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Intuitionistic fuzzy sets in questionnaire analysis

Donata Marasini · Piero Quatto · Enrico Ripamonti

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Abstract Fuzzy sets represent an extension of the concept of set, used to mathematically model veiled and indefinite concepts, such as those of youth, poverty, customer satisfaction and so on. Fuzzy theory introduces a membership function, expressing the degree of membership of the elements to a set. Intuitionistic fuzzy sets and hesitant fuzzy sets are two extensions of the theory of fuzzy sets, in which non-membership degrees and hesitations expressed by a set of experts are, respectively, introduced. In this paper, we apply intuitionistic fuzzy sets to questionnaire analysis, with a focus on the construction of membership, non-membership and uncertainty functions. We also suggest the possibility of considering intuitionistic hesitant fuzzy sets as a valuable theoretical framework. We apply these models to the evaluation of a Public Administration and we assess our results through a sensitivity analysis.

Keywords Intuitionistic fuzzy sets · Hesitant fuzzy sets · Membership functions · Non-membership functions · Uncertainty functions · Questionnaire analysis

1 Introduction

1.1 The general framework

Human language can represent concepts, such as those of poverty and youth, whose definitions and frontiers are, per se, blurry and uncertain. Fuzzy sets (FSs) allow researchers to quantitatively deal with these categorical and dimensional concepts. Recent examples of FSs application are given by da Silva et al. (2014), Chung et al. (2014) and Betti et al. (2011). FS theory extends the mathematical theory of sets, which has been

D. Marasini · P. Quatto · E. Ripamonti (✉)
Statistical Section, Department of Economics, Management and Statistics,
University of Milan-Bicocca, Piazza Ateneo Nuovo 1, 20126 Milan, Italy
e-mail: enrico.ripamonti@unimib.it

formalized by Georg Cantor in 1874, and whose origins can be dated back to the Aristotle's logic. According to set theory, the membership function (also called characteristic function), which defines the belonging of an object to a certain set, is dichotomous: either the object belongs, or it does not belong to the set. The core of the FS theory (Zadeh 1965) is that the mapping function, indicating the membership of objects to a certain set, can take values on the entire unit interval. In Zadeh's viewpoint, membership is conceived in terms of degree of inclusion, i.e., partial belonging to a certain set. Moreover, generalization of basic concepts of classical theory, such as intersection or union, is provided. FS theory is one of the more than 20 theories (see Zimmermann 2010) that have been proposed along with axiomatic probability theory.

The concept of FS was introduced, from a theoretical point of view, by Zadeh (1965) and Goguen (1967), and has found application in various sectors, from quality control to databases, from statistical modeling to economic impact evaluation, from programming tools and machinery to advanced information technology. Despite of its name, fuzzy theory can be conceived as a rigorous, formal and extended mathematical tool to represent phenomena defined by different degrees of class membership (see Bede 2013, for an introduction).

From a historical point of view, Zadeh (1965) originally developed this theory in the sense of a mathematical method aiming at representing and formalizing knowledge and, to this purpose, proposed an extension of the concept of set, the FS, from which it has been consequently extended a fuzzy arithmetic, fuzzy measures and a fuzzy logic. FSs are distinct from classical sets, which, in this context, are defined as crisp (i.e., dichotomous) sets. For a crisp set, the membership function is of Boolean type, i.e., it associates to each element x of the "universe of discourse" X a value, either 1 (true) or 0 (false), depending on whether x belongs to or does not belong to X. Consider, for instance, the set of people who were born in France; this set can be identified through a membership function, that returns the value 1 "true" if and only if a person was actually born in France. There are, however, veiled and indefinite concepts, for which it is neither simple nor immediate to define a membership function. Reflect on the concept of wealth: when does a person can be defined as "rich"? Several definitions could be sketched, for instance: (i) to be above an arbitrarily specified poverty line; (ii) to reach a certain annual income; (iii) to own a given number of assets. For these concepts it is undoubtedly meaningful to establish "how much" an element owns a certain property or, in other words, "how much" such element belongs to a set. Therefore, it is possible to extend the concept of membership mapping, introducing a function that returns a value in the range between 0 and 1; this leads to expand the notion of Boolean membership. Consequently, in FSs, each element can be classified as belonging to the set with a certain degree (i.e., fuzziness) of membership.

Zadeh (1965) defines FSs as an extension to the classical notion of sets, in which each element $x \in X$ of the set is associated with a real number in the interval [0,1], expressing the membership grade of x in a FS. It is worth observing that degrees of membership of the elements of a certain FS do not have to sum up to 1 (differently from probability theory).

The power of the fuzzy approach is that it allows a researcher to mathematically model the semantic, syntactic and pragmatic structure symbolized by human language. FSs can be used to describe statistical and economic concepts, as well as to formalize intuitive and subjective notions. Along with the definition of FS, also a fuzzy logic has been formalized; in this context, propositions are not evaluated through Boolean operators, but, usually, in the interval [0,1]. Thus, logical operators such as negation, disjunction or conjunction are defined so that they can provide consistent information in the context of real values. Atanassov (1986) has generalized the concept of FS introducing the intuitionistic suzzy Set (IFS), which provides both a function of membership and a function of non-membership.

For instance, a subject could express his/her degree of satisfaction with respect to a certain service with a degree of satisfaction of 0.7 and a degree of unsatisfaction of 0.1. As a matter of fact, dissatisfaction is not defined in terms of negation of satisfaction; this implies that there is a degree of uncertainty representing the situation of "neither this nor that". On this point, Li (2014) presents the following example. In an electoral context, a certain candidate can be awarded with degree of 0.5 with respect to his/her election (50 % of people declare to vote for him) and a degree of 0.3 of not-election (30 % of people declare to vote against him); the remaining 0.2 represent the uncertainty of those who do not yet know the candidate for whom they will vote. The introduction of the two functions is therefore used to measure the degree of uncertainty of an assessment. According to Atanassov, the sum of the values assumed by the two functions must not exceed the unit, and we'll follow this approach. However, it has also been proposed a generalization of IFS (Despi et al. 2013) according to which the sum can exceed the unit. Generalizations of FS, such as IFS, have been mainly applied in the literature as far as decision making is concerned (see Li 2014, for a review). Indeed, according to Xu, "During the last few years, more and more researchers have been applying IFSs to multi-attribute decision making under various different situations, and a lot of work has been done" (Xu 2014c, p. 195). Similar conclusions can be drawn as far as HFSs are concerned (Torra 2010; Xu 2014b). According to Rodriguez et al., "HFSs are used by experts to provide their assessments or preferences over the set of criteria and alternatives defined in multicriteria decision making, group decision making, multiexpert multicriteria decision making, and decision support systems" (Rodríguez et al. 2014, p. 519). Finally, also generalizations of IFS and HFS have been developed in order to deal with decision making and multi-criteria decision making (Qian et al. 2013; Zhu et al. 2012). Moreover, Torra (2010) generalized FS with hesitant fuzzy set (HFS), which returns multiple values of membership, indicating, for instance, the hesitation expressed by a set of experts in taking a decision. Many generalizations of IFS and HFS have been proposed (e.g., Rodríguez et al. 2014; Xu 2014a). Zhu et al. (2012) have put forward an extension called dual hesitant fuzzy set (DHFS), whereas a different proposal, which combines IFS with HFS leading to generalized hesitant fuzzy set (G-HFS), has been suggested by Qian et al. (2013). In the following, we'll consider a variant of G-HFS, which keeps into account both the basic properties of IFS and of HFS, that shall be indicated by intuitionistic hesitant fuzzy set (IHFS).

1.2 Application of fuzzy theory: an overview

The main applications of fuzzy theory concern industrial engineering, automation and quality control. Furthermore, this theory has been applied in a number of sectors in which it is necessary to define robust models for qualitative information. From the point of view of statistical research, an important advantage of FS theory is that these models can be normally used also in case the probabilistic models cannot be directly applied. In this paragraph, we'll sketch a general overview in order to illustrate the usefulness of the fuzzy approach also for psychometrics and for the psychological research.

Bellman and Zadeh (1970) analyzed decision making in a fuzzy environment, presenting examples of stochastic/deterministic processes in the context of multistage decision. Decision making processes are a good example of application of fuzzy theory since "much of the decision making in the real world takes place in an environment in which the goals, the constraints and the consequences of possible actions are not known precisely" (p. 141). It is worth noting that the authors, in their seminal paper, suggest that the fuzzy environment can be considered as a better framework than probability and decision theory to quantitatively deal with imprecision. According to these authors, there is "a need for differentiation between randomness and fuzziness, with the latter being a major source of imprecision in many decision processes". Thus, "randomness has to do with uncertainty concerning membership or non membership of an object in a non fuzzy Set. Fuzziness, on the other hand, has to do with classes in which there may be grades of membership intermediate between full membership and non membership" (p. 142). More recently, decision making has been considered also in the framework of IFS (see Li 2014, for a review).

