

# An elementary approach to $6j$ -symbols (classical, quantum, rational, trigonometric, and elliptic)

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Dedicated to Richard Askey.

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**Abstract** Elliptic  $6j$ -symbols first appeared in connection with solvable models of statistical mechanics. They include many interesting limit cases, such as quantum  $6j$ -symbols (or  $q$ -Racah polynomials) and Wilson's biorthogonal  ${}_{10}W_9$  functions. We give an elementary construction of elliptic  $6j$ -symbols, which immediately implies several of their main properties. As a consequence, we obtain a new algebraic interpretation of elliptic  $6j$ -symbols in terms of Sklyanin algebra representations.

**Keywords** Elliptic  $6j$ -symbol · Elliptic hypergeometric series · Biorthogonal rational function · Sklyanin algebra · Eight-vertex model

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## 1 Introduction

The classical  $6j$ -symbols were introduced by Racah and Wigner in the early 1940's [34, 58]. Though they appeared in the context of quantum mechanics, they are natural objects in the representation theory of  $SL(2)$  that can be introduced from purely mathematical considerations. Wilson [59] realized that  $6j$ -symbols are orthogonal polynomials, and that they generalize many classical systems such as Krawtchouk and Jacobi polynomials. This led Askey and Wilson to introduce the more general  $q$ -Racah polynomials [4].

The  $q$ -Racah polynomials belong to the class of basic (or  $q$ -) hypergeometric series [15]. Since the 1980's, there has been a considerable increase of interest in this classical

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subject. One reason for this is relations to solvable models in statistical mechanics, and to the related algebraic structures known as quantum groups.

Kirillov and Reshetikhin [22] found that  $q$ -Racah polynomials appear as  $6j$ -symbols of the  $SL(2)$  quantum group, or *quantum  $6j$ -symbols*. We mention that in the introduction to the standard reference [8], three major applications of quantum groups to other fields of mathematics are highlighted. For at least two of these, namely, invariants of links and three-manifolds [56], and the relation to affine Lie algebras and conformal field theory [11], quantum  $6j$ -symbols play a decisive role.

The  $q$ -Racah polynomials form, together with the closely related Askey–Wilson polynomials, the top level of the Askey Scheme of ( $q$ -)hypergeometric orthogonal polynomials [23]. One reason for viewing this scheme as complete is Leonard’s theorem [31], saying that any finite system of orthogonal polynomials with polynomial duals is a special or degenerate case of the  $q$ -Racah polynomials. However, if one is willing to pass from orthogonal polynomials to biorthogonal rational functions, natural extensions of the Askey Scheme do exist.

One such extension was found by Wilson [60], who constructed a system of biorthogonal rational functions given by  ${}_{10}\phi_9$  (or, more precisely,  ${}_{10}W_9$ ) basic hypergeometric series. These form a generalization of  $q$ -Racah polynomials that seems very natural from the viewpoint of special functions; see also [39].

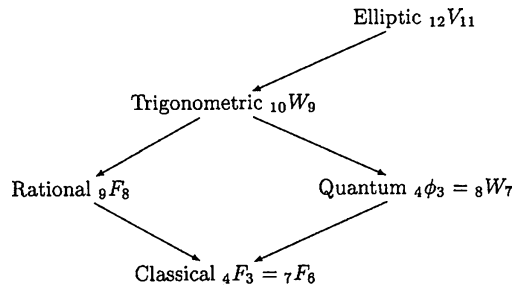
Another indication that natural generalizations of quantum  $6j$ -symbols exist came from statistical mechanics. The solvable models that lead to standard quantum groups appear there as degenerate cases. Typically, the most general case of the models involve elliptic functions. In the 1980’s Date et al. [9, 10] applied a fusion procedure (see [30]) to Baxter’s eight-vertex SOS model [3, 7], obtaining in this way a more general model whose Boltzmann weights generalize quantum and classical  $6j$ -symbols. These were called *elliptic  $6j$ -symbols*. However, no identification of these objects with biorthogonal rational functions was obtained, nor was their nature as generalized hypergeometric sums emphasized.

In the latter direction, Frenkel and Turaev [13] found that the *trigonometric* limit case of elliptic  $6j$ -symbols can be written as  ${}_{10}W_9$ -series. A further limit transition gives *rational  $6j$ -symbols*. Moreover, in [14] it was found that general elliptic  $6j$ -symbols may be expressed as elliptic, or modular, hypergeometric series, a completely new class of special functions. In spite of their intriguing properties, including close relations to elliptic functions and modular forms, such series were never considered in “classical” mathematics, but needed physics for their discovery. We refer to [15] for an introduction to the subject, with further references.

Frenkel and Turaev seem not to have been aware of the work of Wilson. Spiridonov and Zhedanov [49, 50] gave an independent approach to elliptic  $6j$ -symbols, showing in particular that they are biorthogonal rational functions, and that they coincide with Wilson’s functions in the trigonometric limit. More precisely, trigonometric  $6j$ -symbols correspond to certain discrete restrictions on the parameters of Wilson’s functions. Similarly, elliptic  $6j$ -symbols correspond to special parameter choices for Spiridonov’s and Zhedanov’s biorthogonal rational functions. We will be concerned with the larger parameter range, although, for simplicity, we will use the term “ $6j$ -symbol” also in that setting.

To summarize, we have a scheme (in the sense of Askey) consisting of classical, quantum, rational, trigonometric and elliptic  $6j$ -symbols, see Fig. 1. Arrows indicate

**Fig. 1** The hierarchy of  $6j$ -symbols



limit transitions. We also give the hypergeometric type of the systems. (We use Spiridonov’s [46] more logical notation  $_{12}V_{11}$ , see (5.1) below, rather than  $_{10}\omega_9$  as in [14], for the series underlying elliptic  $6j$ -symbols.)

Note that the discrete part of the Askey Scheme lies below the classical and quantum  $6j$ -symbols in Fig. 1. We remark that, once we decide to include biorthogonal rational functions, many further limit cases exist (including biorthogonal polynomials and orthogonal rational functions). It seems desirable to classify all limit cases, along with their continuous relatives. Many known systems (see [1, 2, 16–18, 29, 33, 35–38] for some candidates) should fit into this larger picture.

The aim of the present work is to give a self-contained and elementary approach to  $6j$ -symbols, which works for all five cases. We will show how to obtain many of their properties in an elementary fashion, without using quantum groups or techniques from statistical mechanics (although the approach is certainly related to both). In the exposition we will focus on trigonometric  $6j$ -symbols. We stress that this is not because of any essential difficulties with the elliptic case, but since we want to emphasize the elementary nature of our approach as much as possible.

Our main idea comes from the interpretation of Askey–Wilson and  $q$ -Racah polynomials given in [42]; see [51, 62] for related work. The standard definition of  $6j$ -symbols involves three-fold tensor products of representations. This works equally well in the classical and quantum case. In [42], we gave an interpretation of  $q$ -Racah polynomials involving a single irreducible representation of the  $SL(2)$  quantum group. On the level of polynomials, this means that  $q$ -Racah polynomials appear as  $q$ -analogues of Krawtchouk polynomials rather than of Racah polynomials. Realizing the representation using difference operators on a function space (sometimes called the coherent state method [21]), this yields a kind of generating function for  $q$ -Racah polynomials, see (2.16) below. We may now forget about the quantum group and use the generating function to recover the main properties of  $q$ -Racah polynomials. Our aim is to generalize this approach to include all  $6j$ -symbols in Fig. 1, keeping the underlying quantum group (known as the Sklyanin algebra in the most general case) implicit until the final Section 6.

The plan of the paper is as follows. Section 2 contains preliminaries; in particular we explain in some detail the degenerate cases corresponding to Krawtchouk polynomials and  $q$ -Racah polynomials (or quantum  $6j$ -symbols). In Sections 3 and 4 we generalize this to trigonometric  $6j$ -symbols, and in Section 5 we sketch the straight-forward extension to elliptic  $6j$ -symbols. Although the main point of the paper is to avoid using quantum groups, we give an algebraic interpretation of our construction in

Section 6. It turns out that elliptic  $6j$ -symbols appear as the transition matrix between the solutions of two different generalized eigenvalue problems in a finite-dimensional representation of the Sklyanin algebra.

*Note added in proof:* The bases discussed in Remark 5.2, which play a fundamental role in our approach, were introduced by Takebe [56] in connection with Baxter’s vertex-IRF transformation for the eight-vertex model. This can be compared with the appearance of vertex-IRF transformations as deformed group elements in the degenerate case treated in [42].

## 2 Preliminaries

### 2.1 Notation

We recall the standard notation for shifted factorials

$$(a)_k = a(a + 1) \cdots (a + k - 1),$$

$$(a_1, \dots, a_n)_k = (a_1)_k \cdots (a_n)_k,$$

for hypergeometric series

$${}_rF_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; x \right] = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r)_k x^k}{(b_1, \dots, b_s)_k k!},$$

for  $q$ -shifted factorials

$$(a; q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}),$$

$$(a_1, \dots, a_n; q)_k = (a_1; q)_k \cdots (a_n; q)_k, \tag{2.1}$$

for  $q$ -binomial coefficients

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(q; q)_N}{(q; q)_k (q; q)_{N-k}},$$

for basic hypergeometric series

$${}_r\phi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k$$

and for very-well-poised series

$$\begin{aligned}
 {}_{r+1}W_r(a; b_1, \dots, b_{r-2}; q, z) &= {}_{r+1}\phi_r \left[ \begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b_1, \dots, b_{r-2} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b_1, \dots, aq/b_{r-2} \end{matrix}; q, z \right] \\
 &= \sum_{k=0}^{\infty} \frac{1 - aq^{2k}}{1 - a} \frac{(a, b_1, \dots, b_{r-2}; q)_k}{(q, aq/b_1, \dots, aq/b_{r-2}; q)_k} z^k.
 \end{aligned}$$

If one of the numerator parameters equals  $q^{-n}$ , with  $n$  a non-negative integer, the series reduces to a finite sum. We are particularly interested in the terminating balanced  ${}_{10}W_9$ , that is, the case when  $r = 9$ , the sum is finite,  $z = q$  and  $a^3q^2 = b_1 \dots b_7$ . The standard reference for all this is [15].

To write our results in standard notation, some routine computation involving  $q$ -shifted factorials is necessary. We will not give the details, but we mention that all that one needs is the elementary identities

$$(a; q)_n = (-1)^n q^{\binom{n}{2}} a^n (q^{1-n}/a; q)_n, \tag{2.2a}$$

$$(a; q)_{n+k} = (a; q)_n (aq^n; q)_k, \tag{2.2b}$$

$$(a; q)_{n-k} = (-1)^k q^{\binom{k}{2}} (q^{1-n}/a)^k \frac{(a; q)_n}{(q^{1-n}/a; q)_k}. \tag{2.2c}$$

### 2.2 An extended example: Krawtchouk polynomials

Our guiding example will be Krawtchouk polynomials, arising as matrix elements of  $SL(2, \mathbb{C})$ . (Incidentally, they also appear as  $6j$ -symbols, namely, of the oscillator algebra [57, Section 8.6.6].) For later comparison, we recall some fundamental facts on this topic [27, 57].

