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Multiple q-zeta functions and multiple q-polylogarithms

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Abstract For every positive integer d we define the q-analog of multiple zeta function of depth d and study its properties, generalizing the work of Kaneko et al. who dealt with the case $d = 1$. We first analytically continue it to a meromorphic function on $\mathbb{C}^d$ with explicit poles. In our Main Theorem we show that its limit when $q \uparrow 1$ is the ordinary multiple zeta function. Then we consider some special values of these functions when $d = 2$. At the end of the paper we also propose the q-analogs of multiple polylogarithms by using Jackson's q-iterated integrals and then study some of their properties. Our definition is motivated by those of Koornwinder and Schlesinger although theirs are slightly different from ours.

Keywords Multiple q-zeta functions · Multiple q-polylogarithms · Shuffle relations · Iterated integrals

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1 Introduction and definitions

Let $0 < q < 1$ and for any positive integer k define its q-analog $[k] = [k]_q = (1 - q^k)/(1-q)$. In [9] Kaneko et al. define a function of two complex variables $f_q(s; t) = \sum_{k=1}^{\infty} q^{kt}/[k]^s$ such that the q-analog of Riemann zeta function is realized as

$$\zeta_q(s) := f_q(s; s-1).$$

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For any property $\mathcal{P}$ let $\mathbb{Z}_{\mathcal{P}}$ be the set of integers satisfying $\mathcal{P}$. After analytically continuing $f_q(s;t)$ to $\mathbb{C}^2$ as a meromorphic function Kaneko et al. proved the following main result

Theorem 1.1 ([9, Proposition 2, Theorem 2]) *One can analytically continue $\zeta_q(s)$ to $\mathbb{C}$ with simple poles at $1+(2\pi i/\log q)\mathbb{Z}$ and $\mathbb{Z}_{\le 0}+(2\pi i/\log q)\mathbb{Z}_{\neq 0}$. Moreover, for any $s\neq 1$ one has*

$$\lim_{q\uparrow 1}\zeta_q(s)=\zeta(s).$$

They also study the special values of $\zeta_q(s)$ at non-negative integers. In this paper we shall generalize these to the (Euler–Zagier) multiple zeta functions, which are defined as nested generalizations of Riemann zeta function $\zeta(s)$:

$$\zeta(s_1,\dots,s_d)=\sum_{0<k_1<\cdots<k_d} k_1^{-s_1}\cdots k_d^{-s_d} \tag{1}$$

for complex variables $s_1,\dots,s_d$ satisfying $\sigma_j+\cdots+\sigma_d>d-j+1$ for all $j=1,\dots,d$. Here and in what follows, whenever $s\in\mathbb{C}$ we always write $\sigma=\Re(s)$, the real part of s. The analytic continuation of multiple zeta functions has been studied independently in [2] and [14]. We know that $\zeta(s_1,\dots,s_d)$ can be extended to a meromorphic function on $\mathbb{C}^d\setminus\mathfrak{S}_d$ where

$$\mathfrak{S}_d=\left\{(s_1,\dots,s_d)\in\mathbb{C}^d\;\middle|\;\begin{array}{l}s_d=1,\, s_{d-1}+s_d=2,1,-2m\ \forall m\in\mathbb{Z}_{\ge 0},\\ \text{and } s_j+\cdots+s_d\in\mathbb{Z}_{\le d-j+1}\ \forall j\le d-2.\end{array}\right\} \tag{2}$$

To find the q-analog of multiple zeta functions we first define an auxiliary function of $2d$ complex variables $s_1,\dots,s_d,t_1,\dots,t_d\in\mathbb{C}$

$$f_q(s_1,\dots,s_d;t_1,\dots,t_d)=\sum_{0<k_1<\cdots<k_d}\frac{q^{k_1t_1+\cdots+k_dt_d}}{[k_1]^{s_1}\cdots[k_d]^{s_d}}$$

which converges if $\Re(t_j+\ldots+t_d)>0$ for all $j=1,\dots,d$ (see Proposition 2.2). In the next section we are going to analytically continue this function to $\mathbb{C}^{2d}$ as a meromorphic function with explicitly defined poles.

We now define the multiple q-zeta function by specialization of f_q:

$$\zeta_q(s_1,\dots,s_d):=f_q(s_1,\dots,s_d;s_1-1,\dots,s_d-1)$$

which will be shown to be the correct q-analog of multiple zeta functions. When $\sigma_j>1$ for all j we can express this by the series

$$\zeta_q(s_1,\dots,s_d)=\sum_{0<k_1<\cdots<k_d}\frac{q^{k_1(s_1-1)+\cdots+k_d(s_d-1)}}{[k_1]^{s_1}\cdots[k_d]^{s_d}}. \tag{3}$$

Note that when $d = 1$ this is the same as the q-analog of the Riemann zeta function defined in [9]. Put

$$\mathfrak{S}'_d = \left\{ (s_1, \ldots, s_d) \in \mathbb{C}^d \,\middle|\, \begin{array}{l} s_d \in 1 + \dfrac{2\pi i}{\log q}\mathbb{Z}, \text{ or } s_d \in \mathbb{Z}_{\le 0} + \dfrac{2\pi i}{\log q}\mathbb{Z}_{\neq 0}, \\ \text{or } s_j + \cdots + s_d \in \mathbb{Z}_{\le d-j+1} + \dfrac{2\pi i}{\log q}\mathbb{Z}, \ j < d \end{array} \right\} \supset \mathfrak{S}_d.$$

Here the last part in $\mathfrak{S}'_d$ is vacuous if $d = 1$. The primary goal of this paper is to prove

Main Theorem. *The multiple q-zeta function $\zeta_q(s_1, \ldots, s_d)$ can be extended to a meromorphic function on $\mathbb{C}^d \setminus \mathfrak{S}'_d$ with simple poles along $\mathfrak{S}'_d$. Further, for all $(s_1, \ldots, s_d) \in \mathbb{C}^d \setminus \mathfrak{S}_d$*

$$\lim_{q} \uparrow 1 \zeta_q(s_1, \ldots, s_d) = \zeta(s_1, \ldots, s_d).$$

In Sect. 7, we propose a new definition of the multiple q-polylogarithms and briefly study their properties. We also review Jackson's q-derivatives and q-definite integrals and define q-iterated integrals as q-analogs of Chen's iterated integrals.

It is known that there're two kinds of shuffle relations among multiple zeta values. The first one is produced by iterated integrals, the second by their power series expansions. In the last section of this paper we will apply our q-iterated integral technique to multiple q-polylogarithms in order to study the q-shuffle relations of the first kind for multiple q-zeta values. For simplicity we will only deal with $\zeta_q(m)\zeta_q(n)$ for positive integers $m \neq n$. These relations reduce to the ordinary ones when $q \uparrow 1$. We thank Prof. Kaneko for his questions relating to this part of our study and sending us his offprint upon which the current work is based. The primitive versions of this work was submitted to arxiv.org as [15] in April-May, 2003. The author thanks the Math Department of Penn State University for its hospitality when he visited there in October 2003. The author is indebted to the referee for simplifying the proofs of Lemma 2.1 and Lemma 2.3 and many other valuable comments to improve this paper. Some of the ideas and results in this paper have been suggested or proved also by Bowman and Bradley [3] and Bradley [4] independently.

2 Analytic continuations of f_q and ζ_q

The purpose of this section is two-fold: first we will use the auxiliary functions f_q introduced in the first section to give a quick analytic continuation of multiple q-zeta functions $\zeta_q(s_1, \ldots, s_d)$, though this is not enough to show it's the right q-analog of the multiple zeta functions. Second, we explicitly write down these expressions involving binomial coefficients which will be used to study special values of $\zeta_q(s_1, \ldots, s_d)$ in Sect. 6.

We need a simple lemma first.

Lemma 2.1 *Let k be a positive integer. For all $0 < q < 1$ we have*

$$1 < [k] < k.$$

Proof It is clear that

$$1 < [k] = \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^k < k. \qquad \square$$

Proposition 2.2 *The function*

$$f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) = \sum_{0<k_1<\cdots<k_d} \frac{q^{k_1 t_1 + \cdots + k_d t_d}}{[k_1]^{s_1} \cdots [k_d]^{s_d}}$$

converges if $\Re(t_j + \ldots + t_d) > 0$ for all $j = 1, \ldots, d$. It can be analytically continued to a meromorphic function over $\mathbb{C}^{2d}$ via the series expansion

$$f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) = (1-q)^{\mathrm{wt}(\vec{s})} \sum_{r_1,\ldots,r_d=0}^{+\infty} \prod_{j=1}^{d} \left[\binom{s_j + r_j - 1}{r_j} \frac{q^{j(r_j+t_j)}}{1 - q^{r_j+t_j+\cdots+r_d+t_d}} \right], \qquad (4)$$

where $\mathrm{wt}(\vec{s}) = s_1 + \cdots + s_d$. It has the following (simple) poles: $t_j + \cdots + t_d \in \mathbb{Z}_{\leq 0} + \frac{2\pi i}{\log q}\mathbb{Z}$.

Proof Assume $|\Re(s_j)| < N_j$ and let $\tau_j = \Re(t_j)$ for all $j = 1, \ldots, d$. By Lemma 2.1

$$\left| f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) \right| < \sum_{0<k_1<\cdots<k_d} \prod_{j=1}^{d} (k_j)^{N_j} q^{k_j \tau_j}. \qquad (5)$$

Let $k = k_{d-1}$ (when $d = 1$ take $k_0 = 0$), $n = k_d$, $N = N_d$, and $\tau = \tau_d$. Then by root test $\sum_{n>k} n^N q^{n\tau}$ converges. Moreover, by setting $y = q^\tau$ we get

$$\sum_{n>k} n^N q^{n\tau} = \left(y \frac{d}{dy} \right)^N \sum_{n>k} y^n = (1-y)^{-N} f(N, y; k) y^k,$$

where $f(N, y; x) = \sum_{l=0}^{N} c_l x^l$ is a polynomial of degree N whose coefficients depend only on the constants N and $y = q^\tau$. Let $c(N, q^\tau) = (N+1)\max\{|c_l| : l = 0, \ldots, N\}$ we get

$$\sum_{n>k} n^N q^{n\tau} \leq \frac{c(N, q^\tau)}{(1-q^\tau)^N} k^N q^{\tau k}. \qquad (6)$$

This proves the first part of the lemma when $d = 1$. In the general case it follows from (5) and (6) that

$$\begin{aligned}&\left|f_q(s_1,\dots,s_d;t_1,\dots,t_d)\right|\\&\quad<\frac{c(N,q^\tau)}{(1-q^\tau)^N}\sum_{0<k_1<\cdots<k_{d-2}<k}\left(\prod_{j=1}^{d-2}(k_j)^{N_j}q^{k_j\tau_j}\right)k^{N_{d-1}+N}q^{k(\tau_{d-1}+\tau)}.\end{aligned}$$

It follows from an easy induction on d that

$$\left|f_q(s_1,\dots,s_d;t_1,\dots,t_d)\right|<\prod_{j=1}^{d}\frac{c(N_j+\cdots+N_d,q^{\tau_j+\cdots+\tau_d})}{(1-q^{\tau_j+\cdots+\tau_d})^{N_j+\cdots+N_d}}.$$

This proves the first part of the lemma.