In the context of customer satisfaction, Chien and Tsai (2000) studied the problem of measuring perceived service quality and put forth a method based on triangular fuzzy numbers, considering consumers' satisfaction degree and importance degree. Kwong and Bai (2002) used fuzzy numbers and fuzzy scales to determine the importance weights of customer requirements in the context of quality function deployment, improving the imprecise ranking of customer requirements. Darestani and Jahromi (2009) proposed a new customer satisfaction index based on FS rules aimed to translate in mathematical terms linguistic statements expressed by experienced people. Castillo and Lorenzana (2010), through the linguistic approach of the FS theory, modelled the opinions of experts combining both quantitative and qualitative performance measures. The authors put forward a method aimed to construct a fuzzy composite indicator by means of the aggregation of multiple performance measures, and applied their results to the business scenarios analysis. Finally, Zani et al. (2013) constructed a fuzzy synthetic index aimed to measure a latent phenomenon, that can be quantified through a set of manifest variables. The authors discuss different criteria to define the membership function and propose a method based on the sampling cumulative function in case of ordinal variables (see also Zani et al. 2012).

Important applications of FS theory have also been proposed as far as the evaluation of poverty is concerned. Traditionally, poverty was investigated through the definition of a certain poverty threshold or line, with respect to a single economic variable such as income or consumption. For instance, a researcher could define as "extremely poor" those individuals who have incomes no higher than 40 % of the national *per capita* income and as "almost poor" those subjects who have income not exceeding 60 % of the national *per capita* income. It is reasonable to assume that people can be classified along a poverty/non-poverty line through a *continuum* and gradual membership, and not through a dichotomy. In summary, defining a poverty line leads to uniquely discriminating between poor and non-poor people, that can be considered as an assumption not totally corresponding to reality. A seminal proposal in order to re-consider the poor/non-poor dichotomy has been given in Cerioli and Zani (1990), who suggested to introduce a transition zone between the two states indicating poverty and non-poverty. In their original paper, the authors considered, for each subject i, her/his equalized income x_i and defined a transition zone (a, b) into the poverty line. Moreover, it has been suggested that the membership degree can be defined through a linear function:

$$\mu(x_i) = \begin{cases} 1 & \text{if } x_i < a \\ \dfrac{b - x_i}{b - a} & \text{if } a \leq x_i \leq b \\ 0 & \text{if } x_i > b \end{cases} \tag{1}$$

Cheli and Lemmi (1995) proposed an alternative framework for the membership function in (1), considering the distribution function $F(x)$ of income linearly transformed taking into account the poorest $(F(x) = 0)$ subject and the richest $F(x) = 1$ subject in the population. Cheli (1995) suggested to model the membership function to the detection of poor people, using a power transformation (for $\alpha \geq 1$) of the normalized distribution function:

$$\mu(x_i) = [1 - F(x_i)]^{\alpha},$$

where x_1 indicates the income of the poorest subject in the target population. The choice of the parameter α is arbitrary and the more α increases, the larger is the weight given to the "poor end" of the distribution (Betti et al. 2006). It has also been suggested to consider, in the membership function, both the normalized distribution function and the Lorenz curve of income $L(x)$ i.e., $\mu(x_i) = [1 - F(x_i)]^{\alpha-1}[1 - L(x_i)]$ (Betti et al. 2006).

A further area of application of fuzzy theories is that of teaching and students' evaluation. In this context, we aim at associating to students' performance the most realistic indicator in terms of grades. For instance, a simple system of grades would be: "Unsatisfactory", "Satisfactory", "Good", "Very good", "Excellent". FS theory has been adopted to translate, in mathematical terms, different degrees of evaluation and to express membership values to students' performance. Biswas (1995) proposed a fuzzy evaluation method based on vector-valued marking approach, and subsequently generalized it to a multidimensional (matrix-based) case. Chen and Lee (1999) extended the original proposal by Biswas, suggesting two new methods for students' answers evaluation; such methods avoid the matching procedures that had been put forth in the original Biswas' paper. Lalla et al. (2005) considered a University courses evaluation questionnaire proposed by the Italian Ministry of Education and compared a traditional analysis with a FS analysis. Moreover, these authors proposed a more general approach, based on item-by-item analysis. As a matter of fact, FSs represent an ideal instrument to rank university courses using students' evaluation questionnaires. Generally, ordinal scales are used to collect these judgments (e.g., the scale used in the Italian university courses evaluation questionnaire: a 4-point Likert-type scale, such as, "Definitely not" (DN), "More no than yes" (MN), "More yes than no" (MY), "Definitely yes" (DY), or the mark scale: "Very insufficient", "Insufficient", "Sufficient", "Good", "Very good"). In the context of measures of attitudes, opinions and feelings, Likert scales are generally used; these scales translate into discrete ordinal scales the judgments expressed by subjects, assuming equidistance between alternatives. Participants are presented with a set of statements, that they have to evaluate according to the alternatives: "Strongly disagree", "Disagree", "Uncertain", "Agree", "Strongly agree" (for a review of the main advantages/disadvantages of these scales, see Lalla et al. 2005). Finally, Crocetta and Delvecchio (2007) proposed a fuzzy method to evaluate the customer satisfaction of University graduates; this evaluation was performed *ex-post*, i.e., graduates were required (via telephone interviews) to express their judgments on University education in terms of employability.

In this paper we present some developments and application of the theory of IFS and IHFS in case of evaluation questionnaires. The manuscript is organized as follows: in Sect. 2 we present the notation, preliminary definitions, examples of FS, IFS and HFS. In Sect. 3 it is presented a case study regarding the evaluation of Public Administrations; in Sect. 4, functions of membership, non-membership and uncertainty are introduced and commented. In Sect. 5 and 6 we put forward our proposal of application of IFS and IHFS in the framework of questionnaires; Sect. 7 deals with an application concerning the evaluation of a Public Administration. In Sect. 8, we discuss our main results, also with reference to other approaches used to address customer satisfaction, such as the choice of membership function proposed in Zani et al. (2012) or the CUB model (D'Elia and Piccolo 2005; Piccolo and D'Elia 2008; Iannario and Piccolo 2012); last, we draw some conclusions as well as general ideas for future research in this field.

2 Fuzzy sets: intuitionistic and hesitant generalizations

Let's consider a universe X and a function $\mu_A(x)$ such that

$$\mu_A(x) \in [0, 1] \tag{2}$$

associating to each $x \in X$ a real value between 0 and 1. The FS is defined as:

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\} \tag{3}$$

where the function $\mu_A(x)$ in (2) measures the degree of membership of x to A (Bede 2013). In case $\mu_A(x) = 0$, x does not belong to A; in case $\mu_A(x) = 1$, x certainly belongs to A. If

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

the set is defined as crisp.

Let X be the universe and consider a membership function as in (2) and another function

$$v_A(x) \in [0, 1] \tag{4}$$

so that it fulfills the condition:

$$0 \le \mu_A(x) + v_A(x) \le 1 (x \in X) \tag{5}$$

The set

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle : x \in X\} \tag{6}$$

is an IFS (Atanassov 2012) and the function $v_A(x)$ measures the degree of non-membership of x to A. It can be observed (Bede 2013) that a FS can be interpreted as an IFS with $v_A(x) = 1 - \mu_A(x)$, but the reciprocal is not necessarily true. With IFS we introduce the measure:

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x) \tag{7}$$

describing the degree of uncertainty. Let us now consider X, and let $h_A(x)$ be a mapping associating to each x subset of the interval [0,1]. The set:

$$A = \{\langle x, h_A(x)\rangle : x \in X\} \tag{8}$$

is HFS (Xu 2014a). In the present context (8) generalizes to the case of IHFS with the set:

$$A = \left\{\left\langle x, \tilde{h}_A(x)\right\rangle : x \in X\right\} \tag{9}$$

where $\tilde{h}_A(x)$ is composed by ordered pairs $\langle \mu_A(x), v_A(x)\rangle$, each of them satisfying the intuitionistic constraint (5). As far as IFSs are concerned, the uncertainty of classification through membership/non-membership functions can be captured by a suitable model for uncertainty function in (7), whereas the IHFS perspective allows us to keep into account the uncertainty arising from possible hesitations among various degrees of membership/non-membership (as in the case of one or more experts who express one or more evaluations). It is worth observing that, if $v_A(x) = 1 - \mu_A(x), \tilde{h}_A(x)$ is equivalent to $h_A(x)$, so that the IHFS reduces to an HFS. For the sake of convenience, we'll only consider, as $\tilde{h}_A(x)$, a finite set of size N. In the case of $N = 1$, $\tilde{h}_A(x)$ is constituted by the only pair $\langle \mu_A(x), v_A(x)\rangle$, and the IHFS becomes an IFS; if $v_A(x) = 1 - \mu_A(x)$, that IFS can be viewed as an ordinary FS.

For the sake of comparison, we briefly consider two different proposals combining HFS with IFS. A first proposal leads to G-HFS, which are defined in concordance with (9), provided the pairs $\langle \mu_A(x), v_A(x) \rangle$ satisfy the constraint (5). It is worth observing that, differently from IHFS, in the case of G-HFS the pair is defined as: (i) $\langle \mu_A(x), 1 - v_A(x) \rangle$, when the function $v_A(x)$ is introduced, leading to the interval $[\mu_A(x), 1 - v_A(x)] \subseteq [0, 1]$; (ii) $\langle \mu_A(x), 1 - \mu_A(x) \rangle$ otherwise. The second proposal is that of DHFS; in this case, membership values belong to the set $h_A(x) \subseteq [0, 1]$, and non-membership values belong to the set $g_A(x) \subseteq [0, 1]$, so that the DHFS is given by:

$$A = \{ \langle x, h_A(x), g_A(x) \rangle : x \in X \}$$

In particular, (5) is replaced by the more stringent constraint:

$$0 \leq \max \mu_A(x) + \max \upsilon_A(x) \leq 1 (x \in X)$$

where max $\mu_A(x)$ and max $\upsilon_A(x)$ indicate the maximum values of the membership/non-membership values in $h_A(x)$ and $g_A(x)$.

