Qual Quant (2015) 49:2345–2368
DOI 10.1007/s11135-014-0117-5

Voters' preference diversity, concepts of agreement and Condorcet's paradox

William V. Gehrlein · Dominique Lepelley · Issofa Moyouwou

Published online: 17 October 2014

Abstract Gehrlein et al. (Math Soc Sci 66:352–365, 2013) have shown that an increase of the voters' preference diversity, as measured by the number k of preference types in a voting situation, implies a decrease in the probability of having a Condorcet Winner. The results offered in this paper indicate that this relationship is far from being so clear when we consider instead the proximity of voting situations to having k distinct preference types. This measure of agreement is compared to other measures of group mutual coherence previously analyzed in Gehrlein (Condorcet's paradox, Springer Publishing, Berlin, 2006). It turns out that our results are completely consistent with the theory introduced by List (Good Soc 11:72–79, 2002) that is based on an important distinction between two different concepts of agreement.

Keywords Voting paradox · Group mutual coherence · Condorcet winner · Probability

1 Introduction

We consider elections on three candidates $\{A, B, C\}$ in which each of n voters has one of the six possible complete preference rankings on the candidates, as shown in Fig. 1.

Here, n_i denotes the number of voters with the associated ith preference ranking on candidates for $1 \leq i \leq 6$ in a given election. A particular *voting situation* is denoted as $\boldsymbol{n}$ and it specifies the number of voters that have each of the possible preference rankings in a

W. V. Gehrlein
University of Delaware, Newark, DE, USA
e-mail: wvg@udel.edu

D. Lepelley (✉)
CEMOI, Université de La Réunion, Saint-Denis Messag Cedex 9, France
e-mail: dominique.lepelley@univ-reunion.fr

I. Moyouwou
Ecole Normale Supérieure, Université de Yaoundé I, Yaoundé, Cameroon
e-mail: imoyouwou2@yahoo.fr

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 1 Complete preference rankings on three candidates

given election, such that $n = \sum_{i=1}^{6} n_i$. Since voters have complete preference rankings, voter indifference between candidates is prohibited and voters never have intransitive, or cyclic, preferences on candidates.

Numerous studies have been conducted to consider various counterintuitive election outcomes that are also called voting paradoxes that could be observed from voting situations in elections with commonly used voting procedures. Nurmi (1999) surveys many of the possible voting paradoxes that have been described in the literature. All of these voting paradoxes are interesting, since each suggests possible scenarios that could be pointed out from election outcomes that might undermine the confidence of the electorate in the voting procedure that was being used. As a result, many studies have been conducted to evaluate the likelihood with which each of these voting paradoxes might be observed, and to consider different natural conditions that might be assumed to hold on the preferences of the electorate that would prohibit the existence of voting situations that lead to these paradoxical outcomes.

The voting paradox that has received the most attention in the literature is Condorcet's Paradox. To describe this widely studied paradox, let $A \succ B$ denote the outcome that a specific voter prefers Candidate A to Candidates B. Then $\boldsymbol{AMB}$ if more voters have $A \succ B$ than those who have $B \succ A$. If there are only two candidates in an election, simple majority rule would elect A if $\boldsymbol{AMB}$. Condorcet (1994) extended the notion of majority rule comparisons to elections on three candidates by looking only at the pairwise majority comparisons on the set of pairs of candidates, based on the relative rankings of the candidates in each of the pairs within the voters' complete ranking on all candidates. Using the rankings from Fig. 1, we have for example $\boldsymbol{AMB}$ if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$. If $\boldsymbol{AMB}$ and $\boldsymbol{AMC}$ in a three-candidate election, then Candidate A is the *Condorcet Winner* (CW), and such a candidate is widely viewed as being the best choice for selection as the winner. As we have defined the CW, this is a Strict CW, since there are no ties in the pairwise majority rule relationships, and this must be true for complete preference rankings when n is odd. Condorcet's Paradox occurs when such a candidate does not exist, and Condorcet (1994) provides an example where the pairwise majority rule is cyclic with $\boldsymbol{AMB}$, $\boldsymbol{BMC}$ and $\boldsymbol{CMA}$. In such a situation, no matter which candidate is selected as a winner, there are a majority of voters who would prefer to have some other candidate to be selected as the winner. Much of what is known about the probability that Condorcet's Paradox might occur is surveyed in Gehrlein (2006).

