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The evaluation of character Euler double sums

J.M. Borwein · I.J. Zucker · J. Boersma

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Abstract Euler considered sums of the form

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{m-1} \frac{1}{n^t}.$$

Here natural generalizations of these sums namely

$$[p,q] := [p,q](s,t) = \sum_{m=1}^{\infty} \frac{\chi_p(m)}{m^s} \sum_{n=1}^{m-1} \frac{\chi_q(n)}{n^t},$$

are investigated, where χ_p and χ_q are characters, and s and t are positive integers. The cases when p and q are either $1, 2a, 2b$ or -4 are examined in detail, and closed-form expressions are found for $t = 1$ and general s in terms of the Riemann zeta function and the Catalan zeta function—the Dirichlet series $L_{-4}(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots$. Some results for arbitrary p and q are obtained as well.

In Memoriam: Between the submission and acceptance of this report we greatly regret that our esteemed colleague John Boersma passed away. This paper is dedicated to his memory.

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J.M. Borwein
Faculty of Computing Science, Dalhousie University, Halifax, NS, B3H 3J5, Canada

I.J. Zucker (✉)
Wheatstone Physics Laboratory, King's College, Strand, London WC2R 2LS, UK
e-mail: jzucker@btinternet.com

J. Boersma
Dept. of Mathematics and Computing Science, Eindhoven University of Technology,
5600 MB Eindhoven, The Netherlands

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1 Introduction

In 1742 Goldbach wrote to Euler posing the problem of finding a closed form for the double sum

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{m-1} \frac{1}{n^t}, \tag{1.1}$$

in which the first sum is unrestricted, whilst the second is bounded. We refer to [2] for a description of this exchange. We shall refer to the first sum as the U-sum and the second as the B-sum. Any double sum so constructed will be referred to as an Euler or E-sum.

The motivation for this investigation is an attempt to generalize the E-sums such that the U- and B-sums are general Dirichlet series, i.e. to consider E-sums of the form

$$[p,q] := [p,q](s,t) = \sum_{m=1}^{\infty} \frac{\chi_p(m)}{m^s} \sum_{n=1}^{m-1} \frac{\chi_q(n)}{n^t}, \tag{1.2}$$

where χ_p and χ_q are characters. Elementary *Dirichlet series* are given by

$$L_p(s:x) = \sum_{m=1}^{\infty} \frac{\chi_p(m)x^m}{m^s} \tag{1.3}$$

and we write $L_p(s:1) := L_p(s)$. In Appendix 1 some properties of characters and Dirichlet series relevant to this communication are summarized. A reflection formula is immediately accessible for (1.2). Write $[p,q](s,t)$ as

$$[p,q](s,t) = \sum_{m>n} \frac{\chi_p(m)}{m^s} \frac{\chi_q(n)}{n^t}. \tag{1.4}$$

Then

$$[q,p](t,s) = \sum_{m<n} \frac{\chi_q(n)}{n^t} \frac{\chi_p(m)}{m^s}. \tag{1.5}$$

Adding (1.4) to (1.5) we get

$$\sum_{m\neq n} \frac{\chi_q(n)}{n^t} \frac{\chi_p(m)}{m^s} = \sum_{m\geq 1, n\geq 1} \frac{\chi_p(m)}{m^s} \frac{\chi_q(n)}{n^t} - \sum_{m\geq 1} \frac{\chi_p(m)\chi_q(m)}{m^{s+t}}, \tag{1.6}$$

and since $\chi_p(m)\chi_q(m) = \chi_{pq}(m)$ we have the reflection formula

$$[p,q](s,t) + [q,p](t,s) = L_p(s)L_q(t) - L_{pq}(s+t). \tag{1.7}$$

The situation when $p=q=d$ is thus of interest. Let $P=p_1p_2\cdots p_k$, where the p_k are all different odd primes, i.e. P is odd and square-free and $\neq 1$. Then if $d=P$ it is simple to show that

$$L_{d^2}(s)=\prod_{m=1}^{k}(1-p_m^{-s})\zeta(s). \tag{1.8}$$

If d is even and P is the odd factor >1 of d then

$$L_{d^2}(s)=(1-2^{-s})\prod_{m=1}^{k}(1-p_m^{-s})\zeta(s), \tag{1.9}$$

and if $d=2^n$ then

$$L_{d^2}(s)=(1-2^{-s})\zeta(s). \tag{1.10}$$

Now consider (1.2) and first rewrite it with dummy indices (m,n) replaced by (k,m), and then interchange the order of summation, thus

$$[p,q](s,t)=\sum_{k=1}^{\infty}\frac{\chi_p(k)}{k^s}\sum_{m=1}^{k-1}\frac{\chi_q(m)}{m^t}=\sum_{m=1}^{\infty}\frac{\chi_q(m)}{m^t}\sum_{k=m+1}^{\infty}\frac{\chi_p(k)}{k^s}.$$

Replace k by $m+n$ and we obtain

$$[p,q](s,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi_q(m)}{m^t}\frac{\chi_p(m+n)}{(m+n)^s}. \tag{1.11}$$

Such representations will be referred to as unrestricted double sums or UD-sums. This form of E-sum is most useful as it allows us to represent it in many cases as a single integral. For let us write

$$(m+n)^{-s}=\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}x^{m+n-1}\mathrm{d}x, \tag{1.12}$$

then

$$[p,q](s,t)=\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi_q(m)}{m^t}\chi_p(m+n)x^{m+n-1}\mathrm{d}x. \tag{1.13}$$

If $\chi_p(m+n)$ can be split multiplicatively, the summations may be done and $[p,q]$ found as a single integral. Sadly, however, we are only able to do this for a few cases, and in the first instance we concentrate on these. The E-sums we shall consider are those involving the *primitive characters* χ_1 and χ_{-4}, and two *imprimitive* forms of χ_1, namely χ_{2a} (counting odd terms) and χ_{2b} (counting all terms alternatingly).

The four series considered here in detail are

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{\chi_1(m)}{m^s}=\sum_{m=1}^{\infty}\frac{1}{m^s}=L_1(s), \tag{1.14}$$

$$\lambda(s) = \sum_{m=1}^{\infty} \frac{\chi_{2a}(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\sin^2(m\pi/2)}{m^s}$$
$$= \sum_{m=1}^{\infty} \frac{1}{(2m-1)^s} = L_{2a}(s) = (1-2^{-s})L_1(s), \tag{1.15}$$

$$\eta(s) = \sum_{m=1}^{\infty} \frac{\chi_{2b}(m)}{m^s} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} = L_{2b}(s) = (1-2^{1-s})L_1(s), \tag{1.16}$$

$$\beta(s) = \sum_{m=1}^{\infty} \frac{\chi_{-4}(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\sin(m\pi/2)}{m^s} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^s} = L_{-4}(s). \tag{1.17}$$

Thus, we will consider double series with any one of the four above as the U-sum or B-sum yielding the following 16 E-sums to be considered. These are shown in Table 1. The first 8 E-sums, of the form $[1, q]$ or $[2a, q]$, are convergent for $s \geq 2$, $t \geq 1$. The second 8 E-sums, of the form $[2b, q]$ or $[-4, q]$, are convergent for $s \geq 1$, $t \geq 1$.

The sum [1,1] is the original E-sum considered by Euler. Euler [4] responded to Goldbach's query (somewhat tardily) and wrote a memoir on the topic in 1775. He showed that [1,1] could be evaluated in terms of the Riemann zeta function for all $s+t \leq 6$, and $s+t \leq 13$ when $s+t$ is odd. He then extrapolated his results to obtain a general formula for all $s+t$ odd. He also proved a formula for $[1,1](s,1)$ for general s. Some 130 years later Nielsen [7] extended Euler's investigation to consider [1,2b], [2b,1] and [2b,2b], and derived a formula for $[1,2b](s,1)$. More recently Sitaramachandrarao [8] found general formulae for $[2b,1](2s,1)$ and $[2b,2b](2s,1)$, and Basu and Apostol [1] introduced a new method for deducing Eulers results. Borwein, Borwein and Girgensohn (see [3]) also investigated $[1,1]$, $[1,2b]$, $[2b,1]$ and $[2b,2b]$. They *proved* Euler's general result for $[1,1]$ with $s+t$ odd and also Nielsen's and Sitaramachandrarao's results. Jordan [5] in a paper unconnected with E-sums *per se* gave formulae for modified forms of $[1,2a](2s,1)$ and $[2a,2a](2s,1)$, defined by

$$[1,2a]^*(2s,1) = \sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sum_{n=1}^{m} \frac{1}{2n-1}, \tag{1.18}$$

$$[2a,2a]^*(2s,1) = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^{2s}} \sum_{n=1}^{m} \frac{1}{2n-1}. \tag{1.19}$$

These last results have been starred to indicate they are not true E-sums in that they do not satisfy (1.7). For whereas

$$\sum_{m=1}^{\infty} \frac{\sin^2(m\pi/2)}{m^s} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^s} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{\sin(m\pi/2)}{m^s} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^s},$$

Table 1 Euler sums considered in this article

$[1,1](s,t)=$	$\sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{m-1}\frac{1}{n^t}$
$[1,2a](s,t)=$	$\sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{m-1}\frac{\sin^2(n\pi/2)}{n^t}$
$[1,2b](s,t)=$	$\sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{m-1}\frac{(-1)^{n-1}}{n^t}$
$[1,-4](s,t)=$	$\sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{m-1}\frac{\sin(n\pi/2)}{n^t}$
$[2a,1](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin^2(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{1}{n^t}$
$[2a,2a](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin^2(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{\sin^2(n\pi/2)}{n^t}$
$[2a,2b](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin^2(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{(-1)^{n-1}}{n^t}$
$[2a,-4](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin^2(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{\sin(n\pi/2)}{n^t}$
$[2b,1](s,t)=$	$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^s}\sum_{n=1}^{m-1}\frac{1}{n^t}$
$[2b,2a](s,t)=$	$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^s}\sum_{n=1}^{m-1}\frac{\sin^2(n\pi/2)}{n^t}$
$[2b,2b](s,t)=$	$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^s}\sum_{n=1}^{m-1}\frac{(-1)^{n-1}}{n^t}$
$[2b,-4](s,t)=$	$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^s}\sum_{n=1}^{m-1}\frac{\sin(n\pi/2)}{n^t}$
$[-4,1](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{1}{n^t}$
$[-4,2a](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{\sin^2(n\pi/2)}{n^t}$
$[-4,2b](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{(-1)^{n-1}}{n^t}$
$[-4,-4](s,t)=$	$\sum_{m=1}^{\infty}\frac{\sin(m\pi/2)}{m^s}\sum_{n=1}^{m-1}\frac{\sin(n\pi/2)}{n^t}$

it is important to note that

$$\sum_{n=1}^{m-1} \frac{\sin^2(n\pi/2)}{n} \neq \sum_{n=1}^{m-1} \frac{1}{2n-1} \quad \text{and} \quad \sum_{n=1}^{m-1} \frac{\sin(n\pi/2)}{n} \neq \sum_{n=1}^{m-1} \frac{(-1)^{n-1}}{2n-1}.$$

Otherwise, we have found very little on general E-sums in the literature.