Let us now apply the definitions of FS, IFS and IHFS to the following example; we are interested in interpreting the expression "very young" as far as the variable "youth" is concerned. As universe of discourse, we consider the real interval [0,70] to identify the age. With regard to "very young" we define the membership in (2) as:

$$\mu_{A_1}(x) = \begin{cases} 1 & \text{if } x < 15 \\ \dfrac{25 - x}{10} & \text{if } 15 \leq x \leq 25 \\ 0 & \text{if } x > 25 \end{cases} \tag{10}$$

where $\mu_{A_1}(x)$, gives the degree of membership of x to the FS A_1 representing "very young" in the universe of discourse [0,70]. In case $x \geq 25$, $\mu_{A_1}(x) = 0$, and these elements are certainly not referred to A_1. In case $x \leq 15$, $\mu_{A_1}(x) = 1$ and these elements are certainly referred to "very young"; in case, for instance $x = 22$, $\mu_{A_1}(22) = 0.30$ and this indicates the degree of membership of the age 22 to A_1. If we associate to (10) the function:

$$v_{A_1}(x) = \begin{cases} 0 & \text{if } x < 15 \\ \left(\dfrac{x - 15}{10} \right)^2 & \text{if } \ 15 \leq x \leq 25 \\ 1 & \text{if } x > 25 \end{cases} \tag{11}$$

A_1 becomes an IFS and it follows that the non-membership for the attribute "very young", in case $x = 22$, is given by $v_{A_1}(22) = 0.49$. According to Eq. 7, the degree of indeterminacy can be synthesized with $\pi_{A_1}(22) = 0.24$, i.e., 0.24 indicates the uncertainty associated with (10) and (11) in considering the age 22 as "very young". The structure of the IFS A_1 is given by (6). Let's now consider the variable "youth" and the attribute "young" with the functions:

$$\mu_{A_2}(x) = \begin{cases} 1 & \text{if } x < 20 \\ \dfrac{35 - x}{15} & \text{if } 20 \leq x \leq 35 \\ 0 & \text{if } x > 35 \end{cases} \tag{12}$$

and

Table 1 Three participants are asked to "quantify" (using values between 0 and 1) their ideas on the membership of age 22 to the categories "very young" and "young"

Participant	"Very young"	"Young"
1	0.40	0.60
2	0.30	0.80
3	0.10	0.70

$$\nu_{A_2}(x) = \begin{cases} 0 & \text{if } x < 20 \\ \left(\dfrac{x-20}{15}\right)^2 & \text{if } 20 \leq x \leq 35 \\ 1 & \text{if } x > 35 \end{cases} \tag{13}$$

describing another IFS A_2 with structure (6). Using (12) and (13) we have $\pi_{A_2}(22) = 0.12$, which indicates a lower degree of uncertainty, when considering the age 22 as "young" versus "very young".

Let's now exemplify the structure of HFS and IHFS. Without reference to the functions above, imagine you ask three people to "quantify" their ideas on the membership of the age 22 to the category "very young" and to the category "young", using values between 0 and 1 and assume that three participants return the values in Table 1.

Values in Table 1 take the form:

$$h_{A_1}(22) = \{0.40, 0.30, 0.10\}$$

$$h_{A_2}(22) = \{0.60, 0.80, 0.70\}$$

with $h_{A_1}(22)$ and $h_{A_2}(22)$ indicating the hesitations expressed by three subjects in classifying the age of 22 as "very young" (A_1) or "young" (A_2).

Suppose now to ask subjects to quantify the idea of not belonging of the age 22 to the aforementioned categories. The corresponding pairs $\langle \mu_{A_i}(x), \nu_{A_i}(x) \rangle$, $i = 1, 2$ relative to the age 22, are summarized in Table 2. Values in Table 2 lead to the following:

$$\tilde{h}_{A_1}(22) = \{\langle 0.40, 0.25\rangle, \langle 0.30, 0.50\rangle, \langle 0.10, 0.80\rangle\}$$

$$\tilde{h}_{A_2}(22) = \{\langle 0.60, 0.20\rangle, \langle 0.80, 0.10\rangle, \langle 0.70, 0.15\rangle\}$$

Consider, for instance $\tilde{h}_{A_2}(22)$. From one side, the age 22 is classified as "young" with hesitations 0.60, 0.80 and 0.70, respectively, which express a high degree of membership. From the other side, the hesitations 0.20, 0.10 and 0.15 express a very low degree of non-membership. To summarize, aiming to quantify the attributes "very young" and "young" and to originate FSs and IFSs, we introduced functions of membership and not

Table 2 Three participants are asked to "quantify" on a 0–1 scale, their ideas on both membership and non-membership of age 22 to the categories "very young" and "young"

Participant	"Very young"	"Young"
1	0.40, 0.25	0.60, 0.20
2	0.30, 0.50	0.80, 0.10
3	0.10, 0.80	0.70, 0.15

membership. Moreover, considering a particular age (e.g., 22 years) as attribute, we used the specific opinion of experts that may not be based on the construction of ad hoc functions. In the following, the identification of IFSs will be operationalized by appropriate membership and non-membership functions, built according to suitable conditions that will be analyzed in Sect. 4; the identification of IHFS, albeit indirectly, will refer to the previous functions.

3 Motivating example

We consider, as motivating example, a national questionnaire developed by the Italian Ministry for the Public Administration, aimed to measure the level of well-being of employees and management in Public Administrations and Offices (The Magellano Project, http://www.magellanopa.it/). The survey is directed to monitoring the views and the opinions of the administrative staff with respect to some significant aspects of their work, in order to improve the same aspects. This analysis is conducted through a questionnaire, prepared in electronic format, and that can be filled via Web; it consists of 48 questions for an average compile-time of 15/20 minutes. Overall, the questionnaire is structured in 12 thematic areas; these are preceded by the detection of information regarding the respondents (gender, age, area of origin, staff position…). In general, the proposed items have four modes of response with the addition of a field indicated with "other, do not know". The confidentiality with respect to the identity of the respondent is guaranteed by the local Evaluation Committee. A communication, in which it is provided information on the survey and on how to complete the questionnaire, is sent by e-mail to all the staff of Public administrations (e.g., Universities, Government Departments). The following thematic areas, leading to latent variables, have been considered in the questionnaire: (i) *The characteristics of the work environment*: this section provides a single question or item ("How would you rate the comfort of the environment in which you work?") divided into some modalities with a rating ranging from poor to good; (ii) *The safety*: the section is dedicated to the perceived safety; (iii) *The relationship with colleagues* (4 questions); (iv) *The efficiency*: this section investigates the efficacy and effectiveness of the Administration and consists of seven questions; (v) *The organization*: this section, consisting of four questions, investigates the perception of the subjects about organizational fairness in the Administration; (vi) *The condition and the quality of work*; (vii) *The management*; (viii) *The characteristics of the work*; (ix) *Positive and negative indicators of organizational well-being*; (x) *The mental and physical well-being*; (xi) *Openness to innovation*; (xii) *Further suggestions*.

In this paper, in order to illustrate the IFS and HFS approaches, we'll consider a latent variable expressing how much, a certain Administration, is open-minded with respect to technological innovation. This is part of Section xi of the questionnaire and we'll consider, for illustrative purposes, item 1: "Comparing our experience with that of other Administrations" and item 2: "Recognize and appropriately address past problems and mistakes".

4 Membership, uncertainty and non-membership functions

From the concept of FS, it follows the definition of linguistic variable (Bede 2013; Zadeh 1965), which is a quintuple (S, T, X, g, l), where S is the name of the variable, T is the name of linguistic terms which express the values of the variable, X is the universe of discourse,

g is a collection of syntactic rules, grammar, that produces correct expressions in T, l is a set of semantic rules that map T into FS in X. In other words, through the interpretation of certain values in terms of FS, a researcher can use the set of fuzzy rules for mapping the actual values of a set to real values, by means of a model expressed in linguistic terms. It is worth noting that membership values assume different meanings according to different universes of discourse. Under this premise, let S be a latent variable, i.e., an unobservable variable; consider a rule g leading to establish M ordered modalities of a Likert scale: $s_m, m = 1, \ldots, M$ that define the set T. To each s_m, a rule l assigns a unique real value, i.e., a crisp set, defined in the universe of discourse X, thus creating a particularization of the linguistic variable that has been previously defined. The choice of the two rules g and l is essential to obtain information on the latent variable. Fixed T, the rule l has to be established. Thus, with respect for instance to the modalities DN, MN, MY, DY (see Sect. 1), the rule could assign the values 1, 2, 3, 4 or 2, 5, 7, 10 (Marasini and Quatto 2011). In the former case, the choice between modes made by each subject is considered equi-spaced.