As mentioned above, much work has been done to consider conditions on voters' preferences that will restrict the probability that various paradoxes will occur. A recent analysis of this type was performed in Gehrlein et al. (2013) as an extension of work that was initially performed in Felsenthal et al. (1990), and it is based on the notion of *voters' preference diversity* in a voting situation. Felsenthal et al. (1990) developed definitions of the various possible categories of voting situations that might exist in terms of the general size relationships between the n_i terms with $K = \{i : n_i > 0,\ for\ 1 \leq i \leq 6\}$. When $k = \#K$, with $\#K$ denoting the cardinality of K, k defines the number of coalitions of voter types that exist with similar preference rankings, and no other voter has a preference ranking on candidates that is in disagreement with these coalitions. Parameter k represents a simple measure of the

Table 1 Values of $P^{CW}(\infty, IAC(k))$ for $1 \leq k \leq 6$

k	$P^{CW}(\infty, IAC(k))$
1	1
2	1
3	39/40 = 0.9750
4	19/20 = 0.9500
5	15/16 = 0.9375
6	15/16 = 0.9375

degree of diversity among voters' preferences, and there is a logical connection between the value of Parameter k and the probability that a CW exists in voting situations.

To see this connection, it is obvious from the preference ranking definitions in Fig. 1 that if $k \leq 2$, then some candidate is never ranked as most preferred by any voter, some candidate is never ranked as least preferred by any voter, and some candidate is never middle-ranked by any voter. It is widely known that a Strict CW must exist under any of these three conditions for three-candidate elections when n is odd (see Black 1958; Vickery 1960; Ward 1965), and much more will be said about this later in this paper[1]. Condorcet's Paradox can therefore only exist if at least one Latin Square triple of rankings is contained in K for $k \geq 3$, such that each of the three candidates will be ranked as most preferred, least preferred or middle-ranked by some voter in a triple of possible preference rankings. There are two possible Latin Square triples of rankings for three-candidate rankings in Fig. 1 with $\{1,4,5\} \subseteq K$ or $\{2,3,6\} \subseteq K$. The existence of a Latin Square is a necessary condition for Condorcet's Paradox to exist, but it is not a sufficient condition. The paradox probability must be nonzero for $k = 3, 4$ since it is possible that only one of the two Latin Squares could be present in the associated voting situations, but it is not necessarily true that a Latin Square must be present in this case. When $k = 5$, one of the Latin Squares, but not both, must necessarily be included in the associated voting situation to increase the likelihood of the existence of a majority rule cycle. And, both of the possible Latin Square triples must be included in the associated voting situations when $k = 6$, to allow the maximum possible number of scenarios for introducing majority rule cycles into a voting situation. Intuition therefore strongly suggests that the probability that Condorcet's Paradox will exist should tend to increase as the degree of diversity of voters' preference increases, as measured by Parameter k. Moreover, increasing values of k should generally tend to introduce more degrees of freedom to allow for any voting paradox to be introduced into voting situations.

The study by Gehrlein et al. (2013) evaluates the strength of this logical relationship between Parameter k and the probability that a CW exists. The probability representations in that study are based on an extension of the Impartial Anonymous Culture Condition (IAC), which assumes that all possible voting situations for a specified n are equally likely to be observed. The assumption $IAC(k)$ instead assumes that Parameter k is specified for a given n, and that all possible voting situations with that specified value of k are equally likely to be observed. This assumption is then used to obtain the limiting probability values $P^{CW}(\infty, IAC(k))$ that a CW exists as $n \to \infty$. Table 1 summarizes the limiting results for each $1 \leq k \leq 6$ from Gehrlein et al. (2013).

