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Arithmetical properties of multiple Ramanujan sums

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Abstract In the present paper, we introduce a multiple Ramanujan sum for arithmetic functions, which gives a multivariable extension of the generalized Ramanujan sum studied by D.R. Anderson and T.M. Apostol. We then find fundamental arithmetic properties of the multiple Ramanujan sum and study several types of Dirichlet series involving the multiple Ramanujan sum. As an application, we evaluate higher-dimensional determinants of higher-dimensional matrices, the entries of which are given by values of the multiple Ramanujan sum.

Keywords Ramanujan sum · Divisor function · Dirichlet convolution · Dirichlet series · Smith determinant · Hyperdeterminant

Mathematics Subject Classification (2000) Primary 11A25 · Secondary 11C20

1 Introduction

In 1918, Ramanujan [16] studied the sum

$$c(k,n) := \sum_{\substack{l \pmod{k} \\ \gcd(l,k)=1}} e(k,nl) = \sum_{d \mid \gcd(k,n)} \mu\left(\frac{k}{d}\right) d, \tag{1.1}$$

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where, in the first sum, l runs over a reduced residue system modulo k with $\gcd(l,k)=1$, $e(r,n):=\exp(2\pi\sqrt{-1}n/r)$, and μ is the Möbius function. The sum $c(k,n)$ is called the Ramanujan sum (or the Ramanujan trigonometric sum) and is widely investigated in connection with, for example, even arithmetic functions [7, 8] and cyclotomic polynomials [13, 14]. See [12] for the arithmetic theory of the Ramanujan sum. Among several generalizations and variations of $c(k,n)$, Anderson and Apostol [1] (see also [2]) considered the sum

$$S_{f,g}(k,n):=\sum_{d|\gcd(k,n)} f\left(\frac{k}{d}\right)g(d), \tag{1.2}$$

where f and g are arithmetic functions. Clearly, $S_{f,g}(k,n)$ extends the right-most expression in formula (1.1) and hence gives a generalization of the Ramanujan sum.

Motivated by the study of the above generalized Ramanujan sum, in the present paper, we examine the following type of multiple sum for arithmetic functions $f_1,\dots,f_{m+1}$;

$$S_{f_1,\dots,f_{m+1}}(n_1,\dots,n_{m+1}) := \sum_{\substack{d_j|\gcd(n_1,\dots,n_{j+1}) \\ (j=1,\dots,m)}} f_1\left(\frac{n_1}{d_1}\right) f_2\left(\frac{d_1}{d_2}\right)\cdots f_m\left(\frac{d_{m-1}}{d_m}\right) f_{m+1}(d_m),$$

where $\gcd(n_1,\dots,n_{j+1})$ is the greatest common divisor of $n_1,\dots,n_{j+1}$. We call this a *multiple Ramanujan sum* for $f_1,\dots,f_{m+1}$. Notice that the above expression gives the generalized Ramanujan sum (1.2) when $m=1$ and, moreover, the Dirichlet convolution $f_1 * \cdots * f_{m+1}$ of $f_1,\dots,f_{m+1}$ in the "diagonal case" $n_1=\cdots=n_{m+1}$.

The present paper is organized as follows. In Sect. 2, we introduce a multiple Ramanujan sum $S^{(\gamma_1,\dots,\gamma_m)}_{f_1,\dots,f_{m+1}}$ with positive integer parameters $\gamma_1,\dots,\gamma_m$ so that $S_{f_1,\dots,f_{m+1}} = S^{(1,\dots,1)}_{f_1,\dots,f_{m+1}}$ and study its fundamental properties as a multivariable arithmetic function, such as the degeneracies and the multiplicativity (see [20] for the theory of multivariable arithmetic functions). Then, since $S^{(\gamma_1,\dots,\gamma_m)}_{f_1,\dots,f_{m+1}}$ belongs to the class of even arithmetic functions (mod n_1) as a function of $n_2,\dots,n_{m+1}$ in the sense of Cohen [8], we calculate its finite Fourier expansion, which any even arithmetic function possesses. This expression is important with respect to Sect. 4. In Sect. 3, we study several types of Dirichlet series having coefficients that are given by $S^{(\gamma_1,\dots,\gamma_m)}_{f_1,\dots,f_{m+1}}$. We treat not only single-variable Dirichlet series but also multivariable Dirichlet series. For instance, an analogue of the formula of Borwein and Choi [3], which contains the classical Ramanujan formula concerning the divisor function σ_a, is obtained. Section 4 is devoted to a higher-dimensional generalization of the so-called Smith determinant [18]. We evaluate higher-dimensional determinants (hyperdeterminants) of higher-dimensional matrices (hypermatrices), the entries of which are given by values of $S^{(\gamma_1,\dots,\gamma_m)}_{f_1,\dots,f_{m+1}}$. In fact, we derive a hyperdeterminant formula for even multivariable arithmetic functions. This includes the results of McCarthy [11] and Bourque and Ligh [4] for the two-dimensional case, that is, the usual determinant case, and partially of Haukkanen [9] for the higher-dimensional case.

We use the following notation. The set of natural numbers, the ring of rational integers, the field of real numbers, and the field of complex numbers are denoted respectively as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$. For $n_1, \ldots, n_k \in \mathbb{N}$, $\gcd(n_1, \ldots, n_k)$ (resp. $\mathrm{lcm}(n_1, \ldots, n_k)$) represents the greatest common divisor (resp. least common multiple) of $n_1, \ldots, n_k$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer not exceeding x. We denote the Möbius function by $\mu(n)$, the Euler totient function by $\varphi(n)$, the power function by $\delta^x(n) := n^x$ for $x \in \mathbb{C}$, and the identity element in the ring of arithmetic functions with respect to the Dirichlet convolution $*$ by $\varepsilon(n) := \lfloor \frac{1}{n} \rfloor (= 1$ if $n = 1$ and 0 otherwise). Note that $\delta^0 * \mu = \varepsilon$. Throughout the paper, a product (resp. a sum) over an empty set always equals 1 (resp. 0).

2 A multiple Ramanujan sum

2.1 Preliminaries: γ-convolutions

Let $\mathcal{A}$ be the set of all complex-valued arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. We always understand that, for $f \in \mathcal{A}$, $f(x) = 0$ if $x \notin \mathbb{N}$. The set of all multiplicative and completely multiplicative arithmetic functions are respectively denoted by $\mathcal{M}$ and $\mathcal{M}^c$. Namely, $f \in \mathcal{M}$ (resp. $f \in \mathcal{M}^c$) means that $f(mn) = f(m)f(n)$ for $\gcd(m,n) = 1$ (resp. for all $m, n \in \mathbb{N}$). As usual, the product $fg \in \mathcal{A}$ of $f, g \in \mathcal{A}$ is defined by $(fg)(n) := f(n)g(n)$.

Let $\gamma \in \mathbb{N}$. We define the γ-convolution of $f, g \in \mathcal{A}$ by

$$(g *_\gamma f)(n) := \sum_{d^\gamma | n} f\left(\frac{n}{d^\gamma}\right) g\big(d^\gamma\big).$$

In particular, $*_1 = *$ stands for the usual Dirichlet convolution. Define the function $a_\gamma \in \mathcal{A}$ by $a_\gamma(n) = 1$ if there exists $d \in \mathbb{N}$ such that $n = d^\gamma$, and 0 otherwise. Then, it is clear that $g *_\gamma f = g^{[\gamma]} * f$, where $g^{[\gamma]} := a_\gamma g$. Note that the product $*_\gamma$ does not satisfy the commutativity or associativity properties unless $\gamma = 1$. We therefore inductively define the $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$-convolution for $\gamma_1, \ldots, \gamma_m \in \mathbb{N}$ of $f_1, \ldots, f_{m+1} \in \mathcal{A}$ by

$$\begin{aligned}
&(f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1)(n) \\
&\quad := \big((f_{m+1} *_{\gamma_m} \cdots *_{\gamma_2} f_2) *_{\gamma_1} f_1\big)(n) \\
&\quad = \sum_{d_m^{\gamma_m} | \cdots | d_1^{\gamma_1} | n} f_1\left(\frac{n}{d_1^{\gamma_1}}\right) f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right) \cdots f_m\left(\frac{d_{m-1}^{\gamma_{m-1}}}{d_m^{\gamma_m}}\right) f_{m+1}\big(d_m^{\gamma_m}\big).
\end{aligned}$$

Let $L(s; f) := \sum_{n=1}^{\infty} f(n) n^{-s}$ be the Dirichlet series attached to $f \in \mathcal{A}$, and $\sigma(f) \in \mathbb{R} \cup \{\infty\}$ be the abscissa of absolute convergence of $L(s; f)$. Define $f^{\langle\gamma\rangle} \in \mathcal{A}$ for $\gamma \in \mathbb{N}$ by $f^{\langle\gamma\rangle}(n) := f(n^\gamma)$. Then, it is clear that $L(\gamma s; f^{\langle\gamma\rangle}) = L(s; f^{[\gamma]})$ with $\sigma(f^{[\gamma]}) \leq \sigma(f)$. For $f_1, \ldots, f_{m+1} \in \mathcal{A}$, one can easily verify the following formula.

Proposition 2.1 *Suppose that* $\gamma_0|\gamma_1|\gamma_2|\cdots|\gamma_m$ *with* $\gamma_0 = 1$. *Then, we have, for* $\mathrm{Re}(s) > \max\{\sigma(f_1), \dots, \sigma(f_{m+1})\}$,

$$L(s; f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1) = \prod_{j=1}^{m+1} L\big(s; f_j^{[\gamma_{j-1}]}\big). \tag{2.1}$$

Remark 2.2 We cannot expect that $L(s; f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1)$ is expressed by a product of Dirichlet series such as (2.1) for general $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m) \in \mathbb{N}^m$.

In the next subsection, we will introduce a multiple Ramanujan sum for $f_1, \dots, f_{m+1} \in \mathcal{A}$, which gives the $\boldsymbol{\gamma}$-convolution $f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1$ in the diagonal case (for more detail, see Proposition 2.3(iv)).