We assume that the rule l modifies as: $l(s_m) = m, m = 1, \ldots, M$ and we propose membership functions shaping those values; namely, through the modeling of $\mu_A(x)$, defined with (2), and $\pi_A(x)$ defined with (7), which, for sake of simplicity, we shall indicate as $\mu(x)$ and $\pi(x)$, and the corresponding non-membership functions are obtained. Similarly, modelling $\nu_A(x)$, defined with (4) and indicated with $\nu(x)$, both $\pi(x)$ and the corresponding membership functions are obtained. It should be noted, however, that such functions are proposed on the basis of a certain applied problem, and there are not general rules. A very comprehensive review of the quantification of modalities and their fuzzy expression has been given in Delgado et al. (2014).

There are at least four main approaches to the construction of a membership function (see Smithson and Verkuilen 2006 for a review). Following a formalist approach, researchers can define membership functions referring to the "best" mathematical function, which, in their opinion can well approximate their empirical problem. It is worth observing that this is a subjective approach, allowing a researcher to choose the functional form to be preferred. Secondly, from a probabilistic point of view, membership functions are defined through the probability of belonging to a certain set. Generally, such probabilities can be calculated as sample proportions or can be estimated through the expertise of external judges. In some cases, the formalist and the probabilistic approaches have been combined. A third perspective is given by the decision-theoretic approach, according to which membership functions are calculated with respect to the utility of combining the elements of the set with certain membership values. Finally, following the axiomatic measurement theory, the definition of membership functions is referred to "a set of qualitative axiomatic conditions that can and should be demonstrated empirically" (Smithson and Verkuilen 2006, p. 25).

It is worth observing that much of the literature has dealt with determining membership functions, but not with non-membership functions that are the core of IFSs. In the following the attention will also be focused on the construction of these functions. Such construction works well for all membership functions regardless of the approaches stated above. Without loss of generality, let the interval $[a, b]$ be the range of l. For each membership function we consider an uncertainty function of the form

$$\pi(x) = \mu(x)^{\eta}[1 - \mu(x)]^{\vartheta} \quad \eta, \vartheta \geq 1, \tag{14}$$

where η and ϑ are parameters that provide flexibility to the modelling of uncertainty. The function in (14) satisfies the intuitionistic constraint $\pi(x) \leq 1 - \mu(x)$ a consequence of (7). Hence, the non-membership function

$$v(x) = 1 - \mu(x) - \pi(x) = [1 - \mu(x)]\left\{1 - \mu(x)^{\eta}[1 - \mu(x)]^{\vartheta - 1}\right\}$$

follows directly from (7).

We now propose some natural conditions that the uncertainty, the membership and the non-membership functions should satisfy: (A) $\pi(x) = 0$ if and only if either $\mu(x) = 0$ or $\mu(x) = 1$; (B) if the range $[a, b]$ contains a value x_0 representing an indifference point of the ordinal scale, then the uncertainty function attains its maximum in correspondence with x_0; (C) the membership function is increasing if and only if the non-membership function is decreasing; (D) $v(x) = 0$ if and only if $\mu(x) = 1$; (E) $v(x) = 1$ if and only if $\mu(x) = 0$. In the sake of the axiomatic approach, the general conditions above can be viewed as five natural axioms that should be satisfied by meaningful uncertainty, membership and non-membership functions. In particular, the monotony property (C) stands for the natural structure of ordinal scales and the maximum property (B) allows us to deal with an indifference point of the Likert scale. Indeed, the uncertainty function (14) attains its maximum at the point:

$$\mu(x_0) = \frac{\eta}{\eta + \vartheta} = 1 - \frac{\vartheta}{\eta + \vartheta} \tag{15}$$

depending on the value of η and ϑ, as shown in Fig. 1a.

The role of the two parameters is that of adapting the uncertainty function to achieve the maximum at the point x_0 expressing indifference (which does not necessarily coincide with the middle point of the interval) and to control the degree of uncertainty.

In particular, if the scale is not balanced on the left side, as it happens, for instance, for the scale "Unsatisfactory", "Satisfactory", "Good", "Very good", "Excellent" it can be put $\eta < \vartheta$ and (15) shows that the maximum point is located to the left of the central value

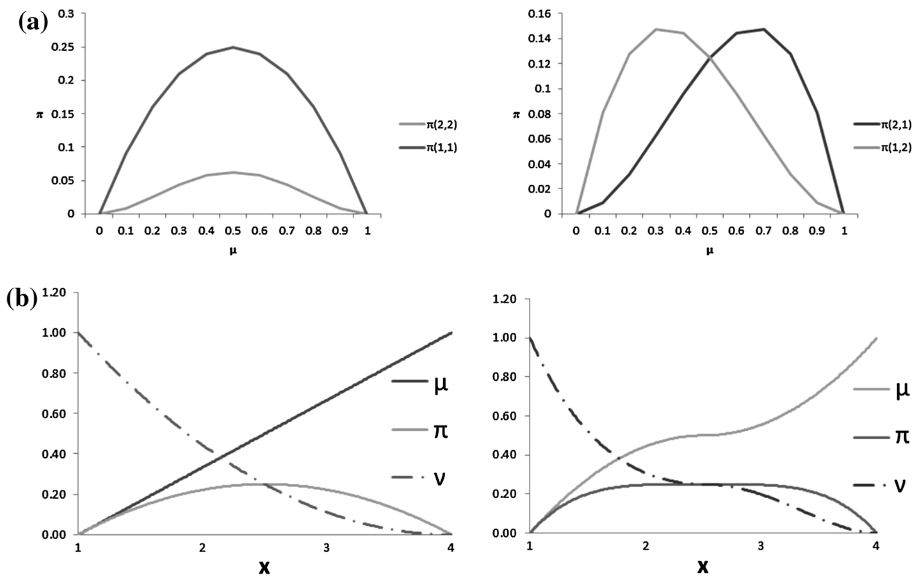

Fig. 1 **a** Uncertainty function, $\pi(x) = \mu(x)^{\eta}[1 - \mu(x)]^{\vartheta}$, for different values of the parameters η and ϑ; **b** Membership functions (μ), uncertainty functions (π), and corresponding non-membership functions (ν). On the left side, linear spline membership function, on the right side, quadratic spline membership function

of the interval [0,1], on which $\mu(x)$ is defined. If the scale is balanced with respect to a central point expressing indifference, such as in the case of the scale "Strongly disagree", "Disagree", "Uncertain", "Agree", "Strongly agree", it can be set $\eta = \vartheta$, having the maximum at the midpoint of the interval. Finally, if the scale is biased towards the right, as it happens for instance with a scale of the type "Very insufficient", "Insufficient", "Almost Sufficient", "Sufficient", "Good", "Very good", a suitable model of uncertainty can be obtained setting $\eta > \vartheta$ so that the maximum point, given by (15), is placed to the right of the central value of the interval. Through the formalist approach we propose a suitable class of "spline" membership functions given by:

$$\mu(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{2} - \frac{1}{2}\left(2\frac{c-x}{b-a}\right)^{\varepsilon} & \text{if } a \le x \le c \\ \frac{1}{2} + \frac{1}{2}\left(2\frac{x-c}{b-a}\right)^{\varepsilon} & \text{if } c < x \le b \\ 1 & \text{if } x > b \end{cases} \tag{16}$$

being $c = \frac{a+b}{2}$ the middle point of $[a, b]$ and $\varepsilon > 0$. If we choose in (14) $\eta = \vartheta = 1$, we obtain

$$\pi(x) = \mu(x)[1-\mu(x)] = \begin{cases} \frac{1}{4} - \frac{1}{4}\left(2\frac{c-x}{b-a}\right)^{2\varepsilon} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and

$$\upsilon(x) = [1-\mu(x)]^2 = \begin{cases} 1 & \text{if } x < a \\ \left[\frac{1}{2} + \frac{1}{2}\left(2\frac{c-x}{b-a}\right)^{\varepsilon}\right]^2 & \text{if } a \le x \le c \\ \left[\frac{1}{2} - \frac{1}{2}\left(2\frac{x-c}{b-a}\right)^{\varepsilon}\right]^2 & \text{if } c < x \le b \\ 0 & \text{if } x > b \end{cases} \tag{17}$$

Specifically, for $\varepsilon = 1$, we have the linear spline used in several applications:

$$\mu(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases} \tag{18}$$

and hence

$$\pi(x) = \mu(x)[1-\mu(x)] = \begin{cases} \frac{(x-a)(b-x)}{(b-a)^2} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$v(x) = [1-\mu(x)]^2 = \begin{cases} 1 & \text{if } x < a \\ \left(\frac{b-x}{b-a}\right)^2 & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases} \tag{19}$$

Figure 1b illustrates the relation among membership, non-membership, and uncertainty functions. For instance, in the case the scale is divided into $M = 4$ ordered modalities (encoded by the first 4 integers) and $\varepsilon = 1$, the use of (18) and (19) for the values $x = 2$, and $x = 3$ provides the measures: $\mu(2) = 0.33$ and $\mu(3) = 0.67$, and, given a degree of indeterminacy of 0.22, in both cases, it follows $v(2) = 0.44$ and $v(3) = 0.11$. The use of (16) and (17) with $\varepsilon = 2$ (corresponding to a quadratic spline) provides instead $\mu(2) = 0.44$ and $\mu(3) = 0.55$. Moreover, given a degree of indeterminacy such as 0.25 in both cases, it follows $v(2) = 0.31$ and $v(3) = 0.20$.