Consider the coefficients  $K_k^l = K_k^l(a, b, c, d; N)$  in

$$(ax + b)^k (cx + d)^{N-k} = \sum_{l=0}^N K_k^l x^l, \tag{2.3}$$

where  $k \in \{0, 1, \dots, N\}$  and we assume, with no great loss of generality, that  $ad - bc = 1$ . Note that  $SL(2)$  acts on polynomials of degree  $\leq N$  by

$$p(x) \mapsto (cx + d)^N p\left(\frac{ax + b}{cx + d}\right), \tag{2.4}$$

and that  $K_k^l$  are the matrix elements of this group action in the standard basis of monomials.

Using the binomial theorem, several different expressions for  $K_k^l$  as hypergeometric sums may be derived. For instance,

$$\begin{aligned} (ax + b)^k (cx + d)^{N-k} &= \left( \frac{1}{d}x + \frac{b}{d}(cx + d) \right)^k (cx + d)^{N-k} \\ &= \sum_{j=0}^k \binom{k}{j} b^{k-j} d^{-k} x^j (cx + d)^{N-j} \\ &= \sum_{j=0}^k \sum_{m=0}^{N-j} \binom{k}{j} \binom{N-j}{m} b^{k-j} c^m d^{N-k-j-m} x^{m+j}. \end{aligned}$$

Thus, writing  $m = l - j$ , we obtain

$$\begin{aligned} K_k^l &= \sum_{j=0}^{\min(k,l)} \binom{k}{j} \binom{N-j}{l-j} b^{k-j} c^{l-j} d^{N-k-l} \\ &= \binom{N}{l} b^k c^l d^{N-k-l} {}_2F_1 \left[ \begin{matrix} -k, -l \\ -N \end{matrix}; -\frac{1}{bc} \right], \end{aligned} \quad (2.5)$$

in standard hypergeometric notation.

Note that the expansion problem inverse to (2.3),

$$x^k = \sum_{l=0}^N \tilde{K}_k^l (ax + b)^l (cx + d)^{N-l} \quad (2.6)$$

is equivalent to the original problem (replace the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  by its inverse). Thus,  $\tilde{K}_k^l$  is given by a similar formula, namely,

$$\tilde{K}_k^l = \binom{N}{l} (-1)^{k+l} a^{N-k-l} b^k c^l {}_2F_1 \left[ \begin{matrix} -k, -l \\ -N \end{matrix}; -\frac{1}{bc} \right].$$

Combining (2.3) and (2.6), we obtain the orthogonality relation

$$\begin{aligned} \delta_{km} &= \sum_{l=0}^N K_k^l \tilde{K}_l^m \\ &= \sum_{l=0}^N \binom{N}{l} \binom{N}{m} (-1)^{l+m} a^{N-l-m} b^{k+l} c^{l+m} d^{N-k-l} \\ &\quad \times {}_2F_1 \left[ \begin{matrix} -k, -l \\ -N \end{matrix}; -\frac{1}{bc} \right] {}_2F_1 \left[ \begin{matrix} -m, -l \\ -N \end{matrix}; -\frac{1}{bc} \right]. \end{aligned} \quad (2.7)$$

Now let us introduce the standard notation

$$K_n(x; p, N) = {}_2F_1 \left[ \begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p} \right].$$

This is a polynomial in  $x$  of degree  $n$ , known as the Krawtchouk polynomial [61]. Writing  $bc = -p, ad = 1 - p$  and  $t = cx/d$ , (2.3) takes the form

$$\left( 1 + \frac{p-1}{p} t \right)^k (1+t)^{N-k} = \sum_{l=0}^N \binom{N}{l} K_l(k; p, N) t^l, \tag{2.8}$$

which is a well-known generating function for Krawtchouk polynomials. Our approach to the  $6j$ -symbols in Fig. 1 will be based on generalizing this identity.

In terms of Krawtchouk polynomials, (2.7) takes the form

$$\sum_{x=0}^N \binom{N}{x} p^x (1-p)^{N-x} K_k(x; p, N) K_m(x; p, N) = \delta_{km} \frac{(1-p)^k}{p^k \binom{N}{k}}.$$

For  $0 < p < 1$ , this is an orthogonality relation for a positive measure, namely, the binomial distribution on a finite arithmetic progression. That we get a genuine orthogonality stems from the fact that the underlying representation is unitarizable for the group  $SU(2)$ .

Several other interesting properties of Krawtchouk polynomials are immediately obtained from (2.3). For instance, one may consider three bases  $e_k, f_k, g_k$ , each being of the form  $(ax + b)^k (cx + d)^{N-k}$ , with different  $a, b, c, d$ . The transition coefficients in

$$e_k = \sum_l K_{kl} g_l = \sum_l K'_{kl} f_l, \quad f_k = \sum_l K''_{kl} g_l$$

are then all given by Krawtchouk polynomials, with different parameter  $p$ . Clearly, they are related by matrix multiplication:

$$K_{nm} = \sum_{k=0}^N K'_{nk} K''_{km}. \tag{2.9}$$

In the case  $e_k = g_k$ , one gets back the orthogonality (2.7).

From the viewpoint of group theory, (2.9) corresponds to representing the group law in an  $(N + 1)$ -dimensional representation. This should be quite familiar when  $N = 1$  and we restrict to  $SO(2)$ , the rotations of the plane, obtaining in this way the addition formulas for sine and cosine. Thus, (2.9) appears as a natural extension of these addition formulas.

From the hypergeometric viewpoint, (2.9) is an instance of Meixner’s formula [32]

$$\sum_{k=0}^{\infty} \frac{(c)_k}{k!} {}_2F_1 \left[ \begin{matrix} -k, a \\ c \end{matrix}; x \right] {}_2F_1 \left[ \begin{matrix} -k, b \\ c \end{matrix}; y \right] z^k = \frac{(1-z)^{a+b-c}}{(1-z+xz)^a(1-z+xz)^b} {}_2F_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; \frac{xyz}{(1-z+xz)(1-z+yz)} \right].$$

More precisely, it is the special case when  $a = -n, b = -m, c = -N$ , with  $m, n, N$  integers such that  $0 \leq m, n \leq N$ .

Another consequence of (2.3) is obtained by exploiting the multiplicative structure of the basis vectors. Namely, expanding both sides of

$$(ax + b)^{k+j}(cx + d)^{M+N-k-j} = (ax + b)^k(cx + d)^{M-k}(ax + b)^j(cx + d)^{N-j} \tag{2.10}$$

into monomials gives

$$\sum_l K_{k+j}^l x^l = \sum_m K_k^m x^m \sum_n K_j^n x^n,$$

or

$$K_{k+j}^l(a, b, c, d; M + N) = \sum_{m+n=l} K_k^m(a, b, c, d; M)K_j^n(a, b, c, d; N). \tag{2.11}$$

In hypergeometric notation, this is

$$\begin{aligned} & \binom{M+N}{l} {}_2F_1 \left[ \begin{matrix} -l, -k-j \\ -M-N \end{matrix}; t \right] \\ &= \sum_{\substack{m+n=l \\ 0 \leq m \leq M \\ 0 \leq n \leq N}} \binom{M}{m} \binom{N}{n} {}_2F_1 \left[ \begin{matrix} -m, -k \\ -M \end{matrix}; t \right] {}_2F_1 \left[ \begin{matrix} -n, -j \\ -N \end{matrix}; t \right]. \end{aligned} \tag{2.12}$$

The group-theoretic interpretation of (2.11) is the following. Let  $V_N$  denote the  $(N + 1)$ -dimensional irreducible representation of  $SL(2)$ , realized on the space of polynomials as above. Then multiplication of polynomials defines a map  $V_M \otimes V_N \rightarrow V_{M+N}$ . The relation (2.10), and thus (2.11), expresses the fact that this map is intertwining, that is, commutes with the group action. This immediately suggests a non-trivial generalization. Namely, one has the equivalence of representations  $V_M \otimes V_N \simeq \bigoplus_{s=0}^{\min(M,N)} V_{M+N-2s}$ , and one may do the same thing for the intertwiners  $V_M \otimes V_N \rightarrow V_{M+N-2s}$ . The corresponding generalization of (2.12) has additional factors of type  ${}_3F_2$  appearing on both sides. From the group-theoretic viewpoint, these are Clebsch–Gordan coefficients and, from the viewpoint of special functions, Hahn polynomials, see [57, Section 8.5.3].



Next we point out that (2.11) may be iterated to

$$\begin{aligned}
 &K_{k_1+\dots+k_n}^l(a, b, c, d; M_1 + \dots + M_n) \\
 &= \sum_{m_1+\dots+m_n=l} K_{k_1}^{m_1}(a, b, c, d; M_1) \cdots K_{k_n}^{m_n}(a, b, c, d; M_n), \tag{2.13}
 \end{aligned}$$

where  $0 \leq k_i, m_i \leq M_i$ . This is especially interesting when  $M_i = 1$  for all  $i$ . Writing the result in hypergeometric form, we get in that case

$$\binom{n}{l} {}_2F_1 \left[ \begin{matrix} -l, -k_1 - \dots - k_n \\ -n \end{matrix}; t \right] = \sum_{\substack{m_1+\dots+m_n=l \\ 0 \leq m_i \leq 1}} \prod_{i=1}^n {}_2F_1 \left[ \begin{matrix} -m_i, -k_i \\ -1 \end{matrix}; t \right].$$

Note that the range of summation may be identified with the  $l$ -element subsets  $L$  of  $N = \{1, \dots, n\}$  (interpreting  $m_i = 1$  as  $i \in L$ ). Similarly,  $(k_1, \dots, k_n)$  labels a subset  $K$  of  $N$  with  $\sum_i k_i$  elements. Since

$${}_2F_1 \left[ \begin{matrix} -m_i, -k_i \\ -1 \end{matrix}; t \right] = \begin{cases} 1-t, & m_i = k_i = 1, \\ 1, & m_i = 0 \text{ or } k_i = 0, \end{cases}$$

the term in the sum is  $(1-t)^{|L \cap K|}$ . Replacing  $t$  with  $1-t$ , we obtain

$$\binom{|N|}{l} {}_2F_1 \left[ \begin{matrix} -l, -|K| \\ -|N| \end{matrix}; 1-t \right] = \sum_{L \subseteq N, |L|=l} t^{|L \cap K|}, \quad K \subseteq N. \tag{2.14}$$

This (not very deep) identity gives a combinatorial interpretation for Krawtchouk polynomials as a generating function for the statistics  $|L \cap K|$  on subsets  $L$  of fixed cardinality. We shall see that the appearance of  $6j$ -symbols in statistical mechanics is via a generalization of this identity.

### 2.3 $q$ -Racah polynomials

In [42], we considered a  $q$ -analogue of the above set-up, leading to general  $q$ -Racah polynomials. The group  $SL(2)$  was replaced by a quantum group, and the basis vectors  $x^k$  and  $(ax + b)^k(cx + d)^{N-k}$  by appropriate  $q$ -shifted products such as

$$\prod_{j=0}^k (axq^j + b) \prod_{j=0}^{N-k} (cxq^j + d).$$

Such bases were interpreted as eigenvectors of Koornwinder’s twisted primitive elements [28], and also as the image of standard basis vectors  $x^k$  under Babelon’s vertex-IRF transformations [6] (called generalized group elements in [42]). Actually, we focused on the case of infinite-dimensional representations, and only mentioned the case of present interest somewhat parenthetically [42, Section 6].

