of (27), (28), and (48), where p runs through the odd integers:

$$R_p(1) = -4(-1)^Q\sum_{r=0}^{[\frac{Q}{2}]-1}\frac{2r+1}{Q}\bigl(\alpha^{Mp(2r+1)} - \alpha^{-Mp(2r+1)}\bigr). \tag{36}$$

Therefore,

$$\begin{aligned}\sum_{p=1}^{\infty}\frac{1}{p}R_p(1) &= -2(-1)^Q\sum_{r=0}^{[\frac{Q}{2}]-1}\frac{2r+1}{Q}\\ &\quad\times\left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right)\right.\\ &\quad\left.- l_1\left(\frac{M(2r+1)}{Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right).\end{aligned} \tag{37}$$

Theorem 3 [1, Theorems 8–15] *Let $\xi = \frac{M}{Q}$ with $Q > 1$. Then we have*

$$\begin{aligned}\log k &= \frac{1}{2}\log y + \frac{M\pi}{2Q}i + \omega(y)\\ &\quad - 2\sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right) - l_1\left(\frac{Mr}{2Q}\right)\right),\end{aligned} \tag{38}$$

$$\log\frac{2K}{\pi} = -\log(1-y) + \omega(y) + \log\frac{\pi}{Q} + \sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right) - 2l_1\left(\frac{Mr}{2Q}\right)\right), \tag{39}$$

and

$$\begin{aligned} -\log k' = \frac{\pi^2}{2Q^2(1-y)} - \frac{\pi^2}{4Q^2} - \log 4 + \omega(y) + 2\sum_{r=0}^{\frac{Q-1}{2}-1}\frac{2r+1}{Q} \\ \times\left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right)\right. \\ \left. - 2l_1\left(\frac{-M(2r+1)}{2Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right) \end{aligned} \tag{40}$$

for Q odd;

$$\begin{aligned} \log k = \frac{1}{2}\log y + \frac{M\pi}{2Q}i + \frac{2\pi^2}{Q^2}\frac{1}{(1-y)} + \omega(y) - \frac{\pi^2}{Q^2} - \log 4 \\ - 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(l_1\left(\frac{Mr}{2Q}\right) - l_1\left(\frac{-Mr}{2Q}\right)\right), \end{aligned} \tag{41}$$

$$\begin{aligned} \log\frac{2K}{\pi} = -\frac{2\pi^2}{Q^2(1-y)} - \log(1-y) + \frac{\pi^2}{Q^2} + \log\frac{8\pi}{Q} + \omega(y) \\ - \sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right)\right. \\ \left. - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right), \end{aligned} \tag{42}$$

and

$$\begin{aligned} -\log k' = -\frac{2\pi^2}{Q^2(1-y)} + \frac{\pi^2}{Q^2} + \log 4 + \omega(y) - 2\sum_{r=0}^{\frac{Q}{2}-1}\frac{2r+1}{Q} \\ \times\left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right)\right. \\ \left. - l_1\left(\frac{M(2r+1)}{Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right) \end{aligned} \tag{43}$$

for Q even and $\frac{Q}{2}$ odd;

$$\log k = \frac{1}{2}\log y - \frac{2\pi^2}{Q^2}\frac{1}{(1-y)} + \omega(y) + \frac{\pi^2}{Q^2} + \log 4 + \frac{M\pi}{2Q}i$$
$$- 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(l_1\left(\frac{Mr}{2Q}\right) - l_1\left(\frac{-Mr}{2Q}\right)\right), \tag{44}$$

$$\log\frac{2K}{\pi} = -\log(1-y) + \log\frac{2\pi}{Q} + \omega(y) + 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}$$
$$\times\left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right), \tag{45}$$

and

$$-\log k' = \omega(y) - 2\sum_{r=0}^{\frac{Q}{2}-1}\frac{2r+1}{Q}$$
$$\times\left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right)\right.$$
$$\left.- l_1\left(\frac{M(2r+1)}{Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right) \tag{46}$$

for $\frac{Q}{2}$ even. Here $2\pi i\frac{M}{Q}$ is one of the values of $\log e^{2\pi i\frac{M}{Q}}$.

Proof Choose the function $R_n(x)$, $x \in [0, 1]$, as (23) if Q is odd and as (24) if Q is even for $\log k$; as

$$R_p(x) = \frac{4\alpha^{Mp}x}{1+\alpha^{Mp}x} - \frac{4x^Q}{1+x^Q} \tag{47}$$

if Q is odd and as (25) or (26) according to $\frac{Q}{2}$ is odd or even for $\log\frac{2K}{\pi}$; and

$$R_p(x) = \begin{cases} \frac{8\alpha^{Mp}x}{1-\alpha^{2Mp}x^2}, & p \not\equiv 0 \pmod{Q},\ x \in [0, 1], \\ \frac{8x}{1-x^2} - \frac{8Qx^Q}{1-x^Q}, & p \equiv 0 \pmod{Q},\ x \in [0, 1), \\ 0, & p \equiv 0 \pmod{Q},\ x = 1, \end{cases} \tag{48}$$

if Q odd and as (27) or (28) according to $\frac{Q}{2}$ is odd or even for $-\log k'$.

