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Jagged Partitions

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Abstract. By jagged partitions we refer to an ordered collection of non-negative integers $(n_1, n_2, \dots, n_m)$ with $n_m \geq p$ for some positive integer p, further subject to some weakly decreasing conditions that prevent them for being genuine partitions. The case analyzed in greater detail here corresponds to $p = 1$ and the following conditions $n_i \geq n_{i+1} - 1$ and $n_i \geq n_{i+2}$. A number of properties for the corresponding partition function are derived, including rather remarkable congruence relations. An interesting application of jagged partitions concerns the derivation of generating functions for enumerating partitions with special restrictions, a point that is illustrated with various examples.

Key words: partitions, generating functions, congruence relations

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1. Introduction

A new type of 'partitions' (dubbed 'jagged' for a reason to be explained shortly) recently arose in the context of a conformal field-theoretical problem, namely the construction of a quasi-particle basis for graded parafermionic theory with $\mathbb{Z}_K$ cyclic symmetry. These jagged partitions are ordered sequences of non-negative integers:

$$(n_1, n_2, \dots, n_m) \qquad n_j \geq 0, \quad n_m \geq 1, \tag{1}$$

satisfying the weakly decreasing conditions

$$n_j \geq n_{j+1} - 1 \qquad \text{and} \qquad n_j \geq n_{j+2}. \tag{2}$$

In their original appearance context, the jagged partitions were further subject to an exclusion condition: $n_i \geq n_{i+K-1} + 1$ or $n_i = n_{i+K-1}$ and $n_{i+1} = n_{i+K-2} + 2$, with K even. The generating function for these restricted jagged partitions has been obtained in [4] while the corresponding one for K odd was found in [10]. This counting problem appeared to be a rather direct generalization of the enumeration of ordinary partitions $(\lambda_1, \lambda_2, \dots, \lambda_m)$ subject to the restriction $\lambda_i \geq \lambda_{i+k-1} + 2$ solved by Andrews [2].

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Our previous results strongly suggest that these new 'partitions' (in their unrestricted versions) could have rather nice properties. One of our goal in this work is to pinpoint some of them. In particular, using the previously obtained generating function for $j(n)$, the number of jagged partitions of n, we derive a recurrence relation together with a number of simple congruence properties for $j(n)$ itself, the most spectacular being that $j(8n+7) \equiv 0$ mod 64. We also show that $j(n)$ is given by the Cauchy product of $p(m-n)$, the number of partitions of $n-m$, and $d(m)$, the number of partitions of m into distinct parts, a result that entails an exact expression for $j(n)$. These results are presented in Section 2. Moreover, by adapting to our case the Ramanujan's method for obtaining the generating function of $p(5n+4)$, we have derived a number of interesting generating functions for $j(rn+s)$ with fixed r and s. These results are reported in Section 3.

Another aim of the present work is to explore the use of jagged partitions as a tool for enumerating standard partitions satisfying special restrictions. Let us explain in which sense that can be done. There exists a simple bijection between unrestricted jagged partitions (that is, vectors satisfying (1) and (2)) and partitions subject to a 'difference-two condition at distance 2' : $\lambda_i \geq \lambda_{i+2} + 2$ (the jagged partitions being simply augmented by the addition of a staircase) [4, 10]. The corresponding generating functions are consequently related in a simple way. Here we stress that phrasing the counting problem in terms of jagged partitions induces an important simplification within our working framework [4]. This method amounts to derive the generating function as the solution of a recurrence relation, a technique inspired from that of Andrews [2] for obtaining recurrence relations for restricted partitions. When formulated in terms of jagged partitions, this method leads to a first-order recurrence relation, while its reformulation directly in terms of ordinary partitions leads to a third-order recurrence relation. This point is discussed in full detail in Section 4, where we also highlight a convenient pictorial tool for deriving recurrence relations.

This suggests that one could similarly count generalized jagged partitions to obtain the generating functions for partitions with other interesting special restrictions.

But in order to see how jagged partitions could naturally be generalized, let us emphasis some salient features of the Definition (1)–(2). We see from (2) that the vector $(n_1, \ldots, n_m)$ is not a genuine partition because the non-decreasing condition is not satisfied: an increment by one unit from n_j to n_{j+1} is allowed. However a further increase by one unit from n_{j+1} to n_{j+2} is ruled out by the second condition in (2). Note also that if the last entry is 1, zero entries are allowed. For instance, the set of jagged partitions of weight ≤ 7 and length 5 (the length and the weight being respectively the number of parts m and the sum of the parts) is

$$\begin{array}{cccccc} (10101) & (20101) & (11101) & (30101) & (21101) & (11111) \\ (12101) & (40101) & (31101) & (22101) & (21111) & (12111) \\ (21201) & (50101) & (41101) & (32101) & (31111) & (22111) \\ (23101) & (21211) & (12121) & (31201) & (22201). & \end{array} \tag{3}$$

Note that the lowest-weight jagged partition of given length has the form $(\cdots 01010101)$. The jagged character of this 'ground-state partition' accounts for their name.

In view of introducing hierarchies of novel jagged partitions, it is appropriate to introduce a more precise and, at the same time, more flexible terminology for those jagged partitions we have been discussing so far. Given that they are characterized by their ground state, these can be conveniently called 01-partitions. The 01 notation indicates that these are the (pseudo) partitions built on the ground state whose period is 01. Natural extensions are thus 02-partitions or 001-partitions, etc. In this terminology, ordinary partitions are 1-partitions. In Section 5, we present the generating functions for 02-, 012-, 001-partitions with fixed length and apply them to special enumeration problems for ordinary partitions.

2. Basic properties of 01-partitions functions

Definition 1. A 01-partition of n is a weakly decreasing sequence of non-negative integers $(n_1, n_2, \cdots)$ where $n = \sum_i n_i$, such that the last entry is non-zero and the conditions (2) are satisfied.

Definition 2. The function $j(n)$ is the number of 01-partitions of n.

Theorem 3. *The generating function for the* 01*-partitions is*

$$J(q) = \sum_{n=0}^{\infty} j(n)q^n = \frac{(-q)_\infty}{(q)_\infty}. \tag{4}$$

Proof: This follows from [4], Eq. (5.17), with $z = 1$, using $(a)_n := (a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. Another proof is presented in Section 4. □

Corollary 4. *The partition function $j(n)$ satisfies the recurrence relation*

$$j(n) = 2\sum_{m\geq 1}(-1)^{m+1} j(n-m^2), \qquad n \geq 1. \tag{5}$$

Proof: This follows from the reexpression of (4) as (cf. [3], the equation following (10.6.6))

$$J(q) = \frac{1}{\theta_4(q)}, \tag{6}$$

where

$$\theta_4(q) = 1 + 2\sum_{m\geq 1}(-1)^m q^{m^2}. \tag{7}$$

One then has

$$1 = J(q)\frac{1}{J(q)} = \sum_{n=0}^{\infty} j(n)q^n \left(1 + 2\sum_{m\geq 1}(-1)^m q^{m^2}\right), \tag{8}$$

from which the result follows. □

Corollary 5. *The partition function $j(n)$ is related to $p(n)$, the number of partitions of n and $d(n)$ the number of partitions of n into distinct parts*, *by*

$$j(n) = \sum_{m=0}^{n} p(n-m)d(m). \tag{9}$$

Proof: The generating functions for $p(n)$ and $d(n)$ are respectively

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q)_\infty}, \qquad D(q) = \sum_{n=0}^{\infty} d(n)q^n = (-q)_\infty. \tag{10}$$

We have thus

$$J(q) = D(q)P(q), \tag{11}$$

which implies (9). □

Remark 6. Given that closed-form expressions are known for both $p(n)$ and $d(n)$, due to Hardy-Ramanujan-Rademacher (cf. theorem 5.1 in [1]) and Hau-Iseki-Hagis respectively (cf. Ex 5.3 therein), this entails a closed-form expression for $j(n)$. We recall that at the origin of the Hardy-Ramanujan collaboration on partition problems is rooted in the fruitful 'Ramanujan's false statement' ([11] p. 9) that the coefficient of q^n in $1/\theta_4(q)$, which is precisely $j(n)$, is given by

$$\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right). \tag{12}$$

This remains a remarkable approximation.

