Ramanujan J (2006) 12:359–378
DOI 10.1007/s11139-006-0149-0

Convolution structure associated with the Jacobi-Dunkl operator on $I\!R$

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Received: 15 October 2003 / Accepted: 13 April 2005

Abstract In this paper, a product formula for the eigenfunction of the Jacobi-Dunkl differential-difference operator is derived. It leads to a uniformly bounded convolution of point measures and a signed hypergroup on $I\!R$.

Keywords Jacobi-Dunkl operator · Jacobi function · Jacobi-Dunkl convolution

2000 Mathematics Subject Classification Primary—34K99, 44A15, 44A35, 43A15

Introduction

We consider the Jacobi-Dunkl differential-difference operator

$$\Lambda_{\alpha,\beta} f(x) = f'(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\left(\frac{f(x)-f(-x)}{2}\right),$$

where $\alpha, \beta \in I\!R$, $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$.
We point out that this operator $\Lambda_{\alpha,\beta}$ coincides with the Heckman-Opdam operator

$$D_\xi = \partial_\xi + \frac{1}{2}\sum_{a\in R_+} k_a a(\xi)\frac{1+e^{-a}}{1-e^{-a}}(1-r_a)$$

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on $I\!R$, with $R_+ = \{2, 4\}$ and suitable choice of k_a's, as mentioned by the anonymous referees.

In [3], the authors have constructed an intertwining operator $V_{\alpha,\beta}$ between $\Lambda_{\alpha,\beta}$ and the usual derivative operator, which is a topological isomorphism from the space of C^∞-functions on $I\!R$ onto itself. This operator is used to define a translation operator only on the space of C^∞-functions as follows

$$\mathcal{T}^x_{\alpha,\beta} f(y) = V^x_{\alpha,\beta} V^y_{\alpha,\beta}\big[V^{-1}_{\alpha,\beta} f(x+y)\big],\ x, y \in I\!R,$$

which gives a convolution on restricted function spaces. This manner does not permit to extend the study of this convolution, for instance, to the space of bounded measures and on the appropriate weighted L^p-spaces.

In this paper, we proceed by a different method for repairing this insufficiency. In fact, we establish a product formula for the eigenfunction of the operator $\Lambda_{\alpha,\beta}$ which permits to define the translation operator on various spaces and consequently to introduce a convolution product of measures and functions.
The eigenfunction $\Psi^{\alpha,\beta}_\lambda$ of $\Lambda_{\alpha,\beta}$ satisfying

$$\begin{cases} \Lambda_{\alpha,\beta}\, u = i\lambda\, u, & \lambda \in \mathbb{C}, \\ u(0) = 1, \end{cases}$$

is related to the Jacobi functions $\varphi^{\gamma,\delta}_\mu$ namely, we have

$$\Psi^{\alpha,\beta}_\lambda(x) = \varphi^{\alpha,\beta}_\mu(x) + i\frac{\lambda}{2(\alpha+1)} \sinh x \cosh x\, \varphi^{\alpha+1,\beta+1}_\mu(x),$$

where $\lambda^2 = \mu^2 + \rho^2$, $\lambda \in \mathbb{C}$, $x \in I\!R$ and $\rho = \alpha + \beta + 1$.
Using the properties of the Jacobi functions $\varphi^{\gamma,\delta}_\mu$, we prove the main result of this work

$$\forall x, y \in I\!R,\ \lambda \in \mathbb{C}, \quad \Psi^{\alpha,\beta}_\lambda(x)\, \Psi^{\alpha,\beta}_\lambda(y) = \int_{I\!R} \Psi^{\alpha,\beta}_\lambda(z) d\mu^{\alpha,\beta}_{x,y}(z),$$

where $\mu^{\alpha,\beta}_{x,y}$ is a real uniformly bounded measure with compact support, which may not be positive.

This product formula permits to define the translation operator

$$\mathcal{T}^x_{\alpha,\beta} f(y) = \int_{I\!R} f(z) d\mu^{\alpha,\beta}_{x,y}(z),\ x, y \in I\!R,$$

here f is a measurable function.

Notice that this translation coincides with the one given in [3] on the space of C^∞-functions. As it is well known, the product formula is an important tool for obtaining a convolution structure. Indeed, the convolution of two bounded measures μ and ν is defined as follows

$$\langle \mu *_{\alpha,\beta} \nu, f\rangle = \int_{IR}\int_{IR} \mathcal{T}^x_{\alpha,\beta} f(y) d\mu(x) d\nu(y).$$

Also the convolution of two functions in $L^1(IR, A_{\alpha,\beta}(y)dy)$ is given by

$$f *_{\alpha,\beta} g(x) = \int_{IR} \mathcal{T}^x_{\alpha,\beta} f(-y) g(y) A_{\alpha,\beta}(y) dy.$$

We prove that this product formula generates a structure of signed hypergroup on IR, in the sense given by M. Rösler in [9], which we call the Jacobi-Dunkl signed hypergroup with parameter (α, β).

The paper is organized as follows. In the first section, we recall some properties of the Jacobi functions, essentially the addition formula and the product formula for these functions. In the second section, we introduce the Jacobi-Dunkl operator, the eigenfunction $\psi_\lambda^{\alpha,\beta}$ and we establish the associated product formula, we give some properties of the measure $\mu_{x,y}^{\alpha,\beta}$, next we provide the real line with a structure of a signed hypergroup. The last section deals with some harmonic analysis associated with the differential-difference operator $\Lambda_{\alpha,\beta}$ essentially, we define the convolution product in appropriate weighted L^p-spaces and the Fourier transform called here Jacobi-Dunkl transform.

1 Preliminaries

In this section we recapitulate some results related to the Jacobi functions which will be used later, for a background on these special functions, one can see, [1, 5, 6].

In [5], M. Flensted-Jensen and T. H. Koornwinder have proved the following addition formula for the Jacobi function $\varphi_\mu^{\alpha,\beta}(x)$:
For $\alpha, \beta \in IR,\ \alpha > \beta > -\frac{1}{2}$ and $(x, y, r, \psi) \in IR \times IR \times [0, 1] \times [0, \pi]$, we have

$$\varphi_\mu^{\alpha,\beta}(arc\cosh(\gamma(x, -y, r, \psi))) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \Phi_{\mu,k,l}^{\alpha,\beta}(x)\ \Phi_{-\mu,k,l}^{\alpha,\beta}(y)\ \chi_{k,l}^{\alpha,\beta}(r, \psi)\ \Pi_{k,l}^{\alpha,\beta}, \tag{1.1}$$

where

$$\gamma(x, y, r, \psi) = |\cosh x \cosh y + r\, e^{i\psi} \sinh x \sinh y|, \tag{1.2}$$

$$\Phi_{\mu,k,l}^{\alpha,\beta}(x) = \frac{c_{\alpha,\beta(-\mu)}}{c_{\alpha+k+l,\beta+k-l(-\mu)}}(2\sinh x)^{k-l}(2\cosh x)^{k+l}\varphi_\mu^{\alpha+k+l,\beta+k-l}(x), \tag{1.3}$$

with

$$c_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma\left(\frac{i\mu+\rho}{2}\right)\Gamma\left(\frac{i\mu+\alpha-\beta+1}{2}\right)}, \tag{1.4}$$

and

$$\chi_{k,l}^{\alpha,\beta}(r,\psi) = R_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)r^{k-l}R_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\psi), \tag{1.5}$$

$R_n^{\alpha,\beta}$ is the normalized Jacobi polynomial.