2 Integral representations of E-sums

As indicated previously, we may use (1.11) and (1.13) to turn the E-sums here into integrals, and we show now how this may be done. Throughout we proceed formally—noting that the sums in question are all absolutely convergent for $s > 1$ and do not explicitly mention the routine justification of rearrangements. We start with the simplest case as a first example. From (1.11) we have

$$[1,1](s,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^t (m+n)^s}. \tag{2.1}$$

Then using (1.13) we have

$$[1,1](s,t) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 (\log x)^{s-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^t} x^{m+n-1} \mathrm{d}x. \tag{2.2}$$

The summation over n gives $1/(1-x)$ and the summation over m may be written in several ways

$$\sum_{m=1}^{\infty} \frac{x^m}{m^t} = \zeta(t:x) = L_1(t:x) = Li_t(x), \tag{2.3}$$

where $Li_t(x)$ is the polylogarithm of Lewin [6]. Thus we have

$$[1,1](s,t) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 (\log x)^{s-1} \frac{Li_t(x)}{1-x} \mathrm{d}x. \tag{2.4}$$

In further examples functions similar to $Li_t(x)$ appear and we list them below:

$$\sum_{m=1}^{\infty} \frac{x^{2m-1}}{(2m-1)^t} = \lambda(t:x) = L_{2a}(t:x) = Thi_t(x)$$

$$= \frac{1}{2} Li_t(x) - \frac{1}{2} Li_t(-x), \tag{2.5}$$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m^t} = \eta(t:x) = L_{2b}(t:x) = -Li_t(-x), \tag{2.6}$$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^{2m-1}}{(2m-1)^t} = \beta(t:x) = L_{-4}(t:x) = Ti_t(x). \tag{2.7}$$

$Ti_t(x)$ is the inverse tangent integral of Lewin [6]. Lewin calls $Thi_t(x)$ Legendre's chi-function, denoted by him as $\chi_t(x)$. However, just as arctan(x) is the progenitor of $Ti_t(x)$, arctanh(x) plays the same role for $Thi_t(x)$, and the latter seems a more apt notation. The following properties of these functions are noted:

$$Li_1(x) = -\log(1-x), \qquad Ti_1(x) = \arctan(x), \qquad Thi_1(x) = \operatorname{arctanh}(x). \quad (2.8)$$

All these functions obey the general rule

$$f_n(x) = \int_0^x \frac{f_{n-1}(y)}{y} \mathrm{d}y, \quad (2.9)$$

and

$$\begin{aligned} Li_n(1) &= \zeta(n), \qquad Li_n(-1) = -\eta(n), \\ Ti_n(1) &= \beta(n), \qquad Thi_n(1) = \lambda(n). \end{aligned} \quad (2.10)$$

As a somewhat more complex example of producing an integral representation for an E-sum consider $[-4, 1](s, t)$. From (1.11) we have

$$\begin{aligned} [-4, 1](s, t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin((m+n)\pi/2)}{m^t(m+n)^s} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin(m\pi/2)\cos(n\pi/2)}{m^t(m+n)^s} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\cos(m\pi/2)\sin(n\pi/2)}{m^t(m+n)^s} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m-1}(-1)^n}{(2m-1)^t(2m+2n-1)^s} \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^m(-1)^{n-1}}{(2m)^t(2m+2n-1)^s}. \end{aligned} \quad (2.11)$$

Then using (1.13) and performing the separate sums over n and m we obtain

$$[-4, 1](s, t) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 (\log x)^{s-1} \left[\frac{2^{-t} Li_t(-x^2)}{1+x^2} - \frac{x\, Ti_t(x)}{1+x^2} \right] \mathrm{d}x. \quad (2.12)$$

In this way all 16 E-sums of Table 1 can be converted into UD-sums and integral representations can be found. These are exhibited in Table 2. It is apparent from these representations that the E-sums displayed in Table 2 are not independent of one another. Indeed 8 relations may be deduced which can be arranged in 4 pairs, namely

$$[1, q] + [2b, q] = 2[2a, q], \quad \text{and} \quad [q, 1] + [q, 2b] = 2[q, 2a], \quad (2.13)$$

where $q = 1, 2a, 2b$ or -4. Within each pair the two relations are equivalent as can be verified by means of the reflection formula (1.7) and the relation

$$L_1(s) + L_{2b}(s) = 2L_{2a}(s) \quad \text{which is equivalent to } \zeta(s) + \eta(s) = 2\lambda(s). \quad (2.14)$$

Table 2 Euler sums transformed into unrestricted sums and integrals

E-sum	UD-sum	Integral representation
$[1,1](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^t(m+n)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Li_t(x)}{1-x}\mathrm{d}x$
$[1,2a](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^t(2m+n-1)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Thi_t(x)}{1-x}\mathrm{d}x$
$[1,2b](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m-1}}{m^t(m+n)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Li_t(-x)}{1-x}\mathrm{d}x$
$[1,-4](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^t(2m+n-1)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Ti_t(x)}{1-x}\mathrm{d}x$
$[2a,1](s,t)$	$\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1+(-1)^{m+n-1}}{m^t(m+n)^s}$	$\frac{1}{2}\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\left[\frac{Li_t(x)}{1-x}+\frac{Li_t(-x)}{1+x}\right]\mathrm{d}x$
$[2a,2a](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^t(2m+2n-1)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{x\,Thi_t(x)}{1-x^2}\mathrm{d}x$
$[2a,2b](s,t)$	$\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m-1}+(-1)^n}{m^t(m+n)^s}$	$-\frac{1}{2}\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}$ $\times\left[\frac{Li_t(-x)}{1-x}+\frac{Li_t(x)}{1+x}\right]\mathrm{d}x$
$[2a,-4](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^t(2m+2n-1)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{x\,Ti_t(x)}{1-x^2}\mathrm{d}x$
$[2b,1](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{m^t(m+n)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Li_t(-x)}{1+x}\mathrm{d}x$
$[2b,2a](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2m-1)^t(2m+n-1)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Thi_t(x)}{1+x}\mathrm{d}x$
$[2b,2b](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{m^t(m+n)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Li_t(x)}{1+x}\mathrm{d}x$
$[2b,-4](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m-1)^t(2m+n-1)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{Ti_t(x)}{1+x}\mathrm{d}x$
$[-4,1](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m-1)^t(2m+2n-1)^s}$ $+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m)^t(2m+2n-1)^s}$	$\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}$ $\times\left[\frac{2^{-t}Li_t(-x^2)-x\,Ti_t(x)}{1+x^2}\right]\mathrm{d}x$
$[-4,2a](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m-1)^t(2m+2n-1)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{x\,Ti_t(x)}{1+x^2}\mathrm{d}x$
$[-4,2b](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m-1)^t(2m+2n-1)^s}$ $-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m)^t(2m+2n-1)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}$ $\times\left[\frac{2^{-t}Li_t(-x^2)+x\,Ti_t(x)}{1+x^2}\right]\mathrm{d}x$
$[-4,-4](s,t)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2m-1)^t(2m+2n-1)^s}$	$-\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1(\log x)^{s-1}\frac{x\,Thi_t(x)}{1+x^2}\mathrm{d}x$

3 Exact expressions for E-sums

We first must make explicit what is meant here by 'exact'. For the E-sums considered previously in [1–8], it clearly meant expressing the result in terms of the constants of classical analysis $\zeta(n)$, $\lambda(n)$ and $\eta(n)$, i.e., essentially the zeta function at integer arguments. However, since we have now added $\beta(n)$ as an extra factor in the E-sums considered here, it would be expected that $\beta(n)$ should appear as an element in any solution. Hence, if we are able to express a general E-sum in terms of the four functions given by (1.14–1.17)—or by like quantities—the expression will be termed *exact*, and said to be given in *fully closed form* in terms of well-known (WK) constants of analysis. If less familiar constants of analysis such as Dirichlet series evaluated with argument $x \neq 1$ appear, and this enables us to express hitherto unknown E-sums 'exactly', these will be said to be given in *semi-closed form*.

Our initial attempts at finding E-sums not previously evaluated were concentrated on evaluating their integral representations as given in Table 2. This was haphazard at best, and generally only for very small values of s and t could the evaluations be accomplished. With the aid of considerable experimental mathematical insight, however, a method was subsequently developed for finding the integrals for general s and $t = 1$.