The expression of the generating function $J(q)$ in terms of the inverse of θ_4 leads immediately to two simple congruence properties.

Corollary 7. *$j(n) \equiv 0$ mod 2 for $n > 1$.*

Proof: Equation (8) can be written as

$$1 = J(q) + 2J(q)\sum_{n\geq 1}(-1)^n q^{n^2}, \tag{13}$$

which implies (using (4))

$$J(q) = 1 + 2J(q)\sum_{n\geq 1}(-1)^{n+1}q^{n^2} = 1 + 2\,\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}(-1)^{n+1}q^{n^2}. \tag{14}$$

01-partition functions are thus always even except for $j(0) = 1$. □

Corollary 8. *$j(n) \equiv 0$ mod 2^{p+1} for all n that cannot be written as a sum of p or less squares.*

Proof: Iterating the first equality in (14), we obtain

$$J(q) = 1 + \sum_{p=1}^{\infty} 2^p \sum_{n_1,n_2,..n_p\geq 1} (-1)^{\sum_{i=1}^{p}(n_i+1)}\, q^{\sum_{i=1}^{p} n_i^2}, \tag{15}$$

from which the statement is immediate.

This result is meaningful only for $p \leq 3$ since, by Lagrange theorem, all numbers can be written as a sum of four squares. Here is a simple application of the corollary: since $(8n+7)$ cannot be written as a sum of less that 4 squares, it readily follows that $j(8n+7) \equiv 0$ mod 16. □

Corollary 9. *If p' stands for the least number of squares into which $rn+s$, with $1 \leq s < r$, is decomposable, then $j(rn+s) \equiv 0$ mod $a\, 2^{p'}$, where $a = \min(c, 2)$ with c being the number of distinct vectors $(n'_1, \ldots, n'_{p'})$ of strictly positive entries which sum to s.*

Proof: The least number of squares into which a number can be decomposed is easily found by inspection of the non-zero values of m^2 modulo r, which are $(1, 4, \ldots, (r-1)^2)$ mod r. The number of solutions of $s = n'_1 + \cdots + n'_{p'}$, where the n'_i are square residues mod r (i.e., $1 \leq n'_i < r$), is a divisor of the number of equivalent terms in the first summation in (15). For $c > 2$, the p'-square contribution might not be the lowest one and we need to consider also the contribution of $p'+1$ squares; therefore the bound obtained by the lowest contributing term is actually $a = \min(c, 2)$.

Let us consider some applications of this corollary. An immediate consequence is that $j(rn+2) \equiv 0$ mod 4 when $r > 2$. Indeed, these numbers can be decomposed in two squares, not less, hence $p' = 2$, and this can be done in one way mod r $(2 = 1+1)$, so that $c = 1$. Similarly, $j(rn+3) \equiv 0$ mod 4 when $r > 3$ since then $p' = 3$ and the combinatorial factor c is again 1. In the same vein, it follows that $j(rn+5) \equiv 0$ mod 8 when $r > 5$ since $rn+5$ can be decomposed as two squares and $s = 4+1 = 1+4$ so that $c = 2$. Our final example deserves to be singled out. □

Corollary 10. $j(8n+7) \equiv 0$ mod 64.

Proof: We have already noticed that the first coefficient of (15) contributing to $j(8n+7)$ comes from the decomposition into 4 squares. But it has a combinatorial factor at least equal to 4 since $7 = 4+1+1+1$ and its three permutations. It is larger than 4 whenever the three residues $n'_i = 1$ are residues of distinct numbers n_i. On the other hand, since $j(8n+7)$ cannot be decomposed into five squares, the factor a in Corollary 9 can be replaced by $a' = \min(c, 4) = 4$.

Here is an illustration of this congruence from a direct computation based on (15):

$$j(15) = -2^4 \cdot 12 - 2^6 \cdot 20 + 2^7 \cdot 7 + 2^9 \cdot 36 - 2^{12} \cdot 12 + 2^{15} = 64 \cdot 23. \tag{16}$$

The number of contributing terms from the first summation is 12 since there are 12 ways of writing $15 = \sum_{i=1}^{4} n_i^2 = 9+4+1+1$ and permutations; modulo 8, this equation reduces to $7 = 4+1+1+1$ and permutations; hence here there are three n'_i equal to 1 but the corresponding n_i's are all different (they are permutations of (9, 4, 1)). □

3. Ramanujan-type generating functions for 01-partitions

In this section, we derive special generating functions analogous to the famous Ramanujan result (cf. [1] chap 10):

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q)_\infty^6}, \tag{17}$$

which implies that $p(5n+4) \equiv 0 \bmod 5$.

We have calculated the generating functions for all 01–partitions $j(rn+s)$ with $2 \le r \le 8$. Those for $r = 2, 3, 4$ take a rather compact form:

Proposition 11. *The generating functions for the* 01*-partitions* $j(rn+s)$, $2 \le r \le 4$ *read*

$$\begin{aligned}
&\sum_{n=0}^{\infty} j(2n)q^n = \frac{(q^4;q^4)_\infty^5}{(q)_\infty^4(q^8;q^8)_\infty^2}, && \sum_{n=0}^{\infty} j(2n+1)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q)_\infty^4(q^4;q^4)_\infty},\\
&\sum_{n=0}^{\infty} j(3n)q^n = \frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^6}{(q)_\infty^8(q^6;q^6)_\infty^3}, && \sum_{n=0}^{\infty} j(3n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3}{(q)_\infty^7},\\
&\sum_{n=0}^{\infty} j(3n+2)q^n = 4\,\frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^3}{(q)_\infty^6}, && \sum_{n=0}^{\infty} j(4n)q^n = \frac{(q^2;q^2)_\infty^{19}}{(q)_\infty^{14}(q^4;q^4)_\infty^6},\\
&\sum_{n=0}^{\infty} j(4n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^{13}}{(q)_\infty^{12}(q^4;q^4)_\infty^2}, && \sum_{n=0}^{\infty} j(4n+2)q^n = 4\,\frac{(q^2;q^2)_\infty^7(q^4;q^4)_\infty^2}{(q)_\infty^{10}},\\
&\sum_{n=0}^{\infty} j(4n+3)q^n = 8\,\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^6}{(q)_\infty^8}.
\end{aligned} \tag{18}$$

Note that from these expressions, we obtain three more formulas for $J(q)$, the simplest one being:

$$\begin{aligned}
J(q) &= \sum_{n=0}^{\infty}[j(2n)q^{2n} + j(2n+1)q^{2n+1}]\\
&= \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^4(q^{16};q^{16})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty}\ .
\end{aligned} \tag{19}$$

The last result we make explicit is the following, which provides an alternative proof of Corollary 10*.*

Proposition 12. *The generating functions for the* 01*-partitions* $j(8n+7)$ *is*

$$\sum_{n=0}^{\infty} j(8n+7)q^n = 64\,\frac{(q^2;q^2)_\infty^{22}}{(q)_\infty^{23}}. \tag{20}$$

Sketch of the proof of Propositions 11 and 12. Our proof of these relations is inspired by the derivation of (17) given in [3] Section 11.7. The proof is divided into three steps.