$\chi_{k,l}^{\alpha,\beta}$ are polynomials in the two variables r^2, $r\cos\psi$, orthogonal (when $\alpha > \beta > -1/2$, $k, l \in I\!N$, $k \geq l \geq 0$) with respect to the measure $m_{\alpha,\beta}$, given by

$$dm_{\alpha,\beta}(r,\psi) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)}(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta}\,rdr\,d\psi. \tag{1.6}$$

Finally,

$$\Pi_{k,l}^{\alpha,\beta} = \left[\int_0^1\int_0^\pi \left(\chi_{k,l}^{\alpha,\beta}(r,\psi)\right)^2 dm_{\alpha,\beta}(r,\psi)\right]^{-1}. \tag{1.7}$$

The double series in (1.1) converges absolutely, uniformly for (x,y,r,ψ) in compact subsets of $I\!R \times I\!R \times [0,1] \times [0,\pi]$.

The authors in [5] pointed out that, if $\alpha = \beta > -\frac{1}{2}$, $r = 1$ or $\alpha > \beta = -\frac{1}{2}$, $\varphi = 0, \pi$ then (1.1) still holds but it degenerates to a single series. The two cases are related by the quadratic transformation,

$$\varphi_\mu^{\alpha,-\frac{1}{2}}(2t) = \varphi_{2\mu}^{\alpha,\alpha}(t). \tag{1.8}$$

Furthermore, the functions $\varphi_\mu^{\alpha,\alpha}(2t)$ can be expressed in terms of Gegenbauer functions

$$\varphi_\mu^{\alpha,\alpha}(t) = \frac{C_{\frac{i\mu-\rho}{2}}^{\alpha+\frac{1}{2}}(\cosh(2t))}{C_{\frac{i\mu-\rho}{2}}^{\alpha+\frac{1}{2}}(1)}, \tag{1.9}$$

(see [4], ch.3).

Also, the functions $\varphi_\mu^{\alpha,\beta}$, $\mu \in \mathbb{C}$, satisfy the following product formula, for $\alpha > \beta > -\frac{1}{2}$

$$\varphi_\mu^{\alpha,\beta}(x)\,\varphi_\mu^{\alpha,\beta}(y) = \int_0^1\int_0^\pi \varphi_\mu^{\alpha,\beta}(arc\cosh\gamma(x,y,r,\psi))dm_{\alpha,\beta}(r,\psi), \; x, y \geq 0. \tag{1.10}$$

A change of integration variables, namely

$$e^{i\chi}\cosh u=\cosh x\cosh y+re^{i\psi}\sinh x\sinh y, \tag{1.11}$$

gives the second form of the product formula

$$\varphi_\mu^{\alpha,\beta}(x)\varphi_\mu^{\alpha,\beta}(y)=\int_0^\infty \varphi_\mu^{\alpha,\beta}(z)W_{\alpha,\beta}(x,y,z)\tilde{A}_{\alpha,\beta}(z)dz,\quad x>0,\ y>0, \tag{1.12}$$

where

$$\tilde{A}_{\alpha,\beta}(x)=2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1},\quad x\geq 0,$$

the function $u\to W_{\alpha,\beta}(x,y,u)$ is nonnegative, symmetric in its three variables and supported in $[|x-y|,x+y]$. It is given by

$$W_{\alpha,\beta}(x,y,u)=2M_{\alpha,\beta}(\sinh x\sinh y\sinh u)^{-2\alpha}\int_0^\pi (g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\, d\chi, \tag{1.13}$$

$$g(x,y,u,\chi)=1-\cosh^2 x-\cosh^2 y-\cosh^2 u+2\cosh x\cosh y\cosh u\cos\chi. \tag{1.14}$$

Here

$$z_+=\begin{cases} z, & \text{if } z>0,\\ 0, & \text{if } z\leq 0,\end{cases}$$

and

$$M_{\alpha,\beta}=\frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)}. \tag{1.15}$$

It satisfies

$$\int_0^\infty W_{\alpha,\beta}(x,y,u)\tilde{A}_{\alpha,\beta}(u)du=1. \tag{1.16}$$

It is remarked in [6], p. 256 that the formula (1.13) can be rewritten as

$$\begin{aligned}W_{\alpha,\beta}(x,y,u)=&\frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x\cosh y\cosh u)^{\alpha-\beta-1}}{\pi^{\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)(\sinh x\sinh y\sinh u)^{2\alpha}}(1-B^2)^{\alpha-\frac{1}{2}}\\ &\times {}_2F_1\left(\alpha+\beta,\alpha-\beta;\alpha+\tfrac{1}{2};\tfrac{1-B}{2}\right),\ |x-y|<u<x+y,\end{aligned} \tag{1.17}$$

where

$$B = \frac{(\cosh x)^2 + (\cosh y)^2 + (\cosh u)^2 - 1}{2 \cosh x \cosh y \cosh u}. \tag{1.18}$$

According to formula (1.17), the product formula given by (1.12) remains valid for $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$.

2 The product formula for the eigenfunction of $\Lambda_{\alpha,\beta}$

In the following, we shall introduce the differential-difference operator $\Lambda_{\alpha,\beta}$ which is a particular case of the operator Λ defined on $I\!R$ by

$$\Lambda f(x) = f'(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1} B(x)$, $\alpha > -1/2$ and B a C^∞-function on $I\!R$ even and positive (see [7]). Also, we remark that this operator coincides with the rank-one Heckman-Opdam operator.

2.1 The differential-difference operator $\Lambda_{\alpha,\beta}$

Definition 2.1. For $\alpha, \beta \in I\!R$, the differential-difference operator $\Lambda_{\alpha,\beta}$ is defined on $C^1(I\!R)$, by

$$\Lambda_{\alpha,\beta} f(x)=f'(x)+[(2\alpha + 1)\coth x + (2\beta + 1)\tanh x]\left(\frac{f(x) - f(-x)}{2}\right), \quad x \in I\!R. \tag{2.1}$$

Notation. For $\alpha, \beta \in I\!R$, $\alpha \geq \beta \geq -\frac{1}{2}$, $\lambda \in \mathbb{C}$ and $x \in I\!R$, we put

$$\diamond\ \Psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \dfrac{i}{\lambda}\dfrac{\partial}{\partial x}\varphi_\mu^{\alpha,\beta}(x), & \text{if } \lambda \neq 0, \\ 1 \qquad , & \text{if } \lambda = 0, \end{cases} \tag{2.2}$$

with $\lambda^2 = \mu^2 + \rho^2$.

