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On an Arithmetical Function

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Abstract. In this paper we introduce an arithmetical function $\delta(n)$, the difference between the number of divisors of n congruent to 1 mod 3 and those congruent to -1 mod 3. This function then is related to the classical function $\sigma(n)$ which is the sum of the divisors of n. In particular we prove the identity

$$3\left(\sum_{n=0}^{\infty}\delta(3n+1)x^n\right)^2 = \sum_{n=0}^{\infty}\sigma(3n+2)x^n.$$

Key words: theta functions, divisor functions

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1. Introduction

In elementary number theory [3] one considers the functions $d(n)$, $\sigma(n)$ which are respectively the number of divisors of the positive integer n and the sum of the divisors of the positive integer n. An additional object of study is the difference between the number of divisors of a positive integer n congruent to 1 or -1 modulo 4. This arises in connection with the number of representations of n as a sum of two squares. In this paper we take a step back and consider for each positive integer n the quantity $\delta(n)$, the difference between the number of divisors of n congruent to 1 or -1 modulo 3. Thus for example we see that

$$\delta(1) = 1, \quad \delta(2) = 0, \quad \delta(3) = 1, \quad \delta(4) = 1, \quad \delta(7) = 2, \quad \delta(10) = 0.$$

As one sees from the list it is clear that for primes p, $\delta(p) = 2$ when $p \equiv 1$ modulo 3 and $\delta(p) = 0$ when $p \equiv 2$ modulo 3.

It is fairly clear that δ is a multiplicative function so that $\delta(mn) = \delta(m)\delta(n)$ whenever $(m, n) = 1$ and thus if $n = 3^\alpha \prod_i p_i^{\alpha_i} \prod_j q_j^{\beta_j}$ with primes $p_i \equiv 1$ modulo 3 and primes $q_j \equiv 2$ modulo 3, then $\delta(n) = \delta(\prod_i p_i^{\alpha_i}) \cdot \epsilon(\beta_1, \ldots \beta_r)$ where $\epsilon(\beta_1, \ldots \beta_r) = 1$ if all β_j are even and $\epsilon(\beta_1, \ldots \beta_r) = 0$ if at least one β_j is odd. The above is not totally obvious although it is obvious that $\delta(3n+2) = 0$. This follows because if we list the divisors as p_i, q_j $i = 1..m$, $j = 1..r$, where p_i, q_j represent divisors congruent to 1 or 2 modulo 3, then the map $d \to \frac{N}{d}$ is a bijection of the set of divisors onto itself and if $N \equiv 2$ modulo 3 it maps the divisors congruent to 1 onto those congruent to 2 modulo 3. Multiplicativity follows from the following remarks.

It is elementary and well known that

$$\sum_{n=1}^{\infty} \frac{n^k x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_k(n) x^n$$

where $\sigma_k(n) = \sum_{d|n} d^k$ and that when $k = 0$, $\sigma_k(n) = d(n)$. From this point of view it is clear that

$$\sum_{n=1}^{\infty} \frac{x^{3n+1}}{1-x^{3n+1}} - \frac{x^{3n+2}}{1-x^{3n+2}} = \sum_{n=1}^{\infty} \delta(n) x^n$$

Hence $\delta(n)$ is multiplicative by Theorem 265 in [3].

As far as we know there is no nontrivial connection between $d(n)$ and $\sigma(n)$. We therefore thought it rather interesting and remarkable to find a nontrivial connection between $\delta(n)$ and $\sigma(n)$. We shall prove the following theorem:

Theorem 1. *For all positive integers n we have*

$$\delta(n) + 3\sum_{j=1}^{n-1} \delta(j)\delta(n-j) = \sigma'(n)$$

where $\sigma'(n) = \sum_{d|n} d$ *and* d *is not a multiple of* 3. *Thus when* $n \equiv 1$ *or* $n \equiv 2$ *modulo* 3, $\sigma'(n) = \sigma(n)$. *If* $n \equiv 0$ *modulo* 3, *and say* $n = 3^\alpha m$, $(m, 3) = 1$, *then* $\sigma'(n) = \sigma(m)$.

2. Theta functions and $\delta(n)$

We begin with a definition:

Definition 1. Let

$$f(z, y) = \exp\left(\frac{\pi i}{6}\right) y^{\frac{1}{24}} z^{\frac{1}{6}} \prod_{n=0}^{\infty} (1 - y^{3n+3})(1 - y^{3n+2}z)(1 - y^{3n+1}/z).$$

The knowlegeable reader will recognize that the function $f(z, y)$ is just the product representation of the first order theta function $\theta\left[\begin{smallmatrix} 1 \\ 3 \\ 1 \end{smallmatrix}\right](\zeta, \tau)$ where $z = \exp(2\pi i\zeta)$ and $y = \exp(\frac{2\pi i \tau}{3})$. For background and more information the reader is referred to [2]