1. We first compute the ratio $J(q^{1/r})/J(q^r)$. We keep the denominator in the form of a product and break the inverse of $J(q^{1/r})$,

$$\frac{1}{J(q^{1/r})} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/r}, \tag{21}$$

into r sums (setting $n = rm + \ell$, with $0 \le \ell \le r-1$) and evaluate each sum by means of the Jacobi triple-product identity:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} z^n = (qz)_\infty \, (z^{-1})_\infty \, (q)_\infty. \tag{22}$$

That leads us to an expression of the form:

$$\frac{J(q^{1/r})}{J(q^r)} = \frac{1}{\sum_{\ell=0}^{r-1} \xi_\ell(q) q^{\ell/r}}. \tag{23}$$

The different $\xi_\ell(q)$'s are now known functions of q. The division by $J(q^r)$ is purely conventional, its role being to simplify the form of the 'coefficients' $\xi_\ell(q)$. In particular, for r odd, it makes $\xi_0(q) = 1$. By construction, the $\xi_\ell(q)$'s are invariant under the transformation $q^{1/r} \to q^{1/r}\omega^k$, with $\omega = e^{2\pi i/r}$.

2. We then observe that by rewriting

$$\frac{1}{\sum_{\ell=0}^{r-1} \xi_\ell(q)\, q^{\ell/r}} = \frac{\prod_{k=1}^{r-1} \sum_{\ell=0}^{r-1} \xi_\ell(q)\, \omega^{k\ell}\, q^{\ell/r}}{\prod_{k=0}^{r-1} \sum_{\ell=0}^{r-1} \xi_\ell(q)\, \omega^{k\ell}\, q^{\ell/r}} =: \frac{\prod_{k=1}^{r-1} \sum_{\ell=0}^{r-1} \xi_\ell(q)\, \omega^{k\ell}\, q^{\ell/r}}{\Omega_r(q)}, \tag{24}$$

the denominator becomes a function of q and not of $q^{1/r}$. This denominator is evaluated from (23), but using this times the product form of both $J(q^{1/r})$ and $J(q^r)$:

$$\frac{1}{\Omega_r(q)} = \frac{1}{[J(q^r)]^r} \prod_{k=0}^{r-1} J(\omega^k q^{1/r}). \tag{25}$$

The evaluation of $\Omega_r(q)$ relies on the lemma that follows this sketch.

3. Finally, in order to select the terms $j(rn - s')$ from $J(q^{1/r})$, we multiply it by $q^{s'/r}$, replace $q^{1/r}$ by $\omega^k q^{1/r}$ and then sum over all values of k. For $J(q^{1/r})$, we use formula (23). The same operation of the rhs selects the $q^{s'/r+m}$ terms in the numerator (with m integer). The final result follows by multiplying both sides by $J(q^r)$.

Lemma 13. *The product* $\prod_{k=0}^{r-1} J(\omega^k q^{1/r})$, *for* r *odd and prime or* $r = 2^p$ *takes the following form*:

$$\prod_{k=0}^{r-1}\prod_{n=1}^{\infty} \frac{1+(\omega^k q^{1/r})^n}{1-(\omega^k q^{1/r})^n} = \frac{(q^2;q^2)_\infty^{r+1}\,(q^r;q^r)_\infty^2}{(q)_\infty^{2r+2}\,(q^{2r};q^{2r})_\infty} \quad \text{for } r \text{ odd and prime,}$$
$$= \frac{(q^2;q^2)_\infty^r}{(q)_\infty^{2r}} \quad \text{for } r = 2^p. \tag{26}$$

Proof: Set $n = rm + s$ with $0 < s \le r$ and $m \ge 0$. In the case $r = 2^p$, for given values of m and $s \ne r$, the r terms (resulting from the product over k) of the numerator are easily seen to be permutations of those of the denominator so that the product is 1; for $s = r$, the k dependence disappear (all r terms are identical) and the product over m yields $(-q)_\infty^r/(q)_\infty^r$. For r prime and $\ne 2$, we find that for m fixed:

$$\prod_{s=1}^{r}\prod_{k=0}^{r-1} \frac{1+(\omega^k q^{1/r})^{rm+s}}{1-(\omega^k q^{1/r})^{rm+s}}$$
$$= \frac{(1+q^{rm+1})\cdots(1+q^{rm+r-1})(1+q^{m+1})^r}{(1-q^{rm+1})\cdots(1-q^{rm+r-1})(1-q^{m+1})^r} \cdot \frac{(1+q^{rm+r})(1-q^{rm+r})}{(1-q^{rm+r})(1+q^{rm+r})}, \tag{27}$$

(where at the end we have introduced a suitable decomposition of 1). The quoted result is obtained by taking the product over all m and reorganizing the sums using repeatedly the simple identity:

$$(-q^c;q^c)_\infty = \frac{(q^{2c};q^{2c})_\infty}{(q^c;q^c)_\infty}. \tag{28}$$

These cases cover all those (namely $r = 3,\ 2^1,\ 2^2$ and 2^3) needed to tackle Propositions 11 and 12. □

Proof of Proposition 11 for $r = 3$. Let us now detail the case $r = 3$. By decomposing the sum in (21) into three sums according to $n = 3m, 3m \pm 1$ and transform them by the Jacobi triple-product identity:

$$\frac{1}{J(q^{1/3})} = \sum_{m=-\infty}^{\infty} (-1)^m q^{3m^2} - 2q^{1/3} \sum_{m=-\infty}^{\infty} (-1)^m q^{3m^2+2m}$$
$$= \frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty} - 2q^{1/3} \frac{(q)_\infty\,(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty\,(q^3;q^3)_\infty}, \tag{29}$$

we end up with

$$\frac{J(q^{1/3})}{J(q^3)} = \frac{1}{1-2q^{1/3}\zeta_1(q)}, \tag{30}$$

where

$$\zeta_1(q) = \frac{(q)_\infty\,(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty\,(q^3;q^3)_\infty^3}. \tag{31}$$

The function $\zeta_1(q)$ is manifestly unaffected by the replacement of $q^{1/3}$ by $q^{1/3}\omega^k$, with $\omega = e^{2\pi i/3}$. We thus have $\xi_0(q) = 1,\ \xi_2(q) = 0$ and we have redefined $\xi_1(q) = -2\zeta_1(q)$. This completes step 1. We now evaluate $\Omega_3(q)$:

$$\Omega_3(q) = \prod_{k=0}^{2}[1 - 2q^{1/3}\omega^k\zeta_1(q)] = 1 - 8q\zeta_1^3(q), \tag{32}$$

while, from (25) and (26), we have

$$\frac{1}{\Omega_3(q)} = \frac{(q^2;q^2)_\infty^4,\,(q^3;q^3)_\infty^8}{(q)_\infty^8(q^6;q^6)_\infty^4}. \tag{33}$$