Using the relation

$$\frac{\partial}{\partial x}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\rho^2 + \mu^2}{2(\alpha + 1)} \sinh x\ \cosh x\ \varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\Psi_\lambda^{\alpha,\beta}$ can be written as follows

$$\Psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{2(\alpha+1)}\cosh x \sinh x\, \varphi_\mu^{\alpha+1,\beta+1}(x). \tag{2.3}$$

One can see by an easy computation, the following result

Proposition 2.2. *For $\alpha \geq \beta \geq -\frac{1}{2}$ and $\lambda \in \mathbb{C}$, the function $\Psi_\lambda^{\alpha,\beta}$ is an eigenfunction of the first-order differential-difference operator $\Lambda_{\alpha,\beta}$ satisfying*

$$\begin{cases} \Lambda_{\alpha,\beta}\Psi_\lambda^{\alpha,\beta} = i\lambda\Psi_\lambda^{\alpha,\beta}, \\ \Psi_\lambda^{\alpha,\beta}(0) = 1. \end{cases}$$

It is noticed immediately that for $\alpha = \beta = -\frac{1}{2}$, the operator $\Lambda_{\alpha,\beta}$ is reduced to the usual first derivative operator and the corresponding Ψ_λ is given by $\Psi_\lambda(x) = e^{i\lambda x}$.

Henceforth, we suppose that $\alpha, \beta \in IR,\ \alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$.

2.2 Product formula for the eigenfunction $\Psi_\lambda^{\alpha,\beta}$

In the following we shall establish the product formula for the eigenfunction $\Psi_\lambda^{\alpha,\beta}$ which will be obtained by using the product formula and the addition formula for the Jacobi functions. We remark that the argumentation follows closely that of [8].

Notations. For $x, y, u \in IR$ and $\chi \in [0,\pi]$ we put

◇ $A_{\alpha,\beta}(x) = 2^{2\rho}(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1},\ \rho = \alpha+\beta+1.$

$$\diamond \quad \sigma_{x,y,u}^{\chi} = \begin{cases} -\dfrac{\cosh u \cos\chi - \cosh x \cosh y}{\sinh x \sinh y}, & \text{if } xy \neq 0, \\ 0 & , \text{if } xy = 0. \end{cases} \tag{2.4}$$

◇ $I_{x,y} = [-|x|-|y|, -||x|-|y||\,] \cup [\,||x|-|y||, |x|+|y|\,].$

$\Psi_{\lambda,o}^{\alpha,\beta}$ (resp. $\Psi_{\lambda,e}^{\alpha,\beta}$) denotes the odd (resp. even) part of $\Psi_\lambda^{\alpha,\beta}$,

$$\Psi_{\lambda,o}^{\alpha,\beta}(x) = i\frac{\lambda}{2(\alpha+1)}\cosh x \sinh x\, \varphi_\mu^{\alpha+1,\beta+1}(x),$$
$$\Psi_{\lambda,e}^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x),$$

with $\lambda, \mu \in \mathbb{C},$ such that $\lambda^2 = \mu^2 + \rho^2$ and $x \in IR$.

In Lemmas 2.3 and 2.4 and in Theorem 2.5, we suppose that $\alpha, \beta \in IR,$ with $\alpha > \beta > -\frac{1}{2}$.

Lemma 2.3. *Let* $x, y \in IR \setminus \{0\}$ *and* $\lambda \in \mathbb{C}$ *then*

$$\Psi_{\lambda,o}^{\alpha,\beta}(x)\,\Psi_{\lambda,e}^{\alpha,\beta}(y) = M_{\alpha,\beta}\int_{I_{x,y}} (\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\,\Psi_{\lambda,o}^{\alpha,\beta}(u)$$
$$\times\int_0^{\pi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sigma_{x,u,y}^{\chi}\,\sin^{2\beta}\chi\,d\chi\,A_{\alpha,\beta}(u)\,du. \qquad (2.5)$$

Proof: Firstly, we show the result for $\lambda \in \mathbb{C}$ with $\lambda^2 = \mu^2 + \rho^2$, $x > 0$, $y > 0$.
Using the formula (1.10), we can write

$$\frac{\partial}{\partial x}\varphi_\mu^{\alpha,\beta}(x)\varphi_\mu^{\alpha,\beta}(y)$$
$$= \int_0^1\int_0^{\pi}\frac{\partial}{\partial x}\left[\varphi_\mu^{\alpha,\beta}(arc\,\cosh\gamma(x,y,r,\psi))\right]dm_{\alpha,\beta}(r,\psi)$$
$$= \int_0^1\int_0^{\pi}\Bigg[\frac{\cosh x\sinh x\cosh^2 y}{\gamma(x,y,r,\psi)\sqrt{[\gamma(x,y,r,\psi)]^2-1}}$$
$$+\frac{r\cos\psi\cosh y\sinh y(\cosh^2 x+\sinh^2 x)+r^2\cosh x\sinh x\sinh^2 y}{\gamma(x,y,r,\psi)\sqrt{[\gamma(x,y,r,\psi)]^2-1}}$$
$$\times\frac{\partial}{\partial x}\left(\varphi_\mu^{\alpha,\beta}\right)(arc\,\cosh\gamma(x,y,r,\psi))\Bigg]dm_{\alpha,\beta}(r,\psi).$$

By the use of the change of integration variables

$$e^{i\chi}\cosh u = \cosh x\cosh y + r\,e^{i\psi}\sinh x\sinh y,$$

we obtain

$$\frac{\partial}{\partial x}\varphi_\mu^{\alpha,\beta}(x)\varphi_\mu^{\alpha,\beta}(y) = 2M_{\alpha,\beta}\int_{|x-y|}^{x+y}(\sinh x\sinh y\sinh u)^{-2\alpha}\frac{\partial}{\partial u}\varphi_\mu^{\alpha,\beta}(u)$$
$$\times\left[\int_0^{\pi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sigma_{x,u,y}^{\chi}\,\sin^{2\beta}\chi\,d\chi\right]A_{\alpha,\beta}(u)\,du.$$

As the functions $u \to \frac{\partial}{\partial u}\varphi_\mu^{\alpha,\beta}(u)$, $u \to \sigma_{x,u,y}^{\chi}$ are odd, the last equality can be rewritten

$$\frac{\partial}{\partial x}\varphi_\mu^{\alpha,\beta}(x)\varphi_\mu^{\alpha,\beta}(y) = M_{\alpha,\beta}\int_{I_{x,y}}(\sinh x\sinh y\sinh|u|)^{-2\alpha}\frac{\partial}{\partial u}\varphi_\mu^{\alpha,\beta}(u)$$
$$\times\left[\int_0^{\pi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sigma_{x,u,y}^{\chi}\,\sin^{2\beta}\chi\,d\chi\right]A_{\alpha,\beta}(u)\,du.$$

Then, for $\lambda \in \mathbb{C} \setminus \{0\}$, (when $\lambda = 0$, the result is clear), and $x, y > 0$, we conclude that

$$\Psi_{\lambda,o}^{\alpha,\beta}(x)\,\Psi_{\lambda,e}^{\alpha,\beta}(y) = M_{\alpha,\beta}\int_{I_{x,y}} (\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\,\Psi_{\lambda,o}^{\alpha,\beta}(u) \times\left[\int_0^{\pi} \sigma_{x,u,y}^{\chi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\,d\chi\right]A_{\alpha,\beta}(u)\,du.$$

Thus the result is proved for $\lambda \in \mathbb{C}$, $x, y > 0$.