We now need to compute. We begin by observing that

$$\begin{aligned} \frac{d}{d\zeta} \log\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (\zeta, \tau) &= 2\pi i z \frac{d}{dz} \log f(z, y) \\ &= 2\pi i \left[\frac{1}{6} + \sum_{n=0}^{\infty} \frac{y^{3n+1}/z}{1 - y^{3n+1}/z} - \frac{y^{3n+2}z}{1 - y^{3n+2}z} \right] \end{aligned}$$

and therefore evaluating at $\zeta = 0$, $z = 1$ we find

$$\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)} = 2\pi i\left[\frac{1}{6}+\sum_{n=1}^{\infty}\delta(n)y^n\right].$$

Differentiatiating once again with respect to ζ and setting $\zeta = 0$ yields

$$\left(\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)}\right)' = -(2\pi i)^2\sum_{n=0}^{\infty}\frac{y^{3n+1}}{1-y^{3n+1}}+\frac{y^{3n+2}}{1-y^{3n+2}}+\frac{y^{6n+2}}{(1-y^{3n+1})^2}+\frac{y^{6n+4}}{(1-y^{3n+2})^2}.$$

The above computation leads us to the following theorem.

Theorem 2. *The power series expansion of*

$$\sum_{n=0}^{\infty}\frac{y^{3n+1}}{1-y^{3n+1}}+\frac{y^{3n+2}}{1-y^{3n+2}}+\frac{y^{6n+2}}{(1-y^{3n+1})^2}+\frac{y^{6n+4}}{(1-y^{3n+2})^2}$$

is given by $\sum_{n=1}^{\infty} a_n y^n$ *with* $a_n = \sum_{d|n}\frac{n}{d}$ *with* d *not a multiple of* 3.

Proof: The contribution from the first two terms in the sum is just the number of divisors of n which are not multiples of 3. We therefore need to look at the contribution of the last two terms.

Since the power series expansion of $\frac{y^{6n+2}}{(1-y^{3n+1})^2}$ is just

$$y^{6n+2}\left(1+\sum_{m=1}^{\infty}(m+1)y^{(3n+1)m}\right)$$

we see that for a given N we are looking for the number of ways we can write $N = 6n+2+m(3n+1) = (m+2)(3n+1)$. Hence the only candidates are those n for which $3n+1$ is a divisor of N. Furthermore the contribution to the coefficient of x^N in this case is just $m+1 = \frac{N}{3n+1}-1$. A similar argument for the power series expansion of $\frac{y^{6n+4}}{(1-y^{3n+2})^2}$ yields a similar result. Hence we see that the total contribution to the coefficient of x^N is given by $\sum_{d|N}(\frac{N}{d}-1)$ with d not a multiple of 3. This is of course the same as $\sum_{d|N}\frac{N}{d}-\{$the number of divisors of N not multiples of 3$\}$ and therefore we have the result required. □

The above theorem has the following simple corollary.

Corollary 1. *If n is congruent to either* 1 *or* 2 *modulo* 3 *then* $a_n = \sigma(n)$. *If n is congruent to* 0 *modulo* 3, *say* $n = 3^r m$ *then,* $a_n = \sum_{d|n} d$ *with d congruent to* 0 *modulo* 3^r *so that* $a_n = 3^r\sigma(m)$.

Proof: The only point that may need some comment is the final statement which follows from the fact that since we are summing $\frac{n}{d}$ with d not a multiple of 3, all our divisors are actually multiples of 3^r. The sum of these is clearly $3^r\sigma(m)$. □

The results we have thus far obtained however seem to have lost the main function we are studying, $\delta(n)$. We need to recover it. In order to do so we shall require one well known additional property of the theta function. The property being that it satisfies the heat equation. We recall the statement which is

$$\frac{d}{d\tau}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau)=\frac{1}{4\pi i}\frac{d^2}{d\zeta^2}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau).$$

We now once again compute

$$\left(\frac{\theta'\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(\zeta,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(\zeta,\tau)}\right)'$$

but now simply observe that at $\zeta = 0$ the above expression reduces to

$$\frac{\theta''\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}-\left(\frac{\theta'\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}\right)^2=4\pi i\frac{\frac{d}{d\tau}\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}-\left(\frac{\theta'\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}\right)^2.$$

The important point now being that we can write each of the terms on the right hand side of the last equation as power series in the variable y. It is clear that

$$\frac{\frac{d}{d\tau}\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)}=\frac{2\pi i}{3}\left[\frac{1}{24}-\sum_{n=0}^{\infty}\frac{ny^n}{1-y^n}\right]$$

so that we can replace the right hand side of the previous equation by

$$\frac{8}{3}(\pi i)^2\left[\frac{1}{24}-\sum_{n=1}^{\infty}\sigma(n)y^n\right]-4(\pi i)^2\left[\frac{1}{6}+\sum_{n=1}^{\infty}\delta(n)y^n\right]^2.$$

The conclusion from the above computation is therefore that

Theorem 3. *We have the following identity between power series.*

$$3\sum_{n=1}^{\infty} a_n y^n = 2\sum_{n=1}^{\infty}\sigma(n)y^n + \sum_{n=1}^{\infty}\delta(n)y^n + 3\sum_{n=2}^{\infty}\sum_{j=1}^{n-1}\delta(j)\delta(n-j)y^n$$

Proof: The proof is using the computations above and the definition of a_n given in Theorem 2. □

It is perhaps better to write this theorem in the following form.