2.2 Definition of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$ and its basic properties

Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m) \in \mathbb{N}^m$ and $\boldsymbol{f} = (f_1, \dots, f_{m+1}) \in \mathcal{A}^{m+1}$. We define the multiple Ramanujan sum $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}} = S_{f_1,\dots,f_{m+1}}^{(\gamma_1,\dots,\gamma_m)}$ of $\boldsymbol{f}$ with parameter $\boldsymbol{\gamma}$ by

$$\begin{aligned} &S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(n_1, \dots, n_{m+1}) \\ &:= \sum_{\substack{d_j^{\gamma_j} | \gcd(n_1,\dots,n_{j+1}) \\ (j=1,\dots,m)}} f_1\left(\frac{n_1}{d_1^{\gamma_1}}\right) f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right) \cdots f_m\left(\frac{d_{m-1}^{\gamma_{m-1}}}{d_m^{\gamma_m}}\right) f_{m+1}\big(d_m^{\gamma_m}\big), \end{aligned}$$

where the sum is taken over all m-tuples $(d_1, \dots, d_m) \in \mathbb{N}^m$ satisfying $d_j^{\gamma_j} | \gcd(n_1, \dots, n_{j+1})$ for each $1 \le j \le m$. Note that the summand vanishes unless $d_m^{\gamma_m} | \cdots | d_1^{\gamma_1} | n_1$. We write $S_{\boldsymbol{f}} := S_{\boldsymbol{f}}^{(1,\dots,1)}$ and understand that $S_f(n) = f(n)$ when $m = 0$. These are the fundamental properties of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$ that are obtained from the elementary properties of the gcd-function.

Proposition 2.3 (i) *For any* $1 \le j \le m+1$, *we have*

$$S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(n_1, \dots, n_{m+1}) = S^{(\gamma_1,\dots,\gamma_{j-1})}_{f_1,\dots,f_{j-1}, S^{(\gamma_j,\dots,\gamma_m)}_{f_j,\dots,f_{m+1}}(\cdot, n_{j+1},\dots,n_{m+1})}(n_1, \dots, n_j). \tag{2.2}$$

(ii) *For any* $1 \le j \le m+1$, *we have*

$$\begin{aligned} &S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(n_1, \dots, n_{j-1}, 1, n_{j+1}, \dots, n_{m+1}) \\ &\quad = f_j(1) \cdots f_{m+1}(1) \cdot S^{(\gamma_1,\dots,\gamma_{j-2})}_{f_1,\dots,f_{j-1}}(n_1, \dots, n_{j-1}). \end{aligned}$$

In particular, we have $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(1, n_2, \dots, n_{m+1}) = f_1(1) \cdots f_{m+1}(1)$.

(iii) *For any* $1 \le j \le m$, *we have*

$$\begin{aligned} &S^{(\gamma_1,\ldots,\gamma_m)}_{f_1,\ldots,f_{j-1},\varepsilon,f_{j+1},\ldots,f_{m+1}}(n_1,\ldots,n_{m+1}) \\ &\quad = S^{(\gamma_1,\ldots,\gamma_{j-1},\gamma_{j+1},\ldots,\gamma_m)}_{f_1,\ldots,f_{j-1},f_{j+1},\ldots,f_{m+1}}(n_1,\ldots,n_{j-1},\gcd(n_j,n_{j+1}), \\ &\qquad\qquad n_{j+2},\ldots,n_{m+1}) \end{aligned} \tag{2.3}$$

and $S^{(\gamma_1,\ldots,\gamma_m)}_{f_1,\ldots,f_m,\varepsilon}(n_1,\ldots,n_{m+1}) = S^{(\gamma_1,\ldots,\gamma_{m-1})}_{f_1,\ldots,f_m}(n_1,\ldots,n_m)$.

(iv) *Let* $n_1|n_j$ *for all* $2 \le j \le m+1$. *Then, we have*

$$S^{\boldsymbol{\gamma}}_{\boldsymbol{f}}(n_1,\ldots,n_{m+1}) = (f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1)(n_1). \tag{2.4}$$

Example 2.4 Let $\boldsymbol{a} = (a_1,\ldots,a_m) \in \mathbb{C}^m$. Define a multiple divisor function $\sigma^{\boldsymbol{\gamma}}_{\boldsymbol{a}}$ with parameter $\boldsymbol{\gamma}$ by

$$\sigma^{\boldsymbol{\gamma}}_{\boldsymbol{a}}(n_1,\ldots,n_{m+1}) := \sum_{\substack{d_m^{\gamma_m}|\cdots|d_1^{\gamma_1}|n_1 \\ d_j^{\gamma_j}|\gcd(n_1,\ldots,n_{j+1})\ (j=1,\ldots,m)}} d_1^{\gamma_1 a_1}\cdots d_m^{\gamma_m a_m}. \tag{2.5}$$

This is a generalization of the usual divisor function $\sigma_a(n) := \sum_{d|n} d^a$; $\sigma_a(n) = \sigma^{(1)}_a(n,n)$. Then, $\sigma^{\boldsymbol{\gamma}}_{\boldsymbol{a}} = S^{\boldsymbol{\gamma}}_{\boldsymbol{f}}$ with $f_j = \delta^{a_0+a_1+\cdots+a_{j-1}}$ for $1 \le j \le m+1$, where $a_0 = 0$. Similarly, for $\boldsymbol{b} = (b_1,\ldots,b_{m+1}) \in \mathbb{C}^{m+1}$, one can see that $\sigma^{\boldsymbol{\gamma}}_{\tilde{\boldsymbol{b}}}(n_1,\ldots,n_{m+1})$ $= n_1^{-b_1} S^{\boldsymbol{\gamma}}_{\delta^{b_1},\ldots,\delta^{b_{m+1}}}(n_1,\ldots,n_{m+1})$, where $\tilde{\boldsymbol{b}} := (b_2-b_1, b_3-b_2,\ldots,b_{m+1}-b_m) \in \mathbb{C}^m$. Note that the sum $Z^{\boldsymbol{\gamma}}_n(a_1,\ldots,a_m) := \sigma^{\boldsymbol{\gamma}}_{\boldsymbol{a}}(n,\ldots,n)$ is studied in [10] and called the multiple finite Riemann zeta function. If $\gamma_0|\gamma_1|\gamma_2|\cdots|\gamma_m$ with $\gamma_0 = 1$, from formula (2.1) we have $L(s; Z^{\boldsymbol{\gamma}}_{\cdot}(a_1,\ldots,a_m)) = \prod_{j=1}^{m+1}\zeta(\gamma_{j-1}(s-a_0-\cdots-a_{j-1}))$, where $\zeta(s) = L(s;\delta^0)$ is the Riemann zeta function.

Example 2.5 Let $f \in \mathcal{A}$. Then, the composition function $f \circ \gcd$ of f and the gcd-function can be expressed in terms of the multiple Ramanujan sum. Actually, from the degeneracy formula (2.3) we have $(f \circ \gcd)(n_1,\ldots,n_{m+1}) = S_{\boldsymbol{f}}(n_1,\ldots,n_{m+1})$ with $f_1 = \delta^0$, $f_2 = \cdots = f_m = \varepsilon$, and $f_{m+1} = f * \mu$.

We next show the multiplicative property of the multiple Ramanujan sum $S^{\boldsymbol{\gamma}}_{\boldsymbol{f}}$. Recall that an arithmetic function $F(n_1,\ldots,n_k)$ of k-variables is called multiplicative if

$$F(m_1n_1,\ldots,m_kn_k) = F(m_1,\ldots,m_k)\cdot F(n_1,\ldots,n_k)$$

for relatively prime k-tuples $(m_1,\ldots,m_k) \in \mathbb{N}^k$ and $(n_1,\ldots,n_k) \in \mathbb{N}^k$. Here, we say that $(m_1,\ldots,m_k)$ and $(n_1,\ldots,n_k)$ are relatively prime if $\gcd(m_i,n_j) = 1$ for all $\le i,j \le k$ or, equivalently, $\gcd(n_1\cdots n_k, m_1\cdots m_k) = 1$ (see [20]). In this case, we have the following Euler product expression:

$$F(n_1,\ldots,n_k) = \prod_p F\left(p^{\alpha_{p,1}},\ldots,p^{\alpha_{p,k}}\right),$$

where $\alpha_{p,j} = \mathrm{ord}_p n_j$ for $1 \leq j \leq k$. Note that this is a finite product since $F(1,\ldots,1) = 1$. Similarly as in the proof of Theorem 1 in [1], we can prove the following result.

Proposition 2.6 *The function $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$ is multiplicative if $f_1,\ldots,f_{m+1} \in \mathcal{M}$.*

2.3 Finite Fourier expansions

An arithmetic function $F(r;n_1,\ldots,n_k)$ of k variables $n_1,\ldots,n_k$ is called periodic (mod r) if $F(r;n_1,\ldots,n_k) = F(r;n_1',\ldots,n_k')$ whenever $n_j \equiv n_j'$ (mod r) for all $1 \leq j \leq k$ (see [12] for the case of $k=1$). It is well known that F is periodic (mod r) if and only if it has an expression of the form

$$F(r;n_1,\ldots,n_k) = \sum_{l_1,\ldots,l_k=1}^{r} a_r(l_1,\ldots,l_k)e(r,n_1l_1)\cdots e(r,n_kl_k) \tag{2.6}$$

and the coefficients $a_r(l_1,\ldots,l_k)$ are uniquely determined by

$$a_r(l_1,\ldots,l_k) = \frac{1}{r^k}\sum_{n_1,\ldots,n_k=1}^{r} F(r;n_1,\ldots,n_k)e(r,-n_1l_1)\cdots e(r,-n_kl_k). \tag{2.7}$$

Moreover, F is called even (mod r) if $F(r;n_1,\ldots,n_k) = F(r;\gcd(n_1,r),\ldots,\gcd(n_k,r))$. Note that F is periodic (mod r) if it is even (mod r). Then, as shown by Cohen [8], F is even (mod r) if and only if it has an expression of the form

$$F(r;n_1,\ldots,n_k) = \sum_{d_1,\ldots,d_k|r} \alpha_r(d_1,\ldots,d_k)c(d_1,n_1)\cdots c(d_k,n_k) \tag{2.8}$$

with

$$\alpha_r(d_1,\ldots,d_k) = \frac{1}{r^k}\sum_{\delta_1,\ldots,\delta_k|r} F(r;\delta_1,\ldots,\delta_k)c\left(\frac{r}{\delta_1},\frac{r}{d_1}\right)\cdots c\left(\frac{r}{\delta_k},\frac{r}{d_k}\right). \tag{2.9}$$

We call expressions (2.6) and (2.8) finite Fourier expansions and coefficients a_r and α_r finite Fourier coefficients of F.