A generating function $G[p,q](w)$ is introduced for the E-sums $[p,q](s,1)$. In general our generating functions are defined by

$$G[p,q](w) = \sum_{s=1\,\mathrm{or}\,2}^{\infty} [p,q](s,1)w^{s-1}. \tag{3.1}$$

Here, the lower limit in the summation is $s = 1$ or $s = 2$ dependent on whether $[p,q](1,1)$ is convergent or divergent. An integral representation for G readily follows from the integral representation for $[p,q](s,1)$ in Table 2. The *even* and *odd parts* of $G(w)$ (even or odd in w) are denoted by

$$Ge[p,q](w) = \frac{1}{2}\{G[p,q](w) + G[p,q](-w)\} = \sum_{s=0\mathrm{or}1}^{\infty} [p,q](2s+1,1)w^{2s}, \tag{3.2}$$

$$Go[p,q](w) = \frac{1}{2}\{G[p,q](w) - G[p,q](-w)\} = \sum_{s=1}^{\infty}[p,q](2s,1)w^{2s-1}. \tag{3.3}$$

One method of evaluating $[p,q](s,1)$ is illustrated below for $[1,1](s,1)$, $[1,2a](s,1)$ and $[1,2b](s,1)$. These E-sums are represented by the integrals

$$[1,1](s,1) = -\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1 \frac{(\log x)^{s-1}}{1-x}\log(1-x)\mathrm{d}x, \tag{3.4}$$

$$[1,2a](s,1) = \frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1 \frac{(\log x)^{s-1}}{1-x}\operatorname{arctanh}(x)\mathrm{d}x, \tag{3.5}$$

$$[1,2b](s,1) = \frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1 \frac{(\log x)^{s-1}}{1-x}\log(1+x)\mathrm{d}x, \tag{3.6}$$

convergent for $s \geq 2$. Since $\operatorname{arctanh}(x) = \frac{1}{2}\log[(1+x)/(1-x)]$, one has the relation

$$[1, 2a](s, 1) = \frac{1}{2}([1, 1](s, 1) + [1, 2b](s, 1)). \tag{3.7}$$

{This result can of course be immediately found from (2.13).}

Introduce the generating functions $G[1, 1](w)$ and $G[1, 2b](w)$ defined and given by

$$G[1, 1](w) = \sum_{s=2}^{\infty}[1, 1](s, 1)w^{s-1} = \int_0^1 \frac{1-x^{-w}}{1-x}\log(1-x)\mathrm{d}x, \tag{3.8}$$

$$G[1, 2b](w) = \sum_{s=2}^{\infty}[1, 2b](s, 1)w^{s-1} = -\int_0^1 \frac{1-x^{-w}}{1-x}\log(1+x)\mathrm{d}x. \tag{3.9}$$

The integrals in (3.8) and (3.9) are convergent for $\Re(w) < 2$ and define $G[1, 1](w)$ and $G[1, 2b](w)$ as real analytic functions of w for $\Re(w) < 2$. The series in (3.8) and (3.9) are convergent for $|w| < 2$. The integral (3.8) for $G[1, 1](w)$ can be expressed as the derivative of a beta function:

$$\begin{aligned} G[1, 1](w) &= \frac{d}{dq}\left[\int_0^1 (1-x^{-w})(1-x)^{q-1}\mathrm{d}x\right]_{q\downarrow 0} \\ &= \frac{d}{dq}\left[\frac{1}{q} - \frac{\Gamma(1-w)\Gamma(q)}{\Gamma(1-w+q)}\right]_{q\downarrow 0}. \end{aligned} \tag{3.10}$$

Expand the latter expression between square brackets in powers of q:

$$\begin{aligned} &\frac{1}{q} - \frac{\Gamma(1-w)\Gamma(q)}{\Gamma(1-w+q)} \\ &\quad = \frac{1}{q} - \frac{1}{q}\bigg\{1 + q[\psi(1) - \psi(1-w)] + \frac{q^2}{2}[\psi^2(1) + \psi'(1) \\ &\qquad - 2\psi(1)\psi(1-w) + \psi^2(1-w) - \psi'(1-w)] + O(q^3)\bigg\}, \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Then $G[1, 1](w)$ is found to be given by

$$G[1, 1](w) = -\frac{1}{2}[\psi(1-w) - \psi(1)]^2 + \frac{1}{2}[\psi'(1-w) - \psi'(1)]. \tag{3.11}$$

We establish the auxiliary expansions

$$\psi(1-w) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\psi^{(k)}(1)w^k = \psi(1) - \sum_{k=1}^{\infty}\zeta(k+1)w^k, \tag{3.12}$$

$$\psi'(1-w) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\psi^{(k+1)}(1)w^k = \psi'(1) + \sum_{k=1}^{\infty}(k+1)\zeta(k+2)w^k, \tag{3.13}$$

which are used to expand $G[1,1](w)$ from (3.11) in powers of w. Then the E-sum $[1,1](s,1)$ is found as the coefficient of w^{s-1} in the expansion of $G[1,1](w)$:

$$[1,1](s,1)=\frac{1}{2}s\zeta(s+1)-\frac{1}{2}\sum_{k=1}^{s-2}\zeta(k+1)\zeta(s-k). \tag{3.14}$$

The integral (3.9) for $G[1,2b](w)$ can be reduced to

$$\begin{aligned}
&G[1,2b](w)\\
&=\int_0^1\frac{1-x^{-w}}{1-x}\log(1-x)\mathrm{d}x-\int_0^1\frac{(1+x)(1-x^{-w})}{1-x^2}\log(1-x^2)\mathrm{d}x\\
&=G[1,1](w)-\frac{1}{2}\int_0^1\frac{1-x^{-w/2}+x^{-1/2}-x^{-1/2-w/2}}{1-x}\log(1-x)\mathrm{d}x\\
&=G[1,1](w)-\frac{1}{2}G[1,1](w/2)-\frac{1}{2}G[1,1](\tfrac{1}{2}+w/2)+\tfrac{1}{2}G[1,1](\tfrac{1}{2}).
\end{aligned} \tag{3.15}$$

By inserting the value of $G[1,1]$ from (3.11) and simplifying it is found that

$$\begin{aligned}
&-\frac{1}{2}G[1,1](\tfrac{1}{2}+w/2)+\tfrac{1}{2}G[1,1](\tfrac{1}{2})\\
&\quad=\frac{1}{4}[\psi(\tfrac{1}{2}-w/2)-\psi(\tfrac{1}{2})]^2-[\psi(\tfrac{1}{2}-w/2)-\psi(\tfrac{1}{2})]\log 2\\
&\qquad-\frac{1}{4}[\psi'(\tfrac{1}{2}-w/2)-\psi'(\tfrac{1}{2})].
\end{aligned} \tag{3.16}$$

We establish the auxiliary expansions

$$\begin{aligned}
\psi(\tfrac{1}{2}-w/2)&=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\psi^{(k)}(\tfrac{1}{2})(w/2)^k\\
&=\psi(\tfrac{1}{2})-2\sum_{k=1}^{\infty}\lambda(k+1)w^k,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\psi'(\tfrac{1}{2}-w/2)&=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\psi^{(k+1)}(\tfrac{1}{2})(w/2)^k\\
&=\psi'(\tfrac{1}{2})+4\sum_{k=1}^{\infty}(k+1)\lambda(k+2)w^k,
\end{aligned} \tag{3.18}$$

which are used to expand the expression (3.16) in powers of w. Then the E-sum $[1,2b](s,1)$ is found as the coefficient of w^{s-1} in the expansion of $G[1,2b](w)$ from

(3.15) in powers of w:

$$[1,2b](s,1) = (1-2^{-s})[1,1](s,1) - s\lambda(s+1) + 2\log 2\lambda(s) + \sum_{k=1}^{s-2}\lambda(k+1)\lambda(s-k). \tag{3.19}$$

Insert the value of $[1,1](s,1)$ from (3.14) and simplify by means of the relations

$$\begin{aligned}
&\frac{1}{2}(1-2^{-s})s\zeta(s+1) - s\lambda(s+1) \\
&\quad = \frac{1}{2}s[1-2^{-s}-2(1-2^{-s-1})]\zeta(s+1) = -\frac{1}{2}s\zeta(s+1), \\
&-\frac{1}{2}(1-2^{-s})\zeta(k+1)\zeta(s-k) + \lambda(k+1)\lambda(s-k) \\
&\quad = \frac{1}{2}[-1+2^{-s}+2(1-2^{-k-1})(1-2^{-s+k})]\zeta(k+1)\zeta(s-k) \\
&\quad = \frac{1}{2}(1-2^{-k})(1-2^{1-s+k})\zeta(k+1)\zeta(s-k) = \frac{1}{2}\eta(k+1)\eta(s-k).
\end{aligned}$$

On setting $\log 2 = \eta(1)$ in (3.19), we obtain as our final result

$$\begin{aligned}
[1,2b](s,1) &= -\frac{1}{2}s\zeta(s+1) + 2\eta(1)\lambda(s) + \frac{1}{2}\sum_{k=1}^{s-2}\eta(k+1)\eta(s-k) \\
&= -\frac{1}{2}s\zeta(s+1) + \eta(1)\zeta(s) + \frac{1}{2}\sum_{k=1}^{s}\eta(k)\eta(s-k+1).
\end{aligned} \tag{3.20}$$

The E-sum $[1,2a](s,1)$ is evaluated by means of (3.7). By addition of (3.14) and the first line of (3.20) we find

$$\begin{aligned}
[1,2a](s,1) &= \eta(1)\lambda(s) - \frac{1}{4}\sum_{k=1}^{s-2}[\zeta(k+1)\zeta(s-k) - \eta(k+1)\eta(s-k)] \\
&= \eta(1)\lambda(s) - \frac{1}{4}\sum_{k=1}^{s-2}(2^{-k}+2^{1-s+k}-2^{1-s})\zeta(k+1)\zeta(s-k).
\end{aligned} \tag{3.21}$$