To be more precise, the expansion problem that yields quantum  $6j$ -symbols ( $q$ -Racah polynomials) is

$$\prod_{j=0}^k (axq^{-j} + b) \prod_{j=0}^{N-k} (cxq^{-j} + d) = \sum_{l=0}^N C_k^l \prod_{j=0}^l (\alpha xq^j + \beta) \prod_{j=0}^{N-l} (\gamma xq^j + \delta). \tag{2.15}$$

For generic parameter values, the polynomials on the right form a basis for the space of polynomials of degree  $\leq N$ , so that the coefficients exist uniquely. For the rest of this section we assume that we are in such a generic situation.

Note that when  $q = 1$ , (2.15) reduces to

$$(ax + b)^k (cx + d)^{N-k} = \sum_{l=0}^N C_k^l (\gamma x + \beta)^l (\gamma x + \delta)^{N-l},$$

which is further reduced to (2.3) by a change of variables. The expansion (2.15) is more rigid. After multiplying with a trivial factor and changing parameters, we may restrict to the case

$$(ax; q^{-1})_k (bx; q^{-1})_{N-k} = \sum_{l=0}^N C_k^l(a, b, c, d; N; q) (cx; q)_l (dx; q)_{N-l}. \tag{2.16}$$

We could dilate  $x$  to get rid of one more parameter, but the remaining 7 parameters, counting  $q$ , enter in a non-trivial fashion. Indeed, we have

$$\begin{aligned} C_k^l(a, b, c, d; N; q) &= q^{l(l-N)} \begin{bmatrix} N \\ l \end{bmatrix}_q \frac{(q^{1-N}b/d; q)_l (q^{1-N}b/c; q)_{N-l} (q^{1-k}a/c; q)_k}{(q^{1-N}c/d; q)_l (q^{-l}d/c; q)_{N-l} (q^{1-N}b/c; q)_k} \\ &\quad \times {}_4\phi_3 \left[ \begin{matrix} q^{-k}, q^{-l}, q^{k-N}b/a, q^{l-N}c/d \\ q^{-N}, c/a, q^{1-N}b/d \end{matrix}; q, q \right]. \end{aligned} \tag{2.17}$$

(In Section 3 we will derive a more general identity in an elementary way.)

Similarly as for (2.3), we may invert (2.16) to get the orthogonality relation

$$\delta_{km} = \sum_{l=0}^N C_k^l(a, b, c, d; N; q) C_l^m(c, d, a, b; N; q^{-1}). \tag{2.18}$$

One may verify that (2.18) gives the orthogonality of  $q$ -Racah polynomials. (If we want a positive measure, some conditions on the parameters must be imposed.)

Note that (2.16) generalizes the generating function (2.8) to the level of  $q$ -Racah polynomials. This identity was obtained, in a related but not identical context, by Koelink and Van der Jeugt [26, Remark 4.11(iii)].

The mixture of base  $q$  and  $q^{-1}$  in (2.16) is crucial. Admittedly, the expansion problem

$$(ax; q)_k (bx; q)_{N-k} = \sum_{l=0}^N D_k^l(a, b, c, d; N; q) (cx; q)_l (dx; q)_{N-l} \tag{2.19}$$

is immediately reduced to (2.16) by a change of variables; explicitly, one has

$$D_k^l(a, b, c, d; N; q) = C_k^l(aq^{k-1}, bq^{N-k-1}, c, d; N; q).$$

However, the relation

$$\delta_{km} = \sum_{l=0}^k D_k^l(a, b, c, d; N; q) D_l^m(c, d, a, b; N; q)$$

is not equivalent to (2.18). It gives a system of biorthogonal rational functions. When  $c\bar{d} = a\bar{b} \in \mathbb{R}$  it is an orthogonal system, found in an equivalent context by Koelink [24, Proposition 9.5]; see also [16, Corollary 4.4] and Remark 4.2 below.

*Remark 2.1.* We conclude the introductory part of the paper with some comments on the relation to Terwilliger’s concept of a Leonard pair; see [55] and references given there. As was mentioned above, if we let  $e_k = (ax; q^{-1})_k (bx; q^{-1})_{N-k}$ ,  $f_k = (cx; q)_k (dx; q)_{N-k}$ , then  $e_k$  and  $f_k$  appear as eigenbases of certain  $q$ -difference operators  $Y_1, Y_2$ , respectively. It is easy to check that each of these operators acts tridiagonally on the eigenbasis of the other, that is,

$$Y_1 f_k \in \text{span}\{f_{k-1}, f_k, f_{k+1}\}, \quad Y_2 e_k \in \text{span}\{e_{k-1}, e_k, e_{k+1}\}.$$

Except for a non-degeneracy condition, this is the definition of  $(Y_1, Y_2)$  being a Leonard pair. Then (2.17) means that  $(Y_1, Y_2)$  is a Leonard pair of “ $q$ -Racah type”, which is the most general kind. This gives a simple model for studying Leonard pairs. For instance, the “split decompositions” [54] are easily understood in this model. A typical split basis between  $e_k$  and  $f_k$  would be  $g_k = (ax; q^{-1})_k (dx; q)_{N-k}$ , which interpolates between the two other bases in the sense that

$$g_k \in \text{span}\{e_k, e_{k+1}, \dots, e_N\} \cap \text{span}\{f_0, f_1, \dots, f_k\}.$$

More generally, we may picture the factors

$$(ax; q^{-1})_k, \quad (bx; q^{-1})_k, \quad (cx; q)_{N-k}, \quad (dx; q)_{N-k}$$

as being attached to the corners of a tetrahedron, with two opposite edges corresponding to the original Leonard pair and the remaining four edges to different split decompositions.

### 3 Trigonometric $6j$ -symbols

It is not hard to check that both  $q$ -Racah polynomials and Koelink’s orthogonal functions are degenerate cases of Wilson’s biorthogonal functions. Thus, if one wants to obtain general trigonometric  $6j$ -symbols in a similar way, it seems necessary to unify the products  $(ax; q)_k$  and  $(ax; q^{-1})_k$ . The correct unification turns out to be

the Askey–Wilson monomials  $h_k(x; a) = h_k(x; a; q)$ , which are the natural building blocks of Askey–Wilson polynomials [5]. They are given by

$$h_k(x; a) = \prod_{j=0}^{k-1} (1 - axq^j + a^2q^{2j}).$$

We will assume that  $q \neq 0$ , but we allow  $q$  to be a root of unity, which case is of special interest in statistical mechanics. We will need the elementary identities

$$h_k(x; a) = q^{k(k-1)} a^{2k} h_k(x; q^{1-k}/a), \tag{3.1}$$

$$h_{k+l}(x; a) = h_k(x; a) h_l(x; aq^k). \tag{3.2}$$

To write  $h_k$  in the notation (2.1) one must introduce an auxiliary variable  $\xi$  satisfying

$$\xi + \xi^{-1} = x; \tag{3.3}$$

then

$$h_k(x; a) = (a\xi, a\xi^{-1}; q)_k. \tag{3.4}$$

(In the context of Askey–Wilson polynomials one usually dilates  $x$  by a factor 2 and writes  $x/2 = \cos \theta, \xi = e^{i\theta}$ .)

It is easy to see that

$$\lim_{t \rightarrow 0} h_k(x/t; at) = (ax; q)_k, \quad \lim_{t \rightarrow 0} t^{2k} h_k(x/t; a/t) = q^{k(k-1)} a^{2k} (x/a; q^{-1})_k. \tag{3.5}$$

Thus, we may unify (2.16) and (2.19), together with several related expansion problems (see Remark 4.2 below), into

$$h_k(x; a) h_{N-k}(x; b) = \sum_{l=0}^N R_k^l(a, b, c, d; N; q) h_l(x; c) h_{N-l}(x; d). \tag{3.6}$$

We will suppress parameters when convenient, writing

$$R_k^l = R_k^l(a, b, c, d; N) = R_k^l(a, b, c, d; N; q).$$

Note that, in contrast to the limit cases considered above, we cannot get rid of any parameters by scaling  $x$ . We shall see that  $R_k^l$  depends on all 8 parameters (counting  $q$ ) in a non-trivial fashion.

Clearly, the coefficients  $R_k^l$  exist uniquely if and only if  $(h_k(x; c) h_{N-k}(x; d))_{k=0}^N$  form a basis for the space of polynomials of degree  $\leq N$ . Although it is not quite necessary for our purposes, we will first settle this question.

**Lemma 3.1.** *The polynomials  $(h_k(x; c)h_{N-k}(x; d))_{k=0}^N$  form a basis for the space of polynomials of degree at most  $N$  if and only if none of the following conditions are satisfied:*

$$c/d \in \{q^{1-N}, q^{2-N}, \dots, q^{N-1}\}, \tag{3.7a}$$

$$cd \in \{q^{1-N}, q^{2-N}, \dots, 1\}, \tag{3.7b}$$

$$c = d = 0. \tag{3.7c}$$

**Proof:** If  $c/d = q^j$  with  $1 - N \leq j \leq 0$ , then all the polynomials have the common zero  $x = d + d^{-1}$ , so they cannot form a basis. Similarly, if  $c/d = q^j$  with  $0 \leq j \leq N - 1$  then  $x = c + c^{-1}$  is a common zero, and if (3.7b) holds then both  $x = c + c^{-1}$  and  $x = d + d^{-1}$  are common zeroes. In the case (3.7c), all the polynomials equal 1 and clearly do not form a basis.

Conversely, assume that none of the conditions (3.7) hold. We need to show that any linear relation

$$\sum_{k=0}^N \lambda_k h_k(x; c)h_{N-k}(x; d) \equiv 0 \tag{3.8}$$

is trivial. By symmetry, we may assume  $c \neq 0$ . Choosing  $x = c + c^{-1}$  in (3.8) gives

$$\lambda_0(dc, d/c; q)_N = 0.$$

Since  $(dc, d/c; q)_N = 0$  only if (3.7a) or (3.7b) holds, we have  $\lambda_0 = 0$ . We may then divide (3.8) with  $1 - cx + c^2$ , giving

$$\sum_{k=1}^N \lambda_k h_{k-1}(x; cq)h_{N-k}(x; d) \equiv 0.$$

By iteration (choosing  $x = cq + (cq)^{-1}$  in the next step) or by induction on  $N$ , we conclude that  $\lambda_i = 0$  for all  $i$ , and thus that the polynomials form a basis. □

We now turn to the problem of computing the coefficients  $R_k^l$ . Recall that our derivation of (2.5) consisted in applying the binomial theorem twice. The same proof should be applicable to (3.6), once we have a generalized binomial theorem of the form

$$h_N(x; a) = \sum_{k=0}^N C_k^N(a, b, c) h_k(x; b)h_{N-k}(x; c). \tag{3.9}$$

In fact, (3.9) is solved by one of the most fundamental results on basic hypergeometric series: Jackson's  ${}_8W_7$  summation [15, 20]. Since we have promised to give a self-contained treatment, we give a straight-forward proof, motivated by our present view of (3.9) as an extension of the binomial theorem.