By definition, the multiple Ramanujan sum $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(n_1,\ldots,n_{m+1})$ is even (mod n_1) as a function of m variables $n_2,\ldots,n_{m+1}$ for any $n_1 \in \mathbb{N}$. Then, let us calculate the finite Fourier expansions of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$. To do so, we add one parameter (or a weight) to $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$. Let ξ be an arithmetic function of m variables. Set

$$\begin{aligned} S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}(n_1,\ldots,n_{m+1}) &= S_{f_1,\ldots,f_{m+1}}^{(\gamma_1,\ldots,\gamma_m),\xi}(n_1,\ldots,n_{m+1}) \\ &:= \sum_{\substack{d_j^{\gamma_j}|\gcd(n_1,\ldots,n_{j+1}) \\ (j=1,\ldots,m)}} \xi\left(d_1^{\gamma_1},\ldots,d_m^{\gamma_m}\right) \\ &\quad \times f_1\left(\frac{n_1}{d_1^{\gamma_1}}\right)f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right)\cdots f_m\left(\frac{d_{m-1}^{\gamma_{m-1}}}{d_m^{\gamma_m}}\right)f_{m+1}\left(d_m^{\gamma_m}\right). \end{aligned}$$

Then, $S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}(n_1,\ldots,n_{m+1})$ is again even (mod n_1) as a function of $n_2,\ldots,n_{m+1}$ for any $n_1\in\mathbb{N}$ and $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}=S_{\boldsymbol{f}}^{\boldsymbol{\gamma},1_m}$, where $1_m(d_1,\ldots,d_m)\equiv 1$. We also write $S_{\boldsymbol{f}}^{\xi}:=S_{\boldsymbol{f}}^{(1,\ldots,1),\xi}$. As is the case of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$, one can show that $S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}$ is multiplicative if $f_1,\ldots,f_{m+1}$ and ξ are multiplicative.

For $\boldsymbol{f}=(f_1,\ldots,f_{m+1})\in\mathcal{A}^{m+1}$, let $G(\boldsymbol{f}):=(\delta^0 f_{m+1},\delta^1 f_m,\ldots,\delta^m f_1)$. Moreover, for $n\in\mathbb{N}$, define the m-variable function $H_n^{\boldsymbol{\gamma},\xi}$ by

$$H_n^{\boldsymbol{\gamma},\xi}(d_1,\ldots,d_m):=\left(\prod_{j=1}^{m}a_{\gamma_j}\left(\frac{n}{d_{m+1-j}}\right)\right)\cdot\xi\left(\frac{n}{d_m},\ldots,\frac{n}{d_1}\right).$$

Remark that $H_n^{(1,\ldots,1),1_m}=1_m$ for any $n\in\mathbb{N}$. Then, we obtain the following theorem. Roughly speaking, the finite Fourier coefficients of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}$ can again be written as a multiple Ramanujan sum.

Theorem 2.7 *Write the finite Fourier expansions of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}$ as*

$$\begin{aligned}S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}(n_1,\ldots,n_{m+1})&=\sum_{l_2,\ldots,l_{m+1}=1}^{n_1}a_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}(l_2,\ldots,l_{m+1})e(n_1,n_2l_2)\cdots e(n_1,n_{m+1}l_{m+1})\\&=\sum_{d_2,\ldots,d_{m+1}\mid n_1}\alpha_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}(d_2,\ldots,d_{m+1})c(d_2,n_2)\cdots e(d_{m+1},n_{m+1}).\end{aligned}$$

Then, the finite Fourier coefficients $a_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}$ and $\alpha_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}$ of $S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}$ are respectively given as

$$a_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}(l_2,\ldots,l_{m+1})=n_1^{-m}S_{G(\boldsymbol{f})}^{H_{n_1}^{\boldsymbol{\gamma},\xi}}(n_1,l_{m+1},\ldots,l_2),\tag{2.10}$$

$$\alpha_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}(d_2,\ldots,d_{m+1})=n_1^{-m}S_{G(\boldsymbol{f})}^{H_{n_1}^{\boldsymbol{\gamma},\xi}}\left(n_1,\frac{n_1}{d_{m+1}},\ldots,\frac{n_1}{d_2}\right).\tag{2.11}$$

In particular, let $a_{\boldsymbol{f},n_1}:=a_{\boldsymbol{f},n_1}^{(1,\ldots,1),1_m}$ and $\alpha_{\boldsymbol{f},n_1}:=\alpha_{\boldsymbol{f},n_1}^{(1,\ldots,1),1_m}$. Then, we have

$$a_{\boldsymbol{f},n_1}(l_2,\ldots,l_{m+1})=n_1^{-m}S_{G(\boldsymbol{f})}(n_1,l_{m+1},\ldots,l_2),\tag{2.12}$$

$$\alpha_{\boldsymbol{f},n_1}(d_2,\ldots,d_{m+1})=n_1^{-m}S_{G(\boldsymbol{f})}\left(n_1,\frac{n_1}{d_{m+1}},\ldots,\frac{n_1}{d_2}\right).\tag{2.13}$$

Proof We only verify formula (2.11) (formula (2.10) can be similarly obtained; moreover, since $H_n^{(1,\ldots,1),1_m}=1_m$ and $S_{\boldsymbol{f}}^{1_m}=S_{\boldsymbol{f}}$, one immediately obtains formulas (2.12) and (2.13) from (2.10) and (2.11), respectively). By formula (2.9), $\alpha_{\boldsymbol{f},n_1}^{\boldsymbol{\gamma},\xi}(d_2,\ldots,d_{m+1})$ is given as

$$n_1^{-m}\sum_{\delta_2,\ldots,\delta_{m+1}\mid n_1}S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}(n_1,\delta_2,\ldots,\delta_{m+1})c\left(\frac{n_1}{\delta_2},\frac{n_1}{d_2}\right)\cdots c\left(\frac{n_1}{\delta_{m+1}},\frac{n_1}{d_{m+1}}\right).$$

Further, by changing the order of the summation, this expression is equivalent to

$$n_1^{-m}\sum_{e_m^{\gamma_m}|\cdots|e_1^{\gamma_1}|n_1}\xi\left(e_1^{\gamma_1},\dots,e_m^{\gamma_m}\right)f_1\left(\frac{n_1}{e_1^{\gamma_1}}\right)f_2\left(\frac{e_1^{\gamma_1}}{e_2^{\gamma_2}}\right)\cdots f_m\left(\frac{e_{m-1}^{\gamma_{m-1}}}{e_m^{\gamma_m}}\right)f_{m+1}\left(e_m^{\gamma_m}\right)$$
$$\times\left(\sum_{e_1^{\gamma_1}|\delta_2|n_1}c\left(\frac{n_1}{\delta_2},\frac{n_1}{d_2}\right)\right)\cdots\left(\sum_{e_m^{\gamma_m}|\delta_{m+1}|n_1}c\left(\frac{n_1}{\delta_{m+1}},\frac{n_1}{d_{m+1}}\right)\right). \tag{2.14}$$

Here, we use the following identity. Let $e|n$ and $d|n$. Then, we have

$$\sum_{e|\delta|n}c\left(\frac{n}{\delta},\frac{n}{d}\right)=\begin{cases}\frac{n}{e} & \text{if } d|e,\\ 0 & \text{otherwise.}\end{cases} \tag{2.15}$$

Actually, one can obtain this formula from the right-most expression (1.1) of the Ramanujan sum $c(k,n)$ and formula $\delta^0*\mu=\varepsilon$. Then, applying formula (2.15), we see that (2.14) can be written as

$$n_1^{-m}\sum_{d_2|e_1^{\gamma_1}|n_1}\ \sum_{d_3|e_2^{\gamma_2}|e_1^{\gamma_1}}\cdots\sum_{d_{m+1}|e_m^{\gamma_m}|e_{m-1}^{\gamma_{m-1}}}\xi\left(e_1^{\gamma_1},\dots,e_m^{\gamma_m}\right)$$
$$\times f_1\left(\frac{n_1}{e_1^{\gamma_1}}\right)f_2\left(\frac{e_1^{\gamma_1}}{e_2^{\gamma_2}}\right)\cdots f_m\left(\frac{e_{m-1}^{\gamma_{m-1}}}{e_m^{\gamma_m}}\right)f_{m+1}\left(e_m^{\gamma_m}\right)\frac{n_1}{e_1^{\gamma_1}}\cdots\frac{n_1}{e_m^{\gamma_m}}.$$

Changing variables $e'_k=n_1/e_k^{\gamma_k}$ for all $1\le k\le m$, we have $a_{\gamma_k}(n_1/e'_k)=1$. Moreover, it holds that $e'_k|\frac{n_1}{d_{k+1}}$ for $1\le k\le m$ and $e'_1|\cdots|e'_m|n_1$ because $e_m^{\gamma_m}|\cdots|e_1^{\gamma_1}|n_1$. Hence, we can rewrite the above expression as

$$n_1^{-m}\sum_{\substack{e'_m|n_1\\ e'_m|\frac{n_1}{d_{m+1}}}}\ \sum_{\substack{e'_{m-1}|e'_m\\ e'_{m-1}|\frac{n_1}{d_m}}}\cdots\sum_{\substack{e'_1|e'_2\\ e'_1|\frac{n_1}{d_2}}}a_{\gamma_1}\left(\frac{n_1}{e'_1}\right)\cdots a_{\gamma_m}\left(\frac{n_1}{e'_m}\right)\cdot\xi\left(\frac{n_1}{e'_1},\dots,\frac{n_1}{e'_m}\right)$$
$$\times f_1(e'_1)f_2\left(\frac{e'_2}{e'_1}\right)\cdots f_m\left(\frac{e'_m}{e'_{m-1}}\right)f_{m+1}\left(\frac{n_1}{e'_m}\right)e'_1\cdots e'_m. \tag{2.16}$$

It is easy to see that (2.16) coincides with the right-hand side of formula (2.11). This completes the proof of the theorem. □

Remark 2.8 Suppose that $\gamma_0|\gamma_1|\cdots|\gamma_m$ with $\gamma_0=1$. Then, we have $S_{G(\boldsymbol{f})}^{H_{n_1}^{\boldsymbol{\gamma},\xi}}=S_{G^{[\boldsymbol{\gamma}]}(\boldsymbol{f})}^{H_{n_1}^{\xi}}$, where $H_n^{\xi}:=H_n^{(1,\dots,1),\xi}$ and $G^{[\boldsymbol{\gamma}]}(\boldsymbol{f})=(\delta^0 f_{m+1}^{[\gamma_m]},\delta^1 f_m^{[\gamma_{m-1}]},\dots,\delta^m f_1^{[\gamma_0]})$. In fact, since the condition above means that $a_{\gamma_k}(n_1/e'_k)=a_{\gamma_k}(e'_{k+1}/e'_k)$ for all $1\le k\le m-1$, the summand in (2.16) is written as

$$\xi\left(\frac{n_1}{e'_1},\dots,\frac{n_1}{e'_m}\right)f_1(e'_1)f_2^{[\gamma_1]}\left(\frac{e'_2}{e'_1}\right)\cdots f_m^{[\gamma_{m-1}]}\left(\frac{e'_m}{e'_{m-1}}\right)f_{m+1}^{[\gamma_m]}\left(\frac{n_1}{e'_m}\right)e'_1\cdots e'_m.$$

Hence, the claim follows.