In the latter sum the first and second constituents are equal by symmetry, so that

$$\begin{aligned}
[1,2a](s,1) &= \eta(1)\lambda(s) - \frac{1}{4}\sum_{k=1}^{s-2}(2\cdot 2^{-k}-2^{1-s})\zeta(k+1)\zeta(s-k) \\
&= \eta(1)\lambda(s) - \sum_{k=1}^{s-2}2^{-k-1}\zeta(k+1)\lambda(s-k).
\end{aligned} \tag{3.22}$$

The E-sum $[2b, -4](s, 1)$ is evaluated in a different manner by finding its generating function directly from its UD representation. From Table 2 we have

$$[2b, -4](s, 1) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m+n-1}}{(2m-1)(2m+n-1)^s}. \tag{3.23}$$

The generating function $G[2b, -4](w)$ is then given by

$$G[2b, -4](w) = \sum_{s=1}^{\infty}[2b, -4](s, 1)w^{s-1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m+n-1}}{(2m-1)(2m+n-1-w)}. \tag{3.24}$$

Decomposing the summand in (3.24) according to

$$\frac{1}{(2m-1)(2m+n-1-w)} = \frac{1}{(2m-1)(n-w)} - \frac{1}{(2m+n-1-w)(n-w)}$$

and using the result

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} = \beta(1) = \frac{\pi}{4},$$

it follows that

$$G[2b, -4](w) = -\beta(1)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n-w} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{(2m+n-1-w)(n-w)}. \tag{3.25}$$

Denoting the double sum in (3.25) by $D(w)$, we split $D(w)$ into a double sum over all $m \geq 1$ and even $n \geq 1$, and a double sum over all $m \geq 1$ and odd $n \geq 1$:

$$D(w) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^m}{(2m+2n-1-w)(2n-w)} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^m}{(2m+2n-2-w)(2n-1-w)}. \tag{3.26}$$

Introducing the new summation variable $k = m + n$ we obtain

$$D(w) = \sum_{k=2}^{\infty} \frac{(-1)^k}{2k-1-w}\sum_{n=1}^{k-1} \frac{(-1)^n}{2n-w} - \sum_{k=2}^{\infty} \frac{(-1)^k}{2k-2-w}\sum_{n=1}^{k-1} \frac{(-1)^n}{2n-1-w}. \tag{3.27}$$

By interchanging the order of summation, the second double sum in (3.27) is reduced to

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1-w}\sum_{k=n+1}^{\infty} \frac{(-1)^k}{2k-2-w} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1-w}\sum_{k=n}^{\infty} \frac{(-1)^k}{2k-w} =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1-w}\sum_{n=k}^{\infty} \frac{(-1)^n}{2n-w}, \tag{3.28}$$

where in the final step the letters n and k were interchanged. By combining (3.28) with the first double sum in (3.27) we find

$$D(w)=\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1-w}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-w}, \tag{3.29}$$

which is to be inserted into (3.25). Expand the single series occurring in (3.25) and (3.29) in powers of w:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-w}=\sum_{n=1}^{\infty}(-1)^{n-1}\sum_{s=1}^{\infty}\frac{w^{s-1}}{n^s}=\sum_{s=1}^{\infty}\eta(s)w^{s-1},$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1-w}=\sum_{k=1}^{\infty}(-1)^k\sum_{s=1}^{\infty}\frac{w^{s-1}}{(2k-1)^s}=-\sum_{s=1}^{\infty}\beta(s)w^{s-1},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-w}=\sum_{n=1}^{\infty}(-1)^n\sum_{s=1}^{\infty}\frac{w^{s-1}}{(2n)^s}=-\sum_{s=1}^{\infty}2^{-s}\eta(s)w^{s-1}.$$

Thus we are led to the following representation for the generating function $G[2b,-4](w)$:

$$G[2b,-4](w)=-\beta(1)\sum_{s=1}^{\infty}\eta(s)w^{s-1}+\sum_{m=1}^{\infty}\beta(m)w^{m-1}\sum_{n=1}^{\infty}2^{-n}\eta(n)w^{n-1}. \tag{3.30}$$

Finally, the E-sum $[2b,-4](s,1)$ is found as the coefficient of w^{s-1} in the expansion in the right-hand side of (3.30):

$$[2b,-4](s,1)=-\beta(1)\eta(s)+\sum_{k=1}^{s}2^{-k}\eta(k)\beta(s-k+1). \tag{3.31}$$

(We are indebted to Larry Glasser for his implicit suggestion to base the derivation on the representation of the generating function by a UD-sum rather than by an integral.)

The methods just described are somewhat limited in their approach. Next, a more illuminating scheme has been developed which is illustrated below by an example.

4 E-sums that can only be evaluated for either even or odd *s*

It may be observed that pairs of the 16 E-sums in Table 2 have similar structures which are reflected in their generating functions. Thus consider the E-sums $[2a,-4](s,1)$ and $[-4,-4](s,1)$ whose generating functions are

$$G[2a,-4](w)=\int_0^1\frac{x^{-w}-1}{1-x^2}x\arctan(x)\,\mathrm{d}x, \tag{4.1}$$

$$G[-4,-4](w) = -\int_0^1 \frac{x^{-w+1}}{1+x^2}\operatorname{arctanh}(x)\mathrm{d}x. \tag{4.2}$$

Let C be the contour made up of three parts: the line segment $z = x$, $0 \le x \le 1$; the quarter circle $z = e^{i\theta}$, $0 \le \theta \le \pi/2$; and the line segment $z = iy$, $1 \ge y \ge 0$. We start from the contour integral

$$\int_C \frac{z^{-w}-1}{1-z^2} z\arctan(z)\mathrm{d}z = 0. \tag{4.3}$$

Evaluation of this integral along the three parts of C yields

$$G[2a,-4](w) + \frac{1}{2}\int_0^{\pi/2} (1-e^{-iw\theta})[\cot\theta + i]\left[\pi/4 + \frac{1}{2}i\log(\tan(\pi/4+\theta/2))\right]\mathrm{d}\theta$$
$$- ie^{-\pi iw/2}G[-4,-4](w) + i[-4,-4](1,1) = 0. \tag{4.4}$$

Let w be real. By successively taking real parts and even parts in (4.4) we obtain

$$Ge[2a,-4](w) + \frac{1}{2}\int_0^{\pi/2} (1-\cos(w\theta))\left[(\pi/4)\cot\theta - \frac{1}{2}\log(\tan(\pi/4+\theta/2))\right]\mathrm{d}\theta$$
$$- \sin(\pi w/2)Go[-4,-4](w) = 0. \tag{4.5}$$

Likewise, by successively taking imaginary parts and odd parts in (4.4) we obtain

$$\frac{1}{2}\int_0^{\pi/2} \sin(w\theta)\left[(\pi/4)\cot\theta - \frac{1}{2}\log(\tan(\pi/4+\theta/2))\right]\mathrm{d}\theta$$
$$- \cos(\pi w/2)Go[-4,-4](w) = 0. \tag{4.6}$$

The integrals in (4.5) and (4.6) are reduced as follows:

$$\begin{aligned} \int_0^{\pi/2} (1-\cos(w\theta))\cot\theta\mathrm{d}\theta &= \frac{1}{2}C(w/2), \\ \int_0^{\pi/2} \sin(w\theta)\cot\theta\mathrm{d}\theta &= \frac{1}{2}S(w/2), \end{aligned} \tag{4.7}$$

where $C(w)$ and $S(w)$ are defined by

$$\begin{aligned} C(w) &= \int_0^{\pi} (1-\cos(w\theta))\cot(\theta/2)\mathrm{d}\theta, \\ S(w) &= \int_0^{\pi} \sin(w\theta)\cot(\theta/2)\mathrm{d}\theta; \end{aligned} \tag{4.8}$$

$$\int_0^{\pi/2} (1-\cos(w\theta))\log(\tan(\pi/4+\theta/2))\mathrm{d}\theta$$
$$= -\int_0^{\pi/2} [1-\cos(\pi w/2)\cos(w\theta)-\sin(\pi w/2)\sin(w\theta)]\log(\tan(\theta/2))\mathrm{d}\theta$$
$$= \int_0^{\pi/2} \frac{\theta}{\sin\theta}\mathrm{d}\theta - \frac{\cos(\pi w/2)}{w}\int_0^{\pi/2}\frac{\sin(w\theta)}{\sin\theta}\mathrm{d}\theta$$
$$- \frac{\sin(\pi w/2)}{w}\int_0^{\pi/2}\frac{1-\cos(w\theta)}{\sin\theta}\mathrm{d}\theta, \tag{4.9}$$

$$\int_0^{\pi/2} \sin(w\theta)\log(\tan(\pi/4+\theta/2))\mathrm{d}\theta$$
$$= -\int_0^{\pi/2} [\sin(\pi w/2)\cos(w\theta)-\cos(\pi w/2)\sin(w\theta)]\log(\tan(\theta/2))\mathrm{d}\theta$$
$$= \frac{\sin(\pi w/2)}{w}\int_0^{\pi/2}\frac{\sin(w\theta)}{\sin\theta}\mathrm{d}\theta - \frac{\cos(\pi w/2)}{w}\int_0^{\pi/2}\frac{1-\cos(w\theta)}{\sin\theta}\mathrm{d}\theta, \tag{4.10}$$

by the substitution $\theta \to \pi/2-\theta$, and the use of

$$\int_0^{\pi/2} \log(\tan(\theta/2))\mathrm{d}\theta = -\int_0^{\pi/2}\frac{\theta}{\sin\theta}\mathrm{d}\theta = -2\beta(2), \tag{4.11}$$
$$\int_0^{\pi/2} \cos(w\theta)\log(\tan(\theta/2))\mathrm{d}\theta = -\frac{1}{w}\int_0^{\pi/2}\frac{\sin(w\theta)}{\sin\theta}\mathrm{d}\theta, \tag{4.12}$$
$$\int_0^{\pi/2} \sin(w\theta)\log(\tan(\theta/2))\mathrm{d}\theta = -\frac{1}{w}\int_0^{\pi/2}\frac{1-\cos(w\theta)}{\sin\theta}\mathrm{d}\theta, \tag{4.13}$$

found by integrating by parts. The evaluation of all the integrals involved in these formulae is given in Appendix 2. Also in Appendix 3 a list of generating functions for the set WK of constants $\zeta(m)$, $\eta(m)$, $\lambda(m)$ and $\beta(m)$ is given.