We can thus write

$$\frac{1}{1 - 2q^{1/3}\zeta_1(q)} = \frac{\prod_{k=1}^{2}\left[1 - 2q^{1/3}\omega^k\zeta_1(q)\right]}{1 - 8q\zeta_1^3(q)} = \frac{1 + 2q^{1/3}\zeta_1(q) + 4q^{2/3}\zeta_1^2(q)}{\Omega_3(q)}, \tag{34}$$

and by (30), this is equal to $J(q^{1/3})/J(q^3)$:

$$J(q^{1/3}) = \sum_{n=0}^{\infty} j(n)\,q^{n/3} = \frac{J(q^3)}{\Omega_3(q)}\big(1 + 2q^{1/3}\zeta_1(q) + 4q^{2/3}\zeta_1^2(q)\big). \tag{35}$$

Let us now replace $q^{1/3}$ by $q^{1/3}\omega^k$ on both sides and sum the resulting equation over k. On the lhs, we get

$$\sum_{k=0}^{2}\sum_{n=0}^{\infty} j(n)\,(q^{1/3}\omega^k)^n = \sum_{n=0}^{\infty} j(n)\,q^{n/3}(1 + \omega^n + \omega^{2n}) = 3\sum_{m=0}^{\infty} j(3m)\,q^m, \tag{36}$$

while on the rhs, we find

$$\sum_{k=0}^{2}\big(1 + 2q^{1/3}\,\omega^k\,\zeta_1(q) + 4q^{2/3}\,\omega^{2k}\,\zeta_1^2(q)\big) = 3, \tag{37}$$

so that

$$\sum_{m=0}^{\infty} j(3m)\,q^m = \frac{(q^2;q^2)_\infty^4\,(q^3;q^3)_\infty^6}{(q)_\infty^8\,(q^6;q^6)_\infty^3}. \tag{38}$$

By multiplying (35) by $q^{1/3}$, replacing $q^{1/3}$ by $q^{1/3}\omega^k$ on both sides and summing over k, we get

$$\sum_{m=0}^{\infty} j(3m-1)\,q^m = q\sum_{m=0}^{\infty} j(3m+2)\,q^m = 4q\,\frac{(q^2;q^2)_\infty^2\,(q^6;q^6)_\infty^3}{(q)_\infty^6}. \tag{39}$$

The generating function for $j(3n+1)$ is obtained in a similar way, by multiplying (35) by $q^{2/3}$, $q^{1/3} \to q^{1/3}\omega^k$ and sum over k.

Proof of Proposition 12. It is convenient to start with $J(q^{1/4})$:

$$\frac{1}{J(q^{1/4})} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/4} = \sum_{n=-\infty}^{\infty} q^{4n^2} - 2q^{1/4}\sum_{n=-\infty}^{\infty} q^{2n(2n+1)} + q\sum_{n=-\infty}^{\infty} q^{4n(n+1)}. \tag{40}$$

Using the Jacobi triple-product identity, this becomes

$$\frac{1}{J(q^{1/4})} = \frac{(q^8;q^8)_\infty^5}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2} - 2q^{1/4}\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} + 2q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}. \tag{41}$$

Therefore $J(q^8)/J(q^{1/8})$ can be written as:

$$\frac{J(q^8)}{J(q^{1/8})} = \zeta_0 - 2q^{1/8}\zeta_1 + 2q^{1/2}\zeta_4, \tag{42}$$

with

$$\zeta_0 = \frac{(q^4;q^4)_\infty^5(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4}, \quad \zeta_1 = \frac{(q^2;q^2)_\infty^2(q^{16};q^{16})_\infty}{(q;q)_\infty(q^8;q^8)_\infty^2}, \quad \zeta_4 = \frac{(q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}. \tag{43}$$

Using (24), we can write

$$\frac{J(q^{1/8})}{J(q^8)} = \frac{q^{7/8}\alpha(\zeta_0,\zeta_1,\zeta_4)+\cdots}{\Omega_8(q)}, \tag{44}$$

where we only write the coefficient of the terms $q^{7/8+m}$ for m integer, which is

$$\alpha(\zeta_0,\zeta_1,\zeta_4) = 64\big(2\zeta_1^7 - \zeta_0^3\zeta_1^3\zeta_4 - 4q\zeta_0\zeta_1^3\zeta_4^3\big), \tag{45}$$

while

$$\frac{1}{\Omega_8(q)} = \frac{(q^2;q^2)_\infty^8(q^8;q^8)_\infty^{16}}{(q;q)_\infty^{16}(q^{16};q^{16})_\infty^8}. \tag{46}$$

Note that $\Omega_8(q)$ is actually a function of q. For $j(8n+7)$, we have thus:

$$\frac{(q^8;q^8)_\infty^2}{(q^{16};q^{16})_\infty}\sum_{n=0}^{\infty} j(8n+7)q^n = 64\big(2\zeta_1^7 - \zeta_0^3\zeta_1^3\zeta_4 - 4q\zeta_0\zeta_1^3\zeta_4^3\big)\frac{(q^2;q^2)_\infty^8(q^8;q^8)_\infty^{16}}{(q;q)_\infty^{16}(q^{16};q^{16})_\infty^8}. \tag{47}$$

The result follows from the identity

$$\zeta_0^3\zeta_1^3\zeta_4 + 4q\zeta_0\zeta_1^3\zeta_4^3 = \zeta_1^7, \tag{48}$$

which is established in the appendix. □

4. Generating function for 01-partitions of prescribed length

In this section, we first rederive the generating function for 01-partitions by using a modified version of a technique introduced in App. A of [4] . We first motivate the method by deriving a recurrence relation for ordinary partitions $p(m, n)$ of n into m parts in the language of Ferrer graphs. Every partition of length m of n can be represented by graphs of n dots distributed among m rows whose length (specified by the parts of the partition) does not increase when read from top to bottom. The set of such Ferrer graphs of m rows can be decomposed into the disjoint union of those graphs with precisely one dot on the last row and those with more that one dot on this m-th row. The former set is characterized by the fact that removing the single dot on the m-th row leaves a Ferrer graph with $m-1$ rows (and of course a total of $n-1$ dots). Similarly, the graphs with two dots or more on the m-th row are characterized by the fact that they still have m rows if we take out the first column (reducing n to $n-m$). That leads directly to the following recurrence relation

$$p(m, n) = p(m-1, n-1) + p(m, n-m). \tag{49}$$

Instead of working with graphs, we will introduce a pictorial representation that captures the notion of 'building on a ground state with prescribed structure'. Let us then represent $p(m, n)$ as

$$p(m, n): \ (\cdots 11111^+) \qquad (m \text{ entries}), \tag{50}$$

where the symbol $+$ indicates that we can build up the partition on the indicated ground state up to the position of the $+$. It is clear that this 'filling process' can be broken as follows:

$$(\cdots 11111^+) = (\cdots 1111^+1) + (\cdots 22222^+), \tag{51}$$

(from now on, all vectors are supposed to have m entries). In the first term of the left hand side, we isolate those partitions that have at least one 1. What is left is completely taken into account by the set $(\cdots 22222^+)$, which describes all partitions without any 1. But this pictorial relation is nothing but the Ferrer graph retranscription of the decomposition considered above. Indeed, elements of the set $(\cdots 22222^+)$ are in a one-to-one correspondence with those enumerated by $p(m, n-m)$: the correspondence is obtained by subtracting the partition (1^m) from elements of the former set. Similarly, there is a one-to-one correspondence between elements of $(\cdots 1111^+1)$ and the partitions of length $m-1$ and weight $n-1$: one simply adds a 1 at the extremity of the elements of the latter set, an operation that preserves the non-increasing character of the partition. Therefore (51) is equivalent to (49).