In the foregoing, we derived a non-membership function starting from a membership function and a uncertainty function through (14). Conversely, for each $\upsilon(x)$, we can consider the uncertainty function

$$\pi(x) = \upsilon(x)^{\eta}[1 - \upsilon(x)]^{\vartheta} \tag{20}$$

(with $\eta, \vartheta \geq 1$) under the constraint $\pi(x) \leq 1 - \upsilon(x)$, leading to

$$\mu(x) = 1 - \upsilon(x) - \pi(x) = [1 - \upsilon(x)]\left\{1 - \upsilon(x)^{\eta}[1 - \upsilon(x)]^{\vartheta - 1}\right\}$$

In analogy, we may consider the following class of "spline" non-membership functions:

$$\upsilon(x) = \begin{cases} 1 & \text{if } x < a \\ \frac{1}{2} + \frac{1}{2}\left(2\frac{c - x}{b - a}\right)^{\varepsilon} & \text{if } a \leq x \leq c \\ \frac{1}{2} - \frac{1}{2}\left(2\frac{x - c}{b - a}\right)^{\varepsilon} & \text{if } c < x \leq b \\ 0 & \text{if } x > b \end{cases} \tag{21}$$

(with $c = \frac{a+b}{2}$ and $\varepsilon > 0$, from which, choosing $\eta = \vartheta = 1$ in (20):

$$\pi(x) = v(x)[1 - v(x)] = \begin{cases} \frac{1}{4} - \frac{1}{4}\left(2\frac{c - x}{b - a}\right)^{2\varepsilon} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu(x) = [1 - v(x)]^2 = \begin{cases} 0 & \text{if } x < a \\ \left[\frac{1}{2} - \frac{1}{2}\left(2\frac{c - x}{b - a}\right)^{\varepsilon}\right]^2 & \text{if } a \leq x \leq c \\ \left[\frac{1}{2} + \frac{1}{2}\left(2\frac{x - c}{b - a}\right)^{\varepsilon}\right]^2 & \text{if } c < x \leq b \\ 1 & \text{if } x > b \end{cases} \tag{22}$$

Modelling the non-membership function (instead of the membership function) can be appropriate in certain applied context (as we will show in Sect. 7).

5 A proposal of intuitionistic fuzzy sets for evaluation questionnaires

Let us consider a latent variable S which can be expressed by K observable variables, i.e., items of a questionnaire, through which information on S has been obtained. Questionnaires have been compiled by N subjects (e.g., students, physicians, employees, etc.), who,

Table 3 N subjects have compiled a questionnaire composed by K items, choosing their answer x_{ik} among M modalities of a Likert scale

	1	…	k	…	K
1	…	…	$\mu(x_{1k}), v(x_{1k})$	…	…
…	…	…	…	…	…
i	$\mu(x_{i1}), v(x_{i1})$		$\mu(x_{ik}), v(x_{ik})$		$\mu(x_{iK}), v(x_{iK})$
…	…	…	…	…	…
N	…	…	$\mu(x_{Nk}), v(x_{Nk})$	…	…

To each x_{ik}, we associate values of membership $\mu(x_{ik})$ and non-membership $v(x_{ik})$

for each item, have chosen their answer among M modalities. Let's indicate with x_{ik} the answer of participant i to item k, on the Likert scale 1,…, M. According to the criteria proposed in Sect. 4, we define the functions of membership and uncertainty, from which non-membership functions can be derived. Therefore, to each $x_{ik} \in \{1, .., M\}$, are associated the values $\mu(x_{ik})$ and $v(x_{ik})$ that can be represented in a matrix (see Table 3).

The K columns in Table 3 can be considered as K IFS and the k-th column has the following structure:

$$F_k = \{\langle i, \mu(x_{ik}), v(x_{ik})\rangle : i = 1, \ldots, N\} \tag{23}$$

By comparing (6) and (23), it emerges that the x values defined in the former are substituted by x_{ik} in the latter, and that the functions of membership/non-membership of elements x to A in (6), in (23) are functions of membership/non-membership of subject i to latent variable S through the item k. In other words, $\mu(x_{ik})$ and $v(x_{ik})$ indicate the degree of membership/non-membership of the latent variable S, associated to subject i, who has marked x_{ik} to the k-th item.

We now aim at aggregating the K items in a unique construct representing the variable S. Several aggregators have been put forward in the literature (see Li 2014, for a literature review); we propose to use the IWAM (Intuitionistic Weighted Mean) aggregator (Beliakov et al 2011) relative to K pairs $\alpha_k = \langle \mu_{A_k}(x), v_{A_k}(x)\rangle (k = 1, \ldots, K)$:

$$IWAM_w(\alpha_1, \ldots, \alpha_K) = \left\langle \sum_{k=1}^{K} w_k \mu_{A_k}(x), \sum_{k=1}^{K} w_k v_{A_k}(x) \right\rangle \tag{24}$$

with

$$0 \le w_k \le 1; \sum_{k=1}^{K} w_k = 1 \tag{25}$$

as suitable weights. Our choice is justified by the fact that, in the same questionnaire, certain items are more significant than others in obtaining information on the latent variables, and this can be mathematically expressed in terms of weights. Consider, for instance, the two items presented in the case study: an expert may judge that item 1 (comparing our experience with that of other administration) is more explanatory than item 2 (recognize and appropriately address past problems and mistakes) as far as the dimension of technological innovation is concerned, thus assigning a larger weight to item 1 than to item 2. The IWAM aggregator leads to a set of pairs that we can interpret as the IFS:

Table 4 (a) Synthesis of the K items presented in Tables 1 and 2, by means of the IWAM aggregator (Beliakov et al. 2011). (b) Illustration of the use of the IWAM aggregator using the motivating example. Three employees of a Public Administration have expressed their judgments with respect to two items of a questionnaire. Using formulas (13) and (14) with $a = 1$ and $b = 4$, membership and non-membership values are obtained. (c) Aggregation through IWAM with suitable weights: $w_1 = 0.70$ and $w_2 = 0.30$ of the membership/non-membership values presented in Table 3b

(a)

	S
1	$\mu(x_1), \nu(x_1)$
…	…
i	$\mu(x_i), \nu(x_i)$
…	…
N	$\mu(x_N), \nu(x_N)$

(b)

	Item 1	Item 2
1	0.33, 0.44	0.66, 0.11
2	0.33, 0.44	1, 0
3	0.66, 0.11	0.33, 0.44

(c)

	S
1	0.43, 0.34
2	0.53, 0.31
3	0.56, 0.21

$$F = \{\langle i, \mu(x_i), \nu(x_i)\rangle : i = 1, \ldots, N\}$$

so that Table 3 is synthesized in Table 4a.

In order to appropriately describe the variable S. we now aim to further synthesize Table 4a in a single degree of membership/non-membership to variable S of the group of N subjects. Each pair $\langle \mu(x_i), \nu(x_i)\rangle$ can be conceived as an intuitionistic fuzzy singleton, i.e., an IFS constituted by the only one pair, which, for the i-th subject, is given by $\{\langle x_i, \mu(x_i), \nu(x_i)\rangle\}$ so that the aggregator IWAM can be used on the N singleton of Table 4a, with $w_i = 1/N$, that is to say to every subject is assigned the same weight; we obtain another IFS singleton formed by the only pair:

$$\langle \bar{\mu}, \bar{\nu}\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N} \mu(x_i), \frac{1}{N}\sum_{i=1}^{N} \nu(x_i)\right\rangle \tag{26}$$

where the values in (26) represent the average degree of membership/non-membership to variable S of the set of N subjects.

To illustrate such a situation, let us consider Table 4b, which expresses the judgments given by three employees of a Public Administration, with respect to the two items described in the motivating example. In particular, using (18) and (19) with $a = 1$ and $b = 4$, the values in Table 4b have been obtained, supposing that the evaluators have marked

$\{2;2;3\}$ to item 1 and $\{3;4;2\}$ to item 2. Using (24) with, for instance, $w_1 = 0.70$, and $w_2 = 0.30$ the IFS in Table 4c is obtained, where subject 1 evaluates the capability of the company to be open to technological innovation with 0.43 (and 0.34 indicates the non-membership); the presence of the second item has increased the overall positive evaluation of subject 1 and has reduced negative rating. The use of (26) leads to the pair $\langle 0.51, 0.29\rangle$, which indicates that the set of three employees considers the openness to innovation of their company at a fair level (0.51), and the not opening at a modest level (0.29) with indeterminacy given by 0.20.

6 A proposal of Intuitionistic hesitant fuzzy sets for evaluation questionnaires

The judgments expressed by N subjects in relation to item k, are now interpreted in terms of hesitations on the item k; as a matter of fact, in Table 4b, item 2 is characterized by three pairs of hesitations, one of which, i.e., $\langle 0, 1\rangle$, actually represents a "certainty"; however, the existence of the other two pairs creates hesitation on item 2. With respect to item 1, the two pairs are equal; it is worth observing that, in the literature on HFS, repetitions are generally deleted, while in the present case they are taken into consideration, since each pair can be appropriately labeled. In this context, IHFS are adopted to analyze questionnaires in fuzzy terms, shifting the universe of discourse from the N subjects, as in (23), to the K items. In fact, the values of the data matrix in Tables 1 and 2 can be represented by the following IHFS:

$$G = \left\{\left\langle k, \tilde{h}(x_k)\right\rangle : k = 1, \ldots, K\right\} \tag{27}$$

where the k-th item is labelled by x_k, $\tilde{h}(x_k)$ represents the hesitation, in terms of functions of membership/non-membership, as measured by the N pairs $\langle \mu(x_{ik}), v(x_{ik})\rangle$, for the k-th item of the variable S.