By binomial expansion $(1-x)^{-s}=\sum_{r=0}^{\infty}\binom{s+r-1}{r}x^r$ we get

$$\begin{aligned}&f_q(s_1,\dots,s_d;t_1,\dots,t_d)\\&\quad=(1-q)^{\mathrm{wt}(\vec{s})}\sum_{0<k_1<\cdots<k_d}\sum_{r_1,\dots,r_d=0}^{+\infty}\left(\prod_{j=1}^{d}\binom{s_j+r_j-1}{r_j}q^{k_j(r_j+t_j)}\right).\end{aligned}$$

As $0<q<1$ the series converges absolutely by Stirling's formula so we can exchange the summations. The proposition follows immediately from the next lemma by taking $x_j=q^{t_j+r_j}$ for $j=1,\dots,d$. □

Lemma 2.3 *Let $x_j\in\mathbb{C}$ such that $|x_j|<1$ for $j=1,\dots,d$. Then*

$$\sum_{0<k_1<\cdots<k_d}\prod_{j=1}^{d}x_j^{k_j}=\prod_{j=1}^{d}\frac{x_j\cdots x_d}{1-x_j\cdots x_d}=\prod_{j=1}^{d}\frac{x_j^j}{1-x_j\cdots x_d}. \tag{7}$$

Proof By re-indexing $k_j=m_1+\cdots+m_j$ we have

$$\sum_{0<k_1<\cdots<k_d}\prod_{j=1}^{d}x_j^{k_j}=\sum_{m_1=1}^{\infty}\cdots\sum_{m_d=1}^{\infty}x_1^{m_1}x_2^{m_1+m_2}\cdots x_d^{m_1+\cdots+m_d}=\prod_{j=1}^{d}\frac{x_j\cdots x_d}{1-x_j\cdots x_d}.$$

The second equation in the lemma follows immediately. □

Recall that $\zeta_q(s_1,\dots,s_d)=f_q(s_1,\dots,s_d;s_1-1,\dots,s_d-1)$. Hence we have the following immediate consequence.

Theorem 2.4 *The multiple q-zeta function $\zeta_q(s_1,\dots,s_d)$ can be extended to a meromorphic function with simple poles lying along $\mathfrak{S}'_d$:*

$$\begin{aligned}&\zeta_q(s_1,\dots,s_d)\\&\quad=(1-q)^{\mathrm{wt}(\vec{s})}\sum_{r_1,\dots,r_d=0}^{+\infty}\prod_{j=1}^{d}\left[\binom{s_j+r_j-1}{r_j}\frac{q^{j(r_j+s_j-1)}}{1-q^{r_j+s_j+\cdots+r_d+s_d-d+j-1}}\right].\end{aligned}$$

To see the effect of taking different specializations of t_j in f_q we define the shifting operators $\mathcal{S}_j$ $(1 \le j \le d)$ on the multiple zeta functions as follows:

$$\mathcal{S}_j \zeta(s_1, \dots, s_d) = \zeta(s_1, \dots, s_d) + (1-q)\zeta(s_1, \dots, s_j - 1, \dots, s_d). \tag{8}$$

It is obvious that these operators commute.

Proposition 2.5 *Let $n_1, \dots, n_d$ be non-negative integers. Then we have*

$$\begin{aligned} & f_q(s_1, \dots, s_d; s_1 - 1 - n_1, \dots, s_d - 1 - n_d) \\ &\quad = \mathcal{S}_1^{n_1} \circ \cdots \circ \mathcal{S}_d^{n_d} \zeta(s_1, \dots, s_d) \\ &\quad = \sum_{r_1=0}^{n_1} \cdots \sum_{r_d=0}^{n_d} \left(\prod_{j=1}^{d} \binom{n_j}{r_j} (1-q)^{r_j} \right) \zeta_q(s_1 - r_1, \dots, s_d - r_d). \end{aligned}$$

Proof We only sketch the proof in the case $n_1 = \cdots = n_{d-1} = 0$. The general case is completely similar. In the rest of the paper we always let $\mathcal{S}$ be the shifting operator on the last variable. Suppose $n_d = n = 1$. Then

$$\begin{aligned} & f_q(s_1, \dots, s_d; s_1 - 1, \dots, s_{d-1} - 1, s_d - 2) \\ &\quad = \sum_{0<k_1<\cdots<k_d} \frac{q^{k_1(s_1-1)+\cdots+k_{d-1}(s_{d-1}-1)+k_d(s_d-2)}}{[k_1]^{s_1} \cdots [k_d]^{s_d}} \\ &\quad = \sum_{0<k_1<\cdots<k_d} \frac{q^{k_1(s_1-1)+\cdots+k_{d-1}(s_{d-1}-1)}}{[k_1]^{s_1} \cdots [k_{d-1}]^{s_{d-1}}} \cdot \frac{q^{k_d(s_d-2)}(1-q^{k_d}) + q^{k_d(s_d-1)}}{[k_d]^{s_d}} \\ &\quad = \mathcal{S}\zeta(s_1, \dots, s_d). \end{aligned}$$

The rest follows easily by induction. □

The next corollary answers an implicit question in [9].

Corollary 2.6 *Let n be a positive integer. The specialization of t in $f_q(s;t)$ to $s-1-n$ is*

$$f_q(s; s-1-n) = \mathcal{S}^n \zeta_q(s) = \sum_{r=0}^{n} \binom{n}{r} (1-q)^r \zeta_q(s-r).$$

We observe that one effect of the shifting operator is to bring in more poles. Essentially, $\mathcal{S}^n$ shifts all the poles of $\zeta_q(s)$ by n to the right on the complex plane.

Remark 2.7 In [10], Kawagoe et al. showed that if $\varphi(s)$ is a meromorphic function of s then $\lim_{q\uparrow 1} f_q(s, \varphi(s)) = \zeta_q(s)$ if and only if $\varphi(s) = s - \nu$ for some $\nu \in \mathbb{N}$. We don't know in depth $d > 1$ case what one should expect of $\varphi_j(\vec{s})$ if $\lim_{q\uparrow 1} f_q(\vec{s}; \varphi_1(\vec{s}), \dots, \varphi_d(\vec{s})) = \zeta_q(\vec{s})$.

3 Analytic continuation of multiple zeta functions

Let's begin with a review of some classical results on Bernoulli polynomials $B_k(x)$ which are defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}.$$

Let $\tilde{B}_k(x)$ be the "periodic Bernoulli polynomial"

$$\tilde{B}_k(x)=B_k(\{x\}),\quad x\geq 1,$$

where $\{x\}$ is the fractional part of x. Then we have ([13, Chap. IX, Misc. Ex. 12])

$$\tilde{B}_k(x)=-k!\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi inx}}{(2\pi in)^k}. \tag{9}$$

Recall that the Bernoulli numbers satisfy $B_k=B_k(1)$ if $k\geq 2$ while $B_0=1$ and $B_1=1/2$.

Lemma 3.1 *For every positive integer $M\geq 2$ and $x>1$ we have*

$$\left|\tilde{B}_M(x)\right|\leq\frac{4M!}{(2\pi)^M}.$$

Proof It follows from the fact that $\zeta(M)\leq\zeta(2)=\pi^2/6<2$ for $M\geq 2$. □

We know that one can analytically continue the multiple zeta functions as independently presented in [14] and [2] by different methods. Moreover, $\zeta(s_1,\dots,s_d)$ has singularities on the hyperplanes in $\mathfrak{S}_d$ defined by (2). However, neither approach is suitable for our purpose here. So we follow the idea in [9] to provide a third approach in the rest of this section. The same idea will also be used to deal with the multiple q-zeta functions.

Let's recall the classical Euler-Maclaurin summation formula [13, 7.21]. Let $f(x)$ be any (complex-valued) C^∞ function on $[1,\infty)$ and let m and M be two positive integers. Then we have

$$\sum_{n=1}^{m}f(n)=\int_1^m f(x)\,dx+\frac{1}{2}(f(1)+f(m))+\sum_{r=1}^{M}\frac{B_{r+1}}{(r+1)!}\big(f^{(r)}(m)-f^{(r)}(1)\big)$$
$$-\frac{(-1)^{M+1}}{(M+1)!}\int_1^m\tilde{B}_{M+1}(x)f^{(M+1)}(x)\,dx. \tag{10}$$

To simplify our notation, in definition (1) we replace s_d, k_{d-1}, k_d by s, k and n, respectively. Taking $f(x)=1/x^s$ and $m=k$ and ∞ in (10) we have:

$$\sum_{n>k}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}-\sum_{n=1}^{k}\frac{1}{n^s}$$

$$= \int_k^\infty f(x)\,dx - \frac{1}{2}f(k) - \sum_{r=1}^{M} \frac{B_{r+1}}{(r+1)!} f^{(r)}(k)$$

$$- \frac{(-1)^{M+1}}{(M+1)!} \int_k^\infty \tilde{B}_{M+1}(x) f^{(M+1)}(x)\,dx$$

$$= \frac{1}{(s-1)k^{s-1}} - \frac{1}{2k^s} + \sum_{r=1}^{M} \frac{B_{r+1}}{(r+1)!} \frac{(s)_r}{k^{s+r}} - \frac{(s)_{M+1}}{(M+1)!} \int_k^\infty \frac{\tilde{B}_{M+1}(x)}{x^{s+M+1}}\,dx.$$

Here we have used the fact that $B_k = 0$ if $k \geq 3$ is odd. By definition (1) we have

Theorem 3.2 *For all* $(s_1, \ldots, s_d) \in \mathbb{C}^d \setminus \mathfrak{S}_d$ *and* $M > 1 + |\sigma_d| + |\sigma_{d-1}|$ *we have*

$$\zeta(s_1, \ldots, s_d) = \sum_{r=0}^{M+1} \frac{B_r}{r!} (s_d)_{r-1} \cdot \zeta(s_1, \ldots, s_{d-1} + s_d + r - 1)$$

$$- \frac{(s_d)_{M+1}}{(M+1)!} \sum_{0<k_1<\cdots<k_d} \frac{1}{k_1^{s_1} \cdots k_{d-1}^{s_{d-1}}} \int_{k_{d-1}}^\infty \frac{\tilde{B}_{M+1}(x)}{x^{s_d+M+1}}\,dx, \quad (11)$$

where we set $(s)_0 = 1$ *and* $(s)_{-1} = 1/(s-1)$. *This provides an analytic continuation of* $\zeta(s_1, \ldots, s_d)$ *to* $\mathbb{C}^d \setminus \mathfrak{S}_d$.

Proof We only need to show that the series in (11) converges. Lemma 3.1 implies (if $d = 2$ then take $k_0 = 1$)

$$\sum_{k_{d-1}=k_{d-2}}^{\infty} \left| \frac{1}{k_{d-1}^{s_{d-1}}} \int_{k_{d-1}}^\infty \frac{\tilde{B}_M(x)}{x^{M+s_d+1}}\,dx \right| \leq \frac{4M!}{(2\pi)^M (M - |\sigma_d|)} \sum_{k_{d-1}=k_{d-2}}^{\infty} \frac{1}{k_{d-1}^{M-|\sigma_{d-1}|-|\sigma_d|}}$$

which converges absolutely whenever $M > 1 + |\sigma_d| + |\sigma_{d-1}|$. □

4 Proof of Main Theorem

Fix $(s_1, \ldots, s_d) \in \mathbb{C}^d$ such that $\sigma_j + \cdots + \sigma_d < d - j + 1$ for all $j = 1, \ldots, d$. When $d = 1$ this is Theorem 1.1 due to Kaneko et al. We now assume $d \geq 2$ and proceed by induction. The key is a recursive formula for $\zeta_q(s_1, \ldots, s_d)$ similar to (11) for $\zeta(s_1, \ldots, s_d)$. To derive this formula we appeal to the Euler-Maclaurin summation formula (10) again. Hence we set

$$F(x) = \frac{q^{x(s-1)}}{(1-q^x)^s}$$

as in [9]. Then

$$F'(x) = (\log q) q^{x(s-1)} \frac{s - 1 + q^x}{(1-q^x)^{s+1}},$$

$$F''(x) = (\log q)^2 q^{x(s-1)} \frac{s(s+1) - 3s(1-q^x) + (1-q^x)^2}{(1-q^x)^{s+2}}.$$