For $x, y \in IR \setminus \{0\}$, we write

$$\begin{aligned}\Psi_{\lambda,o}^{\alpha,\beta}(x)\,\Psi_{\lambda,e}^{\alpha,\beta}(y) &= \mathrm{sgn}(x)\Psi_{\lambda,o}^{\alpha,\beta}(|x|)\,\Psi_{\lambda,e}^{\alpha,\beta}(|y|)\\ &= \mathrm{sgn}(x)M_{\alpha,\beta}\int_{I_{x,y}} (\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\,\Psi_{\lambda,o}^{\alpha,\beta}(u)\\ &\quad\times\left[\int_0^{\pi} \sigma_{|x|,u,|y|}^{\chi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\,d\chi\right]A_{\alpha,\beta}(u)\,du\\ &= M_{\alpha,\beta}\int_{I_{x,y}} (\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\,\Psi_{\lambda,o}^{\alpha,\beta}(u)\\ &\quad\times\left[\int_0^{\pi} \sigma_{x,u,y}^{\chi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\,d\chi\right]A_{\alpha,\beta}(u)\,du.\end{aligned}$$

□

Lemma 2.4. *Let* $x, y \in IR \setminus \{0\}$ *and* $\lambda \in \mathbb{C}$, *then*

$$\Psi_{\lambda,o}^{\alpha,\beta}(x)\,\Psi_{\lambda,o}^{\alpha,\beta}(y) = -M_{\alpha,\beta}\int_{I_{x,y}} \Psi_{\lambda,e}^{\alpha,\beta}(u)(\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha} \times\left[\int_0^{\pi} \sigma_{x,y,u}^{\chi}(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\,d\chi\right]A_{\alpha,\beta}(u)\,du. \tag{2.6}$$

Proof: We begin by proving the result for $x > 0$, $y > 0$.

Using the formula (1.1) and the fact that the functions, $(r,\psi) \to \chi_{k,l}^{\alpha,\beta}(r,\psi)$ are orthogonal with respect to the measure $m_{\alpha,\beta}$, we obtain

$$\begin{aligned}&\int_0^1\int_0^{\pi} \varphi_{\mu}^{\alpha,\beta}(arc\cosh\gamma(x,y,r,\psi))r\cos\psi\,dm_{\alpha,\beta}(r,\psi)\\ &\quad= -\frac{\lambda^2}{16(\alpha+1)^2}\sinh(2x)\,\sinh(2y)\,\varphi_{\mu}^{\alpha+1,\beta+1}(x)\,\varphi_{\mu}^{\alpha+1,\beta+1}(y),\end{aligned}$$

with $\lambda^2 = \mu^2 + \rho^2$.

Using the change of integration variables defined by the formula (1.11), we obtain

$$
\begin{aligned}
&-2M_{\alpha,\beta}\int_{|x-y|}^{x+y}\varphi_\mu^{\alpha,\beta}(u)(\sinh x\sinh y\sinh u)^{-2\alpha}\\
&\quad\times\int_0^\pi (g(x,y,r,\psi))_+^{\alpha-\beta-1}\sigma_{x,y,u}^\chi \sin^{2\beta}\chi\, d\chi\, A_{\alpha,\beta}(u)\, du\\
&\quad=-\frac{\lambda^2}{16(\alpha+1)^2}\sinh(2x)\ \sinh(2y)\,\varphi_\mu^{\alpha+1,\beta+1}(x)\,\varphi_\mu^{\alpha+1,\beta+1}(y).
\end{aligned}
$$

Using the fact that $u\to\sigma_{x,y,u}^\chi$ is even, we obtain for $\lambda\in\mathbb{C},\ x,y\in\mathbb{R}$ and $x>0,\ y>0$,

$$
\begin{aligned}
&-M_{\alpha,\beta}\int_{I_{x,y}}\Psi_{\lambda,e}^{\alpha,\beta}(u)(\sinh x\sinh y\sinh|u|)^{-2\alpha}\\
&\quad\times\left[\int_0^\pi (g(x,y,r,\psi))_+^{\alpha-\beta-1}\sigma_{x,y,u}^\chi\sin^{2\beta}\chi\, d\chi\right]A_{\alpha,\beta}(u)\, du\\
&\quad=\Psi_{\lambda,o}^{\alpha,\beta}(x)\Psi_{\lambda,o}^{\alpha,\beta}(y).
\end{aligned}
$$

For $x,y\in\mathbb{R}\setminus\{0\}$, we conclude the result by using the equality

$$
\Psi_{\lambda,o}^{\alpha,\beta}(x)\Psi_{\lambda,o}^{\alpha,\beta}(y)=\operatorname{sgn}(xy)\Psi_{\lambda,o}^{\alpha,\beta}(|x|)\Psi_{\lambda,o}^{\alpha,\beta}(|y|),\ \ x,y\in\mathbb{R}.
$$

□

Notations. For $x,y,u\in\mathbb{R}$ and $\chi\in[0,\pi]$, we denote by

$$
\diamond\quad \varrho^\chi(x,y,u)=1-\sigma_{x,y,u}^\chi+\sigma_{u,y,x}^\chi+\sigma_{u,x,y}^\chi. \tag{2.7}
$$

$$
\begin{aligned}
\diamond\quad \mathcal{K}_{\alpha,\beta}(x,y,u)&=M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\,1_{I_{x,y}}(u)\\
&\quad\times\int_0^\pi \varrho^\chi(x,y,u)(g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\, d\chi,
\end{aligned}\tag{2.8}
$$

$1_{I_{x,y}}$ denotes the indicator function of the set $I_{x,y}$.

$$
\diamond\quad d\mu_{x,y}^{\alpha,\beta}(u)=\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,u)\,A_{\alpha,\beta}(u)\,du, & \text{if } xy\neq 0,\\ \delta_x\quad , & \text{if } y=0,\\ \delta_y\quad , & \text{if } x=0.\end{cases}\tag{2.9}
$$

As a consequence of the previous results and the relation

$$
\begin{aligned}
\Psi_\lambda^{\alpha,\beta}(x)\,\Psi_\lambda^{\alpha,\beta}(y)&=\Psi_{\lambda,e}^{\alpha,\beta}(x)\Psi_{\lambda,e}^{\alpha,\beta}(y)+\Psi_{\lambda,e}^{\alpha,\beta}(x)\Psi_{\lambda,o}^{\alpha,\beta}(y)\\
&\quad+\Psi_{\lambda,o}^{\alpha,\beta}(x)\Psi_{\lambda,e}^{\alpha,\beta}(y)+\Psi_{\lambda,o}^{\alpha,\beta}(x)\Psi_{\lambda,o}^{\alpha,\beta}(y),\quad x,y\in\mathbb{R},\lambda\in\mathbb{C},
\end{aligned}
$$

we give the main result of this paper, namely the product formula for the eigenfunction $\Psi_\lambda^{\alpha,\beta}$ in the following Theorem.

Theorem 2.5. *Let $x, y \in IR$, $\lambda \in \mathbb{C}$, then we have*

$$\Psi_\lambda^{\alpha,\beta}(x)\,\Psi_\lambda^{\alpha,\beta}(y) = \int_{IR} \Psi_\lambda^{\alpha,\beta}(u)d\mu_{x,y}^{\alpha,\beta}(u). \tag{2.10}$$

Remark. The product formula established in Theorem 2.5 is available in the case where $\alpha > \beta > -\frac{1}{2}$, using the properties of the Jacobi functions (formula (1.8)), we can deduce that

$$\Psi_\lambda^{\alpha,-1/2}(2x) = \Psi_{2\lambda}^{\alpha,\alpha}(x). \tag{2.11}$$

So, to show that the formula (2.10) extends to the case where $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$, it is sufficient to study the case where $\alpha = \beta > -1/2$. In this case, we know that (see [6])

$$\varphi_\mu^{\alpha,\alpha}(x)\varphi_\mu^{\alpha,\alpha}(y) = \int_{|x-y|}^{x+y} \varphi_\mu^{\alpha,\alpha}(u)W_{\alpha,\alpha}(x, y, u)A_{\alpha,\alpha}(u)du, \tag{2.12}$$

where $W_{\alpha,\alpha}$ is given by (1.17) with $\alpha = \beta$.

