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Distribution of rational numbers in short intervals

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Abstract A simplified proof for a well-distribution property for rational numbers is given and a connection with Riemann's Hypothesis is pointed out. More precisely, we consider rational numbers with denominators of a given order of magnitude and show that the number of such numbers lying in a short interval of given length is normally close to its expectation in a mean square sense. The proof is elementary, using only Fourier series and Ramanujan sums. At the end of the paper, a variant of the circle method is discussed as an application.

Keywords Distribution of rational numbers · Circle method

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1 Introduction

By a classical theorem of Franel and Landau (see [2, Sect. 12.2]), Riemann's Hypothesis is equivalent to a certain "global" well-distribution property for the Farey fractions. We considered in [5] an analogous "local" problem: how uniformly are the rationals with denominators of given order distributed in short intervals? The rationals will be written as a/q with $(a,q)=1$ and $q \geq 1$. We are counting these with weights $w(q) \in [0,1]$, where the function w vanishes outside a certain interval $[Q, 2Q]$. Letting $v(x)$ be another weight function of support $[-\Delta, \Delta]$, where Δ is a small positive parameter. Define

$$\lambda(x) = \sum_{a/q} w(q) v\left(\frac{a}{q} - x\right). \tag{1.1}$$

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This is a periodic function of period 1. Statistically, there is a certain expectation λ for $\lambda(x)$, and we are interested in the variance

$$V = \int_I (\lambda(x) - \lambda)^2 \, dx,$$

where $I = [0, 1]$.

For orientation, a typical case is that $v(x)$ is the characteristic function of the interval $[-\Delta, \Delta]$ and $w(q)$ takes values 0 or 1 for $Q \le q \le 2Q$. Thus some denominators in this range are accepted and the others are ignored. In [5] we proved that in this case

$$V \ll \Delta Q^{2+\varepsilon}, \tag{1.2}$$

for any fixed $\varepsilon > 0$; here and elsewhere, the constant implied by the notation may depend on ε, and ε is not necessarily the same at each occurrence. (Actually, the version given in [5] contains also a second term caused by the assumption that a/q should lie in the unit interval I, but we do not now restrict the summation in (1.1) in such a way). A statistical interpretation of (1.2) is that the number of our rationals in the interval $[x - \Delta, x + \Delta]$ is usually close to its expectation.

The present paper has three goals:

(1) To give a simplified proof of a generalized version of (1.2).
(2) To analyze the relevance of Riemann's Hypothesis in this problem.
(3) To reconsider a variant of the Circle Method presented in [6].

The first two topics will be discussed in Sect. 2, and the results corresponding to (1.2) are formulated in Theorem 1. The method mentioned as the third topic has already found some applications to convolutions of Fourier coefficients of cusp forms (see [3, 4, 7]). Its underlying principle is to decompose an integral over the unit interval into a main term and an error term; the main term involves integrals over a system of more or less overlapping short intervals, all of the same length, while the error term depends on the variance V and on the function to be integrated. In Sect. 3, this decomposition will be restated as an *identity* (Theorem 2), and such an explicit formulation of the method is potentially more flexible in applications. This aspect will be commented in the end of the paper in the light of the above mentioned convolutions and additive divisor problems.

2 The variance

Recall that the weight function $w(q)$ should satisfy $w(q) \in [0, 1]$ for all natural numbers q, and $w(q) = 0$ for $q \notin [Q, 2Q]$, where Q is a large positive parameter. Then

$$L = \sum_{q=1}^{\infty} w(q)\varphi(q) \tag{2.1}$$

represents the weighted number of the admissible rationals a/q in the unit interval.

Next, as to $v(x)$, we suppose that it is an even non-negative function of support $[-\Delta, \Delta]$ whose restriction to its support is continuous. Also, we suppose that it is bounded by an absolute constant and non-increasing in $[0, \Delta]$. Thus $v(x)$ is continuous up to jumps at $x = \pm\Delta$ if $v(\Delta) \neq 0$. Here Δ is a parameter with $0 < \Delta \leq 1/3$; in fact we may suppose that $\Delta \geq Q^{-2}$, for otherwise $\lambda(x)$ is bounded, and with λ defined by (2.3) below, the inequality (2.4) in Theorem 1—a slightly more precise version of (1.2)—is trivially true. For convenience, we normalize the mean value of $v(x)$ to be 1 in the interval $[-\Delta, \Delta]$, thus

$$\frac{1}{2\Delta} \int_{-\Delta}^{\Delta} v(x)\, dx = 1. \tag{2.2}$$

The expectation for the function $\lambda(x)$ defined in (1.1) will be then

$$\lambda = 2\Delta L. \tag{2.3}$$

Our version of (1.2) now reads as follows.