Also in this case, we can aggregate $\tilde{h}(x_k)$ adopting a simplified version of the aggregator quasi hesitant fuzzy weighted aggregator (*QHFWA*; Xu 2014a; Xia et al. 2013) used for HSF and keeping into account the non-membership function. This is given by:

$$QHFWA\,(h_1, \ldots, h_K) = \bigcup_{\gamma_1 \in h_1, \ldots, \gamma_K \in h_K} \left\{ \sum_{k=1}^{K} w_k \gamma_k \right\}$$

where γ_k indicates one of the possible values of the HFS $h_k (k = 1, \ldots, K)$In order to explain the use of QHFWA, let's consider the values $\mu(x_{ik})$ listed in Table 4b and let's fix the coefficients $w_1 = 0.70$ and $w_2 = 0.30$, for items 1 and 2, respectively. It follows:

$$\begin{aligned} QHFWA(h_1, h_2) = \{&0.70 \times 0.33 + 0.30 \times 0.66, 0,70 \times 0.33 + 0.30 \times 1, 0.70 \times 0.33 + 0.30 \\ &\times 0.33, 0.70 \times 0.33 + 0.30 \times 0.66, 0,70 \times 0.33 + 0.30 \times 1, 0.70 \\ &\times 0.33 + 0.30 \times 0.33, 0.70 \times 0.66 + 0.30 \times 0.66, 0.70 \times 0.66 \\ &+ 0.30 \times 1, 0.70 \times 0.66 + 0.30 \times 0.33\} \end{aligned}$$

i.e.,

$$QHFWA(h_1, h_2) = \{0.43, 0, 53, 0.33, 0.43, 0.53, 0.33, 0.66, 0.76, 0.56\}$$

which contains $3 \times 3 = 9$ elements, expressing the hesitation of three employees on the variable *openness to innovation* for their company. Such aggregator in the present context is defined as intuitionistic hesitant weighted mean (*IHWM*) that has the structure:

$$IHWM = \bigcup_{i=1}^{N} \left\{ \left\langle \sum_{k=1}^{K} w_k \mu(x_{ik}),\ \sum_{k=1}^{K} w_k v(x_{ik}) \right\rangle \right\} = \bigcup_{i=1}^{N} \{ \langle \mu(x_i), v(x_i) \rangle \} \tag{28}$$

shown in Table 4a. The use of the proposed IHWM in (23), only with respect to the values of the membership function, leads instead to the synthesis shown in Table 4c. It can be observed that the generalization of the QHFWA to the case IHFS would have produced 9 other values for the hesitations on the non-membership, which would have led to a difficult interpretation of the aggregation in relation to the latent variable *S*. In order to synthesize these results, a generalization of the score proposed for HFS (Xu 2014a, b, c) can be used:

$$s(\tilde{h}_A) = \left\langle \mu = \frac{1}{N} \sum_{i=1}^{N} \mu(x_i), v = \frac{1}{N} \sum_{i=1}^{N} v(x_i) \right\rangle = \langle \bar{\mu}, \bar{v} \rangle \tag{29}$$

Using (28) and (29) the IHFS (27) is reduced to the only pair in (26). We may note that such score aims at synthetizing the hesitations expressed on a certain item, differently from an aggregator, whose function is to combine several items.

With regard to the score, here are two other versions proposed in the literature, namely:

$$s(x_{ik}) = \mu(x_{ik}) - v(x_{ik})$$

introduced as far as IFSs are concerned (Beliakov et al. 2011) and:

$$s_{G-HFS}(\tilde{h}_A) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2} [\mu(x_i) + 1 - v(x_i)] = \frac{\bar{\mu} + 1 - \bar{v}}{2} \tag{30}$$

in the case of G-HFS (Qian et al. 2013), which synthesizes the central values of the intervals $[\mu(x_i), 1 - v(x_i)]$, in the corresponding arithmetic mean. We may note that $s_{G-HFS}(\tilde{h}_A) = \bar{\mu} + \frac{\bar{\pi}}{2}$, being $\bar{\pi} = 1 - \bar{\mu} - \bar{v}$. Using $s_{GHFS}(\tilde{h}_A)$ in Table 4c would have led to the value $\frac{0.51+(1-0.29)}{2} = 0.61$, with respect to the pair $\langle 0.51, 0.29 \rangle$ obtained through (29).

Our case study provides an opportunity to show how the evaluation of the latent variable regarding the *openness to business innovation* can be dealt with two types of fuzzy approaches, i.e., the IFS and the IHFS. Our analysis starts with a questionnaire with an ordinal scale modeled with the two functions of membership and non-membership. In both cases the mathematical modelling is based on the subjects who express their evaluation. Nevertheless, considering the data matrix in Table 2, IFS leads to the synthesis (26) from a "column view" analysis, whereas, in the case of IHFS, we obtain the synthesis (29) (that is equal to (26)) from the "row point of view" (representing *K* items). Thus, the two approaches are equivalent, in the sense that both provide an "average" evaluation of *openness to innovation*. It should be observed that the membership/non-membership functions adopted in the case of questionnaire analysis represent theoretical models that associate the same value to the same answer on a Likert scale. As a matter of fact, in the example, all subjects who responded 3 on the scale have associated membership/non-membership degree equal to 0.66 and 0.11, respectively.

In evaluation contexts in which questionnaires are not available, experts can provide assessments, without referring to underlying functions of membership/non-membership. As a general comment, we suggest that the *model-based* approach has the advantage of a simple and general coding, while the expert-based approach takes properly into account the amount of expertise.

7 Application

7.1 Fuzzy analysis on the Magellano Project Dataset

As stated in Sect. 4, we have applied IFS to the analysis of a national questionnaire developed by the Italian Ministry of the Public Administration, named *The Magellano*

Table 5 Questionnaires of the Magellano project compiled by 214 employees

(a)	S_1
Are the objectives of the structure in which you work clear and well defined?	0.58, 0.26
Are there the resources to properly perform your work?	0.57, 0.24
Is your institution able to find adequate solutions to the problems that it routinely faces?	0.58, 0.24
Do you feel satisfied at the end of your daily work?	0.56, 0.25
Do the work conditions in your institution allow the emergence of the personal qualities of each professional?	0.44, 0.39
When you need information, do you know who to ask them?	0.58, 0.24
Are the organizational roles and work tasks clearly and well defined?	0.54, 0.30

(b)	S_3
Over the past 6 months did you happen to feel:	
Headache and concentration difficulties	0.42, 0.44
Stomach pain, gastritis	0.30, 0.60
Nervousness, restlessness, anxiety	0.50, 0.36
Sense of excessive fatigue	0.45, 0.38
Asthma, respiratory difficulties	0.11, 0.83
Muscle and articular aches	0.38, 0.49
Sleeping disorders, insomnia	0.39, 0.49
Depression	0.31, 0.59
At which degree do you attribute the disorders you reported to your job?	0.48, 0.34

(c)	$\langle \bar{\mu}, \bar{v} \rangle$	$S_{G\text{-}HFS}$
Efficiency	$\langle 0.55, 0.27 \rangle$	0.64
Management	$\langle 0.51, 0.33 \rangle$	0.59
Well-being	$\langle 0.37, 0.46 \rangle$	0.45
Innovation	$\langle 0.45, 0.37 \rangle$	0.54

(a) Section (iv): efficiency (S_1), membership and non-membership values. (b) Section (x): mental and physical well-being (S_3), membership and non-membership values. (c) Synthesis of the four latent variables considered S_1–S_4, through (29) and score calculated with (30)

Project. Such questionnaire, filled via the web, is composed by 48 questions organized in 12 thematic areas. For purposes of application, we have considered the following latent variables: (iv) The efficiency (S_1); (vii) The management (S_2); (x) The mental and physical well-being (S_3); (xi) The openness to innovation (S_4). The section on efficiency comprises seven items that investigate as far as the objectives, the resources and the organizational roles in the Public Administration are concerned. The section on management is composed by five items regarding the capacity of Departments' Directors in dealing with Human Resources and Organization. In the section concerning the mental and physical well-being (nine items) it is investigated the presence of physical and psychological symptoms, such as asthma, depression, anxiety. Finally, in the section regarding the Openness to Innovation (nine items), employees are asked to express an evaluation on the openness of their institution as far as new organizational structures and the use of new technologies are concerned. It is worth noting that in the case of S_3 (see Table 5b), the questions investigate the departure from the standard well-being state (e.g., suffering from gastritis). In such a way, it naturally emerges the need of modelling the non-membership function (instead of the membership one). In this thematic area, subjects are indeed requested to express their not *well-being* states (i.e., their non-membership with respect to the FS involved).