By using the change of variable

$$\cosh u = |\cosh x \cosh y + e^{i\psi}\ \sinh x \sinh y|,$$

we obtain

$$\varphi_\mu^{\alpha,\alpha}(x)\varphi_\mu^{\alpha,\alpha}(y)$$
$$= M_\alpha \int_0^\pi \varphi_\mu^{\alpha,\alpha}(arc\ \cosh(|\cosh x \cosh y + e^{i\psi}\ \sinh x \sinh y|))(\sin\psi)^{2\alpha}d\psi, \tag{2.13}$$

where $M_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}$.

The addition formula in this case is

$$\varphi_\mu^{\alpha,\alpha}(arc\cosh(\gamma(x, -y, 1, \psi))) = \sum_{k=0}^{\infty} \Phi_{\mu,k,0}^{\alpha,\alpha}(x)\ \Phi_{-\mu,k,0}^{\alpha,\alpha}(y)\ \chi_{k,0}^{\alpha,\alpha}(1, \psi)\ \Pi_{k,0}^{\alpha,\alpha}. \tag{2.14}$$

Then, the same technique used above (for $\alpha > \beta > -1/2$), gives the following product formula

$$\Psi_\lambda^{\alpha,\alpha}(x)\,\Psi_\lambda^{\alpha,\alpha}(y) = \int_{IR} \Psi_\lambda^{\alpha,\alpha}(u)d\mu_{x,y}^{\alpha,\alpha}(u), \tag{2.15}$$

where

$$d\mu_{x,y}^{\alpha,\alpha}(u) = \begin{cases} \mathcal{K}_{\alpha,\alpha}(x,y,u)\, A_{\alpha,\alpha}(u)\, du, & \text{if } xy \neq 0, \\ \delta_x \quad , & \text{if } y = 0, \\ \delta_y \quad , & \text{if } x = 0, \end{cases} \tag{2.16}$$

with

$$\mathcal{K}_{\alpha,\alpha}(x,y,u) = \frac{1}{2}(1 - \sigma_{x,y,u} + \sigma_{u,y,x} + \sigma_{u,x,y})\, W_{\alpha,\alpha}(x,y,u)\, 1_{I_{x,y}}(u),$$

and

$$\sigma_{x,y,u} = \begin{cases} -\dfrac{(\cosh u)^2 - (\cosh x)^2(\cosh y)^2 - (\sinh x)^2(\sinh y)^2}{2\sinh x \cosh x \cosh y \sinh y}, & \text{if } xy \neq 0, \\ 0 \quad , & \text{if } xy = 0. \end{cases} \tag{2.17}$$

Obviously, we have the following properties.

Proposition 2.6. *Let $x, y \in IR \setminus \{0\}$ and $u \in IR$, we have*

(i) $\mathcal{K}_{\alpha,\beta}(x,y,u) = \mathcal{K}_{\alpha,\beta}(y,x,u)$.
(ii) $\mathcal{K}_{\alpha,\beta}(x,y,u) = \mathcal{K}_{\alpha,\beta}(-x,u,y)$.
(iii) $\mathcal{K}_{\alpha,\beta}(x,y,-u) = \mathcal{K}_{\alpha,\beta}(-x,-y,u)$.

Next, we shall give some properties of the measures $\mu_{x,y}^{\alpha,\beta}$, which are the same as in the rank-one Dunkl setting, see [8].

Proposition 2.7. *For every $x, y \in IR$ we have the following properties*

(i) $\mu_{x,y}^{\alpha,\beta}(IR) = 1$.
(ii) $\|\mu_{x,y}^{\alpha,\beta}\| \leq 4$.
(iii) $supp\,(\mu_{x,y}^{\alpha,\beta}) \subset I_{x,y}$.
(iv) *In general the measures $\mu_{x,y}^{\alpha,\beta}$ are not positive.*

Proof: In our proof, we will be interested only in the case where $\alpha > \beta > -\frac{1}{2}$, (since the other case is obvious).

(i) By replacing λ by 0 in the formula (2.10), we find the result.
(ii) For $x, y \in IR$ with $xy \neq 0$, $u \in I_{x,y}$ and $\chi \in [0,\pi]$, we have $|\sigma_{x,y,u}^{\chi}| \leq 1$, because $\sigma_{x,u,y}^{\chi} = -r\cos\psi$ in the notion of (1.11). By using the fact

$$u \in I_{x,y} \Leftrightarrow x \in I_{u,y} \Leftrightarrow y \in I_{u,x},$$

we obtain $|\sigma_{x,u,y}^{\chi}| \leq 1$ and $|\sigma_{y,u,x}^{\chi}| \leq 1$.
According to the previous results and the formula (1.13), we deduce (ii).

(iii) evident.
(iv) It suffices to show that there exist $x_0, y_0 \in IR$ and a borelian set $V \subset IR$ such that $\mu^{\alpha,\beta}_{x_0,y_0}(V) < 0$. □

A calculation shows that, for $x > 0$ and $\chi \in [0, \pi]$:

$$\varrho^{\chi}\left(x, x, -\frac{x}{2}\right) \leq \varrho^{0}\left(x, x, -\frac{x}{2}\right) \leq -\frac{3}{8} \quad \text{and} \quad g\left(x, x, -\frac{x}{2}, 0\right) > 0.$$

By using continuity argumentations, we can deduce the existence of a compact neighborhood V of $-\frac{x}{2}$ in $I_{x,x}$ such that $\mu^{\alpha,\beta}_{x,x}(V) < 0$.

Also, one can see in the same way the existence of a compact neighborhood V of $-x$ such that $\mu^{\alpha,\beta}_{x,y}(V) < 0$, for $x > y > 0$.

Note that $\mu^{\alpha,\beta}_{x,-x}$ is positive for all $x \in IR$.

Notations. We put

◇ $h^e(x, y, r, \psi) = 1 + r\cos\psi$.
◇ $\delta(x, y, r, \psi) = \sinh(x+y)(\cosh x \cosh y + r\cos\psi \cosh(x+y) + r^2 \sinh x \sinh y)$.
◇ $h^o(x, y, r, \psi) = \begin{cases} \dfrac{\delta(x, y, r, \psi)}{\gamma(x, y, r, \psi)\sqrt{(\gamma(x, y, r, \psi))^2 - 1}}, & \text{if } x \neq -y, \\ 0, & \text{if } x = -y. \end{cases}$
◇ f_e (resp.) f_o denotes the even (resp. odd) part of f.
◇ $C_b(IR)$ denotes the space of continuous functions and bounded on IR.