Theorem 1 *With the above assumptions and notations, we have*

$$V \ll \Delta Q^2 \log^3(1/\Delta). \tag{2.4}$$

Also, on Riemann's Hypothesis and with $w(q) = 1$ *for all* $q \in [Q, 2Q]$, *it holds*

$$V \ll \min(Q^{1+\varepsilon}, \Delta Q^2 \log^3(1/\Delta)). \tag{2.5}$$

Proof We construct a function with period 1 restricting $v(x)$ first to the interval $[-1/2, 1/2]$ and extending it then periodically. The sum function of its Fourier series, say

$$v^*(x) = \sum_{n=-\infty}^{\infty} a_n e(nx), \tag{2.6}$$

where

$$a_n = \int_{-\Delta}^{\Delta} v(x) e(-nx)\, dx$$

with $e(\alpha) = e^{2\pi i\alpha}$ as usual, equals $v(x)$ in the interval $[-1/2, 1/2]$ except that $v^*(\pm\Delta) = \frac{1}{2} v(\Delta)$. This slight deviation will be unimportant in the sequel. In particular, $a_0 = 2\Delta$ by our assumption (2.2). Also,

$$|a_n| \ll \min(\Delta, |n|^{-1}) \quad \text{for } n \neq 0 \tag{2.7}$$

by the properties of the function $v(x)$. Define

$$\lambda^*(x) = \sum_{a/q \in I} w(q) v^*\left(\frac{a}{q} - x\right).$$

Then $\lambda(x) = \lambda^*(x)$ almost everywhere, namely everywhere up to the possible points of discontinuity of $\lambda(x)$, that is the points $a/q \pm \Delta$ with $w(q) \neq 0$. By (2.1), (2.3), and (2.6), we have the Fourier series

$$\lambda^*(x) = \lambda + \sum_{n\neq 0} a_n \sum_{a/q\in I} w(q) e\left(n\left(\frac{a}{q} - x\right)\right). \tag{2.8}$$

Then, by Parseval's formula,

$$\begin{aligned} V &= \int_I (\lambda(x) - \lambda)^2\, dx = \int_I (\lambda^*(x) - \lambda)^2\, dx \\ &= \sum_{n\neq 0} |a_n|^2 \sum_{a_1/q_1\in I} w(q_1) \sum_{a_2/q_2\in I} w(q_2) e\left(n\left(\frac{a_1}{q_1} - \frac{a_2}{q_2}\right)\right) \\ &= \sum_{n\neq 0} |a_n|^2 \sum_{q_1=1}^{\infty} w(q_1) C_{q_1}(n) \sum_{q_2=1}^{\infty} w(q_2) C_{q_2}(n) \\ &= \sum_{n\neq 0} |a_n|^2 \sum_{q_1=1}^{\infty} w(q_1) \sum_{q_2=1}^{\infty} w(q_2) \sum_{d_1|(q_1,n)} \sum_{d_2|(q_2,n)} d_1 d_2 \mu\left(\frac{q_1}{d_1}\right)\mu\left(\frac{q_2}{d_2}\right), \end{aligned} \tag{2.9}$$

where

$$C_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(an/q) = \sum_{d|(q,n)} d\mu(q/d)$$

stands for Ramanujan sums. Hence, changing the order of the summations and summing trivially over q_1 and q_2, we see that

$$V \ll Q^2 \sum_{d_1,d_2\le 2Q} \sum_{\substack{n\neq 0 \\ [d_1,d_2]|n}} |a_n|^2. \tag{2.10}$$

Since d_1 and d_2 are divisors of n, a given term $|a_n|^2$ may occur here at most $d^2(n)$ times. Thus finally by (2.7) and (2.10) we have

$$V \ll Q^2 \sum_{n=1}^{\infty} \min(\Delta^2, n^{-2}) d^2(n) \ll \Delta Q^2 \log^3(1/\Delta).$$

This completes the proof of (2.4).