The recurrence relation (49) is far from being new: it has first been obtained by Euler (see e.g., [12] Eq. (97.7) but quoted with a misprint) from the generating function $P_m(q) =$

$\sum_n p(m,n)q^n = q^m/(q)_m$. The relation follows from the identity $(1-q^m)P_m = qP_{m-1}$. Euler used it to compute recursively the $p(m,n)$'s. Our aim is just the opposite: we want to find generating functions by solving recurrence relations.

Our 'pictorial' derivation of (49) is inspired by the method used by Andrews to find the generating function for partitions with prescribed length and 'difference-two at distance $k-1$' [2]. The idea is to split the set we want to enumerate into distinct sets characterized by their boundary condition. The method is here adapted to the case where the restriction is washed out.

Definition 14. The function $j(m,n)$ yields the number of 01-partitions of n having exactly m parts (i.e., of length m).

Lemma 15. *The function $j(m,n)$ satisfies the recurrence relations*:

$$\begin{aligned} j(m,n) &= j(m-2,n-1) + k(m,n) \\ k(m,n) &= k(m-1,n-1) + j(m,n-m), \end{aligned} \tag{52}$$

where $k(m,n)$ stands for the number of 01-partitions of n having exactly m parts ≥ 1 (i.e., having no 0).

Proof: The functions $j(m,n)$ and $k(m,n)$ have the following pictorial representations:

$$j(m,n):\ (\cdots 010101^+), \qquad k(m,n):\ (\cdots 111111^+). \tag{53}$$

(This last set should not be confused with the set of ordinary partitions of n of length m; $k(m,n)$ really counts special 01-partitions: for instance (1212) is an element of the set whose cardinality is $k(4,6)$.) The following relations are satisfied

$$\begin{aligned} (\cdots 010101^+) &= (\cdots 0101^+01) + (\cdots 111111^+) \\ (\cdots 111111^+) &= (\cdots 11111^+1) + (\cdots 121212^+). \end{aligned} \tag{54}$$

In the first relation, we isolate the terms with at least one pair 01 at the end, i.e., containing at least one 0. What is left is the set of 01-partitions without parts equal to 0, hence built on $(\cdots 111)$. The second relation is similar. These are equivalent to the recurrence relations (52). Let us explain the last term: the jagged partitions $(\cdots 121212^+)$ can be characterized by the following property: subtracting from these elements the partition (1^m) (reducing the weight by m but without affecting the length) makes them elements of the set enumerated by $j(m,n-m)$. The functional retranscription of the other terms is obvious. □

Theorem 16. *The generating function of $j(m,n)$ is*

$$J(z;q) = \frac{(-zq)_\infty}{(z^2q)_\infty}. \tag{55}$$

Proof: Let us introduce the two generating functions

$$J(z;q) = \sum_{m,n=0}^{\infty} j(m,n) z^m q^n, \qquad K(z;q) = \sum_{m,n=0}^{\infty} k(m,n) z^m q^n. \tag{56}$$

The recurrence relations (52) are then lifted to

$$\begin{aligned} J(z;q) &= z^2 q J(z;q) + K(z;q) \\ K(z;q) &= zq K(z;q) + J(zq;q). \end{aligned} \tag{57}$$

Since the q-dependence is never affected, it is usually omitted. The solution of the above relations is easily obtained

$$J(z) = \frac{K(z)}{1-z^2 q} = \frac{J(zq)}{(1-zq)(1-z^2 q)} = \cdots = \frac{(-zq)_\infty}{(z^2 q)_\infty}, \quad K(z) = \frac{(-zq)_\infty}{(z^2 q^2)_\infty}, \tag{58}$$

(where the dots stand for the infinite iteration of the preceding result).

Note that in contrast with $p(m,n)$, the function $j(m,n)$ does not necessarily vanish if $n < m$. For instance $j(4,2) = 1$ and the corresponding 01-partition is (0101). As a further illustration, the q^3 coefficient of $J(z)$ obtained by expanding (55) and the corresponding 01-partitions are:

$$z + 2z^2 + 2z^3 + z^4 + z^5 + z^6: \quad \{(3), (21), (12), (201), (111), (1101), (10101), (010101)\}. \tag{59}$$

The following lemma presents an application of the previous result to the enumeration of 01-partitions with at most m parts. Recall that $p_m(n)$, the number of partitions of n into at most m parts, is equal to $p(m, n+m)$. The analogous relation in the jagged case is given in the following lemma. □

Lemma 17. *The partition function $j_m(n)$ that counts the number of 01-partitions of n with at most m parts satisfies*

$$j_m(n) = j(m, n+m) - j(m-2, n+m-1) = k(m, n+m), \tag{60}$$

where $k(m,n)$ is defined in Lemma 15.

Proof: We have

$$j(m, n+m) = j_m(n) + j(m-2, n+m-1). \tag{61}$$

Indeed, by adding the partition (1^m) to those counted by $j_m(n)$ we obtain all 01-partitions of weight $n+m$ having no 0. To generate the whole set of 01-partitions of weight $n+m$ we simply need to add to this set all those 01-partitions that contain at least one 0, that is, one pair of 01. By stripping this tail 01, we see that all these elements are in correspondence with those counted by $j(m-2, n+m-1)$. This yields the above relation. The derivation makes clear that $j_m(n)$ is exactly the set $k(m, n+m)$. Moreover, (60) is simply the first relation in (52).

Let us recall the bijection introduced in [10]: by adding the staircase $(m-1, m-2, \ldots, 1, 0)$ to the vector $(n_1, \cdots n_m)$, we transform it into an ordinary partition. With $\lambda_j = n_j + m - j$, the weakly decreasing conditions (2) become

$$\lambda_j \geq \lambda_{j+1} \qquad \text{and} \qquad \lambda_j \geq \lambda_{j+2} + 2. \tag{62}$$

To transform a generating function for jagged partitions to one for partitions subject to (62), we replace z^N by $z^N q^{N(N-1)/2}$ within the sum defining the generating function. We thus reproduce the result of Andrews [2] (his $F_{3,3}(z;q)$) as follows (cf. Corollary 9 of [10]): □

Corollary 18. *The generating function for the number of partitions satisfying $\lambda_j \geq \lambda_{j+2} + 2$ is given by*

$$\sum_{m,n\geq 0} j(m,n) z^m q^{m(m-1)/2+n} = \sum_{m_0,m_1\geq 0} \frac{q^{(m_0+m_1)^2+m_1^2} z^{m_0+2m_1}}{(q)_{m_0}(q)_{m_1}}. \tag{63}$$

Remark 19. Let us argue that the jagged-partition formulation of the problem is efficient by reworking the counting problem directly at the level of partitions. Let us then introduce three sets of partitions subject to (62):

$$a(m,n) : (\cdots 553311^+), \quad b(m,n) : (\cdots 55331^+), \quad c(m,n) : (\cdots 765432^+). \tag{64}$$

These sets obey the recurrence relations

$$\begin{aligned} a(m,n) &= a(m, n-m) + b(m-1, n-1) \\ b(m,n) &= a(m-1, n-2m+1) + c(m,n) \\ c(m,n) &= a(m, n-2m) + c(m-1, n-m-1). \end{aligned} \tag{65}$$

For their generating functions (defined in the obvious way), this becomes

$$\begin{aligned} A(z) &= A(zq) + zqB(z) \\ B(z) &= zqA(zq^2) + C(z) \\ C(z) &= A(zq^2) + zq^2C(zq). \end{aligned} \tag{66}$$

Solving for $A(z)$, we end up with

$$A(z) = (1+zq)A(zq) + z^2q^2A(zq^2) - z^3q^5A(zq^3). \tag{67}$$

Although there are ways to solve this relation (using suitable transformations), ending up with a third-order q-equation instead of a first-order one is enough to illustrate our point: reformulating the problem in terms of jagged partitions bear some magic simplifications.