Definition 2.8. Let $x \in IR$, the translation of the function $f \in C_b(IR)$ (or a suitable function f) denoted $\mathcal{T}^x_{\alpha,\beta} f$ is defined on IR by

$$\mathcal{T}^x_{\alpha,\beta} f(y) = \mu^{\alpha,\beta}_{x,y}(f) = \int_{IR} f \, d\mu^{\alpha,\beta}_{x,y}.$$

Proposition 2.9. *Let $f \in C_b(IR)$ and $x, y \in IR$, then*

(i) *for $\alpha > \beta > -1/2$, we have*

$$\mu^{\alpha,\beta}_{x,y}(f) = \int_0^1\int_0^{\pi} f_e(arc\ \cosh(\gamma(x, y, r, \psi))h^e(x, y, r, \psi)dm_{\alpha,\beta}(r, \psi) + \int_0^1\int_0^{\pi} f_o(arc\ \cosh(\gamma(x, y, r, \psi))h^o(x, y, r, \psi)dm_{\alpha,\beta}(r, \psi).$$

(ii) *for $\alpha > -1/2$, we have*

$$\mu^{\alpha,\alpha}_{x,y}(f) = M_\alpha \int_0^{\pi} f_e(arc\ \cosh(\gamma(x, y, 1, \psi))h^e(x, y, 1, \psi)(\sin\psi)^{2\alpha}\, d\psi + M_\alpha \int_0^{\pi} f_o(arc\ \cosh(\gamma(x, y, 1, \psi))h^o(x, y, 1, \psi)(\sin\psi)^{2\alpha}\, d\psi.$$

Proof: (i) The result is clear when $x = 0$ or $y = 0$. Then, we suppose that $xy \neq 0$,

$$\mu_{x,y}^{\alpha,\beta}(f) = 2M_{\alpha,\beta} \int_{||x|-|y||}^{|x|+|y|} f_e(u)(\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha} \int_0^{\pi} \left(1 - \sigma_{x,y,u}^{\chi}\right)$$
$$\times (g(x, y, u, \chi))_+^{\alpha-\beta-1} \sin^{2\beta}\chi\, d\chi\, A_{\alpha,\beta}(u)\, du$$
$$+2M_{\alpha,\beta} \int_{||x|-|y||}^{|x|+|y|} f_o(u)(\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha} \int_0^{\pi} \left(\sigma_{x,u,y}^{\chi} + \sigma_{y,u,x}^{\chi}\right)$$
$$\times (g(x, y, u, \chi))_+^{\alpha-\beta-1} \sin^{2\beta}\chi\, d\chi\, A_{\alpha,\beta}(u)\, du.$$

Using the change of variables

$$\exp(i\chi)\cosh u = \cosh x \cosh y + r \exp(i\psi) \sinh x \sinh y,$$

we can conclude the result (i). We prove (ii) in the same way as (i). □

We achieve this subsection by stating the following standard Lemma which can be proved in the same way as Lemma 3.3 in [8], by using the injectivity of the Fourier-Stieltjes transform in the Jacobi hypergroup.

Lemma 2.10. *Suppose* $\mu \in M_b(IR)$ *with* $\int_{IR} \Psi_\lambda^{\alpha,\beta}(x)d\mu(x) = 0$ *for all* $\lambda \geq \rho$, *then* $\mu = 0$.

2.3 Structure of the Jacobi-Dunkl signed hypergroup

We recall in the following the definition of a signed hypergroup (see [8, 9, 10]).

Definition 2.11. Let X be a locally compact, σ-compact Hausdorff space and m a positive Radon measure on it with $\operatorname{supp} m = X$. Further, let $\omega : X \times X \to M_b(X)$, $(x, y) \to \delta_x * \delta_y$, be a τ_*-continuous mapping, where $M_b(X)$ the space of bounded Radon measures on X, here, τ_*-topology on $M_b(X)$ denotes the weak- $*$-topology $\sigma(M_b(X), C_0(X))$.

Then the triple (X, m, ω) is called a signed hypergroup, if the following axioms are satisfied:

(A1) For each $x \in X$ and $f \in C_b(X)$, the translates $T^x f : y \to \delta_x * \delta_y(f)$ and $T_x f : y \to \delta_y * \delta_x(f)$ again belong to $C_b(X)$. Furthermore, for $f \in C_c(X)$ and any compact subset $K \subset X$, the set $\cup_{x\in K}(\operatorname{supp}(T^x f) \cup \operatorname{supp}(T_x f))$ is relatively compact in X.
(A2) $||\delta_x * \delta_y|| \leq C$ for all $x, y \in X$ with some constant $C > 0$.
(A3) The canonical continuation of ω is associative.
(A4) There exists a neutral element $e \in X$, such that

$$\delta_e * \mu = \mu * \delta_e = \mu, \quad \text{for all } \mu \in M_b(X).$$

(A5) There exists an involution homeomorphism $^-$ on X such that

$$(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}, \text{ for all } x, y \in X,$$

where for a Borel measure $\mu \in X$, the measure μ^- on X is defined by $\mu^-(A) = \mu(A^-)$, $A \subset X$ any Borel set.

(A6) For all $f, g \in C_c(X)$ and $x \in X$ the following adjoint relation holds:

$$\int_X (T^x f) g dm = \int_X f(T^{x^-} g) dm.$$

The signed hypergroup is said commutative if we have

$$\forall x, y \in X, \ \delta_x * \delta_y = \delta_y * \delta_x.$$

In our situation, we put $\omega(x, y) = \omega_{\alpha,\beta}(x, y) = \delta_x *_{\alpha,\beta} \delta_y = \mu^{\alpha,\beta}_{x,y}$.

Theorem 2.12. *For* $\alpha, \beta \in IR$, $\alpha \geq \beta \geq -\frac{1}{2}$ *and* $\alpha \neq -\frac{1}{2}$, *the triple* $(IR, \omega_{\alpha,\beta}, A_{\alpha,\beta})$ *is a commutative signed hypergroup with neutral element* 0, *involution* $x \to -x$ *and* $A_{\alpha,\beta}(x)dx$ *as the Haar measure.*

Proof: It is clear that the $\mu^{\alpha,\beta}_{x,y}$ are real and $\mu^{\alpha,\beta}_{x,y} = \mu^{\alpha,\beta}_{y,x}$. We suppose $\alpha > \beta > -\frac{1}{2}$, the case $\alpha = \beta > -\frac{1}{2}$ can be treated in the same way.

We begin by proving that for $f \in C_b(IR)$ the mapping $(x, y) \to \mu^{\alpha,\beta}_{x,y}(f)$ is continuous on IR^2. This yields τ_*-continuity of $(x, y) \to \mu^{\alpha,\beta}_{x,y}(f)$ and also that $\mathcal{T}^x_{\alpha,\beta} f \in C_b(IR)$ for $f \in C_b(IR)$, by norm-boundedness of $\mu^{\alpha,\beta}_{x,y}$. Two cases are discussed according to the parity of the function f.

If f is even, according to Proposition 2.9, we have

$$\mu^{\alpha,\beta}_{x,y}(f) = \int_0^1 \int_0^\pi f(arc\ \cosh(\gamma(x, y, r, \psi))(1 + r\cos\psi) dm_{\alpha,\beta}(r, \psi).$$

This integral is a continuous function in $(x, y) \in IR^2$.