In definition (3) we replace s_d, k_{d-1}, and k_d by s, k and n, respectively. We now take $M=1$, $f(x)=F(x+k-1)$ and let $m\to\infty$ in (10) and get

$$\begin{aligned}\sum_{n>k}&\frac{q^{n(s-1)}}{[n]^s}\\ &=(1-q)^s\left(-F(k)+\sum_{n=1}^{\infty}f(n)\right)\\ &=(1-q)^s\left(\int_k^\infty F(x)\,dx-\frac{1}{2}F(k)-\frac{1}{12}F'(k)-\frac{1}{2}\int_1^\infty \tilde{B}_2(x)f''(x)\,dx\right)\\ &=\frac{(q-1)}{(s-1)\log q}\frac{q^{k(s-1)}}{[k]^{s-1}}-\frac{1}{2}\frac{q^{k(s-1)}}{[k]^s}+\frac{1}{12}\frac{\log q}{q-1}\frac{q^{k(s-1)}(s+q^k-1)}{[k]^{s+1}}\\ &\quad-\frac{(1-q)^s(\log q)^2}{2}\int_k^\infty \tilde{B}_2(x)q^{x(s-1)}\frac{s(s+1)-3s(1-q^x)+(1-q^x)^2}{(1-q^x)^{s+2}}\,dx\end{aligned}\tag{12}$$

because $\tilde{B}_2(x+k-1)=\tilde{B}_2(x)$ by periodicity. By the same argument as in [9], setting the incomplete beta integrals

$$b_t(\alpha,\beta)=\int_0^t u^{\alpha-1}(1-u)^{\beta-1}\,du,$$

we can obtain from (9) the following expression for the last term involving the integral in (12):

$$-\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{(1-q)^s\log q}{(2\pi in)^2}\sum_{\nu=\pm1,0}a_\nu(s)b_{q^k}(s-1+\delta n,-s+\nu),\tag{13}$$

where $\delta=2\pi i/\log q$, $a_{-1}(s)=s(s+1)$, $a_0(s)=-3s$, and $a_1(s)=1$. Repeatedly applying integration by parts on these incomplete beta integrals we get for $\nu=\pm1,0$ and positive integer $M\geq 2$

$$\begin{aligned}b_{q^k}&(s-1+\delta n,-s+\nu)\\ &=\sum_{r=1}^{M-1}(-1)^{r-1}\frac{(s+1-\nu)_{r-1}}{(s-1+\delta n)_r}\frac{q^{k(s+r-2)}}{(1-q^k)^{s+r-\nu}}\\ &\quad+(-1)^{M-1}\frac{(s+1-\nu)_{M-1}}{(s-1+\delta n)_{M-1}}b_{q^k}(s-2+M+\delta n,-s-M+1+\nu).\end{aligned}\tag{14}$$

Set $\vec{s}'=(s_1,\ldots,s_{d-2})$ if $d\geq 3$ and $\vec{s}'=\emptyset$ if $d=2$. Putting (3), (12), (13) and (14) together and applying Proposition 2.5 we get

$$\zeta_q\left(\vec{s}',s_{d-1},s_d\right)=\frac{(q-1)}{(s_d-1)\log q}\zeta_q\left(\vec{s}',s_{d-1}+s_d-1\right)-\frac{1}{2}\mathcal{S}\zeta_q\left(\vec{s}',s_{d-1}+s_d\right)$$

$$+\frac{s_d}{12}\frac{\log q}{q-1}\mathcal{S}^2\zeta_q\big(\vec{s}',s_{d-1}+s_d+1\big)$$
$$+\frac{\log q}{12}\mathcal{S}\zeta_q\big(\vec{s}',s_{d-1}+s_d\big)-\sum_{\nu=\pm1,0}(C_\nu+D_\nu), \tag{15}$$

where C_ν and D_ν are the contributions from the sum involving $b_{q^k}(\dots,-s+\nu)$ (note that $s=s_d$). Explicitly they are computed as follows. Write

$$T(q,s,n,r)=\prod_{j=0}^{r-1}(2\pi in+(s-1+j)\log q)^{-1}. \tag{16}$$

Then

$$C_{-1}=\sum_{r=1}^{M-1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{T(q,s,n,r)}{(2\pi in)^2}\left(\frac{\log q}{q-1}\right)^{r+1}(s)_{r+1}\cdot\mathcal{S}^3\zeta_q(\vec{s}',s_{d-1}+s_d+r+1),$$

$$\begin{aligned}D_{-1}&=-\sum_{0<k_1<\cdots<k_{d-2}<k_{d-1}}\frac{q^{k_1(s_1-1)+\cdots+k_{d-1}(s_{d-1}-1)}}{[k_1]^{s_1}\cdots[k_{d-1}]^{s_{d-1}}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{T(q,s,n,M-1)}{(2\pi in)^2}\\&\quad\times\left(\frac{\log q}{q-1}\right)^{M+1}(s)_{M+1}\int_{k_{d-1}}^{\infty}e^{2\pi inx}q^{x(s-2+M)}\left(\frac{1-q^x}{1-q}\right)^{-s-M-1}dx\\&=:\sum_{0<k_1<\cdots<k_{d-2}}\frac{q^{k_1(s_1-1)+\cdots+k_{d-2}(s_{d-2}-1)}}{[k_1]^{s_1}\cdots[k_{d-2}]^{s_{d-2}}}\sum_{k=k_{d-2}+1}^{\infty}R(M,q,k,s_{d-1},s),\end{aligned}$$

where we replace the index k_{d-1} by k. Similarly,

$$C_0=3\log q\sum_{r=1}^{M-1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{T(q,s,n,r)}{(2\pi in)^2}\left(\frac{\log q}{q-1}\right)^{r}(s)_r\cdot\mathcal{S}^2\zeta_q(\vec{s}',s_{d-1}+s_d+r),$$

$$\begin{aligned}D_0&=-3\log q\sum_{0<k_1<\cdots<k_{d-2}<k_{d-1}}\frac{q^{k_1(s_1-1)+\cdots+k_{d-1}(s_{d-1}-1)}}{[k_1]^{s_1}\cdots[k_{d-1}]^{s_{d-1}}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{T(q,s,n,M-1)}{(2\pi in)^2}\\&\quad\times\left(\frac{\log q}{q-1}\right)^{M}(s)_M\int_{k_{d-1}}^{\infty}e^{2\pi inx}q^{x(s-2+M)}\left(\frac{1-q^x}{1-q}\right)^{-s-M}dx,\end{aligned}$$

and

$$\begin{aligned}C_1&=(\log q)^2\sum_{r=1}^{M-1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{T(q,s,n,r)}{(2\pi in)^2}\left(\frac{\log q}{q-1}\right)^{r-1}(s)_{r-1}\\&\quad\times\mathcal{S}\zeta_q(\vec{s}',s_{d-1}+s_d+r-1),\end{aligned}$$

$$D_1=-(\log q)^2\sum_{0<k_1<\cdots<k_{d-2}<k_{d-1}}\frac{q^{k_1(s_1-1)+\cdots+k_{d-1}(s_{d-1}-1)}}{[k_1]^{s_1}\cdots[k_{d-1}]^{s_{d-1}}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{T(q,s,n,M-1)}{(2\pi in)^2}$$

$$\times \left(\frac{\log q}{q-1}\right)^{M-1} (s)_{M-1} \int_{k_{d-1}}^{\infty} e^{2\pi i n x} q^{x(s-2+M)} \left(\frac{1-q^x}{1-q}\right)^{-s-M+1} dx.$$

The crucial step next is to control the summations over k_{d-1} and show that they converge uniformly with respect to q. When $0 < q \le 1/2$ this is clear. The only non-trivial part is when $q \uparrow 1$. So we assume $1/2 < q < 1$. Note that

$$\lim_{q\uparrow 1} T(q,s,n,r) = \frac{1}{(2\pi i n)^r}, \qquad \lim_{q\uparrow 1} \frac{\log q}{q-1} = 1.$$

Lemma 4.1 *Let* $s_d = s = \sigma + i\tau$. *Let* $q_0 = \max\{1/2, e^{(6-2\pi)/\tau}\}$ *if* $\tau > 0$ *and let* $q_0 = 1/2$ *if* $\tau \le 0$. *Then for all* $1 > q > q_0$ *and positive integer* k *we have*

$$\left|\frac{\log q}{q-1}\right| < 2, \quad \textit{and} \quad |T(q,s,n,r)| < \frac{1}{(6n)^r}.$$

Proof Let $f(q) = 2(1-q) + \log q$. Then $f'(q) = -2 + 1/q < 0$ whenever $q > q_0$. So $f(q) > f(1) = 0$ whenever $1 > q > q_0$. This implies that $2(1-q) > -\log q$ whence $\log q/(q-1) < 2$.

To bound $T(q,s,n,r)$ we consider each of its factors in definition (16). For each $0 \le j < r$ we have

$$\begin{aligned} &\left|2\pi i n + (s-1+j)\log q\right|^2 \\ &\quad = ((\sigma-1+j)\log q)^2 + (2\pi n + \tau \log q)^2 \ge (2\pi n + \tau \log q)^2 \end{aligned}$$

which is independent j. If $\tau \le 0$ then clearly $|2\pi i n + (s-1+j)\log q| > 6n$. If $\tau > 0$ then it follows from $q > e^{(6-2\pi)\tau}$ that

$$2\pi n + \tau \log q > 2\pi n + 6 - 2\pi \ge 6n,$$

as desired. □

Next we want to bound the integral terms in D_{-1}. Let $|\sigma_d| < N$ and $|\sigma_{d-1}| < N'$ for some positive integers N and N'. Fix an arbitrary $x > k$ and a positive integer $M > 16 + 2N + 6\sum_{j=1}^{d-1}(N_j + 1)$. Then

$$\begin{aligned} &q^{-k(M/6-N'-1)} \left| q^{k(s'-1)} q^{x(s-2+M)} \left(\frac{1-q^x}{1-q}\right)^{-s-M-1} \right| \\ &\quad < q^{x(M-N-2)-kM/6} \left(\frac{1-q}{1-q^x}\right)^{M-N+1}. \end{aligned}$$

Denote by $g(q)$ the right hand side of the above inequality.

Lemma 4.2 *Let* $1/2 < q < 1$. *Then* $g(q)$ *is increasing as a function of* q *so that*

$$g(q) \le \lim_{q\uparrow 1} g(q) = \frac{1}{x^{M-N+1}}.$$

Proof Taking the logarithmic derivative of $g(q)$ we have

$$\frac{g'(q)}{g(q)} = \frac{x(M-N-2)-kM/6}{q} + (M-N+1)\frac{xq^{x-1}(1-q)-(1-q^x)}{(1-q)(1-q^x)}$$
$$= \frac{(1-q)(1-q^x)(x(M-N-2)-kM/6)+(M-N+1)(xq^x(1-q)-q+q^{x+1})}{q(1-q)(1-q^x)}$$

whose numerator is denoted by $h(q)$. Then

$$h'(q) = \big((x+1)q^x - xq^{x-1} - 1\big)(x(M-N-2)-kM/6))$$
$$+ (M-N+1)\big(x\big(xq^{x-1}-(x+1)q^x\big) - 1 + (x+1)q^x\big).$$

Clearly $h'(1)=0$ and moreover

$$\begin{aligned}
q^{2-x}h''(q) &= (x(x+1)q - x(x-1))\big(x(M-N-2)-kM/6\big) \\
&\quad + (M-N+1)(x(x(x-1)-x(x+1)q)+x(x+1)q) \\
&= x(x-1)(kM/6+3x) + qx(x+1)(M-N+1-(kM/6+3x)) \\
&= (kM/6+3x)(x^2(1-q)-x(1+q)) + qx(x+1)(M-N+1) \\
&> qx(x+1)(M-N+1) - x(1+q)(kM/6+3x) \qquad (\text{since } 1>q) \\
&\ge qx^2\left\{M-N+1-\frac{1+q}{q}(3+M/6)\right\} \quad (\text{since } k<x) \\
&> qx^2(M/2-N-8) > 0,
\end{aligned}$$

where we used the fact that if $q>1/2$ then $(1+q)/q<3$. This implies that $h'(q)$ is increasing so that $h'(q)<0$ for all $1>q>1/2$ (recall that $h'(1)=0$). It follows that $h(q)$ is decreasing. But $h(1)=0$ so we know $h(q)>0$ for all such q. Thus $g'(q)>0$ and therefore $g(q)$ is increasing. This completes the proof of the lemma. □

We now can bound the innermost sum of D_{-1}. From Lemma 2.1, Lemma 4.1 and Lemma 4.2 we have (if $d=2$ then take $k_0=1$)

$$\begin{aligned}
&\sum_{k=1+k_{d-2}}^{\infty} |R(M,q,k,s',s)| \\
&\quad < \sum_{k=1+k_{d-2}}^{\infty} (2k)^{N'}\frac{2\zeta(M+1)}{4\pi^2 6^{M-1}}2^{M+1}(N)_{M+1}q^{k(M/6-N'-1)}\int_k^\infty \frac{dx}{x^{M-N+1}} \\
&\quad < \frac{(M+1)!}{M-N}\binom{M+N}{M+1}\sum_{k=1+k_{d-2}}^{\infty}\frac{q^{k(M/6-N'-1)}}{k^{M-N-N'}} \\
&\quad < (M+1)!(2M)^{M+1}\sum_{k=1+k_{d-2}}^{\infty} q^{k(M/6-N'-1)}
\end{aligned}$$

since $2^{N'+M+2}\zeta(M+1) < 4\pi^2 6^{M-1}$ and $M - N > 2$. Therefore by Lemma 2.1

$$|D_{-1}| < (2M)^{2M} \sum_{0<k_1<\cdots<k_{d-2}} \left(\prod_{l=1}^{d-2} k_l^{N_l} q^{k_l(-N_l-1)}\right) q^{k_{d-2}(M/6-N_{d-1}-1)}$$

which converges as proved in Proposition 2.2.