With respect to the use of membership and non-membership functions, we modeled the modes of responses in Sects. (iv), (vii) and (xi) using the quadratic splines (16) and (17) both with $\varepsilon = 2$; as for Section (x), we have chosen instead to use the linear splines (21) and (22) both with $\varepsilon = 1$. In fact, in the former case subjects expressed their opinion using the following scale: "Decidedly No", "More no than yes", "More yes than no", "Decidedly yes". In the latter case, the scale was: "Never", "Few times", "Enough time", "Much time". We suggest that, in the former case, the central modes of response ("More no than yes", "More yes than no") can be considered as more veiled, indefinite, vague and, in a sense, "close" to each other. Therefore, the quadratic spline has allowed us to appropriately model the terminology used in the questionnaire. It's important to emphasize that this is a subjective choice, which largely depends on the expertise of the researcher. It's indeed possible that different choices of the functional form can adequately represent the terminology used in the questionnaire. In the present context, the focus is on the adaptability and flexibility of FS and IFS as a mathematical tool, allowing a researcher to rigorously model a formal translation of human language (adopted by the questionnaire) in suitable functional forms.

We considered the questionnaires compiled by 214 employees (95 males) at the University of Milan-Bicocca in 2013. 48 employees were aged <35 years., 59 were aged 36–40 years., 73 were aged 41–50 years., and 34 were aged >50 years. 4 employees had 8 years. of education, 72 had a high school diploma and 138 were graduated. Moreover, 201 employees had a permanent position and 190 had a full-time contract. In Table 5a and b we present, for the sake of convenience, the items and the means (membership and non-membership functions) obtained by the group of 214 subjects, as far as the section efficiency (S_1) and Well-being (S_3) are concerned. Table 5c presents a synthesis, leading to (30) through (29) in case of the four latent variables we have considered.

It is interesting to note that the employees of this Public Administration expressed an opinion (which we translated in terms of values of membership/non-membership) quite high with regard to the efficiency of their institution and slightly lower than that of management. The evaluation expressed for openness to innovation is lower, implying that this dimension ought to be focused by the management of the University in the development of the administration's objectives. Finally, the level of membership with respect to well-being is quite low, which means, however, that −given the structure of the questionnaire (see Table 3)—the general welfare of the employees in this administration can be considered discrete.

7.2 Sensitivity analysis

In order to study the impact of different choices of indeterminacy functions on our models, we conducted a sensitivity analysis. In fact, in Sect. 4, we introduced formulas for functions of membership/non-membership for values of the parameter $\eta = \vartheta = 1$ in $\pi(x) = \mu(x)^{\eta}[1 - \mu(x)]^{\vartheta}$ (see Eq. 19). We used the dataset of the Magellano project, modifying the initial conditions of our problem, and considering six different scenarios. In three cases, we started from $\eta = \vartheta = 1$ and in other three cases, we started from $\eta = \vartheta = 2$. Across these situations, we run three analyses: (i) η fixed and ϑ assuming different values; (ii) ϑ fixed and η assuming different values; (iii) both η and ϑ assuming different values. The results of these analyses are reported in Table 6, which shows the values of non-membership degree (ν) in such scenarios.

These results demonstrate that our model is indeed robust with respect to the choice of the parameters. For instance, in the first three cases, it is necessary to significantly change the values of η, ϑ or both (e.g. $\eta = \vartheta = 2$) to considerably change the values of the non-membership function. Besides, we observe that in Table 6 the ranking of the ν values is invariant with respect to the changing of the shape parameters, and this can be directly generalized to the score values. The outcomes of such sensitivity analyses confirm that interpreting questionnaire data through *a model-based* IFS approach is only partially dependent from the choice of the mathematical form for the function of indeterminacy.

8 Discussion

In this paper, we have introduced the use of IFS and IHFS in the analysis of questionnaire. Contemporary theory of FS is very advanced from a theoretical point of view, and there are many applications concerning engineering, information technology, economics, psychology and social sciences. Relevant theoretical advances have also appeared with regard to IFS and HFS, which, however, apart from the case of decision making, are still less relevant from an application point of view. In our view, both IFS and IHFS can be considered as a useful theoretical framework for the analysis of questionnaires. In fact, the interpretation of questionnaire results based on the definition of values of membership and not-membership in IFS can be very important, also from a communication point of view. The fuzzy approach can be viewed as a useful way to describe latent constructs such as those of satisfaction or performance through a mathematical formalization (see Zani et al. 2012). Moreover, the Intuitionistic framework we have used in this paper can lead to appropriately quantify, as well as satisfaction or high levels of performance, also dissatisfaction or low levels of performance, and uncertainty. In this way, it was possible to introduce a two-dimensional analysis, thus simultaneously considering both membership and non-membership degrees, and identifying areas of excellence and critical issues. A relevant issue of our analysis has concerned the choice of the functions of membership and non-membership. As regards the former, we have chosen to introduce in our model either a linear or a quadratic spline. As concerns the latter, non-membership has been defined starting from the choice of a specific uncertainty function, which we have also studied in terms of sensitivity with respect to the choice of shape parameters. We have applied our

Table 6 Sensitivity analysis: values of non-membership functions (ν) across six different scenarios, changing the values of the parameters η/ϑ in the uncertainty function

Parameters	η ϑ	1 1	1 1.01	1 1.02	1 1.05	1 1.5	1 2
ν	Efficiency	0.276	0.277	0.278	0.282	0.327	0.363
	Management	0.330	0.331	0.332	0.336	0.378	0.413
	Well-being	0.466	0.467	0.467	0.469	0.494	0.513
	Innovation	0.369	0.370	0.371	0.375	0.422	0.459
Parameters	η ϑ	1 1	1.01 1	1.02 1	1.05 1	1.5 1	2 1
ν	Efficiency	0.276	0.277	0.278	0.282	0.325	0.359
	Management	0.330	0.331	0.332	0.336	0.376	0.409
	Well-being	0.466	0.467	0.468	0.470	0.499	0.521
	Innovation	0.369	0.370	0.371	0.375	0.425	0.463
Parameters	η ϑ	1 1	1.01 1.01	1.02 1.02	1.05 1.05	1.5 1.5	2 2
ν	Efficiency	0.276	0.278	0.281	0.287	0.361	0.404
	Management	0.330	0.332	0.335	0.341	0.411	0.452
	Well-being	0.466	0.468	0.469	0.474	0.520	0.545
	Innovation	0.369	0.371	0.374	0.381	0.462	0.508
Parameters	η ϑ	2 2	2 2.01	2 2.02	2 2.05	2 2.5	2 3
ν	Efficiency	0.404	0.404	0.404	0.405	0.416	0.425
	Management	0.452	0.452	0.452	0.453	0.463	0.472
	Well-being	0.545	0.545	0.545	0.546	0.551	0.555
	Innovation	0.508	0.508	0.509	0.510	0.521	0.531
Parameters	η ϑ	2 2	2.01 2	2.02 2	2.05 2	2.5 2	3 2
ν	Efficiency	0.404	0.404	0.404	0.405	0.416	0.425
	Management	0.452	0.452	0.452	0.453	0.463	0.471
	Well-being	0.545	0.545	0.546	0.546	0.553	0.557
	Innovation	0.508	0.508	0.509	0.510	0.522	0.532
Parameters	η ϑ	2 2	2.01 2.01	2.02 2.02	2.05 2.05	2.5 2.5	3 3
ν	Efficiency	0.404	0.405	0.405	0.407	0.425	0.436
	Management	0.452	0.452	0.453	0.455	0.472	0.482
	Well-being	0.545	0.546	0.546	0.547	0.557	0.563
	Innovation	0.508	0.509	0.510	0.511	0.531	0.543

results to a *case-study* concerning a national Italian survey on the assessment of Public Administration (*The Magellano Project*). The theory of FS, IFS and HFS has allowed us to suitably translate the problem of evaluation into mathematical terms, by modeling the linguistic terms used in the questionnaire.

Let's now briefly recall and underline the main advantages of the fuzzy approach, also in connection with the analysis of survey data. The basic point to enlighten is that FS is a mathematical approach dealing with vagueness, thus analyzing in quantitative terms the meaning underpinned by words and sentences. Questionnaires are constituted by sentences in which respondents are usually requested to quantify vague notion, such as in the case of customer satisfaction. Taking into account and quantifying these aspects is a peculiarity of the fuzzy approach. FSs allow a researcher to deal with the conflict between the linguistic terms used in a scale (considered as a finite set of statements) and their numerical translation (Dubois et al. 2000). In the context of the survey analysis, the attractiveness of the fuzzy approach lays in attaching continuous consistency profiles (Black 1937), to the linguistic terms adopted by the scale. Later, Zadeh (1965) referred to consistency profiles in terms of membership functions. In this paper, we precisely focused on the choice of such models. It is also typical of questionnaires used in evaluation to use categorical and dimensional concepts; for instance, those underpinning latent constructs such as customer satisfaction, performance or well-being. Another reason to consider the fuzzy approach to the analysis of questionnaire data, is that such techniques can be also extended to other multivariate models, such as latent class analysis (Manton et al. 1994) and multilevel models (Goldstein et al. 2000).