We will follow the standard inductive proof of the binomial theorem based on Pascal’s triangle. First we write

$$h_{N+1}(x; a) = h_N(x; a)(1 - aq^N x + a^2 q^{2N}).$$

To get a recurrence for  $C_k^N$ , we split the factor  $1 - axq^N + a^2 q^{2N}$  into parts that attach to the right-hand side of (3.9), namely, as

$$1 - aq^N x + a^2 q^{2N} = A_k(1 - bq^k x + b^2 q^{2k}) + B_k(1 - cq^{N-k} x + c^2 q^{2(N-k)}). \tag{3.10}$$

We compute

$$A_k = \frac{(1 - acq^{2N-k})(1 - aq^k/c)}{(1 - bcq^N)(1 - bq^{2k-N}/c)},$$

$$B_k = \frac{(1 - abq^{N+k})(1 - aq^{N-k}/b)}{(1 - bcq^N)(1 - cq^{N-2k}/b)}, \tag{3.11}$$

assuming that the denominators are non-zero. For the elliptic extension discussed in Section 5 it is important to note that this uses the elementary identity

$$\begin{aligned} &\frac{v}{x} (1 - xy)(1 - x/y)(1 - uv)(1 - u/v) \\ &= (1 - ux)(1 - u/x)(1 - vy)(1 - v/y) \\ &\quad - (1 - uy)(1 - u/y)(1 - vx)(1 - v/x), \end{aligned} \tag{3.12}$$

with

$$(u, v, x, y) \mapsto (cq^{N-k}, bq^k, aq^N, \xi).$$

Combining (3.9) and (3.10) yields the generalized Pascal triangle

$$C_k^{N+1} = B_k C_k^N + A_{k-1} C_{k-1}^N, \tag{3.13}$$

with boundary conditions

$$C_0^0 = 1, \quad C_{-1}^N = C_{N+1}^N = 0.$$

Iterating (3.13), one quickly guesses that

$$C_k^N = q^{k(k-N)} \left[ \begin{matrix} N \\ k \end{matrix} \right]_q \frac{(a/c, q^{N-k} ac; q)_k (a/b, q^k ab; q)_{N-k}}{(q^{k-N} b/c; q)_k (q^{-k} c/b; q)_{N-k} (bc; q)_N}. \tag{3.14}$$

To verify the guess, we plug (3.14) into (3.13). After cancelling common factors, we are left with

$$\begin{aligned} & q^{k-N-1}(1-q^N)(1-q^N ab)(1-q^N ac)(1-q^{N+1-2k}c/b) \\ &= (1-q^k)(1-q^{k-1}ab)(1-q^{2N-k+1}ac)(1-q^{-k}c/b) \\ & \quad - (1-q^{k-N-1})(1-q^{N+k}ab)(1-q^{N-k}ac)(1-q^{N+1-k}c/b), \end{aligned}$$

which is another instance of (3.12), this time with

$$(u, v, x, y) \mapsto (q^{N+\frac{1}{2}}\sqrt{ac}, q^{-\frac{1}{2}}\sqrt{ac}, q^{N-k+\frac{1}{2}}\sqrt{ac}, q^{k-\frac{1}{2}}b\sqrt{a/c}).$$

This shows that, for generic parameters, (3.9) holds with the coefficients given by (3.14). As one expects from Lemma 3.1,  $C_k^N$  has poles precisely if

$$b/c \in \{q^{1-N}, q^{2-N}, \dots, q^{N-1}\}, \quad bc \in \{1, q^{-1}, \dots, q^{1-N}\} \quad \text{or} \quad b = c = 0.$$

Other singularities, such as when  $q$  is a root of unity, are removable.

*Remark 3.2.* Plugging (3.14) into (3.9) and rewriting the result in standard notation gives

$${}_8W_7(q^{-N}b/c; q^{-N}, q^{1-N}/ac, a/c, b\xi, b\xi^{-1}; q, q) = \frac{(cb, c/b, a\xi, a\xi^{-1}; q)_N}{(ab, a/b, c\xi, c\xi^{-1}; q)_N}.$$

This is Jackson’s summation. Essentially the same method was used in [43] to obtain extensions of Jackson’s summation to multiple elliptic hypergeometric series related to the root systems  $A_n$  and  $D_n$ .

We may now compute the coefficients  $R_k^l$  in (3.6) by applying the “binomial theorem” (3.9) twice. For instance, using (3.2) we may write

$$\begin{aligned} h_k(x; a)h_{N-k}(x; b) &= \sum_{j=0}^k C_j^k(a, c, bq^{N-k})h_j(x; c)h_{N-j}(x; b) \\ &= \sum_{j=0}^k \sum_{m=0}^{N-j} C_j^k(a, c, bq^{N-k})C_m^{N-j}(b, cq^j, d)h_{j+m}(x; c)h_{N-j-m}(x; d). \end{aligned}$$

This gives

$$R_k^l = \sum_{j=0}^{\min(k,l)} C_j^k(a, c, bq^{N-k})C_{l-j}^{N-j}(b, cq^j, d).$$

Plugging in the expressions from (3.14) and rewriting the result in standard form one finds that the sum is a balanced  ${}_{10}W_9$  series.

**Theorem 3.3.** For generic values of the parameters, the coefficients  $R_k^l$  in (3.6) exist uniquely and are given by

$$\begin{aligned}
 &R_k^l(a, b, c, d; N; q) \\
 &= q^{l(l-N)} \begin{bmatrix} N \\ l \end{bmatrix}_q \frac{(ac, a/c; q)_k (q^{N-l}bd, b/d; q)_l (b/c; q)_{N-k} (b/c; q)_{N-l} (bc; q)_{N-k}}{(q^{l-N}c/d; q)_l (q^{-l}d/c; q)_{N-l} (cd; q)_N (b/c; q)_N (bc; q)_l} \\
 &\quad \times {}_{10}W_9(q^{-N}c/b; q^{-k}, q^{-l}, q^{k-N}a/b, q^{l-N}c/d, cd, q^{1-N}/ab, qc/b; q, q).
 \end{aligned}$$

*Remark 3.4.* The limit case  $d = 0$  of Theorem 3.3 was recently obtained by Ismail and Stanton [19, Theorem 3.1] using different methods.

*Remark 3.5.* From the definition, it is clear that  $R_k^l$  has the symmetries

$$R_k^l(a, b, c, d; N) = R_{N-k}^l(b, a, c, d; N) = R_k^{N-l}(a, b, d, c; N), \tag{3.15a}$$

and from (3.1) we have moreover that

$$R_k^l(a, b, c, d; N) = q^{-k(k-1)} a^{-2k} R_k^l(q^{1-k}/a, b, c, d; N). \tag{3.15b}$$

Combining these symmetries with Theorem 3.3 gives further expressions for  $R_k^l$  as  ${}_{10}W_9$  sums. These are related via Bailey’s classical  ${}_{10}W_9$  transformations [15]. On the other hand, the explicit expression in Theorem 3.3 implies many symmetries for  $R_k^l$  that are not obvious from the definition.

## 4 Elementary properties

### 4.1 Biorthogonality

It is clear from (3.6) that the coefficients  $R_k^l$  satisfy

$$\delta_{nm} = \sum_{k=0}^N R_n^k(a, b, c, d; N; q) R_k^m(c, d, a, b; N; q). \tag{4.1}$$

We will now show that (4.1) gives a system of biorthogonal rational functions, which is identical to the one obtained by Wilson [60].



To facilitate comparison with Wilson’s result, we rewrite (4.1) in terms of the functions

$$\begin{aligned}
 R_n(\mu(k)) &= \frac{q^{k(N-k)} (q^{-N}; q)_n (q^{k-N} c/d, bc; q)_k (q^{-k} d/c, bd; q)_{N-k} (cd; q)_N}{(cd)^n \begin{bmatrix} N \\ k \end{bmatrix}_q (b/d; q)_k (bc, bd; q)_{N-n} (b/c; q)_{N-k}} \\
 &\quad \times R_n^k(a, b, c, d; N; q) \\
 &= \frac{(q^{-N}, ac, q^{1-N}/bd, a/c; q)_n}{(q^{1-N} c/b; q)_n} \\
 &\quad \times {}_{10}W_9(q^{-N} c/b; q^{-n}, q^{n-N} a/b, q^{-k}, q^{k-N} c/d, cd, q^{1-N}/ab, cq/b; q, q)
 \end{aligned}$$

and

$$\begin{aligned}
 S_m(\mu(k)) &= \frac{q^{m(N-m)} (q^{-N}, ac, ad, q^{m-N} a/b; q)_m (q^{-m} b/a; q)_{N-m} (ab; q)_N}{(ab)^m \begin{bmatrix} N \\ m \end{bmatrix}_q (ac, c/a; q)_k (ad, d/a; q)_{N-k}} \\
 &\quad \times R_k^m(c, d, a, b; N; q) \\
 &= \frac{(q^{-N}, ac, q^{1-N}/bd, d/b; q)_m}{(q^{1-N} a/d; q)_m} \\
 &\quad \times {}_{10}W_9(q^{-N} a/d; q^{-m}, q^{m-N} a/b, q^{-k}, q^{k-N} c/d, ab, q^{1-N}/cd, aq/d; q, q),
 \end{aligned}$$

where

$$\mu(k) = q^{-k} + q^{k-N} c/d.$$

Note that  $R_n$  has the form

$$\begin{aligned}
 R_n(\mu(k)) &= \sum_{j=0}^n \sigma_j \frac{(q^{-k}, q^{k-N} c/d; q)_j}{(q^{1-N+k} c/b, q^{1-k} d/b; q)_j} \\
 &= \sum_{j=0}^n \sigma_j \prod_{t=0}^{j-1} \frac{1 - q^t \mu(k) + q^{2t-N} c/d}{1 - q^{t+1} \mu(k) d/b + q^{2t+2-N} cd/b^2},
 \end{aligned}$$

with  $\sigma_j$  independent of  $k$ , and is thus a rational function in  $\mu(k)$  of degree  $n/n$ . Similarly,

$$S_m(\mu(k)) = \sum_{j=0}^m \tau_j \prod_{t=0}^{j-1} \frac{1 - q^t \mu(k) + q^{2t-N} c/d}{1 - q^{t+1} \mu(k) a/c + q^{2t+2-N} a^2/cd}.$$

In terms of these functions, (4.1) takes the form

$$\sum_{k=0}^N w_k R_n(\mu(k)) S_m(\mu(k)) = C_n \delta_{nm}, \tag{4.2}$$

where

$$w_k = \frac{1 - q^{2k-N}c/d}{1 - q^{-N}c/d} \frac{(q^{-N}c/d, q^{-N}, ac, q^{1-N}/bd, b/d, c/a; q)_k}{(q, qc/d, q^{1-N}/ad, bc, q^{1-N}c/b, q^{1-N}a/d; q)_k} q^k$$

and

$$C_n = \frac{(ba, b/a, dc, d/c; q)_N}{(bc, b/c, da, d/a; q)_N} \times \frac{1 - q^{-N}a/b}{1 - q^{2n-N}a/b} \frac{(q, q^{-N}, ac, ad, q^{1-N}/bc, q^{1-N}/bd, aq/b; q)_n}{(q^{-N}a/b; q)_n} q^{-n}.$$

Thus, we have indeed a system of biorthogonal rational functions.