Exactly the same argument applies to the integral terms in D_0 and D_1, which we leave to the interested readers. These convergence results imply two things. First we can show by induction on d that (15) gives rise to an analytic continuation of $\zeta_q(s_1,\dots,s_d)$ as a meromorphic function on $\mathbb{C}^d \setminus \mathfrak{S}'_d$. Second, also by induction on d, we now can conclude that it's legitimate to take the limit $q \uparrow 1$ inside the sums of C_ν and D_ν to get (note that $\lim_{q\uparrow 1} \mathcal{S}^n \zeta_q(\vec{s}) = \zeta(\vec{s})$ for any $\vec{s} \in \mathbb{C}^{d-1} \setminus \mathfrak{S}'_{d-1}$ and any positive integer n)

$$\begin{aligned}
&\lim_{q\uparrow 1} \zeta_q\left(\vec{s}', s_{d-1}, s_d\right) \\
&\quad= \frac{1}{s-1}\zeta\left(\vec{s}', s_{d-1}+s_d-1\right) - \frac{1}{2}\zeta\left(\vec{s}', s_{d-1}+s_d\right) + \frac{s}{12}\zeta\left(\vec{s}', s_{d-1}+s_d+1\right) \\
&\qquad - \sum_{r=1}^{M-1} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i n)^{r+2}} (s)_{r+1} \cdot \zeta\left(\vec{s}', s_{d-1}+s_d+r+1\right) \\
&\qquad + \sum_{0<k_1<\cdots<k_{d-1}} \frac{1}{k_1^{s_1}\cdots k_{d-1}^{s_{d-1}}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(s)_{M+1}}{(2\pi i n)^{M+1}} \int_{k_{d-1}}^{\infty} e^{2\pi i n x} x^{-s-M-1}\,dx \\
&\quad= \sum_{r=0}^{M+1} \frac{B_r}{r!} (s_d)_{r-1} \cdot \zeta(s_1,\dots,s_{d-1}+s_d+r-1) \\
&\qquad - \frac{1}{(M+1)!} \sum_{0<k_1<\cdots<k_d} \frac{1}{k_1^{s_1}\cdots k_{d-1}^{s_{d-1}}} \int_{k_{d-1}}^{\infty} \tilde{B}_{M+1}(x) \frac{(s)_{M+1}}{x^{s+M+1}}\,dx
\end{aligned}$$

by (9) and its specialization with $x = 1$

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i n)^{r+2}} = -\frac{\tilde{B}_{r+2}(1)}{(r+2)!} = -\frac{B_{r+2}}{(r+2)!}.$$

The main theorem now mostly follows from Theorem 3.2. The poles at $s_d = m - \frac{2\pi i}{\log q} n$ are given by the first term in formula (15) when $m = 1$ and $n = 0$ and by the terms $T(q, s_d, n, r)$ as defined in (16) if $m \le 1$ and $n \ne 0$. The location of the other poles are obtained by induction using those poles of the q-Riemann zeta function presented in Theorem 1.1 for the initial step. This completes the proof of our main theorem.

5 Series q-stuffle relations

The classical multiple zeta functions satisfy the stuffle relations (i.e., stuff and shuffle) originating from their series representations. In [6] Hoffman first studied this type of relations and called them quasi-shuffle relations. For example,

$$\zeta(s_1)\zeta(s_2)=\zeta(s_1,s_2)+\zeta(s_2,s_1)+\zeta(s_1+s_2). \tag{17}$$

In general we can define a stuffle operation $*$ on finite words so that for any complex numbers a, b (regarded as letters) and words w_1 and w_2 of complex numbers

$$aw_1*bw_2=a(w_1*bw_2)+b(aw_1*w_2)+(a+b)(w_1*w_2)$$

so that treating complex variables as words we have

$$\zeta(w_1)\zeta(w_2)=\zeta(w_1*w_2).$$

To generalize these relations we first define shifting operators on words of complex variables:

$$(s_1,\dots,s_{j-1},\mathcal{S}(s_j),s_{j+1},\dots,s_d)=\mathcal{S}_j(s_1,\dots,s_d)$$

and set $\zeta(\mathcal{S}_j(s_1,\dots,s_d))=\mathcal{S}_j\zeta(s_1,\dots,s_d)$, where the shifting operator $\mathcal{S}_j$ is defined by (8). Now we define the q-stuffle operator $*_q$ on words by

$$aw_1*_q bw_2=a(w_1*_q bw_2)+b(aw_1*_q w_2)+\mathcal{S}(a+b)(w_1*_q w_2).$$

Theorem 5.1 *For any two words of complex variables w_1 and w_2 we have*

$$\zeta_q(w_1)\zeta_q(w_2)=\zeta_q(w_1*_q w_2).$$

Proof Induction on the length of w_1w_2. □

For example

$$\zeta_q(s_1)\zeta_q(s_2)=\zeta_q(s_1,s_2)+\zeta_q(s_2,s_1)+\zeta_q(s_1+s_2)+(1-q)\zeta_q(s_1+s_2-1). \tag{18}$$

We can recover the stuffle relation of (17) by taking the limit as $q\uparrow 1$.

Remark 5.2 These relations are studied by D. Bradley in [4] independently.

6 Special values of $\zeta_q(s_1,s_2)$

For integers $n_1,\dots,n_d$ we set

$$\zeta_q(n_1,\dots,n_d)=\lim_{s_1\to n_1}\cdots\lim_{s_d\to n_d}\zeta_q(s_1,\dots,s_d),$$

$$\zeta_q^R(n_1,\dots,n_d)=\lim_{s_d\to n_d}\cdots\lim_{s_1\to n_1}\zeta_q(s_1,\dots,s_d)$$

if the limits exist. One should compare this with the corresponding definitions for multi-zeta values in [1]. In a forthcoming paper [16], using Baxter algebra we will define renormalizations of some special values for our multiple q-zeta functions so that they can still satisfy the q-stuffle relations. Interesting phenomena occur already in the case $d=2$ and these should be generalized to arbitrary depth. By Theorem 2.4 we get

$$\begin{aligned}\zeta_q(s_1,s_2) &= (1-q)^{s_1+s_2}\sum_{r_1,r_2=0}^{+\infty}\binom{s_1+r_1-1}{r_1}\binom{s_2+r_2-1}{r_2}\\ &\quad\times\frac{q^{2s_2+2r_2+s_1+r_1-3}}{(1-q^{s_2+r_2-1})(1-q^{s_2+r_2+s_1+r_1-2})}\\ &= (1-q)^{s_1+s_2}\left\{\frac{q^{2s_2+s_1-3}}{(1-q^{s_2-1})(1-q^{s_2+s_1-2})}+\frac{s_1q^{2s_2+s_1-2}}{(1-q^{s_2-1})(1-q^{s_2+s_1-1})}\right.\\ &\quad+\frac{s_2q^{2s_2+s_1-1}}{(1-q^{s_2})(1-q^{s_2+s_1-1})}+\frac{s_1s_2q^{2s_2+s_1}}{(1-q^{s_2})(1-q^{s_2+s_1})}\\ &\quad\left.+\frac{s_1(s_1+1)q^{2s_2+s_1-1}}{2(1-q^{s_2-1})(1-q^{s_2+s_1})}+\frac{s_2(s_2+1)q^{2s_2+s_1+1}}{2(1-q^{s_2+1})(1-q^{s_2+s_1})}+\cdots\right\}.\end{aligned}$$

Clearly we have

$$\zeta_q(0,0)=\lim_{s_1\to 0}\lim_{s_2\to 0}\zeta_q(s_1,s_2)=\frac{1}{(q^2-1)(q-1)}-\frac{3}{2(q-1)\log q}+\frac{1}{\log^2 q},$$

$$\zeta_q^R(0,0)=\lim_{s_2\to 0}\lim_{s_1\to 0}\zeta_q(s_1,s_2)=\frac{1}{(q^2-1)(q-1)}-\frac{1}{(q-1)\log q}+\frac{q}{2(q-1)\log q}.$$

It is not too hard to find that

$$\lim_{q\uparrow 1}\zeta_q(0,0)=\frac{1}{3},\qquad \lim_{q\uparrow 1}\zeta_q^R(0,0)=\frac{5}{12}.$$

This is consistent with what we found in [14] by using generalized functions (distributions). See also (22) below. In [14] we further showed that near $(0,0)$ the double zeta function has the following asymptotic expansion:

$$\zeta(s_1,s_2)=\frac{4s_1+5s_2}{12(s_1+s_2)}+R(s_1,s_2),$$

where $R(s_1,s_2)$ is analytic at $(0,0)$ and $\lim_{(s_1,s_2)\to(0,0)}R(s_1,s_2)=0$.

Let n, k be two non-negative integers, and $m=k-n-2$. We now consider the double zeta function around $(s_1,s_2)=(-m,-n)$ which has the following expression by Theorem 3.2:

$$\begin{aligned}&\frac{-1}{n+1}\zeta(s_1+s_2-1)-\frac{1}{2}\zeta(s_1+s_2)-\sum_{r=2}^{n+1}\frac{B_r}{r}\binom{n}{r-1}\zeta(s_1+s_2+r-1)\\ &\quad+\alpha(m)(-1)^n\frac{B_k}{k!}n!(k-n-2)!(s_2+n)\zeta(s_2+s_1+k-1),\end{aligned}\tag{19}$$

where $\alpha(m) = 0$ if $m \le -1$ and $\alpha(m) = 1$ if $m \ge 0$. Note that the last term is zero when computing

$$\zeta(-m, -n) = \lim_{s_1 \to -m} \lim_{s_2 \to -n} \zeta(s_1, s_2)$$

while it has possibly nontrivial contribution for $\zeta^R(-m, -n)$ since

$$\lim_{s_2 \to -n} \lim_{s_1 \to -m} (s_2 + n)\zeta(s_2 + s_1 + k - 1) = 1.$$

We get

Table 1 Poles and indeterminacy of double zeta function

k, n	Pole, residue	Indeterminacy, $\zeta = \zeta^R(n+2-k, -n)$
$k = 0$	$-1/(n+1)$	None
$k = 1$	$-1/2$	None
$2 \nmid k, 3 \le k \le n+1$	None	$\frac{B_{k-1}}{2(k-1)} + \sum_{r=k-1}^{n+1} \frac{B_r}{r}\binom{n}{r-1}\zeta(r+1-k)$
$2 \nmid k, k = n+2$	None	$B_{k-1}/(k-1)$
$2 \nmid k, k > n+2$	None	$B_{k-1}/2(k-1)$
$2 \mid k, 2 \le k \le n+1$	$(-1)^{k+1}\frac{B_k}{k}\binom{n}{k-1}$	None

When $m \ge 0$ and $2|k$ the values of $\zeta(-m, -n)$ and $\zeta^R(-m, -n)$ are different in general:

$$\zeta(-m, -n) = \frac{B_k}{k(n+1)} + \sum_{r=1}^{n+1} \frac{B_r}{r}\binom{n}{r-1}\frac{B_{k-r}}{k-r}, \tag{20}$$

$$\zeta^R(-m, -n) = \zeta(-m, -n) + (-1)^n \frac{B_k}{k!} n!(k-n-2)!. \tag{21}$$

Note that the term corresponding to $r = 1$ is non-zero if and only if $k = 2$ (and $n = 0$). From this observation we again recover that

$$\zeta(0, 0) = \frac{1}{3}, \qquad \zeta^R(0, 0) = \frac{5}{12}. \tag{22}$$

We now consider the q-double zeta function.