The integrals in (4.8), (4.12) and (4.13) are given by (8.3), (8.5), (8.10) and (8.12) of Appendix 2, and substituted into (4.5) and (4.6). Then by solving (4.5) and (4.6) for $Ge[2a,-4](w)$ and $Go[-4,-4](w)$ we find

$$Ge[2a,-4](w) = \frac{1}{2}\beta(2) - \frac{\pi}{8}\log 2 - \frac{\pi}{16}\int_0^1 \frac{2-x^{-w/2}-x^{w/2}}{1-x}\mathrm{d}x$$
$$+ \frac{\pi}{16}\frac{1}{\cos(\pi w/2)}\int_0^1 \frac{x^{-w/2}+x^{w/2}}{1+x}\mathrm{d}x$$
$$- \frac{1}{4w}\int_0^1 \frac{x^{-w}-x^{w}}{1+x^2}\mathrm{d}x, \tag{4.14}$$

$$Go[-4,-4](w) = \frac{\pi}{16}\tan(\pi w/2)\int_0^1 \frac{x^{-w/2}+x^{w/2}}{1+x}\mathrm{d}x - \frac{1}{4w}\int_0^1 \frac{x^{-w}+x^{w}-2}{1-x^2}\mathrm{d}x. \tag{4.15}$$

By use of (9.2), (9.5), (9.8), (9.9), (9.11) and (9.12) from Appendix 3, we expand the right-hand sides of (4.14) and (4.15) in even and odd powers of w, respectively, and we obtain

$$[2a,-4](2s+1,1) = -\frac{1}{2}\beta(2s+2) + 2^{-2s-1}\beta(1)\zeta(2s+1) + \sum_{k=0}^{s} 2^{-2k-1}\eta(2k+1)\beta(2s-2k+1), \tag{4.16}$$

and

$$[-4,-4](2s,1) = -\frac{1}{2}\lambda(2s+1) + \sum_{k=0}^{s-1} 2^{-2k-1}\eta(2k+1)\lambda(2s-2k). \tag{4.17}$$

Other pairs of E-sums which may be found in a similar fashion are $[1,1](2s,1)$ (already known) and $[2b,1](2s,1)$; $[1,2b](2s,1)$ (already known) and $[2b,2b](2s,1)$; and $[2a,2a](2s,1)$ and $[-4,2a](2s+1,1)$.

The E-sums $[2a,-4](2s,1)$ and $[-4,-4](2s+1,1)$ are *not* expressible in terms of WK. Instead it has been found that

$$[-4,-4](1,1) = -\frac{\pi^2}{32},$$

$$[-4,-4](3,1) = -\frac{\pi^4}{768} + \frac{1}{8}K[2a,-4](2),$$

$$[-4,-4](5,1) = -\frac{7\pi^6}{61440} + \frac{\pi^2}{64}K[2a,-4](2) - \frac{1}{96}K[2a,-4](4),$$

$$[2a,-4](2,1) = \frac{1}{4}K[2a,-4](1),$$

$$[2a,-4](4,1) = -\frac{\pi^5}{3072} + \frac{\pi}{16}K[2a,-4](2) - \frac{1}{24}K[2a,-4](3),$$

expressed in terms of the typical integral

$$K[2a,-4](n) = \int_0^{\pi/2} \theta^n \cot\theta \log(\tan(\pi/4+\theta/2))\mathrm{d}\theta.$$

Generally, the E-sum $[-4,-4](2s+1,1)$ is expressible in terms of $K[2a,-4](2n)$, $n=1,2,\ldots,s$, while the E-sum $[2a,-4](2s,1)$ is expressible in terms of $K[2a,-4](2n)$, $n=1,2,\ldots,s-1$, and $K[2a,-4](2s-1)$.

The examples given in the previous section and those given here illustrate what has been obtained for the 16 E-sums in Table 1, namely that 4 E-sums can be found for *all* s with $t=1$, whereas the remaining 12 E-sums can only be found in terms of WK either for even s or for odd s as above. In the cases where only even or odd s E-sums can be found the analysis yields integrals which can apparently only be evaluated in terms of WK for either even or odd s.

To summarize our findings, we have found analytically the exact results for $[1,1](s,1)$, $[1,2a](s,1)$, $[1,2b](s,1)$, $[2b,-4](s,1)$, as described in Sect. 3. In the present section we detailed the evaluation of $[2a,-4](2s+1,1)$ and $[-4,-4](2s,1)$. Using the same procedure outlined here we have found $[2a,2a](2s,1)$, $[2b,1](2s,1)$, $[2b,2b](2s,1)$, and $[-4,2a](2s+1,1)$. Again using the quarter-circle contour, the linear combination $[-4,2b](2s+1,1)-[-4,1](2s+1,1)$ was found analytically. The remainder of the 16 E-sums not mentioned can all be found using the relations given in (2.13). In Table 3 all the general results found have been displayed.

5 E-sums *likely not* to be expressible in terms of WK

The five sums $[1,-4]$, $[2a,-4]$, $[-4,1]$, $[-4,2a]$ and $[-4,2b]$—we refer to this set as $\mathcal{A}(s,1)$—seem to be expressible in fully closed form only for *odd* values of s when $t=1$. This was borne out by our experiments with *Integer relation methods* (see [1]). By attempting to solve the integrals involved for the particular case $\mathcal{A}(2,1)$ it was found that a semi-closed form solution (as defined in Sect. 3) could be found with just the addition of one further constant of analysis. This extra constant is $\mathcal{I}(3)$, where

$$\begin{aligned}\mathcal{I}(n) &= \Im Li_n\left(\frac{1+i}{2}\right) = \Im Li_n\left(\frac{e^{i\pi/4}}{\sqrt{2}}\right)\\ &= 2^{-n}L_{-4}\left(n:\frac{1}{2}\right)+\frac{1}{\sqrt{2}}L_{-8}\left(n:\frac{1}{\sqrt{2}}\right).\end{aligned} \tag{5.1}$$

With this extra constant it can be shown that

$$\begin{aligned}
[1,-4](2,1) &= -2\mathcal{I}(3)+\frac{\pi^3}{16}+\frac{\pi}{16}\log^2 2-\frac{1}{2}\beta(2)\log 2,\\
[2a,-4](2,1) &= -\mathcal{I}(3)+\frac{3\pi^3}{128}+\frac{\pi}{32}\log^2 2,\\
[-4,1](2,1) &= -\mathcal{I}(3)+\frac{3\pi^3}{128}+\frac{\pi}{32}\log^2 2-\frac{1}{2}\beta(2)\log 2,\\
[-4,2a](2,1) &= -2\mathcal{I}(3)+\frac{\pi^3}{32}+\frac{\pi}{16}\log^2 2,\\
[-4,2b](2,1) &= -3\mathcal{I}(3)+\frac{5\pi^3}{128}+\frac{3\pi}{32}\log^2 2+\frac{1}{2}\beta(2)\log 2.
\end{aligned} \tag{5.2}$$

Table 3 Closed forms found for Euler sums

$$[1,1](s,1)=\frac{1}{2}s\zeta(s+1)-\frac{1}{2}\sum_{k=2}^{s-1}\zeta(k)\zeta(s-k+1)$$

$$[1,2a](s,1)=\eta(1)\lambda(s)-\sum_{k=2}^{s-1}2^{-k}\zeta(k)\lambda(s-k+1)$$

$$[1,2b](s,1)=-\frac{1}{2}s\zeta(s+1)+\eta(1)\zeta(s)+\frac{1}{2}\sum_{k=1}^{s}\eta(k)\eta(s-k+1)$$

$$[1,-4](2s+1,1)=-\beta(2s+2)+\beta(1)\zeta(2s+1)-\sum_{k=1}^{2s+1}(-2)^{-k}\eta(k)\beta(2s-k+2)$$

$$[2a,1](2s,1)=\frac{1}{2}(2s-1)\lambda(2s+1)-\sum_{k=1}^{s-1}\lambda(2k)\zeta(2s-2k+1)$$

$$[2a,2a](2s,1)=-\frac{1}{2}\lambda(2s+1)+\eta(1)\lambda(2s)-\sum_{k=1}^{s-1}2^{-2k-1}\zeta(2k+1)\lambda(2s-2k)$$

$$[2a,2b](2s,1)=-\frac{1}{2}(2s+1)\lambda(2s+1)+\eta(1)\lambda(2s)+\sum_{k=1}^{s}\lambda(2k)\eta(2s-2k+1)$$

$$[2a,-4](2s+1,1)=-\frac{1}{2}\beta(2s+2)+2^{-2s-1}\beta(1)\zeta(2s+1)$$
$$+\sum_{k=0}^{s}2^{-2k-1}\eta(2k+1)\beta(2s-2k+1)$$

$$[2b,1](2s,1)=-\lambda(2s+1)+s\eta(2s+1)-\sum_{k=1}^{s-1}\eta(2k)\zeta(2s-2k+1)$$

$$[2b,2a](2s,1)=\eta(1)\lambda(2s)-\lambda(2s+1)+\sum_{k=2}^{2s-1}(-2)^{-k}\zeta(k)\lambda(2s-k+1)$$

$$[2b,2b](2s,1)=-\lambda(2s+1)-s\eta(2s+1)+\eta(1)\eta(2s)+\sum_{k=1}^{s}\zeta(2k)\eta(2s-2k+1)$$

$$[2b,-4](s,1)=-\beta(1)\eta(s)+\sum_{k=1}^{s}2^{-k}\eta(k)\beta(s-k+1)$$

$$[-4,1](2s+1,1)=s\beta(2s+2)-2^{-2s-1}\beta(1)\eta(2s+1)-\sum_{k=1}^{s}\zeta(2k+1)\beta(2s-2k+1)$$

$$[-4,2a](2s+1,1)=-\frac{1}{2}\beta(2s+2)+\eta(1)\beta(2s+1)-2^{-2s-1}\beta(1)\eta(2s+1)$$
$$-\sum_{k=1}^{s}2^{-2k-1}\zeta(2k+1)\beta(2s-2k+1)$$