5. Generating function for generalized jagged partitions of prescribed length

We now consider three types of generalized jagged partitions, for which we also find generating functions by solving a first-order recurrence relation. Since the construction of these generating functions follow the same pattern as for the 01-case, we condense the presentation by including the derivations of the recurrence relations within the proof of the theorems in the first two cases and limit ourself to displaying the key recurrence relation and the final result in the last case.

5.1. 02-partitions

Definition 20. 02-partitions are defined as vectors $(n_1, \ldots, n_m)$ of non-negative entries $n_j \geq 0$,with $n_m \geq 2$, subject to the weak ordering conditions:

$$n_j \geq n_{j+1} - 2, \quad n_j \geq n_{j+2}. \tag{68}$$

Theorem 21. *The generating function for $\mathcal{J}(m, n)$, the number of* 02*-partitions of n with length m, is*

$$\tilde{\mathcal{J}}(z) = \sum_{n,m\geq 0} \mathcal{J}(m, n) z^m q^n = \frac{(z^3q^6; q^3)_\infty}{(zq^2)_\infty(z^2q^2)_\infty}. \tag{69}$$

Proof: The first step of the proof amounts to derive appropriate recurrence relations. We introduce the following pictorial representation (function-set bijection) for the $\mathcal{J}(m, n)$ and two auxiliary functions $\mathcal{K}(m, n)$ and $\mathcal{L}(m, n)$

$$\mathcal{J}(m,n) : (\cdots 020202^+), \quad \mathcal{K}(m,n) : (\cdots 121212^+), \quad \mathcal{L}(m,n) : (\cdots 222222^+). \tag{70}$$

again with the understanding that all vectors have m components. While $\mathcal{J}(m, n)$ counts all 02-partitions, $\mathcal{K}(m, n)$ counts those with no 0 and $\mathcal{L}(m, n)$ enumerates those having neither 0 nor 1. The following relations hold between the different 'ground-state fillings':

$$\begin{aligned}
&(\cdots 0202^+) = (\cdots 02^+02) + (\cdots 1212^+)\\
&(\cdots 1212^+) = (\cdots 12^+12) + (\cdots 2222^+) + (\cdots 13^+13)\\
&(\cdots 2222^+) = (\cdots 222^+2) + (\cdots 2323^+).
\end{aligned} \tag{71}$$

These translate into the recurrence relations:

$$\begin{aligned}
\mathcal{J}(m,n) &= \mathcal{J}(m-2, n-2) + \mathcal{K}(m,n)\\
\mathcal{K}(m,n) &= \mathcal{K}(m-2, n-3) + \mathcal{L}(m,n) + \mathcal{J}(m-2, n-m-2)\\
\mathcal{L}(m,n) &= \mathcal{L}(m-1, n-2) + \mathcal{K}(m, n-m).
\end{aligned} \tag{72}$$

As usual, these recurrence relations are then lifted to q-difference equations for their generating functions and these take the form

$$\begin{aligned}
\tilde{\mathcal{J}}(z) &= z^2q^2\tilde{\mathcal{J}}(z) + \tilde{\mathcal{K}}(z)\\
\tilde{\mathcal{K}}(z) &= z^2q^3\tilde{\mathcal{K}}(z) + \tilde{\mathcal{L}}(z) + z^2q^4\tilde{\mathcal{J}}(zq)\\
\tilde{\mathcal{L}}(z) &= zq^2\tilde{\mathcal{L}}(z) + \tilde{\mathcal{K}}(zq).
\end{aligned} \tag{73}$$

The auxiliary functions $\tilde{\mathcal{K}}(z)$ and $\tilde{\mathcal{L}}(z)$ can be eliminated and we get

$$\tilde{\mathcal{J}}(z) = \frac{(1+zq^2+z^2q^4)}{(1-z^2q^2)(1-z^2q^3)}\tilde{\mathcal{J}}(zq) = \frac{(1-z^3q^6)}{(1-zq^2)(1-z^2q^2)(1-z^2q^3)}\tilde{\mathcal{J}}(zq), \quad (74)$$

whose solution is (69).

The expression (69) can be easily expressed as a q-series:

$$\tilde{\mathcal{J}}(z) = \sum_{m_1,m_2,m_3\geq 0} \frac{(-1)^{m_3} q^{2m_1+2m_2+3m_3(m_3+3)/2}\, z^{m_1+2m_2+3m_3}}{(q)_{m_1}(q)_{m_2}(q^3;q^3)_{m_3}}. \quad (75)$$

Although this is not a manifestly positive q-series, the positivity is inherited from the first equality in (74). To illustrate this result, we see that the coefficient of z^5 is $q^6+2q^7+4q^8+7q^9+\cdots$ and the seven corresponding 02-partitions of weight 9 are

$$\mathcal{J}(5,9) = 7 : \{(50202),\ (41202),\ (32202),\ (31212),\ (31302),\ (23202),\ (22212)\}. \quad (76)$$

□

Corollary 22. *The generating function for the number of partitions of n into m parts and subject to the restriction*

$$\lambda_i \geq \lambda_{i+1} \quad \text{and} \quad \lambda_i \geq \lambda_{i+2}+4, \quad (77)$$

is

$$P(z) = \sum_{m_1,m_2,m_3\geq 0} \frac{(-1)^{m_3} q^{2m_1+2m_2+3m_3(m_3+3)/2+(m_1+2m_2+3m_3)(m_1+2m_2+3m_3-2)}\, z^{m_1+2m_2+3m_3}}{(q)_{m_1}(q)_{m_2}(q^3;q^3)_{m_3}}. \quad (78)$$

Proof: 02-partitions are transformed into ordinary partitions by adding the staircase $(\cdots,9,7,5,3,1,-1)$:

$$\lambda_i = n_i + 2(m-i) - 1. \quad (79)$$

It is immediate to verify that the λ_i's satisfy (77). Hence, by modifying the power of q within $\tilde{\mathcal{J}}(z)$ to take into account the addition of the staircase, which amounts to replace $q^n z^m \to q^{n+m(m-2)} z^m$, we obtain the generating function $P(z)$ for partitions satisfying (77) in the form (78).

This expression appears to be a new result. It differs from the generating functions pertaining to the same counting problem presented in Theorem 5.14 of [7] and Theorem 2.7 of [6].