Example 2.9 Retaining the notation in Example 2.4, one can easily see that $G(\boldsymbol{f}) = (\delta^{a_1+\cdots+a_m}, \delta^{a_1+\cdots+a_{m-1}+1}, \ldots, \delta^{a_1+m-1}, \delta^m)$. Hence, the finite Fourier coefficient $\alpha_{\boldsymbol{f},n_1}$ of the multiple divisor function $\sigma_{\boldsymbol{a}} := \sigma_{\boldsymbol{a}}^{(1,\ldots,1)}$ is given by

$$\alpha_{\boldsymbol{f},n_1}(d_2, \ldots, d_{m+1}) = n_1^{a_1+\cdots+a_m-m}\sigma_{\mathbf{1}-{}^t\boldsymbol{a}}\left(n_1, \frac{n_1}{d_{m+1}}, \ldots, \frac{n_1}{d_2}\right),$$

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{C}^m$ and ${}^t\boldsymbol{a} := (a_m, \ldots, a_1) \in \mathbb{C}^m$.

Example 2.10 Retaining the notation in Example 2.5, we have $S_{\boldsymbol{f}} = f \circ \gcd$. Again, by the degeneracy formula of (2.3), the finite Fourier coefficient $\alpha_{\boldsymbol{f},n_1}$ of $f \circ \gcd$ is given by

$$\alpha_{\boldsymbol{f},n_1}(d_2, \ldots, d_{m+1}) = n_1^{-m} S_{f*\mu,\delta^m}\left(n_1, \gcd\left(\frac{n_1}{d_2}, \ldots, \frac{n_1}{d_{m+1}}\right)\right).$$

3 Dirichlet series attached to $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$

In this section, we examine both single-variable and multivariable Dirichlet series, the coefficients of which are given by the multiple Ramanujan sum $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$.

3.1 Single-variable Dirichlet series

We first examine a single-variable Dirichlet series. Recall the following well-known formula concerning the Ramanujan sum $c(k, n)$ (see, e.g., [19]):

$$\sum_{k=1}^{\infty} c(k, n)k^{-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} \quad (\mathrm{Re}(s) > 1). \tag{3.1}$$

In this subsection, we give a generalization of this formula. For $j = 1, 2, \ldots, m+1$, let

$$\Phi_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(s; \check{\boldsymbol{n}}_j) := \sum_{n_j=1}^{\infty} S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(n_1, \ldots, n_{m+1})n_j^{-s},$$

where $\check{\boldsymbol{n}}_j := (n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{m+1}) \in \mathbb{N}^m$. The following proposition says that the series $\Phi_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(s; \check{\boldsymbol{n}}_j)$ is written as the product of a Dirichlet series and a finite sum, which is again given by the multiple Ramanujan sum.

Proposition 3.1 (i) *For $j = 1$, we have*

$$\Phi_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(s; \check{\boldsymbol{n}}_1) = L(s; f_1)S_{F_1}(n_2) = L(s; f_1)F_1(n_2) \quad (\mathrm{Re}(s) > \sigma(f_1)), \tag{3.2}$$

where $F_1 = F_{1,\boldsymbol{f},\check{\boldsymbol{n}}_1}^{\boldsymbol{\gamma},s} := \delta^{-s}(S_{f_2,\ldots,f_{m+1}}^{(\gamma_2,\ldots,\gamma_m)}(\cdot, n_3, \ldots, n_{m+1}) *_{\gamma_1} \delta^s)$.

(ii) *For $2 \le j \le m+1$, we have*

$$\Phi_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(s; \check{\boldsymbol{n}}_j) = \zeta(s)S_{f_1,\ldots,f_{j-2},F_j}^{(\gamma_1,\ldots,\gamma_{j-2})}(n_1, \ldots, n_{j-1}) \quad (\mathrm{Re}(s) > 1), \tag{3.3}$$

where $F_j = F_{j,\boldsymbol{f},\check{\boldsymbol{n}}_j}^{\boldsymbol{\gamma},s} := \delta^{-s}(S_{f_j,\ldots,f_{m+1}}^{(\gamma_j,\ldots,\gamma_m)}(\cdot, n_{j+1}, \ldots, n_{m+1}) *_{\gamma_{j-1}} (\delta^s f_{j-1}))$.

Proof Suppose that $d_k^{\gamma_k}|\gcd(n_1,\ldots,n_{k+1})$ for all $k=1,\ldots,m$. Then, n_j is a multiple of $\mathrm{lcm}(d_1^{\gamma_1},\ldots,d_m^{\gamma_m})=d_1^{\gamma_1}$ if $j=1$, and $\mathrm{lcm}(d_{j-1}^{\gamma_{j-1}},\ldots,d_m^{\gamma_m})=d_{j-1}^{\gamma_{j-1}}$ if $2\le j\le m+1$ (note that we only consider m-tuples $(d_1,\ldots,d_m)$ such that $d_m^{\gamma_m}|\cdots|d_1^{\gamma_1}$). Hence, for $j=1$ and $\mathrm{Re}(s)>\sigma(f_1)$, $\Phi_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(s;\check{\boldsymbol{n}}_j)$ is expressed as

$$\sum_{\substack{d_k^{\gamma_k}|\gcd(n_2,\ldots,n_{k+1})\\(k=1,\ldots,m)}}\sum_{l=1}^{\infty}f_1\left(\frac{d_1^{\gamma_1}l}{d_1^{\gamma_1}}\right)f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right)\cdots f_m\left(\frac{d_{m-1}^{\gamma_{m-1}}}{d_m^{\gamma_m}}\right)f_{m+1}\left(d_m^{\gamma_m}\right)\left(d_1^{\gamma_1}l\right)^{-s},$$

and, for $2\le j\le m+1$ and $\mathrm{Re}(s)>1$,

$$\sum_{\substack{d_k^{\gamma_k}|\gcd(n_1,\ldots,n_{k+1})\\(k=1,2,\ldots,j-2)}}f_1\left(\frac{n_1}{d_1^{\gamma_1}}\right)f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right)\cdots f_{j-2}\left(\frac{d_{j-3}^{\gamma_{j-3}}}{d_{j-2}^{\gamma_{j-2}}}\right)\sum_{d_{j-1}^{\gamma_{j-1}}|d_{j-2}^{\gamma_{j-2}}}f_{j-1}\left(\frac{d_{j-2}^{\gamma_{j-2}}}{d_{j-1}^{\gamma_{j-1}}}\right)$$

$$\times\sum_{\substack{d_k^{\gamma_k}|\gcd(d_{j-1}^{\gamma_{j-1}},n_{j+1},\ldots,n_{k+1})\\(k=j,\ldots,m)}}\sum_{l=1}^{\infty}f_j\left(\frac{d_{j-1}^{\gamma_{j-1}}}{d_j^{\gamma_j}}\right)\cdots f_m\left(\frac{d_{m-1}^{\gamma_{m-1}}}{d_m^{\gamma_m}}\right)f_{m+1}\left(d_m^{\gamma_m}\right)\left(d_{j-1}^{\gamma_{j-1}}l\right)^{-s}.$$

Thus, it is easy to see that these coincide, respectively, with (3.2) and (3.3). This completes the proof. □

Example 3.2 For small m, the series $\Phi_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(s;\check{\boldsymbol{n}}_j)$ is explicitly given as follows. For $m=1$, we have

$$\Phi_{f_1,f_2}^{(\gamma_1)}(s;n_2)=L(s;f_1)\sum_{d^{\gamma_1}|n_2}f_2\left(d^{\gamma_1}\right)d^{-\gamma_1 s},\tag{3.4}$$

$$\Phi_{f_1,f_2}^{(\gamma_1)}(s;n_1)=\zeta(s)\sum_{d^{\gamma_1}|n_1}f_1\left(\frac{n_1}{d^{\gamma_1}}\right)f_2\left(d^{\gamma_1}\right)d^{-\gamma_1 s},\tag{3.5}$$

and for $m=2$, we have

$$\Phi_{f_1,f_2,f_3}^{(\gamma_1,\gamma_2)}(s;n_2,n_3)=L(s;f_1)\sum_{d_1^{\gamma_1}|n_2}\sum_{d_2^{\gamma_2}|\gcd(d_1^{\gamma_1},n_3)}f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right)f_3\left(d_2^{\gamma_2}\right)d_1^{-\gamma_1 s},$$

$$\Phi_{f_1,f_2,f_3}^{(\gamma_1,\gamma_2)}(s;n_1,n_3)=\zeta(s)\sum_{d_1^{\gamma_1}|n_1}\sum_{d_2^{\gamma_2}|\gcd(d_1^{\gamma_1},n_3)}f_1\left(\frac{n_1}{d_1^{\gamma_1}}\right)f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right)f_3\left(d_2^{\gamma_2}\right)d_1^{-\gamma_1 s},$$

$$\Phi_{f_1,f_2,f_3}^{(\gamma_1,\gamma_2)}(s;n_1,n_2)=\zeta(s)\sum_{d_1^{\gamma_1}|\gcd(n_1,n_2)}\sum_{d_2^{\gamma_2}|d_1^{\gamma_1}}f_1\left(\frac{n_1}{d_1^{\gamma_1}}\right)f_2\left(\frac{d_1^{\gamma_1}}{d_2^{\gamma_2}}\right)f_3\left(d_2^{\gamma_2}\right)d_2^{-\gamma_2 s}.$$

Formulas (3.4) and (3.5) were obtained in [1] for the case of $\gamma_1=1$.