Theorem 6.1 *Let k, n be two non-negative integers, and $m = k - n - 2$. If $m \le -1$ then the q-double zeta function $\zeta_q(s_1, s_2)$ has a pole at $(-m, -n)$ with residue given by*:

$$\frac{\mathrm{Res}_{(s_1,s_2)=(-m,-n)}\zeta_q(s_1, s_2)}{-(1-q)^{2-k}(\log q)^{-1}}$$

$$= \begin{cases} \sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} / (q^{n+1-r} - 1) & \textit{if } k \le n, \\ \sum_{r=0}^{n} (-1)^r \binom{n}{r} / (q^{n+1-r} - 1) - \dfrac{(-1)^n}{(n+1)\log q} & \textit{if } k = n+1. \end{cases}$$

Proof Use Theorem 2.4. □

Corollary 6.2 *Let n be a non-negative integer. Then*

$$\mathrm{Res}_{(s_1,s_2)=(1,-n)} \zeta_q(s_1, s_2) = \frac{q-1}{\log q} \zeta_q(-n) \tag{23}$$

and

$$\lim_{q \uparrow 1} \zeta_q(-n) = -\frac{B_{n+1}}{n+1} = \mathrm{Res}_{(s_1,s_2)=(1,-n)} \zeta(s_1, s_2). \tag{24}$$

Proof Equation (23) follows from the case $m = -1$ in the above theorem and [9, (6)]:

$$\zeta_q(-n) = (1-q)^{-n} \left\{ \sum_{r=0}^{n} (-1)^r \binom{n}{r} / (q^{n+1-r} - 1) - \frac{(-1)^n}{(n+1)\log q} \right\}.$$

The first equality in (24) is [9, Theorem 1] and the second equality follows from Table 1. □

Corollary 6.3 *Let k, n be two non-negative integers, $m = k - n - 2 \le -2$. Then*

$$\lim_{q \uparrow 1} \mathrm{Res}_{(s_1,s_2)=(-m,-n)} \zeta_q(s_1, s_2) = \mathrm{Res}_{(s_1,s_2)=(-m,-n)} \zeta(s_1, s_2). \tag{25}$$

Proof By Theorem 6.1 and Table 1 we only need to prove

$$\lim_{q \uparrow 1} (1-q)^{1-k} \sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} / (q^{n+1-r} - 1)$$
$$= \begin{cases} \dfrac{-1}{n+1} & \text{if } k = 0, \\ (-1)^{1-k} \dfrac{B_k}{k} \dbinom{n}{k-1} & \text{if } 1 \le k \le n. \end{cases} \tag{26}$$

First by generating function of the Bernoulli numbers

$$\frac{1}{q^{n+1-r} - 1} = \frac{1}{e^{(n+1-r)\log q} - 1} = \sum_{l=0}^{\infty} \frac{B_l}{l!} ((n+1-r)\log q)^{l-1}.$$

Plugging this into the left hand side of (26), replacing $1-q$ by $-\log q$, and exchanging the summation we get

$$(-\log q)^{1-k}\sum_{r=0}^{k}(-1)^r\binom{n+1-r}{k-r}\binom{n}{r}/(q^{n+1-r}-1)$$
$$=(-1)^{1-k}\sum_{l=0}^{\infty}\frac{B_l}{l!}(\log q)^{l-k}\sum_{r=0}^{k}(-1)^r\binom{n+1-r}{k-r}\binom{n}{r}(n+1-r)^{l-1}. \quad (27)$$

Then the inner sum over r is the coefficient of x^k of the following polynomial

$$f_l(x)=\sum_{k=0}^{n+1}\sum_{r=0}^{k}(-1)^r\binom{n+1-r}{k-r}\binom{n}{r}(n+1-r)^{l-1}x^k$$
$$=\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n+1-r)^{l-1}\sum_{k=r}^{n+1}\binom{n+1-r}{k-r}x^k$$
$$=(x+1)^{n+1}\sum_{r=0}^{n}(-y)^r\binom{n}{r}(n+1-r)^{l-1}, \quad (28)$$

where $y=x/(x+1)$. When $l=0$ this expression becomes

$$f_0(x)=\frac{(x+1)^{n+1}}{n+1}\big[(1-y)^{n+1}-(-y)^{n+1}\big]=\frac{1}{n+1}\big[1-(-x)^{n+1}\big].$$

Note that $k\le n$ we see the coefficient of x^k in $f_0(x)$ is 0 if $k>0$ and it's $1/(n+1)$ if $k=0$. If $k=0$ then only the constant term $-1/(n+1)$ in (27) remains when $q\uparrow 1$ which proves the corollary in this case. So we can assume $l,k>0$. Then

$$f_l(x)=\left(z\frac{d}{dz}\right)^{l-1}\left\{(x+1)^{n+1}\sum_{r=0}^{n}(-y)^r\binom{n}{r}z^{n+1-r}\right\}\Bigg|_{z=1}$$
$$=\left(z\frac{d}{dz}\right)^{l-1}\left\{(x+1)^{n+1}z(z-y)^n\right\}\Big|_{z=1}.$$

Note that highest degree term in $f_l(x)$ is contained in

$$(x+1)^{n+1}\left(\frac{d}{dz}\right)^{l-1}(z-y)^n\Bigg|_{z=1}=n(n-1)\cdots(n-l+2)(x+1)^{n+1}(1-y)^{n-l+1}$$
$$=n(n-1)\cdots(n-l+2)(x+1)^l.$$

If $l=1$ one can easily modify this to get just $x+1$. If $l<k$ then the coefficient of x^k in $f_l(x)$ is 0. If $l=k$ it is equal to

$$n(n-1)\cdots(n-k+2)=(k-1)!\binom{n}{k-1}.$$

The last express is valid even for $k = l = 1$. Thus the range of l in the outer sum of (27) starts from k. Moreover, the first term of (27) is

$$(-1)^{1-k}\frac{B_k}{k}\binom{n}{k-1}$$

as desired. This completes the proof of the corollary. □

Proposition 6.4 *Let k, n be two non-negative integers such that $n \geq k$ and k is even. Let $m = k - n - 2$. Then*

$$\mathrm{Res}_{(s_1,s_2)=(-m,-n)}\zeta_q(s_1,s_2) = \frac{-f(q)(q-1)/\log q}{D(q)}, \quad D(q) = \prod_{j=n+1-k}^{n+1} F(q,j)^{\epsilon_j},$$

where $f(q) \in \mathbb{Z}[q]$ is a palindrome with leading coefficient $\binom{n}{k}$, $F(q,j) \in \mathbb{Z}[q]$ is a factor of $(q^j - 1)/(q-1)$, $\epsilon_j = 0$ or 1, and $\deg_q D(q) = n + \deg_q f(q)$, such that

$$\lim_{q\uparrow 1}\mathrm{Res}_{(s_1,s_2)=(-m,-n)}\zeta_q(s_1,s_2) = \mathrm{Res}_{(s_1,s_2)=(-m,-n)}\zeta(s_1,s_2).$$

Proof The computational proof is left as an exercise for the interested readers. □

Example 6.5 By Theorem 6.1 we find with the help of Maple

$$\mathrm{Res}_{(s_1,s_2)=(4,-4)}\zeta_q(s_1,s_2) = \frac{-2q^3(3q^2+4q+3)(q-1)/\log q}{P_1(q,2)P_1(q,3)P_1(q,4)},$$

where $P_a(q,m) = \sum_{j=0}^{m} q^{aj}$. Moreover we can check that

$$\lim_{q\uparrow 1}\mathrm{Res}_{(s_1,s_2)=(4,-4)}\zeta_q(s_1,s_2) = \mathrm{Res}_{(s_1,s_2)=(4,-4)}\zeta(s_1,s_2) = -\frac{1}{3}$$

by Table 1 with $k = 2$ and $n = 4$.

Example 6.6 From Theorem 6.1 we get

$$\mathrm{Res}_{(s_1,s_2)=(6,-8)}\zeta_q(s_1,s_2) = \frac{-14q^5 g(q)(q-1)/\log q}{P_1(q,4)P_1(q,5)P_1(q,6)P_2(q,3)P_3(q,2)},$$

where $g(q)$ is a polynomial in q of degree 14 satisfying

$$q^{14}f(1/q) = f(q) = 5q^{14} + 6q^{13} + 8q^{12} + 7q^{11} - q^{10} - 20q^9 - 30q^8 - 34q^7 - \cdots.$$

Then we can compute with Maple

$$\lim_{q\uparrow 1}\mathrm{Res}_{(s_1,s_2)=(6,-8)}\zeta_q(s_1,s_2) = \mathrm{Res}_{(s_1,s_2)=(6,-8)}\zeta(s_1,s_2) = \frac{7}{15}$$

by Table 1 with $k = 4$ and $n = 8$.

Example 6.7 Consider the point $(s_1, s_2) = (5, -9)$. We have

$$\mathrm{Res}_{(s_1,s_2)=(5,-9)}\zeta_q(s_1, s_2) = \frac{-42q^4 g(q)(q-1)/\log q}{P_1(q,4)P_1(q,6)P_1(q,7)P_2(q,2)P_3(q,2)A(q,4)},$$

where $A(q,m) = \sum_{j=0}^{m}(-1)^j q^j$ and $g(q)$ is a polynomial in q of degree 18 satisfying

$$\begin{aligned} q^{18}g(1/q) &= g(q) \\ &= 2q^{18} - q^{17} - 7q^{15} - 11q^{14} - 16q^{13} - 4q^{12} + 9q^{11} + 28q^{10} + 30q^9 + \cdots \end{aligned}$$

so that we again have the equality

$$\lim_{q\uparrow 1} \mathrm{Res}_{(s_1,s_2)=(5,-9)}\zeta_q(s_1, s_2) = \mathrm{Res}_{(s_1,s_2)=(5,-9)}\zeta(s_1, s_2) = -\frac{1}{2}$$

by Table 1 with $k = 6$ and $n = 9$.