There is of course a whole tower of $0p$-partitions, with $p \geq 1$. We have seen that for $p=1$ and $p=2$, the generating functions for jagged partitions with specified length are solutions of a first-order q-equation. This does not seem to be the case for $p>2$. For instance, the counting of 03-partitions lead to a second-order equation. □

5.2. 012-partitions

We now turn to the analysis of jagged partitions of another type, namely the 012-partitions.

Definition 23. 012-partitions are defined as vectors $(n_1, \ldots, n_m)$ of non-negative entries $n_j \geq 0$,with $n_m \geq 2$, subject to the weak ordering conditions:

$$n_j \geq n_{j+1} - 1, \quad n_j \geq n_{j+2} - 2, \quad n_j \geq n_{j+3}. \tag{80}$$

Theorem 24. *The generating function for $\mathcal{J}'(m, n)$, the number of 012-partitions of n with length m, is*

$$\tilde{\mathcal{J}}'(z) = \sum_{n,m\geq 0} \mathcal{J}'(m, n) z^m q^n = \frac{(-zq^2)_\infty(-z^2q^3)_\infty}{(z^3q^3)_\infty}. \tag{81}$$

Proof: Introduce the following function-set bijections:

$$\begin{aligned} &\mathcal{J}'(m, n) : (\cdots 012^+), \quad \mathcal{K}'(m, n) : (\cdots 112^+), \quad \mathcal{L}'(m, n) : (\cdots 122^+), \\ &\mathcal{M}'(m, n) : (\cdots 212^+), \quad \mathcal{N}'(m, n) : (\cdots 222^+). \end{aligned} \tag{82}$$

These are related as follows

$$\begin{aligned} (\ldots 012012^+) &= (\ldots 012^+012) + (\ldots 112^+) \\ (\ldots 112112^+) &= (\ldots 112^+112) + (\ldots 122^+12) + (\ldots 122^+) \\ (\ldots 122122^+) &= (\ldots 212^+2) + (\ldots 123^+) \\ (\ldots 212212^+) &= (\ldots 122^+12) + (\ldots 222^+) \\ (\ldots 222222^+) &= (\ldots 222^+2) + (\ldots 223^+). \end{aligned} \tag{83}$$

The functional retranscription of these recurrences reads

$$\begin{aligned} \mathcal{J}'(m, n) &= \mathcal{J}'(m-3, n-3) + \mathcal{K}'(m, n) \\ \mathcal{K}'(m, n) &= \mathcal{K}'(m-3, n-4) + \mathcal{L}'(m-2, n-3) + \mathcal{L}'(m, n) \\ \mathcal{L}'(m, n) &= \mathcal{M}'(m-1, n-2) + \mathcal{J}'(m, n-m) \\ \mathcal{M}'(m, n) &= \mathcal{L}'(m-2, n-3) + \mathcal{N}'(m, n) \\ \mathcal{N}'(m, n) &= \mathcal{N}'(m-1, n-2) + \mathcal{K}'(m, n-m). \end{aligned} \tag{84}$$

leading to

$$\begin{aligned} \tilde{\mathcal{J}}'(z) &= z^3q^3\tilde{\mathcal{J}}'(z) + \tilde{\mathcal{K}}'(z) \\ \tilde{\mathcal{K}}'(z) &= z^3q^4\tilde{\mathcal{K}}'(z) + z^2q^3\tilde{\mathcal{L}}'(z) + \tilde{\mathcal{L}}'(z) \\ \tilde{\mathcal{L}}'(z) &= zq^2\tilde{\mathcal{M}}'(z) + \tilde{\mathcal{J}}'(zq) \\ \tilde{\mathcal{M}}'(z) &= z^2q^3\tilde{\mathcal{L}}'(z) + \tilde{\mathcal{N}}'(z) \\ \tilde{\mathcal{N}}'(z) &= zq^2\tilde{\mathcal{N}}'(z) + \tilde{\mathcal{K}}'(zq). \end{aligned} \tag{85}$$

The auxiliary functions can be eliminated and we get

$$\tilde{\mathcal{J}}'(z) = \frac{(1+z^2q^3)(1-z^4q^8)}{(1-zq^2)(1-z^3q^3)(1-z^3q^4)(1-z^3q^5)}\tilde{\mathcal{J}}'(zq). \tag{86}$$

for which the solution is (81). □

Corollary 25. *The generating function for the number of partitions of n into m parts and subject to the restriction*

$$\lambda_i \geq \lambda_{i+1} \quad \text{and} \quad \lambda_i \geq \lambda_{i+3} + 3, \tag{87}$$

is

$$P'(z) = \sum_{m_1,m_2,m_3 \geq 0} \frac{q^{m_1(m_1+3)/2+m_2(m_2+5)/2+3m_3+(m_1+2m_2+3m_3)(m_1+2m_2+3m_3-3)/2}\, z^{m_1+2m_2+3m_3}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}}. \tag{88}$$

Proof: To generate partitions from 012-partitions of length m, we add the staircase $(m-2,\ldots,1,0,-1)$ of weight $m(m-3)/2$, i.e., $\lambda_j = n_j - m - j - 1$. The weak ordering conditions (80) imply the restrictions (87). Therefore, by replacing z^N by $z^N q^{N(N-3)/2}$ in the generating function for $\mathcal{J}'(m,n)$, whose series expansion reads

$$\mathcal{J}'(z) = \sum_{m_1,m_2,m_3 \geq 0} \frac{q^{m_1(m_1+3)/2+m_2(m_2+5)/2+3m_3}\, z^{m_1+2m_2+3m_3}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}}, \tag{89}$$

we recover the above expression for $P'(z)$.

This expression $P'(z)$ agrees with the suitable specialization of the generating function displayed in Eq. (1.6) of [8] (see also Theorem 9.9 of [9]).

The consideration of 0123-partitions leads to a second-order q-equation. We thus expect that counting jagged partitions with ground state of period $0123\ldots p$ will always lead to higher-order q-equations for $p \geq 3$. □

5.3. 001-partitions

The final class of jagged partitions to be considered are the $0\cdots01$-ones (with p zeros), written for short as 0^p1-partitions.

Definition 26. 0^p1-partitions are defined as vectors $(n_1,\ldots,n_m)$ of non-negative entries $n_j \geq 0$,with $n_m \geq 1$, subject to the weak ordering conditions:

$$n_j \geq n_{j+s} - 1 \quad \text{for } 1 \leq s \leq p, \quad n_j \geq n_{j+p+1}. \tag{90}$$

For all the cases we have considered (namely, $1 \leq p \leq 6$), we always end up with a first-order q-equation, suggesting that this is a generic feature of this class of jagged partitions. Unfortunately, the resulting generating functions we obtain do not have a nice form for $p > 2$, in particular, we cannot write them as a multi-sum. For this reason, we limit ourself to displaying the generating function for 001-partitions. Moreover, since its proof is quite similar to the previous ones, we simply give the recurrence relation underlying its construction:

$$\begin{aligned}(\cdots 001001^+) &= (\cdots 001^+001) + (\cdots 011^+01) + (\cdots 011011^+)\\ (\cdots 011011^+) &= (\cdots 011^+011) + (\cdots 1111^+)\\ (\cdots 111111^+) &= (\cdots 111^+1) + (\cdots 112112^+).\end{aligned} \tag{91}$$

Theorem 27. *The generating function for $\mathcal{J}''(m,n)$, the number of* 001*-partitions of n with length m, is*

$$\tilde{\mathcal{J}}''(z) = \frac{(-z^2q;q^2)_\infty}{(zq)_\infty\,(z^3q;q^3)_\infty\,(z^3q^2;q^3)_\infty}. \tag{92}$$

Again, we can apply this result to the enumeration of partitions subject to some restrictions. In the present case, the restriction is a superposition of a 'difference-1 condition at distance 2' and 'difference-3 condition at distance 3'. The partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is obtained from the vector $(n_1, \ldots, n_m)$ augmented by the staircase $(m-1, \ldots, 1, 0)$.