The main advantage of IFS and IHFS lies in the possibility of taking into account multiple and different sources of uncertainty. In particular, IFS model the uncertainty in choosing among various modes of response, through the π function. In addition, IHFS allows to appreciating the hesitation of each subject. Dealing with uncertainty represents also the aim of non-fuzzy models, like the Combination of Uniform and shifted Binomial random variables (CUB, Iannario and Piccolo 2012). Such a model is provided by a mixture of a Uniform distribution on the range of the linguistic rule l (codifying the response modalities by $1, \ldots, M$) and the distribution of the random variable $B + 1$, being B. Binomial random variable of parameters $(M - 1, p)$, with mixing parameter given by ω and measuring the indecision of the subjects. So, the resulting random variable R allows to calculate the probability of choosing the modality $r = 1, \ldots, M$ as

$$\Pr(R = r) = \frac{\omega}{M} + (1 - \omega)\binom{M-1}{r-1} p^{r-1}(1-p)^{M-r}$$

where the "feeling" parameter p quantifies the adherence to the proposed choice.[1] This model needs to estimate (through EM algorithm) the pair of parameters

$$(p, \omega) \in [0, 1] \times [0, 1)$$

which becomes easily understandable from a communication point of view, because of the fact that p increases when respondents choose high ratings and ω increases with indecision. Also the proposed Intuitionistic fuzzy models are based on a pair of parameters

$$(\mu, \nu) \in [0, 1] \times [0, 1]$$

easy to communicate and measuring, respectively, the agreement and the disagreement with the item considered. As a matter of fact, these parameters are linked to the uncertainty

[1] In order to avoid any conflict of notation, we employed p and ω instead of $(1 - \xi)$ and $(1 - \pi)$ adopted by Iannario and Piccolo (2012).

degree π. From an information quality point of view, IFS and IHFS allow to model the semantic of the ordinal modes of response in a very flexible way by means of suitable membership, non-membership and uncertainty functions (such as the proposed ones). At the price of the flexibility, our fuzzy approach involves a certain level of subjectivity in the choice of functions' shapes. Differently, the CUB model seems like less flexible but more objective, because of the fact that the same probability structure is applied to every possible frame of response modalities. Besides, the CUB feeling parameter can be used to quantify the attribute performance in a context of customer satisfaction to provide a performance analysis (Cugnata and Salini 2013). As a future development of this work, IFS and IHFS as well as CUB will be employed in the framework of performance analysis.

References

Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Set Syst **20**(2), 87–96 (1986)

Atanassov, K.T.: On Intuitionistic Fuzzy Sets Theory. Springer, New York (2012)

Bede, B.: Mathematics of Fuzzy Sets and Fuzzy Logic. Springer, New York (2013)

Beliakov, G., Bustince, H., Goswami, D.P., Mukherjee, U.K., Pal, N.R.: On averaging operators for Atanassov's intuitionistic fuzzy sets. Inform Sciences **181**(6), 1116–1124 (2011)

Bellman, R.E., Zadeh, L.A.: Decision-making in a fuzzy environment. Manag. Sci. **17**(4), B-141 (1970)

Betti, G., Cheli, B., Lemmi, A., Verma, V.: On the construction of fuzzy measures for the analysis of poverty and social exclusion. Stat. Appl. Inf. Sci. **IV**(2), 23–26 (2006)

Betti, G., D'Agostino, A., Neri, L.: Educational mismatch of graduates: a multidimensional and fuzzy indicator. Soc. Indic. Res. **103**(3), 465–480 (2011)

Biswas, R.: An application of fuzzy sets in students' evaluation. Fuzzy Set Syst **74**(2), 187–194 (1995)

Black, M.: Vagueness. An exercise in logical analysis. Philos. Sci. **4**(4), 427–455 (1937)

Castillo, C., Lorenzana, T.: Evaluation of business scenarios by means of composite indicators. Fuzzy Econ. Rev. **15**(2), 3–20 (2010)

Cerioli, A., Zani, S.: A fuzzy approach to the measurement of poverty. In: Dagum, C., Zenga, M. (eds.) Income and wealth distribution, inequality and poverty. Studies in Contemporary Economics, pp. 272–284. Springer, Berlin (1990)

Cheli, B.: Totally fuzzy and relative measures of poverty in dynamic context. Metron **53**, 83–205 (1995)

Cheli, B., Lemmi, A.: A "Totally" fuzzy and relative approach to the multidimensional analysis of poverty. Econ. Notes **24**, 115–134 (1995)

Chen, S.-M., Lee, C.-H.: New methods for students' evaluation using fuzzy sets. Fuzzy Set Syst **104**(2), 209–218 (1999)

Chien, C.-J., Tsai, H.-H.: Using fuzzy numbers to evaluate perceived service quality. Fuzzy Sets Syst. **116**(2), 289–300 (2000)

Chung, S., Choi, H., Lee, S.S.Y.: Measuring social capital in the Republic of Korea with mixed methods: application of factor analysis and fuzzy-set ideal type approach. Soc. Indic. Res. **117**(1), 45–64 (2014)

Crocetta, C., Delvecchio, G.: A fuzzy measure of satisfaction for university education as a key for employment. In: Fabbris, L. (ed.) Effectiveness of University Education in Italy, pp. 11–27. Springer, Berlin (2007)

Cugnata, F., Salini, S.: Model-based approach for importance—performance analysis. Qual. Quant. **48**(6), 3053–3064 (2013)

Darestani, A.Y., Jahromi, A.E.: Measuring customer satisfaction using a fuzzy inference system. J. Appl. Sci. **9**(3), 469–478 (2009)

da Silva, C.F.D., de Araújo Batista, D., de Medeiros, D.D.: A proposed method to evaluate the quality of services using Fuzzy sets theory. Qual. Quant. **48**(2), 871–885 (2014)

Delgado, M., Ruiz, D., Sanchez, D., Vila, A.: Fuzzy quantification: a state of the art. Fuzzy Set Syst. **242**, 1–30 (2014)

D'Elia, A., Piccolo, D.: A mixture model for preferences data analysis. Comput. Stat. Data Anal. **49**(3), 917–934 (2005)

Despi, I., Opris, D., Yalcin, E.: Generalised Atanassov intuitionistic fuzzy sets. In eKNOW 2013, The Fifth International Conference on Information, Process, and Knowledge Management (pp. 51–56) (2013)

Dubois, D., Ostasiewicz, W., Prade, H.: Fuzzy sets: history and basic notions. In: Dubois, D., Prade, H. (eds.) Fundamentals of Fuzzy Sets, pp. 21–124. Springer, New York (2000)
Goldstein, H., Rasbash, J., Browne, W., Woodhouse, G., Poulain, M.: Multilevel models in the study of dynamic household structures. Eur. J. Popul. **16**(4), 373–387 (2000)
Goguen, J.A.: L-fuzzy sets. J. Math. Anal. Appl. **18**(2), 145–174 (1967)
Kwong, C.K., Bai, H.: A fuzzy AHP approach to the determination of importance weights of customer requirements in quality function deployment. J. Intell. Manuf. **13**(5), 367–377 (2002)
Iannario, M., Piccolo, D.: CUB models: Statistical methods and empirical evidence. In: Kennet, R., Salini, S. (eds.) Modern Analysis of Customer Surveys: With Applications using R, pp. 231–258. Wiley, Chichester (2012)
Lalla, M., Facchinetti, G., Mastroleo, G.: Ordinal scales and fuzzy set systems to measure agreement: an application to the evaluation of teaching activity. Qual. Quant. **38**(5), 577–601 (2005)
Li, D.-F.: Decision and Game Theory in Management with Intuitionistic Fuzzy Sets, vol. 308. Springer, New York (2014)
Manton, K.G., Tolley, H.D., Woodbury, M.A.: Statistical Applications Using Fuzzy Sets. Wiley, New York (1994)
Marasini, D., Quatto, P.: Descriptive Analysis of student ratings. J. Appl. Quant. Methods **6**(4), 125–133 (2011)
Piccolo, D., D'Elia, A.: A new approach for modelling consumers' preferences. Food Qual. Prefer. **19**, 247–259 (2008)
Qian, G., Wang, H., Feng, X.: Generalized hesitant fuzzy sets and their application in decision support system. Knowl.-Based Syst. **37**, 357–365 (2013)
Rodríguez, R.M., Martínez, L., Torra, V., Xu, Z.S., Herrera, F.: Hesitant fuzzy sets: state of the art and future directions. Int. J. Intell. Syst. **29**, 495–524 (2014)
Smithson, M., Verkuilen, J.: Fuzzy Set Theory: Applications in the Social Sciences. Sage, London (2006)
Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**(6), 529–539 (2010)
Xia, M., Xu, Z., Chen, N.: Some hesitant fuzzy aggregation operators with their application in group decision making. Group Decis. Negot. **22**(2), 259–279 (2013)
Xu, Z.: Hesitant Fuzzy Sets Theory, vol. 314. Springer, New York (2014a)
Xu, Z.: Hesitant fuzzy Aggregation Operators and Their Applications. Hesitant Fuzzy Sets Theory. Springer, New York (2014b)
Xu, Z.: Intuitionistic Preference and Interactive Decision Making. Springer, New York (2014c)
Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
Zani, S., Milioli, M.A., Morlini, I.: Fuzzy methods and satisfaction indices. In: Kennet, R., Salini, S. (eds.) Modern Analysis of Customer Surveys: With Applications Using R, pp. 439–455. Wiley, Chichester (2012)
Zani, S., Milioli, M.A., Morlini, I.: Fuzzy composite indicators: an application for measuring customer satisfaction. Advances in Theoretical and Applied Statistics, pp. 243–253. Springer, New York (2013)
Zhu, B., Xu, Z., Xia, M.: Dual hesitant fuzzy sets. J. Appl. Math. (2012). doi:10.1155/2012/879629
Zimmermann, H.J.: Fuzzy set theory. Wiley Interdiscip. Rev. **2**(3), 317–332 (2010)