Table 3 (*Continued*)

$$[-4,2b](2s+1,1) = -(s+1)\beta(2s+2) + 2\eta(1)\beta(2s+1) - 2^{-2s-1}\beta(1)\eta(2s+1) + \sum_{k=1}^{s}\eta(2k+1)\beta(2s-2k+1)$$

$$[-4,-4](2s,1) = -\frac{1}{2}\lambda(2s+1) + \sum_{k=0}^{s-1}2^{-2k-1}\eta(2k+1)\lambda(2s-2k)$$

However, so far all attempts at finding any like result for $\mathcal{A}(4,1)$ have failed. It seems unlikely that just one single extra constant, $\mathcal{I}(3)$, will suffice to accommodate all the even s in $\mathcal{A}$, and perhaps higher-order examples of $\mathcal{I}(n)$ will appear as s increases.

The sums $[2a,1]$, $[2a,2a]$, $[2a,2b]$, $[2b,1]$, $[2b,2a]$, $[2b,2b]$ and $[-4,-4]$—call this set $\mathcal{B}(s,1)$—can be expressed in fully closed form only for *even* values of s when $t=1$, with the exception of the case when $s=1$. In this instance the first three members of $\mathcal{B}$ are divergent, but the others represented by the integrals given in Table 2 are particularly simple to evaluate and we have

$$\begin{aligned}
[2b,1](1,1) &= -\frac{1}{2}\eta^2(1) = -\frac{1}{2}\log^2 2,\\
[2b,2a](1,1) &= -\frac{1}{2}\eta(2) = -\frac{\pi^2}{24},\\
[2b,2b](1,1) &= -\eta(2) + \frac{1}{2}\eta^2(1) = -\frac{\pi^2}{12} + \frac{1}{2}\log^2 2,\\
[-4,-4](1,1) &= -\frac{1}{2}\beta^2(1) = -\frac{\pi^2}{32}.
\end{aligned} \tag{5.3}$$

In determining $\mathcal{B}(3,1)$ a situation similar to that of finding $\mathcal{A}(2,1)$ was encountered, namely the addition of one extra constant allowed all $\mathcal{B}(3,1)$ to be found in semi-closed form. The extra constant on this occasion is $c(4)$ where $c(n)$ is defined by

$$c(n) = \sum_{k=0}^{n-2}\frac{(n-2)!}{k!}(\log 2)^k Li_{n-k}\left(\tfrac{1}{2}\right) + \frac{(\log 2)^n}{n}. \tag{5.4}$$

Thus

$$\begin{aligned}
[2a,1](3,1) &= -\frac{1}{2}c(4) + \frac{17\pi^4}{1440},\\
[2a,2a](3,1) &= -\frac{1}{2}c(4) + \frac{23\pi^4}{5760} + \frac{7}{8}\zeta(3)\log 2,\\
[2a,2b](3,1) &= -\frac{1}{2}c(4) - \frac{11\pi^4}{2880} + \frac{7}{4}\zeta(3)\log 2,
\end{aligned}$$

$$[2b,1](3,1) = -c(4) + \frac{\pi^4}{48}, \tag{5.5}$$

$$[2b,2a](3,1) = -c(4) + \frac{19\pi^4}{1440} + \frac{7}{8}\zeta(3)\log 2,$$

$$[2b,2b](3,1) = -c(4) + \frac{\pi^4}{180} + \frac{7}{4}\zeta(3)\log 2,$$

$$[-4,-4](3,1) = -\frac{1}{4}c(4) + \frac{31\pi^4}{23040} + \frac{7}{16}\zeta(3)\log 2.$$

It should be noted that for small values of n, the constant $c(n)$ is expressible in terms of WK, thus

$$c(1) = \log 2, \qquad c(2) = \frac{\pi^2}{12}, \qquad c(3) = \frac{7}{8}\zeta(3). \tag{5.6}$$

So all the results in (5.5) may be viewed as combinations of $c(1)$ to $c(4)$, e.g.

$$[2b,2b](3,1) = -c(4) + 2c(1)c(3) + \frac{4}{5}c^2(2). \tag{5.7}$$

This suggests that $\mathcal{B}(5,1)$ might be various combinations of $c(1)$ to $c(6)$. However, as with $\mathcal{A}(4,1)$, no single value of $\mathcal{B}(5,1)$ has been evaluated in any form, despite much relation hunting. It remains an open question as to whether $\mathcal{A}(2s,1)$ or $\mathcal{B}(2s+1,1)$ is expressible in any reasonable type of closed form.

6 Some further E-sum investigations

A crucial aspect of our investigation has been the study of an integral representation of E-sums, and as pointed out below (1.13), our ability to form the integral depends on whether $\chi_p(m+n)$ can be split multiplicatively as $f(m)g(n)$. This is easily done for $p=1$ when $\chi_1(m+n)=1$ for all m,n, and for $\chi_{2b}(m+n) = (-1)^{m+n-1}$.

So for these cases we have

$$[1,q](s,t) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 (\log x)^{s-1} \frac{L_q(t:x)}{1-x} \mathrm{d}x, \tag{6.1}$$

$$[2b,q](s,t) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 (\log x)^{s-1} \frac{L_q(t:-x)}{1+x} \mathrm{d}x. \tag{6.2}$$

It may also be done with more difficulty for $[2a,q]$ and $[-4,q]$. To simplify our initial investigations the case $s=1$, $t=1$ is considered. Now $[1,q](1,1)$ and $[2a,q](1,1)$ are divergent, so just the three following examples are looked at. These are $[2b,-3]$, $[2b,5]$ and $[-4,-3]$, which involve the L-functions

$$L_{-3}(s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} = \sum_{n=1}^{\infty} \left[\frac{1}{(3n-2)^s} - \frac{1}{(3n-1)^s} \right], \tag{6.3}$$

$$\begin{aligned} L_5(s) &= \sum_{n=1}^{\infty} \frac{\chi_5(n)}{n^s} \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{(5n-4)^s} - \frac{1}{(5n-3)^s} - \frac{1}{(5n-2)^s} + \frac{1}{(5n-1)^s} \right], \end{aligned} \tag{6.4}$$

$$\begin{aligned} L_{12}(s) &= \sum_{n=1}^{\infty} \frac{\chi_{12}(n)}{n^s} \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{(12n-11)^s} - \frac{1}{(12n-7)^s} - \frac{1}{(12n-5)^s} + \frac{1}{(12n-1)^s} \right]. \end{aligned} \tag{6.5}$$

Naturally in evaluating these sums we expect the set of WK has to be enlarged to include $L_{-3}(s)$, $L_5(s)$ and $L_{12}(s)$. Using (6.1) we have

$$[2b, -3](1, 1) = \int_0^1 \frac{L_{-3}(1:-x)}{1+x} \mathrm{d}x \tag{6.6}$$

and

$$[2b, 5](1, 1) = \int_0^1 \frac{L_5(1:-x)}{1+x} \mathrm{d}x. \tag{6.7}$$

The integral for $[-4, -3](1, 1)$ is more complex and is

$$[-4, -3](1, 1) = -\int_0^1 \left[\frac{x L_{12}(1:x)}{1+x^2} + \frac{L_{-3}(1:-x^2)}{2(1+x^2)} \right] \mathrm{d}x. \tag{6.8}$$

The following results may be established:

$$L_{-3}(1:-x) = \frac{2}{\sqrt{3}} \arctan\left(\frac{\sqrt{3}x}{x-2} \right); \tag{6.9}$$

$$\begin{aligned} \sqrt{5} L_5(1:-x) &= \log(x^2 - \omega x + 1) - \log(x^2 + x/\omega + 1), \\ \omega &= (\sqrt{5}+1)/2; \end{aligned} \tag{6.10}$$

$$\sqrt{12} L_{12}(1:x) = \log(x^2 + \sqrt{3}x + 1) - \log(x^2 - \sqrt{3}x + 1). \tag{6.11}$$

First consider $[2b, -3](1, 1)$. Integrating (6.6) by parts gives

$$[2b, -3](1, 1) = -\frac{2\pi \log 2}{3\sqrt{3}} + \int_0^1 \frac{\log(1+x)}{1-x+x^2} \mathrm{d}x. \tag{6.12}$$

The evaluation of the integral in (6.12) may be accomplished using a result given in Lewin [[5], form. (8.18)]. The value of the integral is found to be

$$\int_0^1 \frac{\log(1+x)}{1-x+x^2} \mathrm{d}x = \frac{\pi \log 3}{3\sqrt{3}} - \frac{1}{4} L_{-3}(2).$$

Now from (7.13) it is found that $L_{-3}(1) = \pi/(3\sqrt{3})$, so finally we have

$$[2b, -3](1, 1) = -\frac{1}{4}L_{-3}(2) - L_{-3}(1)\log\left(\frac{4}{3}\right). \tag{6.13}$$

Similarly, and with much more difficulty we obtain the striking results

$$[2b, 5](1, 1) = \frac{1}{2\sqrt{5}}\left\{Li_2\left(\frac{5+\sqrt{5}}{8}\right) - Li_2\left(\frac{5-\sqrt{5}}{8}\right) - \frac{6\pi^2}{25}\right\}, \tag{6.14}$$

and

$$[-4, -3](1, 1) = \frac{\sqrt{3}\pi^2}{36} - \frac{1}{4}L_{12}\left(2 : \frac{2}{3}\right) - \frac{\sqrt{6}}{6}L_8\left(2 : \sqrt{\frac{2}{3}}\right) - \frac{\sqrt{6}}{18}L_8\left(2 : \sqrt{\frac{8}{27}}\right). \tag{6.15}$$

Of these last three results the E-sum $[2b, -3](1, 1)$ is expressed in terms of WK, the other two being in semi-closed form. Next, the sum $[2b, -3](2, 1)$ has been evaluated yielding

$$[2b, -3](2, 1) = \sqrt{3}\left[\frac{43\pi^3}{4860} + \frac{\pi}{60}\log^2 3 - \frac{4}{5}L_{-4}\left(3 : \frac{1}{\sqrt{3}}\right)\right], \tag{6.16}$$

a most surprising result containing the L_{-4} rather than the L_{-3} function. However, the fact that $[2b, -3](1, 1)$ is expressible in terms of WK, whereas $[2b, -3](2, 1)$ is only in semi-closed form, suggests that $[2b, -3]$ is one of those E-sums which might be found in terms of WK when s is odd and $t = 1$.