Theorem 6.8 *Let m, n be two non-negative integers and $k = m + n + 2$. Then $\zeta_q(s_1, s_2)$ has indeterminacy at $(-m, -n)$ such that*

$$\begin{aligned} \zeta_q(-m,-n) = (1-q)^{2-k}\Bigg\{ & \frac{(-1)^k}{(m+1)(n+1)(\log q)^2} \\ & + \sum_{r=0}^{m} \frac{(-1)^{r+n+1}}{(n+1)\log q}\binom{m}{r}\frac{1}{q^{m+1-r}-1} \\ & + \sum_{r=0}^{n} \frac{(-1)^{r+m+1}}{\log q}\frac{m!(n+1-r)!}{(k-r)!}\binom{n}{r}\frac{1}{q^{n+1-r}-1} \\ & + \sum_{r_1=0}^{m}\sum_{r_2=0}^{n}(-1)^{r_1+r_2}\binom{m}{r_1}\binom{n}{r_2}\frac{1}{q^{n+1-r_2}-1}\frac{1}{q^{k-r_1-r_2}-1}\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} \zeta_q^R(-m,-n) = (1-q)^{2-k}\Bigg\{ & \sum_{r=0}^{m} \frac{(-1)^{r+n+1}}{(n+1)\log q}\binom{m}{r}\frac{1}{q^{m+1-r}-1} \\ & + \sum_{r=0}^{m} \frac{(-1)^{r+n}}{\log q}\binom{k-n-2}{r}\frac{n!(m+1-r)!}{(k-r)!}\frac{q^{m+1-r}}{q^{m+1-r}-1} \\ & + \sum_{r_1=0}^{m}\sum_{r_2=0}^{n}(-1)^{r_1+r_2}\binom{m}{r_1}\binom{n}{r_2}\frac{1}{q^{n+1-r_2}-1}\frac{1}{q^{k-r_1-r_2}-1}\Bigg\}. \end{aligned}$$

Proof Use Theorem 2.4. □

Similar to Corollary 6.2 and Corollary 6.3 we have

Corollary 6.9 *Let m and n be two non-negative integers. Then*

$$\lim_{q\uparrow 1}\zeta_q(-m,-n)=\zeta(s_1,s_2),\qquad \lim_{q\uparrow 1}\zeta_q^R(-m,-n)=\zeta^R(s_1,s_2). \tag{29}$$

Proof Set $k=m+n+2$. We consider $\zeta_q^R(-m,-n)$ first. From Theorem 6.8 we get

$$\zeta_q^R(-m,-n)=(q-1)^{-k}\left(\frac{q-1}{\log q}\right)^2(A+B+C),$$

where

$$A=\sum_{j=0}^{\infty}\frac{B_j}{j!}(\log q)^j\sum_{r=0}^{m}\frac{(-1)^{m+r+1}}{n+1}\binom{m}{r}(m+1-r)^{j-1},$$

$$B=\sum_{i=0}^{\infty}\frac{B_i}{i!}(\log q)^i\sum_{r=0}^{m}(-1)^{r+m+i}\binom{m}{r}\frac{n!(m+1-r)!}{(k-r)!}(m+1-r)^{i-1},$$

$$C=\sum_{i,j=0}^{\infty}\frac{B_iB_j}{i!j!}(\log q)^{i+j}\sum_{r_1=0}^{m}\sum_{r_2=0}^{n}(-1)^{k+r_1+r_2}\binom{m}{r_1}\binom{n}{r_2}$$
$$\times(n+1-r_2)^{i-1}(k-r_1-r_2)^{j-1}.$$

We first compute B as follows. Write

$$B=\sum_{i=0}^{\infty}\frac{B_i}{i!}(\log q)^i W(m,n,i),$$

where

$$W(m,n,i)=\sum_{r=0}^{m}(-1)^{r+k+n+i}\frac{m!n!}{r!(k-r)!}(m+1-r)^i.$$

If $i=0$ then we can prove by decreasing induction on n that

$$W(m,n,0)=\sum_{r=0}^{m}(-1)^{r+k+n}\frac{m!n!}{r!(k-r)!}=\frac{1}{k(n+1)}. \tag{30}$$

This is trivial if $n=k-2$. Suppose (30) is true for $n\ge 1$ then we have

$$W(m,n-1,0)=-\frac{k-n-1}{n}W(m,n,0)+\frac{(n-1)!}{(n+1)!}=\frac{1}{kn},$$

as desired.

Similarly, we can compute C as follows. Put

$$C=\sum_{i,j=0}^{\infty}\frac{B_iB_j(\log q)^{i+j}}{i!j!(n+1)}\sum_{r_1=0}^{m}\sum_{r_2=0}^{n}(-1)^{k+r_1+r_2}\binom{m}{r_1}\binom{n+1}{r_2}$$
$$\times(n+1-r_2)^i(k-r_1-r_2)^{j-1}.$$

We now change the upper limit of r_2 from n to $n+1$ in the above. The extra terms correspond to those by setting $i=0$ and $r_2=n+1$, which produce exactly A. Therefore,

$$C=\sum_{i,j=0}^{\infty}\frac{B_iB_j(\log q)^{i+j}}{i!j!}V(m,n,i,j)-A,$$

where

$$V(m,n,i,j)=\sum_{r_1=0}^{m}\sum_{r_2=0}^{n+1}\frac{(-1)^{k+r_1+r_2}}{n+1}\binom{m}{r_1}\binom{n+1}{r_2}(n+1-r_2)^i(k-r_1-r_2)^{j-1}.$$

For $j\ge 1$ we have

$$\begin{aligned}V(m,n,i,j)&=\left(x\frac{d}{dx}\right)^{j-1}\left\{\left(y\frac{d}{dy}\right)^{i}\left\{\sum_{r_1=0}^{m}\sum_{r_2=0}^{n+1}\frac{(-1)^{k+r_1+r_2}}{n+1}\right.\right.\\&\qquad\left.\left.\times\binom{m}{r_1}\binom{n+1}{r_2}y^{n+1-r_2}x^{k-r_1-r_2}\right\}\Bigg|_{y=1}\right\}\Bigg|_{x=1}\\&=\frac{(-1)^k}{n+1}\left(x\frac{d}{dx}\right)^{j-1}\left\{\left(y\frac{d}{dy}\right)^{i}\left\{x(xy-1)^{n+1}(x-1)^{k-n-2}\right\}\Big|_{y=1}\right\}\Big|_{x=1}\\&=\begin{cases}0 & \text{if } i+j<k,\\ \dfrac{(-1)^kn!}{(n+1-i)!}(k-i-1)! & \text{if } i+j=k,\ i\le n+1.\end{cases}\end{aligned}\tag{31}$$

We have used the fact that if $i+j=k$ and $i>n+1$ then $j<k-n-1$ and by exchanging the two operators $x(d/dx)$ and $y(d/dy)$ we can easily show that (31) is zero. So if $l<k$ the total contribution to the coefficient of $(\log q)^l$ from $V(m,n,i,j)$ with $j>0$ is trivial and if $l=k$ it is equal to

$$\sum_{i=0}^{n+1}B_iB_{k-i}\frac{(-1)^kn!(k-i-1)!}{i!(k-i)!(n+1-i)!}=\begin{cases}\dfrac{B_{k-1}}{2(k-1)} & \text{if } 2\nmid k,\\ \dfrac{B_k}{k(n+1)}+\displaystyle\sum_{i=1}^{n+1}\frac{B_iB_{k-i}}{i(k-i)}\binom{n}{i-1} & \text{if } 2\mid k,\end{cases}\tag{32}$$

because $k\ge n+2\ge 2$ and $B_k=0$ if k is odd.

To deal with $V(m,n,i,0)$ note that (31) still makes sense if we interpret the operator $x(d/dx)^{-1}$ as follows:

$$\left(x\frac{d}{dx}\right)^{-1}\{F(x)\}\Big|_{x=1}=\int_0^1\frac{F(x)}{x}dx$$

whenever $F(0)=0$. Thus we get

$$V(m,n,i,0)=\frac{(-1)^k}{n+1}\left(y\frac{d}{dy}\right)^i\left\{\int_0^1 (xy-1)^{n+1}(x-1)^m\,dx\right\}\Big|_{y=1}.$$

Therefore if $i=0$ then we get

$$V(m,n,0,0)=\frac{(-1)^k}{n+1}\int_0^1 (x-1)^{k-1}\,dx=-\frac{1}{k(n+1)}=-W(m,n,0) \tag{33}$$

from (30). If $i\geq 1$ then integrating by parts we get

$$\begin{aligned}
&\int_0^1 (xy-1)^{n+1}(x-1)^m\,dx\\
&=\frac{(xy-1)^{n+2}}{y(n+2)}(x-1)^m\Big|_0^1-\frac{m}{y(n+2)}\int_0^1 (xy-1)^{n+2}(x-1)^{m-1}\,dx\\
&=\cdots\\
&=(-1)^{k+1}\Bigg(\frac{1}{y(n+2)}-\frac{m}{y^2(n+2)(n+3)}+\cdots+(-1)^{m-1}\frac{m!(n+1)!}{y^m(m+n+1)!}\\
&\quad+(-1)^m\frac{m!(n+1)!}{y^m(m+n+1)!}\int_0^1 (xy-1)^{m+n+1}\,dx\Bigg)\\
&=(-1)^{k+n}\frac{m!(n+1)!}{k!}\frac{(y-1)^k}{y^{m+1}}+\sum_{r=0}^{m}(-1)^{r+k+1}\frac{m!(n+1)!}{y^{r+1}(m-r)!(n+2+r)!}.
\end{aligned}$$

It follows from changing the index r to $m-r$ that

$$\begin{aligned}
&V(m,n,i,0)\\
&\quad=(-1)^{k+n}\frac{m!n!}{k!}\left(y\frac{d}{dy}\right)^i\left\{\frac{(y-1)^k}{y^{m+1}}\right\}\Big|_{y=1}\\
&\qquad+\sum_{r=0}^{m}(-1)^{r+n+i+1}\frac{m!n!}{r!(k-r)!}(m+1-r)^i\\
&\quad=\begin{cases}(-1)^{k+1}W(m,n,0) & \text{if } 0<i<k,\\ (-1)^{k+1}W(m,n,0)+(-1)^{k+n}n!(k-n-2)! & \text{if } i=k.\end{cases}
\end{aligned} \tag{34}$$

Thus when $0<i<k$ and k is even we have $V(m,n,i,0)=-W(m,n,i)$. It follows from (32), (33) and (34) that

$$\lim_{q\uparrow 1}\zeta_q^R(-m,-n)=\zeta^R(s_1,s_2)$$

since $B_k=0$ if $k>2$ is odd.

Let's turn to prove the first equality in (29). By Theorem 6.8 we have

$$\zeta_q(-m,-n)=(q-1)^{-k}\left(\frac{q-1}{\log q}\right)^2(D+A+E+C)$$

where A and C are as above and

$$D=\frac{1}{(m+1)(n+1)},\qquad E=\sum_{i=0}^{\infty}\frac{B_i}{i!}(\log q)^i U(m,n,i),$$

where

$$U(m,n,i)=-\sum_{r=0}^{n}(-1)^{r+n}\frac{m!n!}{r!(k-r)!}(n+1-r)^i.$$

Hence

$$U(m,n,0)=W(n,m,0)=\frac{-1}{k(m+1)}=\frac{-1}{k(k-n-1)}=\frac{1}{k(n+1)}-D. \tag{35}$$

We only need to show that

$$U(m,n,i)=\begin{cases}W(m,n,i) & \text{if } 0<i<k,\\ W(m,n,i)-(-1)^n n!(k-n-2)! & \text{if } i=k.\end{cases} \tag{36}$$

Indeed when $i>0$ we have

$$\begin{aligned}
&U(m,n,i)\\
&=(-1)^{n+1}\left(y\frac{d}{dy}\right)^i\left\{\sum_{r=0}^{n}(-1)^r\frac{m!n!}{k!}\binom{k}{r}y^{n+1-r}\right\}\Bigg|_{y=1}\\
&=(-1)^{n+1}\frac{m!n!}{k!}\left(y\frac{d}{dy}\right)^i\left\{y^{n+1-k}(y-1)^k-\sum_{r=n+1}^{k}(-1)^r\binom{k}{r}y^{n+1-r}\right\}\Bigg|_{y=1}\\
&=(-1)^{n+1}\frac{m!n!}{k!}\left(y\frac{d}{dy}\right)^i\left\{\frac{(y-1)^k}{y^{m+1}}\right\}+\sum_{r=n+1}^{k}(-1)^{r+n}\frac{m!n!}{r!(k-r)!}(n+1-r)^i.
\end{aligned}$$

Then first term is 0 if $i<k$ and it's $(-1)^{n+1}m!n!$ if $i=k$. When $r=n+1$ the summand in the second term is zero since $i>0$. So we can let r range only from $n+2$ to k. Then change the index r to $k-r$ (and let r run from 0 to m) we can see immediately that the second term is exactly $W(m,n,i)$. This proves (36) which together with (35) implies the first equation in (29). We thus finish the proof of the corollary. □