Corollary 28. *The generating function for the number of partitions of n into m parts and subject to the restrictions*

$$\lambda_i \geq \lambda_{i+1}, \quad \lambda_i \geq \lambda_{i+2} + 1 \quad \text{and} \quad \lambda_i \geq \lambda_{i+3} + 3, \tag{93}$$

is

$$P''(z) = \sum_{m_0,m_1,m_2,m_3 \geq 0} \frac{q^\beta\, z^{2m_0+m_1+3m_2+3m_3}}{(q^2;q^2)_{m_0}\,(q)_{m_1}\,(q^3;q^3)_{m_2}\,(q^3;q^3)_{m_3}}, \tag{94}$$

with

$$\beta = m_0^2 + m_1 + m_2 + 2m_3 + (2m_0 + m_1 + 3m_2 + 3m_3)(2m_0 + m_1 + 3m_2 + 3m_3 - 1)/2. \tag{95}$$

Appendix A: Proof of identity (48)

We first obtain an identity for $\zeta_0^2 + 4q\zeta_4^2$. Starting from the expression of the ζ_i's given in (43), we have

$$\begin{aligned}\left[\frac{(q^8;q^8)_\infty^2}{(q^{16};q^{16})_\infty}\right]^2 (\zeta_0^2 + 4q\zeta_4^2) &= \left[\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2}\right]^2 + q\left[\frac{2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\right]^2\\ &= \left[\sum_{n=-\infty}^{\infty} q^{2n^2}\right]^2 + q\left[\sum_{n=-\infty}^{\infty} q^{2n(n+1)}\right]^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,m=-\infty}^{\infty} \left[q^{2n^2+2m^2} + q^{2n^2+2n+2m^2+2m+1}\right] \\
&= \sum_{n,m=-\infty}^{\infty} \left[q^{(n+m)^2+(n-m)^2} + q^{(n-m)^2+(n+m+1)^2}\right] \\
&= \sum_{n,m=-\infty}^{\infty} q^{(n-m)^2}\left[q^{(n+m)^2} + q^{(n+m+1)^2}\right] \\
&= \sum_{n,m=-\infty}^{\infty} q^{n^2}\left[q^{(n+2m)^2} + q^{(n+2m+1)^2}\right] \\
&= \sum_{n,m=-\infty}^{\infty} q^{n^2} q^{(n+m)^2} = \sum_{n,m=-\infty}^{\infty} q^{n^2} q^{m^2} \\
&= \left[\sum_{n=-\infty}^{\infty} q^{n^2}\right]^2 = \frac{(q^2;q^2)_\infty^{10}}{(q)_\infty^4\,(q^4;q^4)_\infty^4}.
\end{aligned}
\tag{96}
$$

The starting trick is to multiply the sum $\zeta_0^2 + 4q\zeta_4^2$ by an appropriate factor to transform it into a sum of squares. Each square is then rewritten as an infinite sum by means of the Jacobi triple-product identity. In the fifth line, we replace n by $n - m$. In the sixth one, we see that the second summation is broken into its even and odd parts; grouping them together leads to the seventh line. There, we replace m by $m - n$. At the end, Jacobi triple-product identity is used once more. The core identity in (96) is of course

$$
\left[\sum_{n=-\infty}^{\infty} q^{2n^2}\right]^2 + q\left[\sum_{n=-\infty}^{\infty} q^{2n(n+1)}\right]^2 = \left[\sum_{n=-\infty}^{\infty} q^{n^2}\right]^2, \tag{97}
$$

which is actually well-known[1]. With

$$
\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \quad \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{98}
$$

(97) is nothing but

$$
\theta_2^2(q^2) + \theta_3^2(q^2) = \theta_3^2(q), \tag{99}
$$

an identity that arises naturally from the parametrization of the arithmetic-geometric mean iteration using theta functions - cf. Section 2.1 of [5] (and (97) is precisely Eq. (2.1.8) there). Let us now return to (96) in order to finish the proof of (48). Isolating $\zeta_0^2 + 4q\zeta_4^2$ and multiplying the result by $\zeta_0\zeta_1^3\zeta_4$ yields:

$$
\zeta_0\zeta_1^3\zeta_4\big(\zeta_0^2 + 4q\zeta_4^2\big) = \frac{(q^2;q^2)_\infty^{14}\,(q^{16};q^{16})_\infty^7}{(q)_\infty^7\,(q^8;q^8)_\infty^{14}} = \zeta_1^7, \tag{100}
$$

as desired.

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Note added in proof

A bijection between jagged partitions and overpartitions has also been obtained by K. Mahlburg, *The overpartition function modulo small powers of 2*, *Discrete Math.* **286** (2004), 263–267. This article describes an extension of the congruence mod 64 obtained here.

Note

1. It also has a natural combinatorial interpretation. Write the q-series expansion of (97) as $\sum_{p=0}^{\infty} a_p q^p$. The right hand side shows that a_p is the number of ways p can be written as a sum of two squares. The left hand side shows that for p even, a_p is also the number of ways of writing $p/2$ as a sum of two squares, while for p odd, a_p stands for the number of ways $(p-1)/2$ can be written as a sum of two near squares, i.e., products $n(n+1)$.

References

1. G.E. Andrews, *The theory of partitions*, Cambridge Univ. Press, 1984.
2. G.E. Andrews, "Multiple q-series," *Houston J. Math.* **7** (1981), 11–22.
3. G.E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its applications **71**, Cambridge Univ. Press, 1999.
4. L. Bégin, J.-F. Fortin, P. Jacob, and P. Mathieu, "Fermionic characters for graded parafermions," *Nucl. Phys.* **B659** (2003), 365–386.
5. J.M. Borwein and P.B. Borwein, *Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity*, Wiley, N.Y., 1987.
6. B. Feigin, M. Jimbo, S. Loktev, T. Miwa, and E. Mukhin, "Bosonic formulas for (k, l)-admissible partitions," math.QA/0107054.
7. B. Feigin, M. Jimbo, and T. Miwa, "Vertex operator algebra arising from the minimal series $M(3, p)$ and monomial basis," math.QA/0012193.
8. B. Feigin, M. Jimbo, T. Miwa, E. Mukhinand, and Y. Takeyama, "Particle content of the $(k, 3)$-configurations," math.QA/0212348.
9. B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, and Y. Takeyama, "Fermionic formulas for $(k, 3)$-admissible configurations," math.QA/0212347.
10. J.-F. Fortin, P. Jacob, and P. Mathieu, "Generating function for K-restricted jagged partitions," *The Electronic Journal of Combinatorics*, **12** (2005), R12.
11. G.H. Hardy, *Ramanujan*, Cambridge Univ. Press, 1940.
12. H. Rademacher, *Topics in Analytic Number Theory*, Springer, Verlag, 1973.