Theorem 4. *For all positive integers n we have*

$$3a_n - 2\sigma(n) = \delta(n) + 3\sum_{j=1}^{n-1}\delta(j)\delta(n-j)$$

When stated in this form we have the immediate corollaries:

Corollary 2. *For all positive integers n we have the following: If* $n \equiv 1$ *or if* $n \equiv 2$ *modulo* 3 *we have*

$$\delta(n) + 3\sum_{j=1}^{n-1}\delta(j)\delta(n-j) = \sigma(n).$$

If $n \equiv 0$ *modulo* 3, *say* $n = 3^r m$ *with* $(3, m) = 1$, *then*

$$\delta(n) + 3\sum_{j=1}^{n-1}\delta(j)\delta(n-j) = \sigma'(n).$$

Proof: Since we have already seen that when $n \equiv 1$ or $n \equiv 2$ modulo 3, $a_n = \sigma(n)$, we only need comment on the last statement. Under the hypothesis on n we have $a_n = 3^r\sigma(m)$ so that

$$3a_n - 2\sigma(n) = 3^{r+1}\sigma(m) - 2\sigma(3^r m) = \sigma(m)$$

which is clearly the same as $\sigma'(n)$. □

We shall point out a few more corollaries of these ideas.

Corollary 3.

$$\sigma(3m+2) = 3\sum_{j=0}^{m}\delta(3j+1)\delta(3(m-j)+1)$$

$$\sigma(3m+1) = \delta(3m+1) + 6\sum_{j=0}^{m-1}\delta(3j+1)\delta(m-j)$$

Proof: We use the formula of the previous corollary for $\sigma(n)$ but observe that when $n \equiv 2$ modulo 3 the only j which contribute anything are those which are congruent to 1 modulo 3. Of course, $\delta(3m+2) = 0$. When $n \equiv 1$ modulo 3, the only terms which contribute are those j which are congruent to either 1 or 0 modulo 3. These terms however reinforce to give the result. □

We also give a condition for the primality of either $3n+1$ or $3n+2$ in terms of n.

Corollary 4. *$3n+1$ is prime iff $\sum_{j=0}^{n-1} \delta(3j+1)\delta(n-j) = \frac{n}{2}$. $3n+2$ is prime iff $\sum_{j=0}^{n} \delta(3j+1)\delta(3(n-j)+1) = n+1$.*

Our final corollary is

Corollary 5. *$\sum_{j=1}^{n-1} \delta(j)\delta(n-j) = 0$ iff $n = 3^r$ for some positive integer r.*

This last result is also easily derivable from the definitions and does not really require the relationship with σ although this does make the proof completely transparent.

3. Final remarks

The results obtained so far raise the following question. We have seen above that

$$\sigma(3m+2) = 3\sum_{j=0}^{m} \delta(3j+1)\delta(3(n-j)+1).$$

This gives rise to the identity

$$\sum_{n=0}^{\infty} \sigma(3n+2)x^n = 3\left(\sum_{n=0}^{\infty} \delta(3n+1)x^n\right)^2.$$

There is of course a similar identity related with $\sigma(3n+1)$. The formula raises the question what is the function

$$g(x) = \sum_{n=0}^{\infty} \delta(3n+1)x^n?$$

The referee has kindly pointed out that the expression for the function $g(x)$ we shall give is not new but already appears in [1] and [4]. In a forthcoming publication together with Y. Godin we shall reprove that

$$g(x) = \frac{\prod_{n=1}^{\infty}(1-x^{3n})^3}{\prod_{n=1}^{\infty} 1-x^n}$$

and that

$$f(x) = \frac{\prod_{n=1}^{\infty}(1-x^n)^3}{\prod_{n=1}^{\infty} 1-x^{3n}} = 1 - 3\sum_{n=1}^{\infty} f(n)x^n$$

where

$$\begin{cases} f(3n+1) = \delta(3n+1) \\ f(3n+2) = \delta(3n+2) \\ f(3n) = 2\delta(3n) \end{cases}$$

In the above of course, $\delta(3n+2) = 0$ and $\delta(3n) = \delta(n)$. As a corollary we will be able to express the function δ in terms of the Ramanujan partition function. In addition we shall be able to express $\delta(n)$ in terms of $\delta(k)$ with $k < n$ and will give a formula for the number of integer solutions to the equation $x^2 + 3y^2 = N$.

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