Example 3.3 For $0 \le k \le m+1$ and $\boldsymbol{a} = (a_1, \ldots, a_{m+1-k}) \in \mathbb{C}^{m+1-k}$, we define generalizations of the classical Ramanujan sum $c(n_1, n_2)$ as

$$c^{\boldsymbol{a}}_{m+1,k}(n_1, \ldots, n_{m+1}) := S_{\underbrace{\mu, \ldots, \mu}_{k}, \delta^{a_1}, \ldots, \delta^{a_{m+1-k}}}(n_1, \ldots, n_{m+1}).$$

It is clear that $c(n_1, n_2) = c^{(1)}_{2,1}(n_1, n_2)$. Then, for $1 \le k \le m$, using formulas (3.2) and $L(s; \mu) = 1/\zeta(s)$, we can verify by induction on m and k that

$$\sum_{n_1,\ldots,n_k=1}^{\infty} c^{\boldsymbol{a}}_{m+1,k}(n_1, \ldots, n_{m+1}) n_1^{-s_1} \cdots n_k^{-s_k}$$
$$= \frac{\prod_{j=2}^{k} \zeta(s_j)}{\prod_{j=1}^{k} \zeta(s_1 + \cdots + s_j)} \sum_{d \mid n_{k+1}} d^{a_1 - (s_1 + \cdots + s_k)} \sigma_{\widetilde{\boldsymbol{a}}}(d, n_{k+2}, \ldots, n_{m+1}), \quad (3.6)$$

where $\widetilde{\boldsymbol{a}} := (a_2 - a_1, a_3 - a_2, \ldots, a_{m+1-k} - a_{m-k}) \in \mathbb{C}^{m-k}$ (see Example 2.4). Setting $m = 1$, $k = 1$ (in this case, $\sigma_{\widetilde{\boldsymbol{a}}} \equiv 1$), and $a_1 = 1$, we obtain formula (3.1). See [5] for a multivariable analogue of formula (3.1).

Example 3.4 Retaining the notation in Example 2.5, we have $S_{\boldsymbol{f}} = f \circ \gcd$. Let $\gcd(\check{\boldsymbol{n}}_j) := \gcd(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{m+1})$. Then, since $\gcd(n_1, \ldots, n_{m+1}) = \gcd(n_j, \gcd(\check{\boldsymbol{n}}_j))$, we have the following well-known formula:

$$\sum_{n_j=1}^{\infty} (f \circ \gcd)(n_1, \ldots, n_{m+1}) n_j^{-s}$$
$$= \Phi_{\delta^0, f*\mu}\big(s; \gcd(\check{\boldsymbol{n}}_j)\big) = \zeta(s) \sum_{d \mid \gcd(\check{\boldsymbol{n}}_j)} (f * \mu)(d) d^{-s}.$$

3.2 Multivariable Dirichlet series

For an arithmetic function $F(n_1, \ldots, n_k)$ of k variables, we denote by $L(\boldsymbol{s}; F)$ the multivariable Dirichlet series attached to F, that is,

$$L(\boldsymbol{s}; F) := \sum_{n_1,\ldots,n_k=1}^{\infty} F(n_1, \ldots, n_k) n_1^{-s_1} \cdots n_k^{-s_k}$$

for $\boldsymbol{s} = (s_1, \ldots, s_k) \in \Sigma(F)$, where $\Sigma(F) \subset \mathbb{C}^k$ is the region of absolute convergence for $L(\boldsymbol{s}; F)$. The first goal of this subsection is to calculate $L(\boldsymbol{s}; S^{\boldsymbol{\gamma}}_{\boldsymbol{f}})$ explicitly.

Theorem 3.5 *Let $\gamma_0 | \gamma_1 | \gamma_2 | \cdots | \gamma_m$ with $\gamma_0 \in \mathbb{N}$. Define the function $a_{\gamma_0} S^{\boldsymbol{\gamma}}_{\boldsymbol{f}}$ of $(m+1)$ variables as $(a_{\gamma_0} S^{\boldsymbol{\gamma}}_{\boldsymbol{f}})(n_1, \ldots, n_{m+1}) := a_{\gamma_0}(n_1) S^{\boldsymbol{\gamma}}_{\boldsymbol{f}}(n_1, \ldots, n_{m+1})$. Then, we have*

$$L\big(\boldsymbol{s}; a_{\gamma_0} S^{\boldsymbol{\gamma}}_{\boldsymbol{f}}\big) = \prod_{j=2}^{m+1} \zeta(s_j) \cdot \prod_{j=1}^{m+1} L\big(s_1 + \cdots + s_j; f_j^{[\gamma_{j-1}]}\big) \quad (3.7)$$

in the region

$$\left\{ \boldsymbol{s} = (s_1, \ldots, s_{m+1}) \in \mathbb{C}^{m+1} \,\middle|\, \begin{array}{ll} \mathrm{Re}(s_j) > 1 & (2 \le j \le m+1) \\ \mathrm{Re}(s_1 + \cdots + s_j) > \sigma(f_j) & (1 \le j \le m+1) \end{array} \right\}.$$

In particular, setting $\gamma_0 = 1$, *we have*

$$L\big(\boldsymbol{s}; S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}\big) = \prod_{j=2}^{m+1} \zeta(s_j) \cdot L(s_1; f_1) \prod_{j=2}^{m+1} L\big(s_1 + \cdots + s_j; f_j^{[\gamma_{j-1}]}\big). \tag{3.8}$$

Proof This is proven by induction on m. Suppose that $m = 1$. Then, from formula (3.5) we have that

$$\begin{aligned} L\big((s_1, s_2); a_{\gamma_0} S_{f_1, f_2}^{(\gamma_1)}\big) &= \sum_{n_1=1}^{\infty} \Phi_{f_1, f_2}^{(\gamma_1)}\big(s_2; n_1^{\gamma_0}\big) n_1^{-\gamma_0 s} \\ &= \zeta(s_2) \sum_{n_1=1}^{\infty} \sum_{d^{\gamma_1} | n_1^{\gamma_0}} f_1\left(\frac{n_1}{d^{\gamma_1}}\right) f_2\big(d^{\gamma_1}\big) d^{-\gamma_1 s_2} n_1^{-\gamma_0 s_1}. \end{aligned}$$

Here, since $\gamma_0 | \gamma_1$, $n_1^{\gamma_0}$ is expressed as $n_1^{\gamma_0} = d^{\gamma_1} l^{\gamma_0}$ with $l \in \mathbb{N}$. Hence, this expression is written as

$$\zeta(s_2) \sum_{d=1}^{\infty} \sum_{l=1}^{\infty} f_1\left(\frac{d^{\gamma_1} l^{\gamma_0}}{d^{\gamma_1}}\right) f_2\big(d^{\gamma_1}\big) d^{-\gamma_1 s_2} \big(d^{\gamma_1} l^{\gamma_0}\big)^{-s_1},$$

thus completing the proof for $m = 1$. Next, suppose that $m - 1$. Note that $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}(n_1, \ldots, n_{m+1}) = S^{(\gamma_1)}_{f_1, S^{(\gamma_2, \ldots, \gamma_m)}_{f_2, \ldots, f_{m+1}}(\cdot, n_3, \ldots, n_{m+1})}(n_1, n_2)$ by formula (2.2). Then, by (3.7) for $m = 1$, $L(\boldsymbol{s}; a_{\gamma_0} S_{\boldsymbol{f}}^{\boldsymbol{\gamma}})$ is equal to

$$\begin{aligned} &\sum_{n_3, \ldots, n_{m+1}=1}^{\infty} L\big((s_1, s_2); a_{\gamma_0} S^{(\gamma_1)}_{f_1, S^{(\gamma_2, \ldots, \gamma_m)}_{f_2, \ldots, f_{m+1}}(\cdot, n_3, \ldots, n_{m+1})}\big) n_3^{-s_1} \cdots n_{m+1}^{-s_{m+1}} \\ &= \zeta(s_2) L\big(s_1; f_1^{[\gamma_0]}\big) \\ &\quad \times \sum_{n_3, \ldots, n_{m+1}=1}^{\infty} L\big(s_1 + s_2; a_{\gamma_1} S^{(\gamma_2, \ldots, \gamma_m)}_{f_2, \ldots, f_{m+1}}(\cdot, n_3, \ldots, n_{m+1})\big) n_3^{-s_1} \cdots n_{m+1}^{-s_{m+1}} \\ &= \zeta(s_2) L\big(s_1; f_1^{[\gamma_0]}\big) L\big((s_1 + s_2, s_3, \ldots, s_{m+1}); a_{\gamma_1} S^{(\gamma_2, \ldots, \gamma_m)}_{f_2, \ldots, f_{m+1}}\big). \end{aligned}$$

This completes the proof for m based on the assumption of induction. Hence, we obtain the desired formula. □

Remark that formula (2.1) is regarded as the "diagonally summed version" of (3.8) since $S^{(\gamma_1, \ldots, \gamma_m)}_{f_1, \ldots, f_{m+1}}(n, \ldots, n) = (f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1)(n)$.

Example 3.6 Retaining the notation in Example 2.4, we have

$$L(\boldsymbol{s}; \sigma_{\boldsymbol{a}}) = \zeta(s_1) \prod_{j=2}^{m+1} \zeta(s_j)\zeta(s_1 + \cdots + s_j - a_1 - \cdots - a_{j-1}). \tag{3.9}$$

Example 3.7 Retaining the notation in Example 3.3, we have

$$L\big(\boldsymbol{s}; c_{m+1,k}^{\boldsymbol{a}}\big) = \frac{\prod_{l=2}^{m+1} \zeta(s_l) \prod_{l=1}^{m+1-k} \zeta(s_1 + \cdots + s_{k+l} - a_l)}{\prod_{l=1}^{k} \zeta(s_1 + \cdots + s_l)}.$$

We can also obtain this formula from (3.6) and (3.9). In particular, we have $L((s_1, s_2); c) = \zeta(s_2)\zeta(s_1 + s_2 - 1)/\zeta(s_1)$.

Example 3.8 Retaining the notation in Example 2.5, we have $S_{\boldsymbol{f}} = f \circ \gcd$. In this case,

$$L(\boldsymbol{s}; f \circ \gcd) = \frac{\zeta(s_1) \cdots \zeta(s_{m+1})}{\zeta(s_1 + \cdots + s_{m+1})} L(s_1 + \cdots + s_{m+1}; f).$$

Next, we present another type of multivariable Dirichlet series involving $S_{\boldsymbol{f}}^{\boldsymbol{\gamma}}$. Recall the well-known Ramanujan formula for the divisor function σ_a (see, e.g., [19], also [6]):

$$L(s; \sigma_a\sigma_b) = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

for $\mathrm{Re}(s) > \max\{1, \mathrm{Re}(a)+1, \mathrm{Re}(b)+1, \mathrm{Re}(a+b)+1\}$. This formula was generalized by Borwein and Choi [3] as follows. Let $f_1, f_2, g_1, g_2 \in \mathcal{M}^c$. Then, we have

$$\begin{aligned} &L\big(s; (f_2 *_\gamma f_1)(g_2 *_\gamma g_1)\big) \\ &\quad = \frac{L(s; f_1g_1)L(s; (f_2g_1)^{[\gamma]})L(s; (f_1g_2)^{[\gamma]})L(s; (f_2g_2)^{[\gamma]})}{L(2s; (f_1f_2g_1g_2)^{[\gamma]})} \end{aligned} \tag{3.10}$$

for $\mathrm{Re}(s) > \max\{\sigma(f_1g_1), \sigma(f_2g_1), \sigma(f_1g_2), \sigma(f_2g_2), \sigma(f_1f_2g_1g_2)/2\}$. They gave formula (3.10) for the case $\gamma = 1$, and it is easy to obtain the equation for general $\gamma \in \mathbb{N}$ in the same manner. Similarly, since $S_{f_1,f_2}^{(\gamma)}(n,n) = (f_2 *_\gamma f_1)(n)$, we can regard formula (3.10) as the diagonally summed version of the following formula.