We now compare this result with the work of Wilson [60], who used the notation

$$\begin{aligned} r_n \left( \frac{z + z^{-1}}{2}; a, b, c, d, e, f; q \right) &= \frac{(ab, ac, ad, 1/af; q)_n}{(aq/e; q)_n} {}_{10}W_9(a/e; az, a/z, q/be, q/ce, q/de, q^n/ef, q^{-n}; q, q), \end{aligned} \tag{4.3}$$

where

$$abcdef = q.$$

The normalization is chosen so as to make  $r_n$  symmetric in  $a, b, c, d$ . Assuming  $ab = q^{-N}$  with  $N$  a non-negative integer, Wilson obtained the biorthogonality relation

$$\begin{aligned} \sum_{k=0}^N w_k r_n \left( \frac{aq^k + a^{-1}q^{-k}}{2}; a, b, c, d, e, f; q \right) &\times r_m \left( \frac{aq^k + a^{-1}q^{-k}}{2}; a, b, c, d, f, e; q \right) = C_n \delta_{nm}, \end{aligned} \tag{4.4}$$

where

$$w_k = \frac{1 - a^2q^{2k}}{1 - a^2} \frac{(a^2, ab, ac, ad, ae, af; q)_k}{(q, aq/b, aq/c, aq/d, aq/e, aq/f; q)_k} q^k$$

and

$$C_n = \frac{(a^2q, q/cd, q/ce, q/de; q)_N (q, q^n/ef, ab, ac, ad, bc, bd, cd; q)_n}{(aq/c, aq/d, aq/e, bf; q)_N (q/ef; q)_{2n}} q^{-n}$$

(in [60], the factor  $q^{-n}$  and the exponent 2 in  $a^2q$  are missing in the expression for  $C_n$ ). Note that the case  $m = n = 0$  of (4.4) is the Jackson sum.

It is now easy to check that (4.2) and (4.4) are equivalent. The explicit correspondence of parameters is

$$(a, b, c, d, e, f) \mapsto (q^{-\frac{N}{2}}\sqrt{c/d}, q^{-\frac{N}{2}}\sqrt{d/c}, q^{\frac{N}{2}}a\sqrt{cd}, q^{1-\frac{N}{2}}/b\sqrt{cd}, q^{\frac{N}{2}}b/\sqrt{cd}, q^{\frac{N}{2}}\sqrt{cd}/a)$$

(which is consistent with the relations  $ab = q^{-N}$ ,  $abcdef = q$ ) or, conversely,

$$(a, b, c, d) \mapsto (\sqrt{c/f}, q/d\sqrt{cf}, a\sqrt{cf}, b\sqrt{cf}). \tag{4.5}$$

*Remark 4.1.* Continuous biorthogonality measures for the function  $r_n$  (not assuming  $ab = q^{-N}$ ) were obtained by Rahman [36, 38], see [48] for the elliptic case.

*Remark 4.2.* Note that, in view of the limit relations (3.5), any one of the sixteen expansion problems

$$(ax; q^\pm)_k (bx; q^\pm)_{N-k} = \sum_{l=0}^N C_k^l (cx; q^\pm)_l (dx; q^\pm)_{N-l},$$

with all possible choices of  $\pm$ , may be obtained as a degenerate case of (3.6). It is easy to see from Theorem 3.3 that the coefficients  $C_k^l$  are always given by  ${}_4\phi_3$  or (equivalently, in view of Watson’s transformation [15])  ${}_8W_7$  sums. Gupta and Masson [16] worked out all such degenerate cases of Wilson’s biorthogonal rational functions, finding five different systems. The system in [16, Corollary 4.2] is related to the expansion

$$(ax; q)_k (bx; q^{-1})_{N-k} = \sum_{l=0}^N C_k^l (cx; q)_l (dx; q^{-1})_{N-l},$$

the system in [16, Corollary 4.3] to

$$(ax; q)_k (bx; q^{-1})_{N-k} = \sum_{l=0}^N C_k^l (cx; q)_l (dx; q)_{N-l},$$

the system in [16, Corollary 4.4] is essentially Koelink’s functions [24], related to

$$(ax; q)_k (bx; q)_{N-k} = \sum_{l=0}^N C_k^l (cx; q)_l (dx; q)_{N-l},$$

the system in [16, Corollary 4.5] to

$$(ax; q)_k (bx; q)_{N-k} = \sum_{l=0}^N C_k^l (cx; q)_l (dx; q^{-1})_{N-l},$$

and the system in [16, Corollary 4.6] is the  $q$ -Racah polynomials, related to

$$(ax; q^{-1})_k (bx; q^{-1})_{N-k} = \sum_{l=0}^N C_k^l (cx; q)_l (dx; q)_{N-l}.$$

(For some of the systems Gupta and Masson gave a more general version, with infinite discrete biorthogonality measure.) All other cases may be reduced to one of those five.

### 4.2 Addition formula

By iterating (3.6), one immediately generalizes the biorthogonality relation (4.1) to

$$R_n^m(a, b, e, f; N; q) = \sum_{k=0}^N R_n^k(a, b, c, d; N; q) R_k^m(c, d, e, f; N; q). \tag{4.6}$$

This is an extension of the addition formula (2.9). We do not believe that the general case of (4.6) can be found in the literature, although it can probably be obtained by analytic continuation from the Yang–Baxter equation for trigonometric  $6j$ -symbols [9, 13]. Though in the present approach it seems almost trivial, in a more direct approach, such as defining  $R_k^m$  through the explicit expression in Theorem 3.3, it might not be easy to guess the existence of such an identity.

It may be of interest to rewrite (4.6) in Wilson’s notation (4.3). We introduce  $s = e/a, t = b/f$  as new parameters, and then make the change of variables (4.5). The calculations are essentially the same as those in Section 4.1, and we are content with stating the end result.

**Corollary 4.3.** *For  $abcdef = q, ab = q^{-N}$  and  $s$  and  $t$  arbitrary, Wilson’s functions (4.3) satisfy the addition formula*

$$\begin{aligned} & \sum_{k=0}^N w_k r_n \left( \frac{aq^k + a^{-1}q^{-k}}{2}; a, b, c, d, e, f; q \right) \\ & \quad \times r_m \left( \frac{aq^k + a^{-1}q^{-k}}{2}; a, b, cs, dt, f/s, e/t; q \right) \\ & = X R_n^m(\sqrt{c/f}, q/d\sqrt{cf}, s\sqrt{c/f}, q/t d\sqrt{cf}; q, N) \\ & = Y r_n \left( \frac{Aq^m + A^{-1}q^{-m}}{2}; A, B, C, D, E, F; q \right), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
 w_k &= \frac{1 - a^2 q^{2k}}{1 - a^2} \frac{(a^2, ab, acs, ad, ae, af/s; q)_k}{(q, aq/b, aq/cs, aq/d, aq/e, aqs/f; q)_k} q^k, \\
 X &= \frac{(a^2 q, q/cdst, qt/ces, q/de; q)_N}{(aq/cs, aq/d, aq/e, bf/s; q)_N} (ab, ad, bd; q)_n \\
 &\quad \times \frac{(q, q^m st/ef, acs, bcs, cdst; q)_m}{(qst/ef; q)_{2m}} q^{-n} t^{2m-N} q^{m^2-n^2} (ce^2 f)^{n-m}, \\
 Y &= \frac{(a^2 q, q/de, q/cds, q/ces; q)_N}{(aq/cs, aq/d, aq/e, bf/s; q)_N} \frac{(ad, bd; q)_n}{(q/ces, dt/e; q)_n} \\
 &\quad \times \frac{(ab, t, acs, bcs, dt/e; q)_m}{(qs/df, qs/ef; q)_m},
 \end{aligned}$$

$$(A, B, C, D, E, F) = (\sqrt{st/ef}, ab\sqrt{ef/st}, c\sqrt{es/ft}, d\sqrt{ft/es}, \sqrt{eft/s}, \sqrt{efs/t}).$$

The intermediate expression in (4.7) makes it clear that the special case  $s = t = 1$  gives back (4.4), since then  $R_n^m = \delta_{nm}$ . The presence of square roots is due to Wilson’s choice of parametrization. Writing the identity explicitly in terms of  ${}_{10}W_9$ -series, all square roots combine or cancel.

### 4.3 Convolution formulas

Next we extend (2.11) to the present setting, by exploiting the multiplicative property (3.2) of our basis elements. Because of the shifts appearing in that identity there are several different convolution formulas, which we write compactly as follows.

**Corollary 4.4.** *The coefficients  $R_k^m$  satisfy the convolution formulas*

$$\begin{aligned}
 R_{k+j}^l(a, b, c, d; M + N; q) &= \sum_{m+n=l} R_k^m(aq^{\alpha j}, bq^{\beta(N-j)}, c, d; M; q) \\
 &\quad \times R_j^n(aq^{(1-\alpha)k}, bq^{(1-\beta)(M-k)}, cq^m, dq^{M-m}; N; q) \quad (4.8)
 \end{aligned}$$

for all  $\alpha, \beta \in \{0, 1\}$ , where  $0 \leq k, m \leq M, 0 \leq j, n \leq N$ .

**Proof:** Since, for generic parameters,  $R_k^l$  is determined by (3.6), it suffices to compute

$$\sum_{l=0}^{M+N} C_l h_l(x; c) h_{M+N-l}(x; d),$$

where  $C_l$  is the right-hand side of (4.8). Inside the summation sign, we split the factors as

$$h_{m+n}(x; c)h_{M+N-m-n}(x; d) = h_m(x; c)h_{M-m}(x; d)h_n(x; cq^m)h_{N-n}(x; dq^{M-m}).$$

Performing the summation, using (3.6), gives

$$h_k(x; aq^{\alpha j})h_{M-k}(x; bq^{\beta(N-j)})h_j(x; aq^{(1-\alpha)k})h_{N-j}(x; bq^{(1-\beta)(M-k)}).$$

For any  $\alpha, \beta \in \{0, 1\}$ , these factors combine to

$$h_{k+j}(x; a)h_{M+N-k-j}(x; b),$$

which completes the proof. □

#### 4.4 Combinatorial formulas

To get analogues of (2.13) and (2.14), we first consider all possible extensions of (3.2) to a general sum  $h_{k_1+\dots+k_n}(x; a)$ . These are naturally labelled by permutations  $\sigma$  of  $\{1, \dots, n\}$ :

$$h_{k_1+\dots+k_n}(x; a) = h_{k_{\sigma(1)}}(x; a)h_{k_{\sigma(2)}}(x; aq^{k_{\sigma(1)}}) \dots h_{k_{\sigma(n)}}(x; aq^{k_{\sigma(1)}+\dots+k_{\sigma(n-1)}}).$$

Replacing  $\sigma$  by  $\sigma^{-1}$ , this may be written

$$h_{k_1+\dots+k_n}(x; a) = \prod_{i=1}^n h_{k_i}(x; aq^{|k|_i^\sigma}),$$

where we introduced the notation

$$|k|_i^\sigma = \sum_{\{j; \sigma(j) < \sigma(i)\}} k_j$$

for a multi-index  $k$ . Note that

$$|k|_i^{\text{id}} = k_1 + k_2 + \dots + k_{i-1}. \tag{4.9}$$

Thus, we have an extension of (4.8) labelled by two permutations  $\sigma, \tau$ :

$$\begin{aligned} &R_{k_1+\dots+k_n}^l(a, b, c, d; M_1 + \dots + M_n) \\ &= \sum_{m_1+\dots+m_n=l} \prod_{i=1}^n R_{k_i}^{m_i}(aq^{|k|_i^\sigma}, bq^{|M-k|_i^\tau}, cq^{|m|_i^{\text{id}}}, dq^{|M-m|_i^{\text{id}}}; M_i), \end{aligned} \tag{4.10}$$

where  $0 \leq k_i, m_i \leq M_i$ . (We could replace both occurrences of  $\text{id}$  in (4.10) by an arbitrary permutation, but the resulting identity is immediately reduced to (4.10) by

permuting the  $m_i$ .) In particular, when  $M_1 = \dots = M_n = 1$ , one has

$$\begin{aligned}
 &R_{k_1+\dots+k_n}^l(a, b, c, d; n) \\
 &= \sum_{\substack{m_1+\dots+m_n=l \\ 0 \leq m_i \leq 1}} \prod_{i=1}^n R_{k_i}^{m_i}(aq^{|k|_i^\sigma}, bq^{|1-k|_i^\tau}, cq^{|m|_i^{\text{id}}}, dq^{|1-m|_i^{\text{id}}}; 1). \tag{4.11}
 \end{aligned}$$

Note that on the right-hand side of (4.11), only the elementary coefficients

$$R_k^m = R_k^m(a, b, c, d; 1)$$

given by

$$\begin{pmatrix} R_0^0 & R_0^1 \\ R_1^0 & R_1^1 \end{pmatrix} = \begin{pmatrix} \frac{(1-bc)(1-b/c)}{(1-dc)(1-d/c)} & \frac{(1-bd)(1-b/d)}{(1-cd)(1-c/d)} \\ \frac{(1-ac)(1-a/c)}{(1-dc)(1-d/c)} & \frac{(1-ad)(1-a/d)}{(1-cd)(1-c/d)} \end{pmatrix}$$

appear. We shall see in Section 5.2 that the Eq. (4.11) is closely related to the fusion of  $R$ -matrices developed in [9, 10]. This explains the relation between our construction and the statistical mechanics approach.