We conclude this section by remarking that by the q-stuffle relation (18) we can also analyze $\zeta_q(s_1,s_2)$ at $(-n,n+2-k)$ for any non-negative integers k and n. For example, it's easy to compute directly that

$$\mathrm{Res}_{(s_1,s_2)=(-3,2)}\zeta_q(s_1,s_2) = -\frac{1}{(1-q)\log q}\sum_{r=0}^{3}(-1)^r\binom{3}{r}(r+1)\frac{q^{r+1}}{1-q^{r+1}}$$
$$= \frac{-q(q-1)^2}{(q+1)(q^2+1)(q^2+q+1)\log q}, \tag{37}$$

which can be obtained also by the q-stuffle relation (18) and the expression

$$\mathrm{Res}_{(s_1,s_2)=(2,-3)}\zeta_q(s_1,s_2) = \frac{q(q-1)^2}{(q+1)(q^2+1)(q^2+q+1)\log q}$$

by taking $k=n=3$ in Theorem 6.1. Thus $(-3,2)$ is a simple pole of the double q-zeta function $\zeta_q(s_1,s_2)$. On the other hand the ordinary double zeta function $\zeta(s_1,s_2)$ does not have a pole along $s_1+s_2=-1$. Indeed from (37) we find that

$$\lim_{q\uparrow 1}\mathrm{Res}_{(s_1,s_2)=(-3,2)}\left\{\zeta_q(s_1,s_2)\right\} = 0.$$

7 Multiple q-polylogarithms

It is well known that special values of the multiple zeta function $\zeta(s_1,\dots,s_d)$ at positive integers $(n_1,\dots,n_d)$ can be regarded as single-valued version of multiple polylogarithm $\mathrm{Li}_{n_1,\dots,n_d}(z_1,\dots,z_d)$ evaluated at $z_1=\cdots=z_d=1$. For $|z_j|<1$ these functions can be defined as

$$\mathrm{Li}_{n_1,\dots,n_d}(z_1,\dots,z_d) = \sum_{0<k_1<\cdots<d_d}\frac{z_1^{k_1}\dots z_d^{k_d}}{k_1^{n_1}\dots k_d^{n_d}}.$$

By Chen's iterated integral

$$(-1)^d\int_0^1 \frac{dt_1}{t_1-a_1}\circ\left(\frac{dt_1}{t_1}\right)^{\circ(n_1-1)}\circ\cdots\circ\frac{dt_d}{t_d-a_d}\circ\left(\frac{dt_d}{t_d}\right)^{\circ(n_d-1)}, \tag{38}$$

where $a_j = 1/\prod_{i=j}^d z_i$ for all $j=1,\dots,d$, we can obtain the analytic continuation of this function as a multi-valued function on $\mathbb{C}^d\setminus\mathfrak{D}_d$ where

$$\mathfrak{D}_d = \left\{(z_1,\dots,z_d)\in\mathbb{C}^d : \prod_{i=j}^d z_i = 1,\ j=1,\dots,d\right\}.$$

When $|z_j|<$ we can define its q-analog $(0<q<1)$ by

$$\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d) = \sum_{0<k_1<\cdots<d_d}\frac{z_1^{k_1}\dots z_d^{k_d}}{[k_1]^{n_1}\dots[k_d]^{n_d}}.$$

Clearly when $q \uparrow 1$ we recover the ordinary multiple polylogarithm. Moreover, the special value of multiple q-zeta function at positive integers is related to the multiple q-polylogarithm by

$$\zeta_q(n_1,\dots,n_d) = \mathrm{Li}_{q;n_1,\dots,n_d}\big(q^{n_1-1},\dots,q^{n_d-1}\big).$$

Note that our definition of the multiple q-polylogarithms is different from that of [12]. In case of logarithm and dilogarithm our definitions are different from that of [11]. We want to convince the readers that ours are also good analogs of the ordinary ones.

To begin with, we can mimic the method in Sect. 2 to get the analytic continuation of $\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d)$.

Theorem 7.1 *The multiple q-polylogarithm function $\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d)$ converges if $|z_j|<1$ for all $j=1,\dots,d$. It can be analytically continued to a multi-valued function over $\mathbb{C}^d \setminus \mathfrak{D}_{q;d}$ via the series expansion*

$$\frac{\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d)}{(1-q)^{n_1+\cdots+n_d}} = \sum_{r_1,\dots,r_d=0}^{+\infty}\prod_{j=1}^{d}\left[\binom{n_j+r_j-1}{r_j}\frac{z_j^j q^{jr_j}}{1-(z_j\cdots z_d)q^{r_j+\cdots+r_d}}\right]. \tag{39}$$

Proof The first part of the lemma is obvious. Let's concentrate on the analytic continuation. By binomial expansion $(1-x)^{-n}=\sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$ we get

$$\frac{\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d)}{(1-q)^{n_1+\cdots+n_d}} = \sum_{0<k_1<\cdots<k_d}\ \sum_{r_1,\dots,r_d=0}^{+\infty}\prod_{j=1}^{d}\binom{n_j+r_j-1}{r_j}(z_jq^{r_j})^{k_j}.$$

As $0<q<1$ the series converges absolutely by Stirling's formula so we can exchange the summations. The theorem follows immediately from Lemma 2.3 by taking $x_j=z_jq^{r_j}$. □

However, this analytic continuation is not suitable for comparing with its ordinary counterpart. Therefore we define Jackson's q-differential operator (cf. [7]) by

$$D_{q;z}f(z)=\frac{f(z)-f(qz)}{(1-q)z}.$$

Lemma 7.2 *Let $d,n_1,\dots,n_d$ be positive integers. If $n_j\ge 2$ then we have*

$$D_{q;z_j}\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d)=\frac{1}{z_j}\mathrm{Li}_{q;n_1,\dots,n_j-1,\dots,n_d}(z_1,\dots,z_d);$$

if $d\ge 2$ and $n_j=1$ then

$$\begin{aligned}D_{q;z_j}\mathrm{Li}_{q;n_1,\dots,n_d}(z_1,\dots,z_d) &= \frac{1}{1-z_j}\mathrm{Li}_{q;n_1,\dots,\hat{n}_j,\dots,n_d}(z_1,\dots,z_{j-1}z_j,\dots,z_d)\\ &\quad -\frac{1}{z_j(1-z_j)}\mathrm{Li}_{q;n_1,\dots,\hat{n}_j,\dots,n_d}(z_1,\dots,z_jz_{j+1},\dots,z_d).\end{aligned}$$

Here the second term does not appear if $j=d$. If $d=n_1=1$ then

$$D_{q;z}\mathrm{Li}_{q;1}(z)=\frac{1}{1-z}.$$

Proof Clear. □

The same properties listed in the lemma are satisfied by the ordinary multiple polylogarithms. We note that the first equation in [12, Lemma 1] is valid only for $n_j\geq 2$.

Recall that for any continuous function $f(x)$ on $[a,b]$ Jackson's q-integral (cf. [8]) is defined by

$$\int_a^b f(x)\,d_qx:=\sum_{i=0}^{\infty} f\big(a+q^i(b-a)\big)\big(q^i-q^{i+1}\big)(b-a).$$

Then for every $x>0$ we have

$$D_{q;x}\int_0^x f(t)\,d_qt=f(x),\qquad \int_0^x D_{q;t}f(t)\,d_qt=f(x)-f(0), \tag{40}$$

and

$$\lim_{q\uparrow 1}\int_0^x f(t)\,d_qt=\int_0^x f(t)\,dt. \tag{41}$$

Remark 7.3 Note that in general $\int_a^b f(x)\,d_qx+\int_b^c f(x)\,d_qx\neq\int_a^c f(x)\,d_qx$.

Similar to Chen's iterated integrals one can define the q-iterated integrals as follows:

$$\int_a^b \frac{d_qt_1}{t_1-a_1}\circ\cdots\circ\frac{d_qt_r}{t_r-a_r}$$
$$:=\int_a^b\left(\int_a^{t_r}\cdots\int_a^{t_3}\left(\int_a^{t_2}\frac{d_qt_1}{t_1-a_1}\right)\frac{d_qt_2}{t_2-a_2}\cdots\frac{d_qt_{r-1}}{t_{r-1}-a_{r-1}}\right)\frac{d_qt_r}{t_r-a_r}.$$

This was first introduced by Bowman and Bradley [3].

Set

$$\mathfrak{D}_{q;d}=:\left\{(z_1,\ldots,z_d)\in\mathbb{C}^d:\prod_{i=j}^{d}z_i=q^{-m},\ m\in\mathbb{Z}_{\geq 0},\ j=1,\ldots,d\right\}.$$

Then we have

Corollary 7.4 *We can analytically continue $\mathrm{Li}_{q;n_1,\ldots,n_d}(z_1,\ldots,z_d)$ to $\mathbb{C}^d\setminus\mathfrak{D}_{q;d}$ by the q-iterated integral*

$$(-1)^d\int_0^1\frac{d_qt_1}{t_1-a_1}\circ\left(\frac{d_qt_1}{t_1}\right)^{\circ(n_1-1)}\circ\cdots\circ\frac{d_qt_d}{t_d-a_d}\circ\left(\frac{d_qt_d}{t_d}\right)^{\circ(n_d-1)},$$

where $a_j = 1/\prod_{i=j}^d z_i$ *for all* $j = 1, \dots, d$. *Further, for all* $(z_1, \dots, z_d) \in \mathbb{C}^d$ *such that* $\prod_{i=j}^d z_i \notin [1, +\infty)$ *for* $1 \le j \le d$ *we have*

$$\lim_{q\uparrow 1} \mathrm{Li}_{q;n_1,\dots,n_d}(z_1, \dots, z_d) = \mathrm{Li}_{n_1,\dots,n_d}(z_1, \dots, z_d),$$

where $\mathrm{Li}_{n_1,\dots,n_d}(z_1, \dots, z_d)$ *is defined by* (38) *with the path being the straight line segment from* 0 *to* 1 *in* $\mathbb{C}^1$.

Proof It follows from Lemma 7.2 and (41). The set of singularities $\mathfrak{D}_{q;d}$ is determined by Theorem 7.1 so that for each $j = 1, \dots, d$ the function $1/(t - a_j)$ is continuous on [0, 1]. □

Remark 7.5 Corollary 7.4 is also discovered by Bradley [4] independently and his proof is different from ours. Please note that his notation of multiple q-zeta function is related to ours by $\zeta[s_d, \dots, s_1] = \zeta_q(s_1, \dots, s_d)$.