Theorem 3.9 *Let $f_1, f_2, g_1, g_2 \in \mathcal{M}^c$. Then, we have*

$$\begin{aligned} &L\big((s_1, s_2); S_{f_1,f_2}^{(\gamma)} S_{g_1,g_2}^{(\gamma)}\big) \\ &\quad = \frac{\zeta(s_2)L(s_1; f_1g_1)L(s_1+s_2; (f_2g_1)^{[\gamma]})L(s_1+s_2; (f_1g_2)^{[\gamma]})L(s_1+s_2; (f_2g_2)^{[\gamma]})}{L(2(s_1+s_2); (f_1f_2g_1g_2)^{[\gamma]})} \end{aligned} \tag{3.11}$$

in the region

$$\left\{(s_1,s_2)\in\mathbb{C}^2 \,\middle|\, \begin{array}{l}\mathrm{Re}(s_2)>1,\ \mathrm{Re}(s_1)>\sigma(f_1g_1),\\ \mathrm{Re}(s_1+s_2)>\max\{\sigma(f_2g_1),\sigma(f_1g_2),\sigma(f_2g_2),\sigma(f_1f_2g_1g_2)/2\}\end{array}\right\}.$$

Proof First, note that, for a multiplicative arithmetic function F of k variables, $L(\boldsymbol{s};F)$ has the Euler product expression $L(\boldsymbol{s};F)=\prod_p\sum_{l_1,\dots,l_{k=1}}^{\infty}F(p^{l_1},\dots,p^{l_k})\times p^{-s_1l_1}\cdots p^{-s_kl_k}$.

Let $h_1,h_2\in\mathcal{M}^c$. Then,

$$S^{(\gamma)}_{h_1,h_2}\big(p^{l_1},p^{l_2}\big)=\frac{h_1(p)^\gamma}{h_1(p)^\gamma-h_2(p)^\gamma}h_1(p)^{l_1}\left(1-\left(\frac{h_2(p)}{h_1(p)}\right)^{\gamma\cdot m^\gamma(l_1,l_2)}\right),$$

where $m^\gamma(l_1,l_2):=\lfloor\frac{1}{\gamma}\min\{l_1,l_2\}\rfloor+1$. Hence, from the Euler product expression, we have

$$\begin{aligned}
&L\big((s_1,s_2);S^{(\gamma)}_{f_1,f_2}S^{(\gamma)}_{g_1,g_2}\big)\\
&=\prod_p\sum_{l_1,l_2=0}^{\infty}S^{(\gamma)}_{f_1,f_2}\big(p^{l_1},p^{l_2}\big)S^{(\gamma)}_{g_1,g_2}\big(p^{l_1},p^{l_2}\big)p^{-s_1l_1}p^{-s_2l_s}\\
&=\prod_p(C_p^\gamma)^{-1}\sum_{l_1,l_2=0}^{\infty}(a_1b_1x_1)^{l_1}x_2^{l_2}\\
&\quad\times\left(1-\left(\frac{a_2}{a_1}\right)^{\gamma\cdot m^\gamma(l_1,l_2)}\right)\left(1-\left(\frac{b_2}{b_1}\right)^{\gamma\cdot m^\gamma(l_1,l_2)}\right),
\end{aligned}\tag{3.12}$$

where $a_i:=f_i(p)$, $b_i:=g_i(p)$, $x_i:=p^{-s_i}$ for $i=1,2$, and $C_p^\gamma=C_p^\gamma(f_1,f_2,g_1,g_2):=(a_1^\gamma-a_2^\gamma)(b_1^\gamma-b_2^\gamma)/(a_1b_1)^\gamma$ for each prime p. Write I for the inner sum of the right-most hand side of (3.12). Moreover, divide I into two parts as $\sum_{l_1,l_2=0}^{\infty}=\sum_{l_1\ge l_2}+\sum_{l_1<l_2}=\sum_{l_2=0}^{\infty}\sum_{l_1=l_2}^{\infty}+\sum_{l_1=0}^{\infty}\sum_{l_2=l_1+1}^{\infty}$ and denote by I_1 and I_2 the former and latter sums, respectively. Writing l_2 as $l_2=\gamma l+k$ in I_1 and l_1 as $l_1=\gamma l+k$ in I_2 with $l\in\mathbb{Z}_{\ge0}$ and $0\le k\le\gamma-1$, respectively, we have

$$\begin{aligned}
I_1&=\sum_{k=0}^{\gamma-1}\sum_{l=0}^{\infty}\sum_{l_1=\gamma l+k}^{\infty}(a_1b_1x_1)^{l_1}x_2^{\gamma l+k}\left(1-\left(\frac{a_2}{a_1}\right)^{\gamma(l+1)}\right)\left(1-\left(\frac{b_2}{b_1}\right)^{\gamma(l+1)}\right)\\
&=\frac{1}{1-a_1b_1x_1}\frac{1-(a_1b_1x_1x_2)^\gamma}{1-a_1b_1x_1x_2}M
\end{aligned}$$

and similarly

$$I_2=\frac{x_2}{1-x_2}\frac{1-(a_1b_1x_1x_2)^\gamma}{1-a_1b_1x_1x_2}M,$$

where $M := \sum_{l=0}^{\infty}(a_1b_1x_1x_2)^{\gamma l}(1-(\frac{a_2}{a_1})^{\gamma(l+1)})(1-(\frac{b_2}{b_1})^{\gamma(l+1)})$. Hence, we have

$$I = I_1 + I_2 = \frac{1-(a_1b_1x_1x_2)^{\gamma}}{(1-x_2)(1-a_1b_1x_1)}M.$$

Now, M is straightforwardly calculated as

$$\frac{C_p^{\gamma}(1-(a_1a_2b_1b_2x_1^2x_2^2)^{\gamma})}{(1-(a_1b_1x_1x_2)^{\gamma})(1-(a_2b_1x_1x_2)^{\gamma})(1-(a_1b_2x_1x_2)^{\gamma})(1-(a_2b_2x_1x_2)^{\gamma})},$$

where the desired formula immediately follows from formula (3.12). □

Remark 3.10 For $m \geq 2$, we cannot expect that the single-variable Dirichlet series $L(s;(f_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} f_1)(g_{m+1} *_{\gamma_m} \cdots *_{\gamma_1} g_1))$ has a product expression such as (3.10), and, similarly, the multivariable Dirichlet series $L((s_1,\ldots,s_{m+1}); S_{f_1,\ldots,f_{m+1}}^{(\gamma_1,\ldots,\gamma_m)} S_{g_1,\ldots,g_{m+1}}^{(\gamma_1,\ldots,\gamma_m)})$ has a product expression such as (3.11).

4 A generalization of the Smith determinant

In this section, we evaluate the hyperdeterminants in the sense of Cayley of the hypermatrices, the entries of which are given by values of the multiple Ramanujan sum $S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}$.

4.1 Hyperdeterminant

Recall the definition and some properties of the hyperdeterminants in the sense of Cayley. Let $A = (A(i_1,\ldots,i_k))_{1\leq i_1,\ldots,i_k\leq n}$ be a k-dimensional matrix of order n. For a subset $I \subseteq \{1,\ldots,k\}$, set $\varepsilon_j = 1$ if $j \in I$ and 0 otherwise. Then, the hyperdeterminant $\det_I A$ of A with the signature I is defined as

$$\det_I A := \frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_k\in\mathfrak{S}_n}\prod_{j=1}^{k}\operatorname{sgn}(\sigma_j)^{\varepsilon_j}\sum_{v=1}^{n}A\bigl(\sigma_1(v),\ldots,\sigma_k(v)\bigr), \tag{4.1}$$

where $\mathfrak{S}_n$ is the symmetric group of degree n. We obtain the usual determinant of a matrix A when $k=2$ and $I=\{1,2\}$; $\det_{\{1,2\}} A = \det A$. Note that $\det_I A$ is identically zero if the number of elements in I is odd. For the theory of hyperdeterminants, see, e.g., [15]. We next present some lemmas that will be needed in the subsequent discussion.

Lemma 4.1 *Let $A = (A(i_1,\ldots,i_k))$ be a k-dimensional matrix of order n.*
(i) *For $\pi \in \mathfrak{S}_n$, we have*

$$\det_I\bigl(A\bigl(\pi(i_1),\ldots,\pi(i_k)\bigr)\bigr) = \det_I\bigl(A(i_1,\ldots,i_k)\bigr). \tag{4.2}$$

(ii) *For $\pi \in \mathfrak{S}_k$, set $\pi^{-1}(I) := \{\pi^{-1}(i) | i \in I\}$. Then, we have*

$$\det_I\bigl(A(i_{\pi(1)},\ldots,i_{\pi(k)})\bigr) = \det_{\pi^{-1}(I)}\bigl(A(i_1,\ldots,i_k)\bigr). \tag{4.3}$$

Proof These are immediately obtained from the definition of the hyperdeterminant. □

Let $A = (A(i_1, \dots, i_k))$ and $B = (B(i_1, \dots, i_l))$ be k-dimensional and l-dimensional matrices, respectively, of order n. Then, the Cayley product AB of A and B is a $(k+l-2)$-dimensional matrix of order n given by

$$(AB)(i_1, \dots, i_{k+l-2}) := \sum_{j=1}^{n} A(i_1, \dots, i_{k-1}, j) B(j, i_k, \dots, i_{k+l-2}). \tag{4.4}$$

Lemma 4.2 *Let* $A = (A(i_1, \dots, i_k))$ *and* $B = (B(i_1, \dots, i_l))$ *be* k*-dimensional and* l*-dimensional matrices, respectively, of order* n*. Let* $K \subseteq \{1, 2, \dots, k-1\}$ *and* $L \subseteq \{2, 3, \dots, l\}$ *with odd cardinality. Set* $I := K \cup (L + (k-2))$*, where* $L + (k-2) := \{l + k - 2 | l \in L\}$*. Then, we have*

$$\det_I(AB) = \det_{K \cup \{k\}} A \cdot \det_{\{1\} \cup L} B. \tag{4.5}$$

Proof See [9, 17]. □

4.2 Smith hyperdeterminants

Let $S = \{x_1, \dots, x_n\}$ be a set of distinct positive integers. The goal of this subsection is to evaluate the hyperdeterminant $\det_I(S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}(x_{i_1}, \dots, x_{i_{m+1}}))$ for a factor-closed set S. Here, S is called factor-closed if S contains every divisor of x for any $x \in S$. To accomplish this, we slightly extend the definition of the even arithmetic function examined in Sect. 2.