The combinatorics of the sum (4.11) deserves a separate study, but we will make some further comments here. Note that, in (4.11), a large number of right-hand sides give the same left-hand side. If we only strive for a combinatorial understanding of the coefficients  $R_k^l$ , it may be enough to choose the right-hand side in a particularly simple fashion. For instance, we may take  $\sigma = \tau = \text{id}$ , and choose  $k_i$  as

$$(k_1, \dots, k_n) = (\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{n-k}). \tag{4.12}$$

It seems natural to identify the summation-indices  $m$  with lattice paths starting at  $(0, 0)$  and going right at step  $i$  if  $m_i = 1$  and up if  $m_i = 0$ , thus ending at  $(l, n - l)$ . Suppose that the  $i$ :th step in the path starts at  $(x, y)$ . Then, by (4.9),

$$|m|_i^{\text{id}} = x, \quad |1 - m|_i^{\text{id}} = y.$$

Moreover, if the  $k_i$  are chosen as in (4.12), then

$$|k|_i^{\text{id}} = \begin{cases} i - 1 = x + y, & 1 \leq i \leq k, \\ k, & k + 1 \leq i \leq n, \end{cases}$$

$$|1 - k|_i^{\text{id}} = \begin{cases} 0, & 1 \leq i \leq k, \\ i - 1 - k = x + y - k, & k + 1 \leq i \leq n. \end{cases}$$

Thus, for instance, any one of the first  $k$  steps in the path that goes right contributes a factor

$$R_1^l(aq^{x+y}, b, cq^x, dq^y; 1) = \frac{(1 - q^{x+2y}ad)(1 - q^xa/d)}{(1 - q^{x+y}cd)(1 - q^{x-y}c/d)}$$

to the sum. There are three other types of steps, giving rise to similar factors. After replacing  $n$  by  $N$ , this yields the following result.

**Corollary 4.5.** *The coefficient  $R_k^l(a, b, c, d; N; q)$  is given by the combinatorial formula*

$$\sum_{\text{paths}} \prod_{\text{early right}} \frac{(1 - q^{x+2y}ad)(1 - q^xa/d)}{(1 - q^{x+y}cd)(1 - q^{x-y}c/d)} \prod_{\text{early up}} \frac{(1 - q^{2x+y}ac)(1 - q^ya/c)}{(1 - q^{x+y}cd)(1 - q^{y-x}d/c)} \\ \times \prod_{\text{late right}} \frac{(1 - q^{x+2y-k}bd)(1 - q^{x-k}b/d)}{(1 - q^{x+y}cd)(1 - q^{x-y}c/d)} \prod_{\text{late up}} \frac{(1 - q^{2x+y-k}bc)(1 - q^{y-k}b/c)}{(1 - q^{x+y}cd)(1 - q^{y-x}d/c)},$$

where the sum is over all up-right lattice paths from  $(0, 0)$  to  $(l, N - l)$ , the products are over steps in these paths, the first  $k$  steps being called “early” and the remaining  $N - k$  steps being called “late”. In each factor,  $(x, y)$  denotes the starting point of the corresponding step.

There are many limit cases when Corollary 4.5 takes a simpler form. It might be interesting to investigate the limit cases corresponding to various polynomials in the Askey Scheme. As an example, let us consider the limit

$$L = \lim_{s \rightarrow 0} \lim_{c \rightarrow 0} (q^k d / bs)^{N-l} R_k^l(as, bs, c, d; N; q).$$

It is easy to see from Theorem 3.3 that

$$L = \begin{bmatrix} N \\ l \end{bmatrix}_q (q^{k-l} a/b)^k {}_3\phi_1 \left[ \begin{matrix} q^{-k}, q^{-l}, q^{k-N} a/b \\ q^{-N} \end{matrix}; q, \frac{q^l b}{a} \right],$$

which, by [15, Exercise 1.15], equals

$$\begin{bmatrix} N \\ l \end{bmatrix}_q (q^k a/b)^k {}_3\phi_2 \left[ \begin{matrix} q^{-k}, q^{-l}, q^{-k} b/a \\ q^{-N}, 0 \end{matrix}; q, q \right]. \tag{4.13}$$

(As an alternative, one may first use the symmetries (3.15) to write

$$R_k^l(a, b, c, d; N) = q^{-2\binom{k}{2}} a^{-2k} q^{-2\binom{N-k}{2}} b^{-2(N-k)} R_k^l(q^{1-k}/a, q^{1+k-N}/b, c, d),$$

and then take the termwise limit in the accordingly transformed version of Theorem 3.3, thereby obtaining (4.13) directly.) The quantity (4.13) may be identified with a



$q$ -Krawtchouk or dual  $q$ -Krawtchouk polynomial [23]. On the other hand, starting from Corollary 4.5 gives the combinatorial expression

$$L = \sum_{\text{paths}} \prod_{\text{early right}} 1 \prod_{\text{early up}} \frac{aq^{k+x}}{b} \prod_{\text{late right}} 1 \prod_{\text{late up}} q^x = \sum_{\text{paths}} \prod_{\text{up}} q^x \prod_{\text{early up}} \frac{aq^k}{b}.$$

Note that  $\prod_{\text{up}} q^x = q^{||\lambda||}$ , where  $||\lambda||$  is the number of boxes in the Young diagram to the upper left of the path. Writing  $t = aq^k/b$ , we conclude that

$$\begin{bmatrix} N \\ l \end{bmatrix}_q t^k {}_3\phi_2 \left[ \begin{matrix} q^{-k}, q^{-l}, 1/t \\ q^{-N}, 0 \end{matrix}; q, q \right] = \sum_{\text{paths}} q^{||\lambda||} t^{y(k)},$$

where  $y(k)$  is the number of early ups, that is, the  $y$ -coordinate of the end-point of the  $k$ :th step. This is a simple  $q$ -analogue of (2.14). Like (2.14), it is not very deep, but it gives an idea about what kind of information is contained in (4.11). Note also that when  $k = 0$  or  $t = 1$  we recover the well-known fact

$$\begin{bmatrix} N \\ l \end{bmatrix}_q = \sum_{\text{paths}} q^{||\lambda||}.$$

### 5 Elliptic $6j$ -symbols

#### 5.1 Definition and elementary properties

In this section we discuss the extension of our approach to *elliptic*  $6j$ -symbols, or, more precisely, to their continuation in the parameters studied in [49]. Roughly speaking, this corresponds to replacing everywhere “ $1 - x$ ” with the theta function

$$\theta(x; p) = \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x), \quad |p| < 1.$$

Since  $\theta(x; 0) = 1 - x$ , the case  $p = 0$  will give back Wilson’s functions discussed above. The main difference is that in the elliptic case there is no Askey-type scheme of degenerate cases; these limits require  $p = 0$  to make sense.

We recall the notation [15]

$$\begin{aligned} (a; q, p)_k &= \prod_{j=0}^{k-1} \theta(aq^j; p), \\ \theta(x_1, \dots, x_n; p) &= \theta(x_1; p) \cdots \theta(x_n; p), \\ (a_1, \dots, a_n; q, p)_k &= (a_1; q, p)_k \cdots (a_n; q, p)_k, \end{aligned}$$

where  $q \neq 0$ . Elliptic  $6j$ -symbols may be expressed in terms of the sum [15]

$$\begin{aligned}
 & {}_{12}V_{11}(a; b, c, d, e, f, g, q^{-n}; q, p) \\
 &= \sum_{k=0}^n \frac{\theta(aq^{2k})}{\theta(a)} \frac{(a, b, c, d, e, f, g, q^{-n}; q, p)_k}{(q, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq^{n+1}; q, p)_k} q^k, \quad (5.1)
 \end{aligned}$$

subject to the balancing condition  $a^3q^{n+2} = bcdefg$ . We mention that this function is invariant under a natural action of  $SL(2, \mathbb{Z})$  on  $(q, p)$ -space, cf. [14, 47].

Since

$$\theta(1/x; p) = -\theta(x; p)/x, \quad (5.2)$$

the symbols  $(a; q, p)_n$  satisfy elementary identities similar to (2.2). Moreover, (3.12) has the elliptic analogue (Riemann’s addition formula)

$$\frac{v}{x} \theta(xy, x/y, uv, u/v; p) = \theta(ux, u/x, vy, v/y; p) - \theta(uy, u/y, vx, v/x; p). \quad (5.3)$$

As an extension of (3.4) we introduce the function

$$h_k(x; a) = h_k(x; a; q, p) = (a\xi, a\xi^{-1}; q, p)_k, \quad \xi + \xi^{-1} = x.$$

For  $a \neq 0$ , this is an entire function of  $x$ . (If  $a = 0$ , it does not make sense unless  $p = 0$ .) We may then introduce the coefficients  $R_k^l = R_k^l(a, b, c, d; N; q, p)$  by

$$h_k(x; a)h_{N-k}(x; b) = \sum_{l=0}^N R_k^l(a, b, c, d; N; q, p) h_l(x; c)h_{N-l}(x; d). \quad (5.4)$$

Since the computation leading to Theorem 3.3 only used results that have *verbatim* elliptic extensions, it immediately carries over to the elliptic case.