This has led, for odd $s > 1$, to the following quite complex expression, where we have written $\eta(s) = L_{2b}(s)$ by (1.16):

$$[2b, -3](3, 1) = -\frac{1}{16}L_{-3}(4) + \frac{5}{8}L_{-3}(3)\log 3 - \frac{1}{3}L_{-3}(2)\eta(2) - \frac{22}{27}L_{-3}(1)\eta(3), \tag{6.17}$$

which we have derived analytically. We then discovered experimentally

$$[2b, -3](5, 1) = -\frac{1}{64}L_{-3}(6) + \frac{17}{32}L_{-3}(5)\log 3 - \frac{1}{3}L_{-3}(4)\eta(2) - \frac{13}{27}L_{-3}(2)\eta(4) + \frac{20}{27}L_{-3}(3)\eta(3) - \frac{230}{243}L_{-3}(1)\eta(5). \tag{6.18}$$

The general formula, *empirically* derived through the intensive use of integer relation methods and with symbolic computation, is

$$\begin{aligned}
[2b,-3](2s+1,1) = &-\frac{L_{-3}(2s+2)}{4^{1+s}}+\frac{1+4^{-s}}{2}L_{-3}(2s+1)\log 3 \\
&-\sum_{k=1}^{s}\frac{1-3^{1-2k}}{2}L_{-3}(2s-2k+2)\,\eta(2k) \\
&+\sum_{k=1}^{s}\frac{1-9^{-k}}{1-4^{-k}}\frac{1+4^{-s+k}}{2}L_{-3}(2s-2k+1)\,\eta(2k+1) \\
&-2L_{-3}(1)\,\eta(2s+1)\,.
\end{aligned} \tag{6.19}$$

Whether other such formulae for other E-sums can be found either empirically or analytically remains an open question.

On completion of this work we discovered a paper by Terhune [9]. He derives an expression which is essentially our reflection formula (1.7). Using a totally different approach to us, he derives expressions for (in our notation) $[1,-3](s,t)$ for $(s,t)=(3,1)$ and $(2,2)$, and gives results for $(5,1)$, $(2,4)$, $(4,2)$ and $(3,3)$. He also gives results for $[1,-4](s,t)$ with $(s,t)=(3,1)$, $(5,1)$, $(2,4)$, $(4,2)$ and $(3,3)$. Where our results overlap, i.e. for $[1,-4](s,t)$ with $(s,t)=(3,1)$ and $(5,1)$, they agree.

Appendix 1

Elementary *Dirichlet series* are given by

$$L_d(s:x)=\sum_{n=1}^{\infty}\frac{\chi_d(n)x^n}{n^s}. \tag{7.1}$$

$\chi_d(n)$ is a function which has the additive property $\chi_d(n+d)=\chi_d(n)$ and the multiplicative property $\chi_d(mn)=\chi_d(m)\chi_d(n)$. χ_d is called a *character* modulo d, and d will be referred to as the modulus. Usually we are only concerned with $x=1$, and we write $L_d(s:1):=L_d(s)$. Real *primitive characters*—meaning that all others may be reduced to linear combinations of scalings thereof—are defined as follows. If p is a prime >2 then

$$\chi_p(n)=\chi_p=(n\mid p), \tag{7.2}$$

where $(n|p)$ is the *Legendre-Jacobi-Kronecker symbol*. The suffix p will be *signed* according to whether $\chi_p(p-1)=\pm 1$. Thus, for example

$$\begin{aligned}
\chi_{-3}(n) &= +1 \quad n\equiv 1 \pmod 3, & \chi_5(n) &= +1 \quad n\equiv 1,4 \pmod 5,\\
&= -1 \quad n\equiv 2 \pmod 3, & &= -1 \quad n\equiv 2,3 \pmod 5,\\
&= 0 \quad\;\; n\equiv 0 \pmod 3, & &= 0 \quad\;\; n\equiv 0 \pmod 5.
\end{aligned} \tag{7.3}$$

In addition to these, there are three other primitive characters. These are

$$\begin{aligned}\chi_{-4}(n) &= +1 \quad n \equiv 1 \ (\mathrm{mod}\ 4),\\ &= -1 \quad n \equiv 3 \ (\mathrm{mod}\ 4),\\ &= 0 \quad\ \ (n,4) \neq 1,\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}\chi_{-8}(n) &= +1 \quad n \equiv 1,3 \ (\mathrm{mod}\ 8), & \chi_{8}(n) &= +1 \quad n \equiv 1,7 \ (\mathrm{mod}\ 8),\\ &= -1 \quad n \equiv 5,7 \ (\mathrm{mod}\ 8), & &= -1 \quad n \equiv 3,5 \ (\mathrm{mod}\ 8),\\ &= 0 \quad\ \ (n,8) \neq 1, & &= 0 \quad\ \ (n,8) \neq 1.\end{aligned} \tag{7.5}$$

Further primitive characters may be formed from square-free products of prime characters, and those multiplied by χ_{-4}, χ_{-8} and χ_8. Thus let $P = p_1 p_2 \cdots p_k$ where the p_k are all different odd primes, that is P is odd and square-free. Then $\chi_P = \chi_{p_1}\chi_{p_2}\chi_{p_3}\cdots\chi_{p_k}$ and $\chi_{-4}\chi_P$, $\chi_{-8}\chi_P$ and $\chi_8\chi_P$ are all primitive characters. The character series

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\chi_P(n)}{n^s}, \qquad \sum_{n=1}^{\infty}\frac{\chi_{-4}(n)\chi_P(n)}{n^s},\\ &\sum_{n=1}^{\infty}\frac{\chi_{-8}(n)\chi_P(n)}{n^s}, \qquad \sum_{n=1}^{\infty}\frac{\chi_{8}(n)\chi_P(n)}{n^s},\end{aligned} \tag{7.6}$$

each yield an independent Dirichlet L-function. For example

$$L_1(s) = \sum_{n=1}^{\infty}\frac{\chi_1(n)}{n^s} = 1^{-s} + 2^{-s} + 3^{-s} + \cdots = \zeta(s), \tag{7.7}$$

$$L_{-3}(s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} = 1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + \cdots . \tag{7.8}$$

Two other characters are encountered in our work. These are as follows:

$$\begin{aligned}&\chi_{2a}(n) = +1 \quad n \text{ is odd}, \qquad \chi_{2a}(n) = 0 \quad n \text{ is even}\\ &\chi_{2b}(n) = (-1)^{(n-1)}.\end{aligned} \tag{7.9}$$

These latter *non-primitive characters* are essentially modifications of the principal character, $\chi_1(n) = +1$ for all n. In fact χ_{2b} is strictly not a character since though it has the additive property it is not multiplicative. However, when these non-primitive characters multiply primitive characters, the latter are modified in such a way that the character series produced are L-series of the original primitive character multiplied by a factor. Thus it may be shown that for $d = P$ odd and square-free that

$$\sum_{n=1}^{\infty}\frac{\chi_{2a}(n)\chi_d(n)}{n^s} = [1 - (2 \mid d)2^{-s}]L_d(s), \tag{7.10}$$

$$\sum_{n=1}^{\infty} \frac{\chi_{2b}(n)\chi_d(n)}{n^s} = [1-(2\mid d)2^{1-s}]L_d(s). \tag{7.11}$$

whereas if d is even the original L_d-series is unaltered.

It may also be shown [10] that

$$L_p(2s) = \frac{(-1)^{s-1}2^{2s-1}\pi^{2s}}{\sqrt{p}} \sum_{n=1}^{p} \chi_p(n) \frac{B_{2s}(1-n/p)}{(2s)!}, \tag{7.12}$$

and

$$L_{-q}(2s-1) = \frac{(-1)^{s-1}2^{2s-2}\pi^{2s-1}}{\sqrt{q}} \sum_{n=1}^{q} \chi_{-q}(n) \frac{B_{2s-1}(1-n/q)}{(2s-1)!}, \tag{7.13}$$

where $B_s(x)$ are the Bernoulli polynomials. As both n and d $(= p$ or $q)$ are positive integers, $B_s(1-n/d)$ are rational numbers. Hence for s a positive integer

$$L_p(2s) = R(p)\sqrt{p}\pi^{2s}, \tag{7.14}$$

$$L_{-q}(2s-1) = R(q)\sqrt{q}\pi^{2s-1}, \tag{7.15}$$

where $R(p)$ and $R(q)$ are rational numbers. It is also known that

$$L_p(1) = \frac{2h(p)}{\sqrt{p}} \log \epsilon_0, \tag{7.16}$$

where $h(p)$ is the class number of the binary quadratic form of discriminant p and ϵ_0 is the fundamental unit of the number field $Q(\sqrt{p})$.