Next we show that if $w(q) = 1$ for all $q \in [Q, 2Q]$, then (2.5) holds under Riemann's Hypothesis. We may suppose that $\Delta \ge Q^{-1}$, for otherwise the assertion follows from (2.4) unconditionally. A well-known equivalent formulation of Riemann's Hypothesis is that

$$\sum_{n\le x} \mu(n) \ll x^{1/2+\varepsilon}$$

for all $x \ge 1$ (see [2, p. 261]). Using this estimate to the sums over q_i in (2.9), we get

$$V \ll Q^{1+\varepsilon} \sum_{d_1,d_2\le 2Q} (d_1 d_2)^{1/2} \sum_{\substack{n\neq 0 \\ [d_1,d_2]|n}} |a_n|^2.$$

Now, since $d_1 d_2 \ll \min(n^2, Q^2)$, we infer by (2.7) that

$$V \ll Q^{1+\varepsilon} \sum_{n=1}^{\infty} \min(\Delta^2, n^{-2}) \min(n, Q) d^2(n) \ll Q^{1+\varepsilon} \log^4 Q;$$

recall the assumption $\Delta \geq Q^{-1}$. This completes the proof of (2.5). □

3 The circle method

In the classical version of the Hardy-Littlewood circle method, the integral of a periodic function of period 1 over the unit interval is decomposed according to a system of Farey intervals. In applications of the circle method, the irregularity of the lengths of the Farey intervals is an obstacle called the *levelling problem*. H.D. Kloosterman devised a refinement of the Hardy-Littlewood method to cope with this difficulty, and his argument gave rise to the exponential sums bearing his name. If the periodic function is of the form $e(nx)$ for an integer n, then the circle method gives a decomposition of the δ-function $\delta(n)$ which equals 1 if $n = 0$, and zero otherwise. Another expression for the δ-function was given in [1]. In [6], we presented a variant of the circle method, based on the estimate (1.2) above, where the integral in question is decomposed approximately to a number of integrals over short partially overlapping intervals of the same length. We first recall the idea of that method and then show how to formulate it precisely as an identity.

Let $f(x)$ be a continuous function of period 1 and consider its integral over the unit interval I. We approximate the integral by a sum of integrals over short intervals $[a/q - \Delta, a/q + \Delta]$ letting $\chi_{a/q}(x)$ denote its characteristic function. Then

$$\int_I f(x)\,dx \approx \frac{1}{\lambda} \sum_{a/q \in I} w(q) \int \chi_{a/q}(x) f(x)\,dx,$$

where λ is defined by (2.3) and (2.1), and the weight function $w(q)$ satisfies the same conditions as in the preceding section. This is seen on counting the weight of occurrence of a given value x on the right, and the weight is normally close to 1 by our estimate for the variance V. But we may replace the characteristic function $\chi_{a/q}(x)$ by a more general weight function $v(a/q - x)$, where v satisfies the conditions mentioned in the preceding section. In the previous notation, we now have

$$\begin{aligned} &\frac{1}{\lambda} \sum_{a/q \in I} w(q) \int v\left(\frac{a}{q} - x\right) f(x)\,dx \\ &\quad = \frac{1}{\lambda} \sum_{a/q \in I} w(q) \int_I v^*\left(\frac{a}{q} - x\right) f(x)\,dx \\ &\quad = \frac{1}{\lambda} \int_I \lambda^*(x) f(x)\,dx = \int_I f(x)\,dx + \frac{1}{\lambda} \int_I (\lambda^*(x) - \lambda) f(x)\,dx. \qquad (3.1) \end{aligned}$$

Invoking the Fourier series (2.8) of $\lambda^*(x)$, we may write the last term in terms of the Fourier coefficients a_n of v^* and b_n of f:

$$\frac{1}{\lambda}\int_I (\lambda^*(x)-\lambda) f(x)\,dx = \frac{1}{\lambda}\sum_{n\neq 0} a_n b_n \sum_q w(q) C_q(n). \tag{3.2}$$

Finally we substitute the arithmetic formula for the Ramanujan sums to obtain the following expression for the integral of f over the unit interval, that is for the constant term of its Fourier series.