The following procedure of the proof is similar, and therefore we only show the proofs of (38), (39), (40), (44), (45), and (46).

Noting that the R-function is (23), (47), and (48) for Q odd, we may express (20), (21), and (22) as

$$\log k = \log 4\sqrt{y} + \frac{\xi\pi}{2} i + \sum_{n=1}^{\infty} \frac{1}{n} R_n\left(y^n\right), \tag{49}$$

$$\log \frac{2K}{\pi} = g\left(y^Q\right) + \sum_{p=1}^{\infty} \frac{1}{p} R_p\left(y^p\right), \tag{50}$$

and

$$-\log k' = f\left(y^{Q^2}\right) + \sum_{p=1}^{\infty} \frac{1}{p} R_p\left(y^p\right), \tag{51}$$

where $f(y)$ and $g(y)$ are given by Examples 1 and 2.

We apply Theorem 1 to obtain

$$\log k = \log 4\sqrt{y} + \omega(y) + \frac{\xi\pi}{2} i + \sum_{n=1}^{\infty} \frac{1}{n} R_n(1), \tag{52}$$

$$\log \frac{2K}{\pi} = -\log(1-y) + \log \frac{\pi}{Q} + \omega(y) + \sum_{p=1}^{\infty} \frac{1}{p} R_p(1), \tag{53}$$

and

$$-\log k' = \frac{\pi^2}{2Q^2(1-y)} - \frac{\pi^2}{4Q^2} - \log 4 + \sum_{p=1}^{\infty} \frac{1}{p} R_p(1) + \omega(y). \tag{54}$$

Applying (31) with Q (Q odd), we have

$$\sum_{n=1}^{\infty} \frac{R_n(1)}{n} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} - 2\sum_{r=1}^{q-1} (-1)^r \sum_{n=1}^{\infty} \frac{(-1)^n (\alpha^{2mr})^n}{n}.$$

Now the inner alternate sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\alpha^{2mr})^n$ is just $l_1(\frac{2mr}{q}) - l_1(\frac{mr}{q})$ by (2), the first sum is $-2\log 2$, and we conclude (38).

Substituting (33) into (53), we conclude (39).

Substituting (37) with Q odd into (54), we conclude (40).

Correspondingly to (49), (50), and (51), applying Theorem 1, we obtain

$$\begin{aligned} \log k = \frac{1}{2} \log y + \frac{\xi\pi}{2} i + \sum_{n=1}^{\infty} \frac{1}{n} R_n(1) \\ - \frac{2\pi^2}{Q^2(1-y)} + \frac{\pi^2}{Q^2} + \log 4 + \omega(y), \end{aligned} \tag{55}$$

$$\log \frac{2K}{\pi} = -\log(1-y) + \log \frac{2\pi}{Q} + \omega(y) + \sum_{p=1}^{\infty} \frac{1}{p} R_p(1), \tag{56}$$

and

$$-\log k' = \omega(y) + \sum_{p=1}^{\infty} \frac{1}{p} R_p(1). \tag{57}$$

Substituting the evaluations

$$\sum_{n=1}^{\infty} \frac{1}{n} R_n(1) = 2 \sum_{r=1}^{\frac{Q}{2}-1} (-1)^r \frac{2r}{Q} \left(l_1\left(\frac{Mr}{2Q}\right) - l_1\left(-\frac{Mr}{2Q}\right)\right), \tag{58}$$

$$\sum_{p=1}^{\infty} \frac{1}{p} R_p(1) = 2 \sum_{r=1}^{\frac{Q}{2}-1} (-1)^r \frac{2r}{Q} \times \left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right), \tag{59}$$

$$\sum_{p=1}^{\infty} \frac{1}{p} R_p(1) = -2 \sum_{r=0}^{\frac{Q}{2}-1} \frac{2r+1}{Q} \times \left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right), \tag{60}$$

we conclude (44), (45), and (46), completing the proof. □

Remark 3 For the sake of easier comparison with Arias de Reyna's results, we remark that for (38), (39), and (40), the readers may refer to [1, Theorems 8, 13, and 12 (m even)], for (41), (42), and (43), the readers may refer to [1, Theorem 9, 14, and 12 (m odd)], and for (44), (45), and (46), the readers may refer to [1, Theorem 10, 11, and 15], respectively.

4 Concluding remark

We only investigate the part of Theorems 4 to 7 and Theorems 8 to 15 of [1] of Riemann's fragment developed by Arias de Reyna, but we have not touched the more advanced part of Riemann's fragment such as the Lambert series associated to the linear combinations of the Dedekind eta function and some modified Dedekind sums, also developed in [1]. We shall return to the study of this topic elsewhere.

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