Appendix 2

The integrals $C(w)$ and $S(w)$ defined in (4.8) may be found as follows. Let C denote the contour that consists of the line segment $z = x$, $0 \le x \le 1$; the semi-circle $z = e^{i\theta}$, $0 \le \theta \le \pi$; and the line segment $z = xe^{\pi i}$, $1 \ge x \ge 0$. We start from the contour integrals

$$\int_C \frac{2 - z^{-w} - z^w}{1-z} \mathrm{d}z = 0, \qquad \int_C \frac{z^{-w} - z^w}{1-z} \mathrm{d}z = 0. \tag{8.1}$$

Evaluation of the first integral in (8.1) yields

$$\int_0^1 \frac{2 - x^{-w} - x^w}{1-x} \mathrm{d}x - \int_0^{\pi} (1-\cos(w\theta))[\cot(\theta/2) + i]\mathrm{d}\theta + 2\log 2 - \cos(\pi w)\int_0^1 \frac{x^{-w} + x^w}{1+x} \mathrm{d}x + i\sin(\pi w)\int_0^1 \frac{x^{-w} - x^w}{1+x}\mathrm{d}x = 0. \tag{8.2}$$

Let w be real. Then by taking real parts in (8.2) we find

$$C(w) = 2\log 2 + \int_0^1 \frac{2 - x^{-w} - x^w}{1-x}\mathrm{d}x - \cos(\pi w)\int_0^1 \frac{x^{-w} + x^w}{1+x}\mathrm{d}x. \tag{8.3}$$

Evaluation of the second integral in (8.1) yields

$$\int_0^1 \frac{x^{-w} - x^w}{1-x}\mathrm{d}x + i\int_0^\pi \sin(w\theta)[\cot(\theta/2) + i]\mathrm{d}\theta$$
$$+ \cos(\pi w)\int_0^1 \frac{x^{-w} - x^w}{1+x}\mathrm{d}x - i\sin(\pi w)\int_0^1 \frac{x^{-w} + x^w}{1+x}\mathrm{d}x = 0. \tag{8.4}$$

Let w be real. Then by taking imaginary parts in (8.4) we find

$$S(w) = \sin(\pi w)\int_0^1 \frac{x^{-w} + x^w}{1+x}\mathrm{d}x. \tag{8.5}$$

The integrals

$$\int_0^{\pi/2} \frac{1 - \cos(w\theta)}{\sin\theta}\mathrm{d}\theta \quad \text{and} \quad \int_0^{\pi/2} \frac{\sin(w\theta)}{\sin\theta}\mathrm{d}\theta, \tag{8.6}$$

appearing in (4.12) and (4.13), may be found in a similar manner. We start from the contour integrals

$$\int_C \frac{e^{-\pi i w/2}z^w + e^{\pi i w/2}z^{-w} - 2}{1+z^2}\mathrm{d}z = 0, \tag{8.7}$$

$$\int_C \frac{e^{-\pi i w/2}z^w - e^{\pi i w/2}z^{-w}}{1+z^2}\mathrm{d}z = 0, \tag{8.8}$$

where the contour C is the quarter-circle contour described in Sect. 4. These latter integrals vanish because the integrands are analytic inside C; in particular, the integrands have a removable singularity at $z = i$. Evaluation of the integral in (8.7) yields

$$\cos(\pi w/2)\int_0^1 \frac{x^w + x^{-w}}{1+x^2}\mathrm{d}x + i\sin(\pi w/2)\int_0^1 \frac{x^{-w} - x^w}{1+x^2}\mathrm{d}x - \frac{\pi}{2}$$
$$- i\int_0^{\pi/2} \frac{1 - \cos(w\theta)}{\sin\theta}\mathrm{d}\theta - i\int_0^1 \frac{y^w + y^{-w} - 2}{1-y^2}\mathrm{d}y = 0, \tag{8.9}$$

where in the third integral we replaced the original integration variable θ by $\pi/2 - \theta$. Let w be real. Then from the imaginary part of (8.9) we obtain

$$\int_0^{\pi/2} \frac{1 - \cos(w\theta)}{\sin\theta}\mathrm{d}\theta = -\int_0^1 \frac{y^w + y^{-w} - 2}{1-y^2}\mathrm{d}y$$
$$+ \sin(\pi w/2)\int_0^1 \frac{x^{-w} - x^w}{1+x^2}\mathrm{d}x. \tag{8.10}$$

Evaluation of the integral in (8.8) yields

$$
\begin{aligned}
&-\cos(\pi w/2)\int_0^1 \frac{x^{-w}-x^{w}}{1+x^2}\mathrm{d}x - i\sin(\pi w/2)\int_0^1 \frac{x^{w}+x^{-w}}{1+x^2}\mathrm{d}x\\
&\quad+\int_0^{\pi/2}\frac{\sin(w\theta)}{\sin\theta}\mathrm{d}\theta - i\int_0^1 \frac{y^{w}-y^{-w}}{1-y^2}\mathrm{d}y = 0,
\end{aligned}
\tag{8.11}
$$

where in the third integral we replaced the original integration variable θ by $\pi/2-\theta$. Let w be real. Then from the real part of (8.11) we get

$$
\int_0^{\pi/2}\frac{\sin(w\theta)}{\sin\theta}\mathrm{d}\theta = \cos(\pi w/2)\int_0^1 \frac{x^{-w}-x^{w}}{1+x^2}\mathrm{d}x. \tag{8.12}
$$

Appendix 3

We present here a list of generating functions associated with the set WK of constants $\zeta(m)$, $\eta(m)$, $\lambda(m)$, and $\beta(m)$.

$$
\int_0^1 \frac{x^{-w}-1}{1-x}\mathrm{d}x = \psi(1)-\psi(1-w) = \sum_{m=1}^{\infty}\zeta(m+1)w^m, \tag{9.1}
$$

$$
\begin{aligned}
\frac{1}{2}\int_0^1 \frac{x^{-w}+x^{w}-2}{1-x}\mathrm{d}x &= \frac{1}{2}[2\psi(1)-\psi(1-w)-\psi(1+w)]\\
&= \sum_{m=1}^{\infty}\zeta(2m+1)w^{2m},
\end{aligned}
\tag{9.2}
$$

$$
\frac{1}{2}\int_0^1 \frac{x^{-w}-x^{w}}{1-x}\mathrm{d}x = \frac{1}{2w}-\frac{\pi}{2}\cot(\pi w) = \sum_{m=1}^{\infty}\zeta(2m)w^{2m-1}; \tag{9.3}
$$

$$
\int_0^1 \frac{x^{-w}}{1+x}\mathrm{d}x = \frac{1}{2}[\psi(1-w/2)-\psi(\tfrac{1}{2}-w/2)] = \sum_{m=0}^{\infty}\eta(m+1)w^m, \tag{9.4}
$$

$$
\begin{aligned}
\frac{1}{2}\int_0^1 \frac{x^{-w}+x^{w}}{1+x}\mathrm{d}x &= \frac{1}{4}[\psi(1-w/2)-\psi(\tfrac{1}{2}-w/2)+\psi(1+w/2)\\
&\quad -\psi(\tfrac{1}{2}+w/2)] = \sum_{m=0}^{\infty}\eta(2m+1)w^{2m},
\end{aligned}
\tag{9.5}
$$

$$
\frac{1}{2}\int_0^1 \frac{x^{-w}-x^{w}}{1+x}\mathrm{d}x = -\frac{1}{2w}+\frac{\pi}{2}\frac{1}{\sin(\pi w)} = \sum_{m=1}^{\infty}\eta(2m)w^{2m-1}; \tag{9.6}
$$

$$
\int_0^1 \frac{x^{-w}-1}{1-x^2}\mathrm{d}x = \frac{1}{2}[\psi(\tfrac{1}{2})-\psi(\tfrac{1}{2}-w/2)] = \sum_{m=1}^{\infty}\lambda(m+1)w^m, \tag{9.7}
$$

$$\frac{1}{2}\int_0^1 \frac{x^{-w}+x^{w}-2}{1-x^2}\mathrm{d}x = \frac{1}{4}[2\psi(\tfrac{1}{2})-\psi(\tfrac{1}{2}-w/2)-\psi(\tfrac{1}{2}+w/2)] = \sum_{m=1}^{\infty}\lambda(2m+1)w^{2m}, \tag{9.8}$$

$$\frac{1}{2}\int_0^1 \frac{x^{-w}-x^{w}}{1-x^2}\mathrm{d}x = \frac{\pi}{4}\tan(\pi w/2) = \sum_{m=1}^{\infty}\lambda(2m)w^{2m-1}; \tag{9.9}$$

$$\int_0^1 \frac{x^{-w}}{1+x^2}\mathrm{d}x = \frac{1}{4}[\psi(\tfrac{3}{4}-w/4)-\psi(\tfrac{1}{4}-w/4)] = \sum_{m=0}^{\infty}\beta(m+1)w^{m}, \tag{9.10}$$

$$\frac{1}{2}\int_0^1 \frac{x^{-w}+x^{w}}{1+x^2}\mathrm{d}x = \frac{\pi}{4}\frac{1}{\cos(\pi w/2)} = \sum_{m=0}^{\infty}\beta(2m+1)w^{2m}, \tag{9.11}$$

$$\frac{1}{2}\int_0^1 \frac{x^{-w}-x^{w}}{1+x^2}\mathrm{d}x = \frac{1}{8}[\psi(\tfrac{3}{4}-w/4)-\psi(\tfrac{1}{4}-w/4)-\psi(\tfrac{3}{4}+w/4) + \psi(\tfrac{1}{4}+w/4)] = \sum_{m=1}^{\infty}\beta(2m)w^{2m-1}. \tag{9.12}$$

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