**Theorem 5.1.** *For generic values of the parameters, the coefficients  $R_k^l$  in (5.4) exist uniquely and are given by*

$$\begin{aligned}
 R_k^l(a, b, c, d; N; q, p) &= q^{l(l-N)} \frac{(q; q, p)_N}{(q; q, p)_l(q; q, p)_{N-l}} \\
 &\times \frac{(ac, a/c; q, p)_k (q^{N-l}bd, b/d; q, p)_l (b/c; q, p)_{N-k} (b/c; q, p)_{N-l} (bc; q, p)_{N-k}}{(q^{l-N}c/d; q, p)_l (q^{-l}d/c; q, p)_{N-l} (cd; q, p)_N (b/c; q, p)_N (bc; q, p)_l} \\
 &\times {}_{12}V_{11}(q^{-N}c/b; q^{-k}, q^{-l}, q^{k-N}a/b, q^{l-N}c/d, cd, q^{1-N}/ab, qc/b; q, p).
 \end{aligned}$$

*Remark 5.2.* Like for Theorem 3.3, the existence and uniqueness falls out of the computation, but can also be explained directly. Let  $f$  be any function of the form

$$f(\xi) = \prod_{j=1}^N \theta(a_j \xi, a_j \xi^{-1}; p), \tag{5.5}$$

and let  $F$  be the function

$$F(x) = f(e^{2\pi i x}) = \prod_{j=1}^N \theta(a_j e^{2\pi i x}, a_j e^{-2\pi i x}; p).$$

Then  $F$  is an entire function satisfying

$$F(x + 1) = F(x), \quad F(x + \tau) = e^{-2\pi i N(2x + \tau)} F(x), \quad F(-x) = F(x),$$

where  $p = e^{2\pi i \tau}$ . In classical terminology,  $F$  is an even theta function of order  $2N$  and zero characteristics. It is known that the space  $V_N$  of such functions has dimension  $N + 1$ . We denote by  $W_N$  the space of corresponding functions  $f$ , that is, of holomorphic functions on  $\mathbb{C} \setminus \{0\}$  such that

$$f(\xi) = f(\xi^{-1}), \quad f(p\xi) = (1/p\xi^2)^N f(\xi).$$

(In [40], these are called  $BC_1$  theta functions of degree  $N$ .) Now we observe that, for  $ab \neq 0$ , the functions

$$f_k(\xi) = h_k(x; a)h_{N-k}(x; b), \quad x = \xi + \xi^{-1},$$

are of the form (5.5). With essentially the same proof as for Lemma 3.1, one may check that for

$$p^m a/b \notin \{q^{1-N}, q^{2-N}, \dots, q^{N-1}\}, \quad p^m ab \notin \{1, q^{-1}, \dots, q^{1-N}\}, \quad m \in \mathbb{Z},$$

$(f_k)_{k=0}^N$  form a basis for  $W_N$ . We may then interpret Theorem 5.1 as giving the matrix for a change between two such bases. Note that we need not impose any condition on  $q$  apart from  $q \neq 0$ ; the singularities apparent when  $q$  is a root of unity are removable.

The following Corollary will be used below (at the end of Section 5.2 and in the proof of Proposition 6.2).

**Corollary 5.3.** *If  $m$  and  $n$  are non-negative integers, then*

$$h_k(x; a)h_{N-k}(x; b) \in \text{span}_{k-m \leq l \leq k+n} \{h_l(x; aq^m)h_{N-l}(x; bq^n)\}.$$

**Proof:** We need to consider the coefficient  $R_k^l(a, b, aq^m, bq^n; N; q, p)$ . Isolating the factors containing the quotients  $b/d$  and  $a/c$  in Theorem 5.1 gives

$$R_k^l(a, b, c, d; N; q, p) = \sum_{j=0}^{\min(k,l)} \lambda_j(a/c; q, p)_{k-j}(b/d; q, p)_{l-j},$$

where  $\lambda_j$  collects all other factors. If  $a/c = q^{-m}$  this vanishes unless  $k - j \leq m$ , and if  $b/d = q^{-n}$  unless  $l - j \leq n$ . Thus, the range of summation is restricted to  $\max(k - m, l - n) \leq j \leq \min(k, l)$ , which is indeed empty unless  $k - m \leq l \leq k + n$ .  $\square$

It is clear that the coefficients  $R_k^l$  enjoy similar properties as were obtained above in the special case  $p = 0$ . This applies to the biorthogonality relation (4.1) (allowing us to recover the biorthogonal rational functions of Spiridonov and Zhedanov [49]), the addition formula (4.6), the convolution formulas in Corollary 4.4 and the combinatorial formulas (4.11). In particular, the identity in Corollary 4.5 holds after replacing all factors  $1 - x$  with the theta function  $\theta(x; p)$ .

*Remark 5.4.* Rains [40] has obtained multivariable (Koornwinder–Macdonald-type) extensions of Spiridonov’s and Zhedanov’s biorthogonal rational functions. The approach is different from ours, although there are similarities. Note, for instance, that the one-variable case of the *interpolation functions* in [40, Definition 5] are essentially of the form  $h_k(x; a) / h_k(x; b)$ .

### 5.2 Comparison with statistical mechanics

In this section we compare the coefficients  $R_k^l$  with the elliptic  $6j$ -symbols as defined in [10]. We shall see that the latter correspond to certain discrete restrictions on the parameters of  $R_k^l$ .

We will follow the notation of [9], where elliptic  $6j$ -symbols are denoted

$$W_{MN}(a, b, c, d | u).$$

They depend on four external parameters  $p, \lambda, \xi, K$  and are defined for integers  $a, b, c, d$  such that

$$a - b, c - d \in \{-M, 2 - M, \dots, M\}, \tag{5.6}$$

$$a - d, b - c \in \{-N, 2 - N, \dots, N\}. \tag{5.7}$$

As was observed in [14], these symbols may be expressed in terms of the elliptic hypergeometric series  ${}_{12}V_{11}$ . Using Theorem 5.1, we may then relate them to the

coefficients  $R_k^l$ . For instance, we have

$$\begin{aligned}
 &W_{MN}(j + 2l - N, i + 2k - N, i, j|u) \\
 &= q^{(\xi+j+k+l-N)(l-k)+\frac{1}{2}N(N-u)+\frac{1}{2}(N-2k)(i-j)} \frac{(q^{u+M+1-N}; q, p)_N}{(q; q, p)_N} \\
 &\quad \times R_k^l(a, b, c, d; N; q, p),
 \end{aligned} \tag{5.8}$$

where  $q = e^{\pi i \lambda / K}$  and

$$\begin{aligned}
 &(a, b, c, d) \\
 &= (q^{\frac{1}{2}(u+\xi+i+1-N)}, q^{\frac{1}{2}(u-\xi-i+1-N)}, q^{\frac{1}{2}(u+\xi+j+M+1-N)}, q^{\frac{1}{2}(u-\xi-j+M+1-N)}),
 \end{aligned} \tag{5.9}$$

or, equivalently,

$$(q^M, q^{\xi+i}, q^{\xi+j}, q^u) = (cd/ab, a/b, c/d, abq^{N-1})$$

( $\xi$  is a parameter from [9] that has nothing to do with (3.3)). Note that (5.7) corresponds to the condition  $0 \leq k, l \leq N$  on  $R_k^l$ , while (5.6) gives a further discrete restriction on the parameters.

In view of the large symmetry group of the terminating  ${}_{12}V_{11}$ , there are many different ways to identify elliptic  $6j$ -symbols with the coefficients  $R_k^l$ . We have chosen the representation (5.8) since it explains the relation between fusion of  $R$ -matrices and the combinatorial formulas of Section 4.4. Namely, it is straight-forward to check that if we let  $\sigma = \tau = \text{id}$  in (4.11), specialize the parameters as in (5.9) and replace  $i$  by  $n + 1 - i$  in the product, then (4.11) reduces to [9, Eq. (2.1.21)].

It is interesting to consider the degeneration of the expansion problem (5.4) corresponding to the restriction (5.6). For this we introduce the parameter  $m = (M + j - i)/2$ . The condition on  $c - d$  in (5.6) means that  $m$  is an integer with  $0 \leq m \leq M$ . The coefficients (5.8) appear in the expansion problem

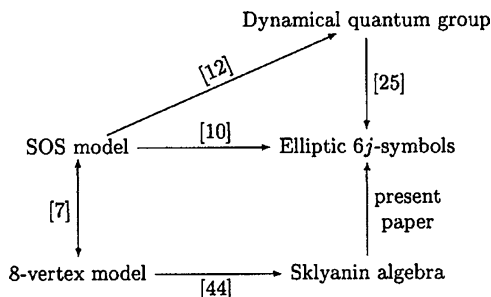
$$h_k(x; a)h_{N-k}(x; b) = \sum_{l=0}^N R_k^l h_l(x; aq^m)h_{N-l}(x; bq^{M-m}). \tag{5.10}$$

By Corollary 5.3,  $R_k^l$  vanishes unless  $k - m \leq l \leq M + k - m$ , which corresponds exactly to the condition on  $a - b$  in (5.6). Then (5.10) reduces to

$$h_k(x; a)h_{N-k}(x; b) = \sum_{l=\max(0, k-m)}^{\min(N, M+k-m)} R_k^l h_l(x; aq^m)h_{N-l}(x; bq^{M-m}),$$

where  $0 \leq m \leq M$ , which is thus the expansion problem solved by the elliptic  $6j$ -symbols of [10].

**Fig. 2** Connections to quantum groups and solvable models



### 6 Sklyanin algebra and generalized eigenvalue problem

As was explained in Section 2.3, our approach was motivated by previous work on relations between the standard  $SL(2)$  quantum group and quantum  $6j$ -symbols. It is natural to ask what “quantum group” is behind the more general case of elliptic  $6j$ -symbols. The answer turns out to be very satisfactory, namely, the Sklyanin algebra [44].

The Sklyanin algebra was first obtained from the  $R$ -matrix of the eight-vertex model. Baxter found that this model is related to a certain SOS (or face) model by a vertex-IRF transformation [7]. The original construction of elliptic  $6j$ -symbols starts from the  $R$ -matrix of the latter model. Moreover, starting from Baxter’s SOS model, Felder and Varchenko constructed a dynamical quantum group [12], which was recently related to elliptic  $6j$ -symbols [25]. We summarize these connections in Fig. 2.

It would be interesting to find a direct link between the approach of [25] and the discussion below. Presumably, this would involve extending Stokman’s paper [51] to elliptic quantum groups. In particular, vertex-IRF transformations should play an important role.

To explain the connection with the Sklyanin algebra we introduce the difference operators

$$\Delta(a, b, c, d)f(\xi) = \frac{\xi^{-2}\theta(a\xi, b\xi, c\xi, d\xi; p)f(q^{\frac{1}{2}}\xi) - \xi^2\theta(a\xi^{-1}, b\xi^{-1}, c\xi^{-1}, d\xi^{-1}; p)f(q^{-\frac{1}{2}}\xi)}{\xi\theta(\xi^{-2}; p)}.$$

Moreover,  $N$  being fixed we write

$$\Delta(a, b, c) = \Delta(a, b, c, q^{-N}/abc).$$

The following observation was communicated to us by Eric Rains, see [41].

**Proposition 6.1** (Rains, Sklyanin). *The operators  $\Delta(a, b, c)$  preserve the space  $W_N$  defined in Remark 5.2. Moreover, they generate a representation of the Sklyanin algebra on that space.*

These representations were found by Sklyanin [45, Theorem 4], except that he used the equivalent space denoted  $V_N$  in Remark 5.2 and  $\Theta_{00}^{2N+}$  in [45]. Let  $\Delta_i$ ,  $i = 0, 1, 2, 3$  be the operators representing Sklyanin’s generators  $S_i$ , pulled over from  $V_N$  to  $W_N$ . Rains observed that every  $\Delta_i$  is given by an operator of the form  $\Delta(a, b, c)$ , with specific choices of the parameters, and that, conversely, every  $\Delta(a, b, c)$  may be expressed as a linear combination of the  $\Delta_i$ . One may view the resulting representation as an elliptic deformation of the group action (2.4).

Next we consider the action of the operators  $\Delta$  on our basis vectors.