If f is odd, again by using Proposition 2.9, we have

$$\mu^{\alpha,\beta}_{x,y}(f) = \int_0^1 \int_0^\pi f(arc\ \cosh(\gamma(x, y, r, \psi)) h^o(x, y, r, \psi) dm_{\alpha,\beta}(r, \psi).$$

It is clear that the integral is a continuous function if $|x| \neq |y|$, (since $\gamma(x, y, r, \psi) > 1$). If $|x| = |y|$, as the weight $h^o(x, y, r, \psi)$ is bounded by 2 and $f(0) = 0$, then we obtain the result.

Concerning axiom A_1, it remains to note that if $f \in C_c(IR)$, with $supp(f) \subset [-a, a]$), then $supp\ \mathcal{T}^x_{\alpha,\beta}(f) \subset [-a - |x|, a + |x|]$), for all $x \in IR$.

We remark that the argumentations for $(A_2) - (A_6)$ are very similar to those in [8], hence we omit their proofs. □

2.4 The dual of the Jacobi-Dunkl signed hypergroup

The dual of the Jacobi signed hypergroup of parameters α, β, denoted $\widehat{X}_{\alpha,\beta}$ is the set of the functions defined on $I\!R$ with values in $\mathbb{C}$, which are multiplicative (i.e. $\mathcal{T}_x^{\alpha,\beta}\chi(y) = \chi(x)\chi(y)$), continuous, bounded and hermitian (i.e. $\chi(x^-) = \overline{\chi}(x)$) (see [9]).

Proposition 2.13. *Let $\alpha, \beta \in I\!R$, $\alpha \geq \beta \geq -\frac{1}{2}$, then*

$$\widehat{X}_{\alpha,\beta} = \left\{\Psi_\lambda^{\alpha,\beta}, \lambda \in I\!R\right\}.$$

Proof: We suppose that $\alpha > \beta > -1/2$, (for $\alpha = \beta > -1/2$ the result can be seen in the same way).

From the product formula, it is clear that $\Psi_\lambda^{\alpha,\beta}$, $\lambda \in \mathbb{C}$ is multiplicative.

On the other hand, let φ be a multiplicative function, then for all $x, y \in I\!R$, $x, y > 0$, we have

$$\begin{aligned}\varphi_e(x)\varphi_e(y) &= 2M_{\alpha,\beta}\int_{|x-y|}^{x+y}\varphi_e(u)(\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\\ &\quad\times\int_0^\pi (g(x,y,u,\chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi\,d\chi\,A_{\alpha,\beta}(u)du,\end{aligned}$$

(φ_e denotes the even part of φ), and

$$\begin{aligned}\varphi_o(x)\varphi_o(y) &= -2M_{\alpha,\beta}\int_{|x-y|}^{x+y}\varphi_e(u)(\sinh|x|\sinh|y|\sinh|u|)^{-2\alpha}\\ &\quad\times\int_0^\pi (g(x,y,u,\chi))_+^{\alpha-\beta-1}\sigma_{x,y,u}^{\chi}\sin^{2\beta}\chi\,d\chi\,A_{\alpha,\beta}(u)du.\end{aligned}$$

(φ_o denotes the odd part of φ).

The first equality shows that φ_e is a multiplicative function on the Jacobi hypergroup. According to [2], there exists $\mu \in \mathbb{C}$ such that $\varphi_e = \varphi_\mu^{\alpha,\beta}$. Replacing φ_e by its value in the second equality and using Lemma 2.4, we obtain

$$\varphi_o(x)\varphi_o(y) = \Psi_{\lambda,o}^{\alpha,\beta}(x)\Psi_{\lambda,o}^{\alpha,\beta}(y),$$

where $\lambda \in \mathbb{C}$ such that $\lambda^2 = \mu^2 + \rho^2$, hence $\varphi_o(x) = \Psi_{\lambda,o}^{\alpha,\beta}(x)$ or $\varphi_o(x) = -\Psi_{\lambda,o}^{\alpha,\beta} = \Psi_{-\lambda,o}^{\alpha,\beta}$. The fact that $\Psi_\lambda^{\alpha,\beta} = \Psi_{\lambda,e}^{\alpha,\beta} + \Psi_{\lambda,o}^{\alpha,\beta}$, gives $\varphi(x) = \Psi_\lambda^{\alpha,\beta}(x)$ or $\varphi(x) = \Psi_{-\lambda}^{\alpha,\beta}$, with $\lambda \in \mathbb{C}$. Consequently, there exists $\lambda \in \mathbb{C}$ such that $\varphi = \Psi_\lambda^{\alpha,\beta}$.

From [3], we know that for $\lambda \in I\!R$ the eigenfunction $\Psi_\lambda^{\alpha,\beta}$ is bounded. Hence, we can see that

$$\left\{\Psi_\lambda^{\alpha,\beta}, \lambda \in I\!R\right\} \subset \widehat{X}_{\alpha,\beta} \subset \left\{\Psi_\lambda^{\alpha,\beta}, \lambda \in \mathbb{C}\right\}.$$

Reciprocally, if $\Psi_\lambda^{\alpha,\beta} \in \widehat{X}_{\alpha,\beta}, \lambda \in \mathbb{C}$ then $\Psi_\lambda^{\alpha,\beta}(-x) = \overline{\Psi_\lambda^{\alpha,\beta}(x)}$, hence

$$\begin{cases} \Psi_{\lambda,e}^{\alpha,\beta}(x) = \overline{\Psi_{\lambda,e}^{\alpha,\beta}(x)}, & x \in I\!R, \\ \Psi_{\lambda,o}^{\alpha,\beta}(x) = -\overline{\Psi_{\lambda,o}^{\alpha,\beta}(x)}, & x \in I\!R, \end{cases}$$

the first equality shows that $\lambda \in I\!R \cup i\,I\!R$. Taking into account this result, the second implies that $\lambda \in I\!R$.

Since $\Psi_\lambda^{\alpha,\beta}$ is continuous and bounded, we conclude the result. □

3 Some Fourier analysis on the Jacobi-Dunkl signed hypergroup

In this section we give some properties of the convolution product associated with $\Lambda_{\alpha,\beta}$. In particular, estimates are given for $\|f *_{\alpha,\beta} g\|_r$ where $f \in L^p(A_{\alpha,\beta})$ and $g \in L^q(A_{\alpha,\beta})$. Also, we deal with the related Jacobi-Dunkl transform introduced in [3].

We remark that some of these definitions and results can be deduced from those stated in the context of general commutative signed hypergroups, see [9].

Definition 3.1. The product of convolution of suitable functions f and g is

$$f *_{\alpha,\beta} g(x) = \int_{I\!R} \mathcal{T}^x_{\alpha,\beta}(f)(-y) g(y) A_{\alpha,\beta}(y) dy, \quad x \in I\!R.$$

Obviously, for $f, g, h \in L^1(A_{\alpha,\beta})$, we have

(i) $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$.
(ii) $(f *_{\alpha,\beta} g) *_{\alpha,\beta} h = f *_{\alpha,\beta} (g *_{\alpha,\beta} h)$.

Proposition 3.2.

(i) For all f in $L^p(A_{\alpha,\beta})$, $p \in [1, \infty]$ the function $\mathcal{T}^x_{\alpha,\beta}(f)$, $x \in I\!R$ is defined almost everywhere on $I\!R$, belongs to $L^p(A_{\alpha,\beta})$, and we have $\|\mathcal{T}^x_{\alpha,\beta}(f)\|_p \leq 4\|f\|_p$.
(ii) Let p, q, r be such that $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$.