8 Iterated integral *q*-shuffle relations

In Sect. 5 we encountered some q-stuffle relations of multiple q-zeta functions. Classically, multiple zeta values satisfy another kind of relation coming from their representations by iterated integrals. In our setting we have seen that special values of multiple q-zeta functions can be also represented by q-iterated integrals. In this last section we would like to study the shuffle relations related to these q-iterated integrals. This was first introduced by Bowman and Bradley in [3, §7] and studied by Bradley in [4, §6] independently. We shall see that they're more involved than their ordinary counterparts. Let us start by writing

$$\mathrm{Shfl}(u_1 \circ \cdots \circ u_r, u_{r+1} \circ \cdots \circ u_{r+s}) = \sum_\sigma u_{\sigma(1)} \circ \cdots \circ u_{\sigma(r+s)},$$

where σ runs through all the permutations of $\{1, \dots, r+s\}$ such that $\sigma^{-1}(a) < \sigma^{-1}(b)$ whenever $1 \le a < b \le r$ or $r+1 \le a < b \le r+s$. For any expressions F_i we put

$$\bigsqcup_{i=1}^r F_i = F_1 \circ \cdots \circ F_r.$$

Lemma 8.1 *Let* $u_i = d_q t/(t - a_i)$ *and* $v_j = d_q t/(t - b_i)$ *where* $|a_i|, |b_j| \le 1$ *for all* $1 \le i \le r$ *and* $1 \le j \le s$. *Let a be any positive number. Then*

$$\int_0^a u_1 \circ \cdots \circ u_r \cdot \int_0^a v_1 \circ \cdots \circ v_s$$
$$= \int_0^a \mathrm{Shfl}(u_1 \circ \cdots \circ u_r, v_1 \circ \cdots \circ v_s) + \sum_{c=1}^{\min(r,s)} (q-1)^c$$

$$\times \sum_{\substack{1\le i_1<\cdots<i_c\le r\\ 1\le j_1<\cdots<j_c\le s}} \int_0^a \bigsqcup_{\alpha=1}^{c+1} \mathrm{Shfl}(u_{1+i_{\alpha-1}}\circ\cdots\circ u_{i_\alpha-1}, v_{1+j_{\alpha-1}}\circ\cdots\circ v_{j_\alpha-1})$$

$$\circ\langle u_{i_\alpha}, v_{j_\alpha}\rangle, \tag{42}$$

where $i_0=j_0=0$, $i_{c+1}=r+1$, $j_{c+1}=s+1$, $\langle u_{r+1}, v_{s+1}\rangle=1$, *and for all* i, j

$$\langle u_i, v_j\rangle = \frac{t d_q t}{(t-a_i)(t-b_j)} = \begin{cases} \dfrac{1}{b_j-a_i}\left(\dfrac{b_j d_q t}{t-b_j} - \dfrac{a_i d_q t}{t-a_i}\right) & \text{if } a_i\neq b_j,\\ \dfrac{d_q t}{t-b} + \dfrac{b d_q t}{(t-b)^2} & \text{if } a_i=b_j=b.\end{cases}$$

Proof This can be proved by induction on $r+s$. We may use (40) and the following key formula

$$D_{q;x}[f(x)g(x)] = [D_{q;x}f(x)]g(x) + f(x)[D_{q;x}g(x)] + x(q-1)[D_{q;x}f(x)][D_{q;x}g(x)].$$

□

We will say the term $\langle u_i, v_j\rangle$ is a *collapse* in the shuffle. The lemma roughly says that q-iterated shuffle relations is different from those produced by regular iterated integrals because collapses may occur. Each collapse produce a factor of $q-1$ and the number of collapses is at most $\min(r,s)$.

Lemma 8.1 implies that if $m, n\ge 2$ are different then

$$\zeta_q(m)\zeta_q(n) = \int_0^1 \frac{d_q t}{t-q^{1-m}}\circ\left(\frac{d_q t}{t}\right)^{\circ(m-1)}\cdot\int_0^1 \frac{d_q t}{t-q^{1-n}}\circ\left(\frac{d_q t}{t}\right)^{\circ(n-1)}$$
$$= A_q(m,n) + A_q(n,m) + B_q(m,n), \tag{43}$$

where

$$A_q(m,n) = \sum_{a=0}^{m-1}\sum_{c=0}^{\min(a,n)} E(a, n-1; c)$$
$$\times\int_0^1 \frac{d_q t}{t-q^{1-m}}\circ\left(\frac{d_q t}{t}\right)^{\circ(m-1-a)}\circ\frac{d_q t}{t-q^{1-n}}\circ\left(\frac{d_q t}{t}\right)^{\circ(n+a-1-c)},$$

and

$$B_q(m,n) = (q-1)\sum_{c=0}^{\min(m,n)-1} E(m-1, n-1; c)$$
$$\times\int_0^1 \frac{t d_q t}{(t-q^{1-m})(t-q^{1-n})}\circ\left(\frac{d_q t}{t}\right)^{\circ(m+n-2-c)}.$$

Here the coefficient $E(r,s;c)$ represents $(q-1)^c$ times the numbers of terms in shuffle of $u_1\circ\cdots\circ u_r$ and $v_1\circ\cdots\circ v_s$ with c collapses (see (42)). It is not hard to see that $E(r,s;0)=\binom{r+s}{r}$. Thus

$$E(r,s;c)=(q-1)^c\sum_{\substack{1\le i_1<\cdots<i_c\le r\\ 1\le j_1<\cdots<j_c\le s}}\prod_{\alpha=1}^{c+1}\binom{i_\alpha+j_\alpha-i_{\alpha-1}-j_{\alpha-1}-2}{i_\alpha-i_{\alpha-1}-1}. \tag{44}$$

Ideally one wants to convert the expressions in (43) into something that is close to combinations of multiple q-zeta functions, together with the shift operator $\mathcal{S}$. Since $m\neq n$ we get

$$\begin{aligned}
&\int_0^1 \frac{t d_q t}{(t-q^{1-m})(t-q^{1-n})}\circ\left(\frac{d_q t}{t}\right)^{\circ(m+n-2-c)}\\
&=\int_0^1\left(\frac{1}{1-q^{m-n}}\frac{d_q t}{t-q^{1-m}}+\frac{1}{1-q^{n-m}}\frac{d_q t}{t-q^{1-n}}\right)\circ\left(\frac{d_q t}{t}\right)^{\circ(m+n-2-c)}\\
&=\frac{1}{q^{m-n}-1}\sum_{k=1}^\infty\frac{q^{(m-1)k}}{[k]^{m+n-1-c}}+\frac{1}{q^{n-m}-1}\sum_{k=1}^\infty\frac{q^{(n-1)k}}{[k]^{m+n-1-c}}.
\end{aligned}$$

Proposition 8.2 *For any positive integers m,n we have*

$$\begin{aligned}
B_q(m,n)=(q-1)&\sum_{c=0}^{\min(m,n)-1}E(m-1,n-1;c)\\
&\times\left(\frac{1}{q^{m-n}-1}\mathcal{S}^{n-1-c}\zeta_q(m+n-1-c)\right.\\
&\left.+\frac{1}{q^{n-m}-1}\mathcal{S}^{m-1-c}\zeta_q(m+n-1-c)\right),
\end{aligned}$$

where E is defined by (44).

Proof It follows from Corollary 2.6. □

To handle $A_q(m,n)$ we need to evaluate

$$\mathrm{Li}_{q;m-\alpha,\beta}(q^{m-n},q^{n-1})=\int_0^1\frac{d_q t}{t-q^{1-m}}\circ\left(\frac{d_q t}{t}\right)^{\circ(m-\alpha-1)}\circ\frac{d_q t}{t-q^{1-n}}\circ\left(\frac{d_q t}{t}\right)^{\circ(\beta-1)},$$

where $0\le\alpha\le m-1$ and $\max(n,\alpha)\le\beta\le n+\alpha$. By Corollary 2.6 we get

$$\mathrm{Li}_{q;m-\alpha,\beta}(q^{m-n},q^{n-1})=\sum_{i=0}^{\beta-n}\binom{\beta-n}{i}(1-q)^i\mathrm{Li}_{q;m-\alpha,\beta-i}(q^{m-n},q^{\beta-i-1}). \tag{45}$$

So we need to evaluate

$$\mathrm{Li}_{q;m-\alpha,\gamma}(q^{m-n},q^{\gamma-1})=\sum_{1\le k<l}\frac{q^{(m-n)k}}{[k]^{m-\alpha}}\frac{q^{(\gamma-1)l}}{[l]^\gamma}.$$

If $n > \alpha$ then by Corollary 2.6 we get

$$\mathrm{Li}_{q;m-\alpha,\gamma}(q^{m-n}, q^{\gamma-1}) = \sum_{j=0}^{n-\alpha-1} \binom{n-\alpha-1}{j} (1-q)^j \zeta_q(m-\alpha-j, \gamma). \quad (46)$$

By taking $\alpha = a$, $\beta = n + a - c$ and $\gamma = \beta - i$ in (45) and (46) we get

Proposition 8.3 *If $n > m$ then*

$$A_q(m,n) = \sum_{a=0}^{m-1} \sum_{c=0}^{a} \sum_{i=0}^{a-c} \sum_{j=0}^{n-a-1} E(a, n-1; c) \binom{a-c}{i} \binom{n-a-1}{j} \times (1-q)^{i+j} \zeta_q(m-a-j, n+a-c-i),$$

where E is defined by (44).

The case $n < m$ can be treated similarly although it's more difficult. We get

Proposition 8.4 *If $n < m$ then*

$$\begin{aligned} A_q(m,n) = {} & \sum_{a=0}^{n-1} \sum_{c=0}^{a} \sum_{i=0}^{a-c} \sum_{j=0}^{n-a-1} E(a, n-1; c) \binom{a-c}{i} \binom{n-a-1}{j} \\ & \times (1-q)^{i+j} \zeta_q(m-a-j, n+a-c-i) \\ & + \sum_{a=n}^{m-1} \sum_{c=0}^{n} \sum_{i=0}^{a-c} E(a, n-1; c) \binom{a-c}{i} (1-q)^i X_{n+a-c-i}(m-a, m-n), \end{aligned}$$

where E is defined by (44) *and*

$$X_\gamma(r,s) = \sum_{i=0}^{r-1} (q-1)^i \binom{i+s-r}{s-r} \zeta_q(r-i, \gamma) + (q-1)^r \sum_{j=0}^{s-r} r^j \sum_{1 \le k < l} \frac{q^{(s-r-j)k + (\gamma-1)l}}{[l]^\gamma}.$$

Putting everything together we arrive at

Theorem 8.5 *Let $m \neq n$ be two positive integers no less than 2. Then*

$$\zeta_q(m)\zeta_q(n) = A_q(m,n) + A_q(n,m) + B_q(m,n),$$

where $A_q(m,n)$ is given by Proposition 8.3 and Proposition 8.4, and $B_q(m,n)$ is given by Proposition 8.2.

For example, we have

$$\begin{aligned}\zeta_q(2)\zeta_q(3) &= 6\zeta_q(1,4) + 3\zeta_q(2,3) + \zeta_q(3,2)\\ &\quad + (1-q)\Bigg(\sum_{i=0}^{2}\sum_{j=2}^{4-i} a(i,j,q)\zeta_q(i,j)\\ &\quad + \sum_{n=2}^{4} b(n,q)\zeta_q(n) + \sum_{n=2}^{4} c(n,q)\varphi_q(n)\Bigg),\end{aligned} \tag{47}$$

where $a(i, j, q)$, $b(n, q)$ and $c(n, q)$ are all polynomials of q with integer coefficients, and

$$\varphi_q(n) = \sum_{k=1}^{\infty}(k-1)\frac{q^{(n-1)k}}{[k]^n}.$$

When $q \uparrow 1$ we recover the relation

$$\zeta(2)\zeta_q(3) = 6\zeta(1,4) + 3\zeta(2,3) + \zeta(3,2).$$

D. Bradley obtains a more elegant formula by using difference operators. In our notation (see Remark 7.5) it reads

Theorem 8.6 ([5, *Theorem* 1]) *If* $s-1$ *and* $t-1$ *are positive integers, then*

$$\begin{aligned}\zeta_q(s)\zeta_q(t) &= \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+t-1}{t-1}\binom{t-1}{b}(1-q)^b\zeta_q(s-a-b,t+a)\\ &\quad + \sum_{a=0}^{t-1}\sum_{b=0}^{t-1-a}\binom{a+s-1}{s-1}\binom{t-1}{b}(1-q)^b\zeta_q(t-a-b,s+a)\\ &\quad - \sum_{j=1}^{\min(s,t)}\frac{(s+t-j-1)!}{(s-j)!(t-j)!}\cdot\frac{(1-q)^j}{(j-1)!}\varphi_q(s+t-j).\end{aligned}$$

When $(s, t) = (2, 3)$ the first two lines agrees with our (47). It is not hard to see that when $q \uparrow 1$ we can recover the ordinary shuffle relations of the multiple zeta values originally produced by iterated integrals from either Theorem 8.5 or Theorem 8.6:

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1}\binom{a+t-1}{t-1}\zeta(s-a,t+a) + \sum_{a=0}^{t-1}\binom{a+s-1}{s-1}\zeta(t-a,s+a).$$

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