**Proposition 6.2.** *With  $x = \xi + \xi^{-1}$  one has*

$$\begin{aligned} &\Delta(a, b, c)h_k(x; q^{\frac{1}{2}}a)h_{N-k}(x; q^{\frac{1}{2}}b) \\ &= \frac{q^{-N}}{abc} \theta(q^k ac, q^{N-k} bc, q^N ab; p) h_k(x; a)h_{N-k}(x; b) \end{aligned} \tag{6.1}$$

and

$$\Delta(a, b, c)h_k(x; \lambda, \mu) \in \text{span}_{k-1 \leq j \leq k+1} \{h_j(x; q^{\frac{1}{2}}\lambda, q^{\frac{1}{2}}\mu)\}. \tag{6.2}$$

The identity (6.1) is an analogue of the fact that, in the situation of Section 2.2, any basis  $((ax + b)^k (cx + d)^{N-k})_{k=0}^N$  is the eigenbasis of a Lie algebra element. Similarly, (6.2) is an analogue of the fact that any other Lie algebra element acts tridiagonally on that basis. The parameter shifts are unavoidable and related to the fact that elliptic  $6j$ -symbols are biorthogonal rational functions rather than orthogonal polynomials, see Remark 6.5 below.

*Remark 6.3.* The identities (6.1) and (6.2) are consistent in view of

$$h_k(x; a)h_{N-k}(x; b) \in \text{span}_{k-1 \leq j \leq k+1} \{h_j(x; aq)h_{N-j}(x; bq)\},$$

which is a special case of Corollary 5.3.

Henceforth we suppress the deformation parameters  $p, q$ , thus writing

$$\theta(x) = \theta(x; p), \quad (a)_k = (a; q, p)_k.$$

When using notation such as  $\theta(a\xi^\pm)$ , we will mean  $\theta(a\xi; p)\theta(a\xi^{-1}; p)$ . The following theta function identity will be used in the proof of Proposition 6.2.

**Lemma 6.4.** *If  $a_1 \dots a_n b_1 \dots b_{n+2} = 1$ , then*

$$\begin{aligned} &\xi^{-n-1} \prod_{j=1}^n \theta(a_j \xi) \prod_{j=1}^{n+2} \theta(b_j \xi) - \xi^{n+1} \prod_{j=1}^n \theta(a_j \xi^{-1}) \prod_{j=1}^{n+2} \theta(b_j \xi^{-1}) \\ &= \frac{(-1)^n \xi \theta(\xi^{-2})}{a_1 \dots a_n} \sum_{k=1}^n \frac{\prod_{j=1}^{n+2} \theta(a_k b_j) \prod_{j=1, j \neq k}^n \theta(a_j \xi^\pm)}{\prod_{j=1, j \neq k}^n \theta(a_k/a_j)}. \end{aligned}$$

**Proof:** This is equivalent to the classical identity [53, p. 34], see also [43],

$$\sum_{k=1}^n \frac{\prod_{j=1}^n \theta(a_k/b_j)}{\prod_{j=1, j \neq k}^n \theta(a_k/a_j)} = 0, \quad a_1 \dots a_n = b_1 \dots b_n.$$

Namely, replace  $n$  with  $n + 2$  and  $b_j$  with  $b_j^{-1}$  in that identity, and put  $a_{n+1} = \xi$ ,  $a_{n+2} = \xi^{-1}$ . Moving the last two terms in the sum to the right gives

$$\sum_{k=1}^n \frac{\prod_{j=1}^{n+2} \theta(a_k b_j)}{\theta(a_k \xi^\pm) \prod_{j=1, j \neq k}^n \theta(a_k/a_j)} = -\frac{\prod_{j=1}^{n+2} \theta(\xi b_j)}{\theta(\xi^2) \prod_{j=1}^n \theta(\xi/a_j)} - \frac{\prod_{j=1}^{n+2} \theta(\xi^{-1} b_j)}{\theta(\xi^{-2}) \prod_{j=1}^n \theta(\xi^{-1}/a_j)}.$$

After multiplying with  $(-1)^n \xi \theta(\xi^{-2}) \prod_{j=1}^n a_j^{-1} \theta(a_j \xi^\pm)$  and using (5.2) repeatedly, one obtains the desired identity.  $\square$

**Proof of Proposition 6.2:** We start with (6.2). Writing out the left-hand side explicitly, collecting common factors and using (5.2) repeatedly gives

$$\begin{aligned} &\Delta(a, b, c) ((\lambda \xi^\pm)_k (\mu \xi^\pm)_{N-k}) \\ &= \frac{1}{\xi \theta(\xi^{-2})} \left\{ \xi^{-2} \theta(a \xi, b \xi, c \xi, d \xi) (q^{\frac{1}{2}} \lambda \xi, q^{-\frac{1}{2}} \lambda \xi^{-1})_k (q^{\frac{1}{2}} \mu \xi, q^{-\frac{1}{2}} \mu \xi^{-1})_{N-k} \right. \\ &\quad \left. - \xi^2 \theta(a \xi^{-1}, b \xi^{-1}, c \xi^{-1}, d \xi^{-1}) (q^{-\frac{1}{2}} \lambda \xi, q^{\frac{1}{2}} \lambda \xi^{-1})_k (q^{-\frac{1}{2}} \mu \xi, q^{\frac{1}{2}} \mu \xi^{-1})_{N-k} \right\} \\ &= \frac{q^{-1} \lambda \mu (q^{\frac{1}{2}} \lambda \xi^\pm)_{k-1} (q^{\frac{1}{2}} \mu \xi^\pm)_{N-k-1}}{\xi \theta(\xi^{-2})} \\ &\quad \times \left\{ \xi^{-4} \theta(a \xi, b \xi, c \xi, d \xi, q^{k-\frac{1}{2}} \lambda \xi, q^{\frac{1}{2}} \lambda^{-1} \xi, q^{N-k-\frac{1}{2}} \mu \xi, q^{\frac{1}{2}} \mu^{-1} \xi) \right. \\ &\quad \left. - \xi^4 \theta(a \xi^{-1}, b \xi^{-1}, c \xi^{-1}, d \xi^{-1}, q^{k-\frac{1}{2}} \lambda \xi^{-1}, q^{\frac{1}{2}} \lambda^{-1} \xi^{-1}, q^{N-k-\frac{1}{2}} \mu \xi^{-1}, q^{\frac{1}{2}} \mu^{-1} \xi^{-1}) \right\}, \end{aligned} \tag{6.3}$$

where  $abcd = q^{-N}$ . We may apply the case  $n = 3$  of Lemma 6.4 to the factor in brackets. Choose  $(b_1, \dots, b_5)$  as  $(a, b, c, d)$  together with any one of the four numbers

$$(q^{k-\frac{1}{2}} \lambda, q^{\frac{1}{2}} \lambda^{-1}, q^{N-k-\frac{1}{2}} \mu, q^{\frac{1}{2}} \mu^{-1})$$



and choose  $(a_1, a_2, a_3)$  as the remaining three of those numbers. As a function of  $\xi$ , our expression then takes the form

$$(q^{\frac{1}{2}}\lambda\xi^\pm)_{k-1}(q^{\frac{1}{2}}\mu\xi^\pm)_{N-k-1} \times \{C_1\theta(a_2\xi^\pm, a_3\xi^\pm) + C_2\theta(a_1\xi^\pm, a_3\xi^\pm) + C_3\theta(a_1\xi^\pm, a_2\xi^\pm)\}.$$

Depending on the choice of  $a_i$ , each term is proportional to one of the six functions

$$\begin{aligned} &(q^{-\frac{1}{2}}\lambda\xi^\pm)_k(q^{-\frac{1}{2}}\mu\xi^\pm)_{N-k}, \quad (q^{-\frac{1}{2}}\lambda\xi^\pm)_{k+1}(q^{\frac{1}{2}}\mu\xi^\pm)_{N-k-1}, \\ &(q^{-\frac{1}{2}}\lambda\xi^\pm)_k(q^{\frac{1}{2}}\mu\xi^\pm)_{N-k}, \quad (q^{\frac{1}{2}}\lambda\xi^\pm)_k(q^{-\frac{1}{2}}\mu\xi^\pm)_{N-k}, \\ &(q^{\frac{1}{2}}\lambda\xi^\pm)_{k-1}(q^{-\frac{1}{2}}\mu\xi^\pm)_{N-k+1}, \quad (q^{\frac{1}{2}}\lambda\xi^\pm)_k(q^{\frac{1}{2}}\mu\xi^\pm)_{N-k}. \end{aligned}$$

By Corollary 5.3, these all belong to

$$\text{span}_{k-1 \leq j \leq k+1} \{(q^{\frac{1}{2}}\lambda\xi^\pm)_j(q^{\frac{1}{2}}\mu\xi^\pm)_{N-j}\}.$$

This completes the proof of (6.2).

If we put  $\lambda = q^{\frac{1}{2}}a, \mu = q^{\frac{1}{2}}b$  in (6.3), the factor  $\theta(a\xi^\pm, b\xi^\pm)$  can be pulled out from the bracket, giving

$$\begin{aligned} \Delta(a, b, c)((q^{\frac{1}{2}}a\xi^\pm)_k(q^{\frac{1}{2}}b\xi^\pm)_{N-k}) &= \frac{(a\xi^\pm)_k(b\xi^\pm)_{N-k}}{\xi\theta(\xi^{-2})} \\ &\times \{\xi^{-2}\theta(c\xi, d\xi, q^k a\xi, q^{N-k} b\xi) - \xi^2\theta(c\xi^{-1}, d\xi^{-1}, q^k a\xi^{-1}, q^{N-k} b\xi^{-1})\}. \end{aligned}$$

The case  $n = 1$  of Lemma 6.4, which is equivalent to (5.3), now gives (6.1). □

*Remark 6.5.* Proposition 6.2 connects our work with the generalized eigenvalue problem (GEVP), which is central to the approach of Spiridonov and Zhedanov [49, 50]. Recall that, roughly speaking, the theory of orthogonal polynomials is equivalent to spectral theory of Jacobi operators, that is, to the eigenvalue problem

$$Ye_k = \lambda_k e_k$$

for a (possibly infinite) tridiagonal matrix  $Y$ . The theory of biorthogonal rational functions similarly corresponds to the GEVP

$$Y_1 e_k = \lambda_k Y_2 e_k \tag{6.4}$$

for two tridiagonal matrices  $Y_1, Y_2$ . Note that (6.1) means that

$$e_k = h_k(x; a)h_{N-k}(x; b)$$

solves the two-parameter family of GEVP:s

$$\Delta_1 e_k = \lambda_k \Delta_2 e_k,$$

where  $\Delta_1 = \Delta(q^{-\frac{1}{2}}a, q^{-\frac{1}{2}}b, c)$ ,  $\Delta_2 = \Delta(q^{-\frac{1}{2}}a, q^{-\frac{1}{2}}b, d)$ , with  $c$  and  $d$  arbitrary. Moreover, if we let  $\Delta_3$  be any operator of the form  $\Delta(e, f, g)$  and we put  $Y_1 = \Delta_3 \Delta_1$ ,  $Y_2 = \Delta_3 \Delta_2$ , we have that  $e_k$  solves (6.4) with  $Y_i$  tridiagonal in the basis  $(e_k)_{k=0}^N$ . Thus, we may view elliptic  $6j$ -symbols as the change of base matrix between the solutions of two different GEVP:s, where the involved tridiagonal operators are appropriate elements of the Sklyanin algebra, acting in a finite-dimensional representation.

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