First, we give some notation. For $\boldsymbol{d} = (d_1, \dots, d_k) \in \mathbb{N}^k$, we set $|\boldsymbol{d}| := \sum_{j=1}^{k} d_j$ and $x_{\boldsymbol{d}} := (x_{d_1}, \dots, x_{d_k})$. Moreover, for $r \in \mathbb{N}$, we set $\gcd(\boldsymbol{d}, r) := (\gcd(d_1, r), \dots, \gcd(d_k, r)) \in \mathbb{N}^k$, write $\boldsymbol{d}|r$ if $d_j|r$ for $1 \le j \le k$, and, in this case, set $r/\boldsymbol{d} := (r/d_1, \dots, r/d_k) \in \mathbb{N}^k$. Let $f(m, n)$ be an arithmetic function of two variables. For $\boldsymbol{m} = (m_1, \dots, m_k) \in \mathbb{N}^k$ and $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, we set $f(\boldsymbol{m}, \boldsymbol{n}) := \prod_{j=1}^{k} f(m_j, n_j)$.

Let $\boldsymbol{r} = (r_1, \dots, r_m) \in \mathbb{N}^m$ and $\boldsymbol{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$. Let $\boldsymbol{n}_j = (n_{j,1}, \dots, n_{j,k_j}) \in \mathbb{N}^{k_j}$ for $j = 1, \dots, m$. We call an arithmetic function $F(\boldsymbol{r}; \boldsymbol{n}_1, \dots, \boldsymbol{n}_m)$ of $|\boldsymbol{k}|$ variables $\boldsymbol{n}_1, \dots, \boldsymbol{n}_m$ even (mod $\boldsymbol{r}^{(\boldsymbol{k})}$) if

$$F\big(\boldsymbol{r}; \gcd(\boldsymbol{n}_1, r_1), \dots, \gcd(\boldsymbol{n}_m, r_m)\big) = F(\boldsymbol{r}; \boldsymbol{n}_1, \dots, \boldsymbol{n}_m).$$

Then, the following lemma is obtained as described in [8] (note that the case of $m = 1$ is nothing more than Theorem 22 in [8]).

Lemma 4.3 *A function* $F(\boldsymbol{r}; \boldsymbol{n}_1, \dots, \boldsymbol{n}_m)$ *is even* (mod $\boldsymbol{r}^{(\boldsymbol{k})}$) *if and only if it possesses a representation of the form*

$$F(\boldsymbol{r}; \boldsymbol{n}_1, \dots, \boldsymbol{n}_m) = \sum_{\substack{\boldsymbol{d}_j | r_j \\ (j=1,\dots,m)}} \alpha_{\boldsymbol{r}}(\boldsymbol{d}_1, \dots, \boldsymbol{d}_m) \prod_{j=1}^{m} c(\boldsymbol{d}_j, \boldsymbol{n}_j), \tag{4.6}$$

where the sum is taken over all $\boldsymbol{d}_j = (d_{j,1}, \ldots, d_{j,k_j}) \in \mathbb{N}^{k_j}$ *for* $j = 1, \ldots, m$ *satisfying* $\boldsymbol{d}_j | r_j$. *Moreover, the finite Fourier coefficients* $\alpha_{\boldsymbol{r}}(\boldsymbol{d}_1, \ldots, \boldsymbol{d}_m)$ *are uniquely determined by*

$$\alpha_{\boldsymbol{r}}(\boldsymbol{d}_1, \ldots, \boldsymbol{d}_m) = \frac{1}{r_1^{k_1} \cdots r_m^{k_m}} \sum_{\substack{\boldsymbol{\delta}_j | r_j \\ (j=1,\ldots,m)}} F(\boldsymbol{r}; \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_m) \prod_{j=1}^{m} c\left(\frac{r_j}{\boldsymbol{\delta}_j}, \frac{r_j}{\boldsymbol{d}_j}\right), \tag{4.7}$$

where the sum is also taken over all $\boldsymbol{\delta}_j = (\delta_{j,1}, \ldots, \delta_{j,k_j}) \in \mathbb{N}^{k_j}$ *for* $j = 1, \ldots, m$ *satisfying* $\boldsymbol{\delta}_j | r_j$.

Now, we obtain the following proposition.

Proposition 4.4 *Let* $S = \{x_1, \ldots, x_n\}$ *be a factor-closed set, and let* $F(\boldsymbol{r}; \boldsymbol{n}_1, \ldots, \boldsymbol{n}_k)$ *be an even function* (mod $\boldsymbol{r}^{(\boldsymbol{k})}$). *Define the* $(m + |\boldsymbol{k}|)$*-dimensional matrix* B *of order* n *as*

$$\begin{aligned} &B(i_1, \ldots, i_{m+|\boldsymbol{k}|}) \\ &\quad = F(x_{i_1}, \ldots, x_{i_m}; \underbrace{x_{i_{m+1}}, \ldots, x_{i_{m+k_1}}}_{k_1}, \ldots, \underbrace{x_{i_{m+k_1+\ldots+k_{m-1}+1}}, \ldots, x_{i_{m+|\boldsymbol{k}|}}}_{k_m}). \end{aligned}$$

Let I *be a subset of* $\{1, 2, \ldots, m + |\boldsymbol{k}|\}$ *with even cardinality such that* $\{m + 1, m + 2 \ldots, m + |\boldsymbol{k}|\} \subseteq I$, *and set* $\varepsilon_j = 1$ *if* $j \in I$ *and* 0 *otherwise. Define the subset* $\widetilde{I}$ *of* $\{1, 2, \ldots, m\}$ *as* $j \in \widetilde{I}$ *if and only if* $\varepsilon_j + k_j$ *is odd for* $1 \le j \le m$. *Then, we have*

$$\det{}_I B = (x_1 \cdots x_n)^{|\boldsymbol{k}|} \det{}_{\widetilde{I}}\big(\alpha_{x_{i_1},\ldots,x_{i_m}}(\underbrace{x_{i_1}, \ldots, x_{i_1}}_{k_1}, \ldots, \underbrace{x_{i_m}, \ldots, x_{i_m}}_{k_m})\big)_{1 \le i_1,\ldots,i_m \le n}. \tag{4.8}$$

Here, $\alpha_{x_{i_1},\ldots,x_{i_m}}$ *is the finite Fourier coefficient of* F *given by* (4.7).

We need the following lemma.

Lemma 4.5 *Let* $A = (A(i_1, \ldots, i_k))$ *and* $C = (C(i_1, i_2))$ *be* k*-dimensional and two-dimensional matrices, respectively, of order* n. *For* $l = 0, 1, \ldots, k$, *define the* k*-dimensional matrices* $A_C^{(l)}$ *of order* n *by the following recursion formula*:

$$\begin{cases} A_C^{(0)}(i_1, \ldots, i_k) := A(i_1, \ldots, i_k) & (l = 0), \\ A_C^{(l)}(i_1, \ldots, i_k) := (A_C^{(l-1)} C)(i_2, \ldots, i_k, i_1) & (l = 1, 2, \ldots, k), \end{cases}$$

where $A_C^{(l-1)} C$ *is the Cayley product of* $A_C^{(l-1)}$ *and* C. *Then, we have*

$$A_C^{(l)}(i_1, \ldots, i_k) = \sum_{j_1,\ldots,j_l=1}^{n} A(i_{l+1}, \ldots, i_k, j_1, \ldots, j_l) \prod_{h=1}^{l} C(j_h, i_h). \tag{4.9}$$

Proof This is shown by induction on l from the definition of the Cayley product (4.4). □

Proof of Proposition 4.4 From formula (4.2) we can assume that $x_1 < x_2 < \cdots < x_n$. Define the $(m+|\boldsymbol{k}|)$-dimensional matrix A and two-dimensional matrix C of order n, respectively, as

$$A(i_1,\dots,i_{m+|\boldsymbol{k}|}) := \alpha_{x_{i_1},\dots,x_{i_m}}(x_{i_{m+1}},\dots,x_{i_{m+|\boldsymbol{k}|}}) \prod_{j=1}^{m}\prod_{l_j=1}^{k_j} e_{i_{m+k_1+\cdots+k_{j-1}+l_j},i_j},$$

$$C(i_1,i_2) := c(x_{i_1},x_{i_2}),$$

where $e_{j,i}=1$ if $x_j|x_i$ and 0 otherwise. Let $\rho=(1,2,\cdots,m+|\boldsymbol{k}|)\in\mathfrak{S}_{m+|\boldsymbol{k}|}$ be the cyclic permutation of order $(m+|\boldsymbol{k}|)$. Then, we have

$$B(i_1,\dots,i_{m+|\boldsymbol{k}|}) = A_C^{(|\boldsymbol{k}|)}(i_{\rho^m(1)},\dots,i_{\rho^m(m+|\boldsymbol{k}|)}). \tag{4.10}$$