Theorem 2 *Let f be a continuous function of period* 1 *with Fourier coefficients b_n, let the weight functions $w(q)$ and $v(x)$ satisfy the conditions in Sect.* 2, *and let* (2.6) *be the Fourier series of the periodic extension of v. Then*

$$\begin{aligned} b_0 = \int_I f(x)\,dx &= \frac{1}{\lambda}\sum_{a/q\in I} w(q)\int v\left(\frac{a}{q}-x\right) f(x)\,dx \\ &\quad - \frac{1}{\lambda}\sum_{d=1}^{\infty} d \sum_{m\neq 0} a_{dm} b_{dm} \sum_{r=1}^{\infty} w(dr)\mu(r). \end{aligned} \tag{3.3}$$

Remark 1 The rightmost terms in (3.1), (3.2), or (3.3), which represent the same quantity in different shapes, are of the order

$$\ll \lambda^{-1}\|f\|_2 Q\Delta^{1/2}\log^{3/2}(1/\Delta); \tag{3.4}$$

to see this, apply Cauchy's inequality and (2.4) in the first mentioned term.

Remark 2 If v is smooth, then $|a_n|$ decays rapidly as $|n|$ exceeds Δ^{-1}, which means that the n-sum in (3.2) can be truncated at Δ^{-1} or so with a small error, or that the d-sum in (3.3) can be truncated similarly. Suppose now that ΔQ is large and $w(q)=1$ for $q\in[Q,2Q]$. Then Riemann's Hypothesis implies cancellation in the r-sum in (3.3) in the same way as in Theorem 1. Or alternatively one may use the generating function of the Ramanujan sums (see [8, (1.5.4)]) to show cancellation in the q-sum in (3.2) under Riemann's Hypothesis.

Remark 3 To illustrate the structure of the formula (3.3) by an example, let us consider briefly sums of the additive divisor problem type

$$C(m) = \sum_{n=1}^{\infty} \varphi(n)\varphi(n+m)c(n)c(n+m),$$

where $c(n)$ is a real-valued arithmetic function, $\varphi(n)$ is a smooth weight function of compact support in $(0,\infty)$, and m is a given positive integer. The cases we have in mind are the (normalized) Fourier coefficients of a cusp form, or the divisor function $d(n)$. Let

$$S(x) = \sum_{n=1}^{\infty} \varphi(n)c(n)e(nx),$$

$$f(x) = |S(x)|^2 e(-mx).$$

Then $b_n = C(n+m)$ in the notation of Theorem 2, in particular $b_0 = C(m)$.

Write (3.3) as $b_0 = A - B$ for short. Then B is a linear combination of sums $C(m+\nu)$ with $\nu \neq 0$. In A, the interesting contribution b_0 arises if $f(x)$ is replaced by the constant term of its Fourier series, and the role of the remaining terms in A is simply to compensate the "error term" B. But A can be treated also differently. Namely, in the cases mentioned above, we may transform the exponential sum $S(x)$ in a neighborhood of a rational a/q by Voronoi summation. Then Kloosterman sums emerge, and sums of these can be estimated by spectral methods. In the case of the Fourier coefficients of a cusp form, we worked out this argument in [7]. The term B was estimated by (3.4) using a known good bound for $S(x)$. However, in the case $c(n) = d(n)$ we should be able to extract a *main term* for $C(m)$. Now B cannot be treated simply as an error term; it is a linear combination of sums $C(m+\nu)$ with $\nu \neq 0$, and each of these involves a main term contributing to the main term of $C(m)$. But in principle it is easier to deal with an average of sums rather than an individual sum. The term A can be handled much as before except that the non-negativity of $d(n)$ is a new feature to be taken into account. All in all, though Theorem 2 opens an approach even to the additive divisor problem, it is not quite straightforward to carry out an analogue of the corresponding argument in [7].

References

1. Duke, W., Friedlander, J.B., Iwaniec, H.: A quadratic divisor problem. Invent. Math. **115**, 209–217 (1994)
2. Edwards, H.M.: Riemann's Zeta-Function. Academic, New York (1974)
3. Harcos, G.: New bounds for automorphic L-functions. Ph.D. Thesis, Princeton University (2003)
4. Harcos, G.: An additive problem in the Fourier coefficients of Maass forms. Math. Ann. **326**, 347–365 (2003)
5. Jutila, M.: Transformations of exponential sums. In: Proc. Amalfi Conference on Analytic Number Theory, Maiori, 1989, pp. 263–270. Univ. Salerno, Salerno (1992)
6. Jutila, M.: A variant of the circle method. In Sieve Methods, Exponential Sums, and their Applications in Number Theory, pp. 245–254. Cambridge Univ. Press (1996)
7. Jutila, M.: Convolutions of Fourier coefficients of cusp forms. Publ. Inst. Math. (Beograd) (N.S.) **65**(79), 31–51 (1999)
8. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function. 2nd edn., revised by Heath-Brown, D.R. Clarendon Press, Oxford (1986)