If $f \in L^p(A_{\alpha,\beta})$ and $g \in L^q(A_{\alpha,\beta})$, then $f *_{\alpha,\beta} g \in L^r(A_{\alpha,\beta})$ and

$$\|f *_{\alpha,\beta} g\|_r \leq 4\|f\|_p \, \|g\|_q.$$

Definition 3.3.

(i) The Jacobi-Dunkl transform of a bounded Radon measure μ on IR is defined by

$$\mathcal{F}_{\alpha,\beta}(\mu)(\lambda) = \int_{IR} \Psi_{-\lambda}^{\alpha,\beta}(x) d\mu(x).$$

(ii) The Jacobi-Dunkl transform for a suitable function f, denoted $\mathcal{F}_{\alpha,\beta}(f)$ is defined on IR by

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_{IR} f(x)\Psi_{-\lambda}^{\alpha,\beta}(x) A_{\alpha,\beta}(x) dx.$$

In [3], The authors have established the following Proposition

Proposition 3.4.

(i) (Plancherel Formula) $\forall f \in \mathcal{D}(IR)$, the space of C^{∞}-functions with compact supports, we have

$$\int_{IR} |f(x)|^2 A_{\alpha,\beta}(x) dx = \int_{IR} |\, \mathcal{F}_{\alpha,\beta}(f)(\lambda)\,|^2 d\Pi_{\alpha,\beta}(\lambda).$$

(ii) The Jacobi-Dunkl transform extends uniquely to an unitary isomorphism from $L^2(A_{\alpha,\beta})$ onto $L^2(\Pi_{\alpha,\beta})$.

$d\Pi_{\alpha,\beta}(\lambda) = \frac{|\lambda| d\lambda}{8\pi\sqrt{\lambda^2-\rho^2}\,|c_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2}\,)|^2} 1_{IR\setminus]-\rho,\rho[}(\lambda)$, denotes the spectral or Plancherel measure.

Proposition 3.5.

(i) Let $f \in L^1(A_{\alpha,\beta})$ and $x \in IR$ then

$$\mathcal{F}_{\alpha,\beta}\big(\mathcal{T}_{\alpha,\beta}^{x} f\big)(\lambda) = \Psi_{\lambda}^{\alpha,\beta}(x)\mathcal{F}_{\alpha,\beta}(f)(\lambda), \qquad \forall \lambda \in IR.$$

(ii) Let f be in $L^2(A_{\alpha,\beta})$ and $x \in IR$ then

$$\mathcal{F}_{\alpha,\beta}\big(\mathcal{T}_{\alpha,\beta}^{x} f\big)(\lambda) = \Psi_{\lambda}^{\alpha,\beta}(x)\mathcal{F}_{\alpha,\beta}(f)(\lambda), \qquad \Pi_{\alpha,\beta} - a.e.$$

Proposition 3.6.

(i) For μ and ν two Radon measures on IR, we have

$$\mathcal{F}_{\alpha,\beta}(\mu *_{\alpha,\beta} \nu)(\lambda) = \mathcal{F}_{\alpha,\beta}(\mu)(\lambda)\mathcal{F}_{\alpha,\beta}(\nu)(\lambda), \qquad \forall \lambda \in IR.$$

(ii) If $f, g \in L^1(A_{\alpha,\beta})$, then

$$\mathcal{F}_{\alpha,\beta}(f *_{\alpha,\beta} g)(\lambda) = \mathcal{F}_{\alpha,\beta}(f)(\lambda)\mathcal{F}_{\alpha,\beta}(g)(\lambda), \qquad \forall \lambda \in IR.$$

(iii) If $f \in L^1(A_{\alpha,\beta})$ and $g \in L^2(A_{\alpha,\beta})$, then

$$\mathcal{F}_{\alpha,\beta}(f *_{\alpha,\beta} g)(\lambda) = \mathcal{F}_{\alpha,\beta}(f)(\lambda)\mathcal{F}_{\alpha,\beta}(g)(\lambda), \qquad \Pi_{\alpha,\beta} - \ a.e.$$

Remarks.

(i) From the definition of the Jacobi-Dunkl transform and the fact that $|\Psi_\lambda^{\alpha,\beta}| \leq 1, (\lambda \in I\!R)$, it follows that:
If $f \in L^1(A_{\alpha,\beta})$ then $\mathcal{F}_{\alpha,\beta}(f) \in L^\infty(\Pi_{\alpha,\beta})$ and we have

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_\infty \leq \|f\|_1.$$

On the other hand, the Plancherel formula says that

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_2 = \|f\|_2.$$

Then the Riesz-Thorin interpolation theorem permits to extend $\mathcal{F}_{\alpha,\beta}$ from $L^p(A_{\alpha,\beta})$ $1 < p < 2$, into $L^q(\Pi_{\alpha,\beta})$, where q is the conjugate exponent of p, and we have the following estimate

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_q \leq \|f\|_p.$$

(ii) If p and q are conjugate exponents such that $p \in [1, 2[, \ q \in]2, +\infty]$, then using the estimate given in Proposition 3.2 in [3] :

$$\forall x \in I\!R, |\lambda| \geq \rho, |\Psi_\lambda^{\alpha,\beta}(x)| \leq M\frac{(1+\rho)^2}{\rho}(1+|x|)e^{-\rho|x|},$$

where M is a positive constant, we deduce that for $|\lambda| \geq \rho$, the function $x \to \Psi_\lambda^{\alpha,\beta}(x)$ belongs to $L^q(A_{\alpha,\beta})$ and $||\Psi_\lambda^{\alpha,\beta}||_q$ is bounded independently of λ, $|\lambda| \geq \rho$.
Also, for ϕ in $L^p(A_{\alpha,\beta})$, the function $\mathcal{F}_{\alpha,\beta}(\phi)$ satisfies

$$\forall \lambda \in I\!R, |\lambda| > \rho, |\mathcal{F}_{\alpha,\beta}(\phi)(\lambda)| \leq \|\Psi_\lambda^{\alpha,\beta}\|_q \ \|\phi\|_p.$$

Consequently $\mathcal{F}_{\alpha,\beta}(\phi) \in L^\infty(\Pi_{\alpha,\beta})$.

(iii) For f in $L^p(A_{\alpha,\beta})$, $1 < p < 2$ and $x \in I\!R$, we have

$$\mathcal{F}_{\alpha,\beta}\big(\mathcal{T}_{\alpha,\beta}^x f\big)(\lambda) = \Psi_\lambda^{\alpha,\beta}(x)\mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad \forall \lambda \in supp\,(\Pi_{\alpha,\beta}).$$

Also, for $f \in L^1(A_{\alpha,\beta})$ and $g \in L^p(A_{\alpha,\beta})$, $1 < p < 2$, we have

$$\mathcal{F}_{\alpha,\beta}(f *_{\alpha,\beta} g)(\lambda) = \mathcal{F}_{\alpha,\beta}(f)(\lambda)\mathcal{F}_{\alpha,\beta}(g)(\lambda), \ \forall \lambda \in supp\,(\Pi_{\alpha,\beta}).$$

Acknowledgment The authors are very grateful to the anonymous referees for their valuable comments and interesting suggestions.

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