In fact, by (4.9), the right-hand side of (4.10) is equal to

$$\begin{aligned}
&A_C^{(|\boldsymbol{k}|)}(i_{m+1},\dots,i_{m+|\boldsymbol{k}|},i_1,\dots,i_m)\\
&= \sum_{\substack{d_{j,1},\dots,d_{j,k_j}=1\\(j=1,\dots,m)}}^{n} \alpha_{x_{i_1},\dots,x_{i_m}}(\underbrace{x_{d_{1,1}},\dots,x_{d_{1,k_1}}}_{k_1},\dots,\underbrace{x_{d_{m,1}},\dots,x_{d_{m,k_m}}}_{k_m})\\
&\quad\times\prod_{j=1}^{m}\prod_{l_j=1}^{k_j} e_{d_{j,l_j},i_j}\, c(x_{d_{j,l_j}},x_{i_{m+k_1+\cdots+k_{j-1}+l_j}})\\
&= \sum_{\substack{x_{\boldsymbol{d}_j}|x_{i_j}\\(j=1,\dots,m)}} \alpha_{x_{i_1},\dots,x_{i_m}}(x_{\boldsymbol{d}_1},\dots,x_{\boldsymbol{d}_m})\prod_{j=1}^{m}\prod_{l_j=1}^{k_j} c(x_{d_{j,l_j}},x_{i_{m+k_1+\cdots+k_{j-1}+l_j}}),
\end{aligned}$$

where $\boldsymbol{d}_j := (d_{j,1},\dots,d_{j,k_j})$ for $1\le j\le m$. This clearly coincides with the left-hand side of formula (4.10) from the expression of (4.6) of F, thus completing the proof. Note that, since S is factor-closed, every divisor of x_j can again be written as x_d for some d. Next, using formula (4.3), we have

$$\begin{aligned}
\det_I B &= \det_I\big(A_C^{(|\boldsymbol{k}|)}(i_{\rho^m(1)},\dots,i_{\rho^m(m+|\boldsymbol{k}|)})\big)\\
&= \det_I\big(\big(A_C^{(|\boldsymbol{k}|-1)}C\big)(i_{\rho^m(2)},\dots,i_{\rho^m(m+|\boldsymbol{k}|)},i_{\rho^m(1)})\big)\\
&= \det_I\big(\big(A_C^{(|\boldsymbol{k}|-1)}C\big)(i_{\rho^{m+1}(1)},\dots,i_{\rho^{m+1}(m+|\boldsymbol{k}|)})\big)\\
&= \det_{\rho^{-m-1}(I)}\big(\big(A_C^{(|\boldsymbol{k}|-1)}C\big)(i_1,\dots,i_{m+|\boldsymbol{k}|})\big).
\end{aligned}$$

Here, note that $m+1\in I$ implies $m+|\boldsymbol{k}|=\rho^{-m-1}(m+1)\in\rho^{-m-1}(I)$. Therefore, using formula (4.5) with $K=\rho^{-m-1}(I)\setminus\{m+|\boldsymbol{k}|\}$ and $L=\{2\}$, we see that the right-hand side of the above equation is equivalent to

$$\begin{aligned}
&\det_{(\rho^{-m-1}(I)\setminus\{m+|\boldsymbol{k}|\})\cup\{m+|\boldsymbol{k}|\}}\big(A_C^{(|\boldsymbol{k}|-1)}\big)\cdot\det_{\{1,2\}}C\\
&\quad=\det C\cdot\det_{\rho^{-m-1}(I)}\big(\big(A_C^{(|\boldsymbol{k}|-2)}C\big)(i_2,\dots,i_{m+|\boldsymbol{k}|},i_1)\big)\\
&\quad=\det C\cdot\det_{\rho^{-m-2}(I)}\big(\big(A_C^{(|\boldsymbol{k}|-2)}C\big)(i_1,\dots,i_{m+|\boldsymbol{k}|})\big).
\end{aligned}$$

Under the condition $m+1,\dots,m+|\boldsymbol{k}|\in I$, the above procedure yields

$$\det_I B=(\det C)^{|\boldsymbol{k}|}\cdot\det_{\rho^{-m-|\boldsymbol{k}|}(I)}\big(A_C^{(0)}\big)=(\det C)^{|\boldsymbol{k}|}\cdot\det_I A.$$

By Corollary 3 in [4], we have $\det C=\det(c(x_{i_1},x_{i_2}))=x_1\cdots x_n$. Hence, it is sufficient to calculate $\det_I A$. Since $\varepsilon_j=1$ for $j=m+1,\dots,m+|\boldsymbol{k}|$, we have

$$\begin{aligned}
\det_I A=\frac{1}{n!}\sum_{\boldsymbol{\sigma}}\sum_{\boldsymbol{\sigma}_1,\dots,\boldsymbol{\sigma}_m}\prod_{j=1}^{m}\operatorname{sgn}(\sigma_j)^{\varepsilon_j}\prod_{j=1}^{m}\prod_{l_j=1}^{k_j}\operatorname{sgn}(\sigma_{j,l_j})\\
\times\prod_{v=1}^{n}\alpha_{x_{\boldsymbol{\sigma}(v)}}(x_{\boldsymbol{\sigma}_1(v)},\dots,x_{\boldsymbol{\sigma}_m(v)})\prod_{j=1}^{m}\prod_{l_j=1}^{k_j}e_{\sigma_{j,l_j}(v),\sigma_j(v)}.
\end{aligned}$$

Here, the sum is taken over all $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_m)\in(\mathfrak{S}_n)^m$ and $\boldsymbol{\sigma}_j=(\sigma_{j,1},\dots,\sigma_{j,k_j})\in(\mathfrak{S}_n)^{k_j}$ for $1\le j\le m$. For $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_k)\in(\mathfrak{S}_n)^k$ and $v\in\{1,\dots,n\}$, we set $\boldsymbol{\sigma}(v):=(\sigma_1(v),\dots,\sigma_k(v))$. Replacing the variables $\tau_{j,l_j}=\sigma_{j,l_j}\sigma_j^{-1}$ with $1\le j\le m$ and $1\le l_j\le k_j$ and writing $\boldsymbol{\tau}_j=(\tau_{j,1},\dots,\tau_{j,k_j})$, we have

$$\begin{aligned}
\det_I A=\frac{1}{n!}\sum_{\boldsymbol{\sigma}}\prod_{j=1}^{m}\operatorname{sgn}(\sigma_j)^{\varepsilon_j+k_j}\sum_{\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_m}\prod_{j=1}^{m}\prod_{l_j=1}^{k_j}\operatorname{sgn}(\tau_{j,l_j})\\
\times\prod_{v=1}^{n}\alpha_{x_{\boldsymbol{\sigma}(v)}}(x_{\boldsymbol{\tau}_1\sigma_1(v)},\dots,x_{\boldsymbol{\tau}_m\sigma_m(v)})\prod_{j=1}^{m}\prod_{l_j=1}^{k_j}e_{\tau_{j,l_j}\sigma_j(v),\sigma_j(v)},
\end{aligned}$$

where $\boldsymbol{\tau}_j\sigma_j=(\tau_{j,1}\sigma_j,\dots,\tau_{j,k_j}\sigma_j)$ for $1\le j\le m$. Note that $e_{\tau_{j,l_j}\sigma_j(v),\sigma_j(v)}=1$ for all $1\le v\le n$ if and only if $\tau_{j,l_j}=1$ since $x_1<x_2<\cdots<x_n$. Therefore, $\det_I A$ is equal to

$$\begin{aligned}
&\frac{1}{n!}\sum_{\boldsymbol{\sigma}}\prod_{j=1}^{m}\operatorname{sgn}(\sigma_j)^{\varepsilon_j+k_j}\prod_{v=1}^{n}\alpha_{x_{\boldsymbol{\sigma}(v)}}(x_{\boldsymbol{\sigma}_1(v)},\dots,x_{\boldsymbol{\sigma}_m(v)})\\
&\quad=\det_{\tilde{I}}\big(\alpha_{x_{i_1},\dots,x_{i_m}}(\underbrace{x_{i_1},\dots,x_{i_1}}_{k_1},\dots,\underbrace{x_{i_m},\dots,x_{i_m}}_{k_m})\big)_{1\le i_1,\dots,i_m\le n}.
\end{aligned}$$

This completes the proof of the proposition. □

Theorem 4.6 *Let $S = \{x_1, \dots, x_n\}$ be a factor-closed set, and let $F(r; n_1, \dots, n_k)$ be an even function* $(\mathrm{mod}\ r)$ *with respect to the variables $n_1, \dots, n_k$. Let $I = \{2, 3, \dots, k+1\}$ if k is even and $\{1, 2, \dots, k+1\}$ otherwise. Then, we have*

$$\det{}_I\big(F(x_{i_1}; x_{i_2}, \dots, x_{i_{k+1}})\big)_{1\le i_1,\dots,i_{k+1}\le n} = (x_1 \cdots x_n)^k \prod_{v=1}^{n} \alpha_{x_v}(x_v, \dots, x_v), \quad (4.11)$$

where α_{x_v} is the finite Fourier coefficient given by (2.9).

Proof Setting $m = 1$ in Proposition 4.4, we immediately obtain this theorem. □

Remark 4.7 The case of $k = 1$ is obtained in Theorem 2 in [4].

Corollary 4.8 *Let $S = \{x_1, \dots, x_n\}$ be a factor-closed set. Let $I = \{2, 3, \dots, m+1\}$ if m is even and $\{1, 2, \dots, m+1\}$ otherwise. Let ξ be an arbitrary arithmetic function of m variables. Then, we have*

$$\det{}_I\big(S_{\boldsymbol{f}}^{\boldsymbol{\gamma},\xi}(x_{i_1}, \dots, x_{i_{m+1}})\big)_{1\le i_1,\dots,i_{m+1}\le n}$$
$$= \big(f_1(1)\cdots f_m(1)\big)^n \prod_{v=1}^{n} \xi(x_v, \dots, x_v) f_{m+1}^{[\mathrm{lcm}(\gamma_1,\dots,\gamma_m)]}(x_v). \quad (4.12)$$

In particular, we have

$$\det{}_I\big(S_{\boldsymbol{f}}(x_{i_1}, \dots, x_{i_{m+1}})\big)_{1\le i_1,\dots,i_{m+1}\le n} = \big(f_1(1)\cdots f_m(1)\big)^n \prod_{v=1}^{n} f_{m+1}(x_v). \quad (4.13)$$

Proof From formula (2.11) we have $\alpha_{\boldsymbol{f},x_v}^{\boldsymbol{\gamma},\xi}(x_v, \dots, x_v) = x_v^{-m} S_{G(\boldsymbol{f})}^{H_{x_v}^{\boldsymbol{\gamma},\xi}}(x_v, 1, \dots, 1)$. Furthermore, using the identity $a_{\gamma_1}(x_v)\cdots a_{\gamma_m}(x_v) = a_{\mathrm{lcm}(\gamma_1,\dots,\gamma_m)}(x_v)$, one can see that the right-hand side above is equal to $x_v^{-m}\xi(x_v, \dots, x_v) f_1(1)\cdots f_m(1) \times f_{m+1}^{[\mathrm{lcm}(\gamma_1,\dots,\gamma_m)]}(x_v)$. Hence we obtain (4.12) and immediately (4.13). This completes the proof. □

Example 4.9 Retaining the notation in Example 2.5, we have $S_{\boldsymbol{f}} = f \circ \gcd$. Hence, by formula (4.13), we have, for a factor-closed set $S = \{x_1, \dots, x_n\}$,

$$\det{}_I\big((f \circ \gcd)(x_{i_1}, \dots, x_{i_{m+1}})\big)_{1\le i_1,\dots,i_{m+1}\le n} = \prod_{v=1}^{n} (f * \mu)(x_v).$$

This is a part of the result of Haukkanen [9].

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