[1] When $k \leq 2$, there are at most two distinct complete rankings that can represent voters' preferences, so the more preferred ranking of these two represents the preferences of a strict majority of voters for odd n. The candidate that is most-preferred in that ranking must therefore be the CW.

The results of Table 1 indicate that the expected general relationship between the diversity of voters' preferences, as measured by Parameter k, and the probability that A CW exists does indeed hold up, since $P^{CW}(\infty, IAC(k))$ never increases as k increases.

It would certainly be possible to perform this analysis by using other assumptions that apply to the likelihood that various voting situations are observed. One commonly used assumption of this type in the literature is the Impartial Culture Condition (IC), which assumes that each voter is independently and equally likely to have any of the possible complete preference rankings on the candidates. We have chosen not to consider IC as a basis in our analysis for two reasons. First, when the number of rankings on candidates is reduced from the situation in which all possible complete preference rankings are allowable, the resulting limiting IC-based probability representations typically converge to values of just zero or one. Second, and most importantly, recent analysis in Gehrlein and Plassmann (2014) indicates that IAC can be expected to produce probability estimates that exhibit the same patterns of behavior for studies of this type as those patterns that are observed in data sets that are taken from actual election results.

The primary focus of this current paper is to determine if the nature of this broad general relationship between Parameter k and the probability that a CW exists will be observed continues to remain valid under additional scrutiny. We find that the answer is that this is not always the case, and we then analyze what additional restrictions must be assumed in order for this relationship to become true. Results of the analysis give strong support to theories that have been presented previously in the work of Christian List and his co-authors. That is, there are multiple levels at which voters' preference rankings in a voting situation might tend to reflect degrees of mutual agreement, and the strongest levels of mutual agreement are associated with scenarios in which the voters are generally able to come to an agreement on a consistent listing of candidates along some common dimension of comparison.

2 Measuring the proximity of voting situations to meeting strict conditions

It was mentioned above that a sufficient restriction on three candidate voting situations to require the existence of a CW is that some candidate is never ranked as least preferred by any voter. Arrow (1963) showed that this restriction is equivalent to the well-known condition of single-peaked preferences from Black (1958), who proved that single-peakedness was sufficient to ensure the existence of a CW in three-candidate elections.

The notion of single-peaked preferences follows from a very natural model to describe how individual voters form their preference rankings on candidates. Each voter has a measure of utility that is associated with each of the given candidates. Each voter then obtains their associated complete preference ranking on candidates in the same order that the candidates are ranked according to that voter's decreasing utility values. For single-peakedness to apply, all voters must be able to agree on a common ordering of candidates from left to right on a given dimension, and each voter has some most preferred candidate with maximum utility along this common ordering of candidates. Each voter's utility for candidates must then continuously decrease for candidates when moving in either to the left or right away from their most preferred candidate in the common ordering. We note that 'left' and 'right' do not necessarily correspond to the commonly used definitions of political leanings of candidates in this discussion. It only refers to the relative geometric position of candidates along a line that represents some dimension or characteristic.

Niemi (1969) performed an empirical study to try to explain why Condorcet's Paradox was not observed very frequently in empirical studies with a small number of candidates. The basic premise was that voting situations only had to be 'close' to being perfectly single-peaked to have a very high probability that a CW exists. His measure of 'closeness' or 'proximity' of a voting situation to being perfectly single-peaked was simply given as the minimum proportion of all voters whose preferences must be ignored in order for the remaining voters to have perfectly single-peaked preferences. If the preferences of a very small proportion of voters must be ignored to meet this condition, the likelihood that a CW exists should be expected to be quite high. As this minimum necessary proportion to be removed increases, which means that voting situations are becoming farther removed from having perfectly single-peaked preferences, the probability that a CW will be observed is then expected to decrease consistently.

We apply the same concept from Niemi (1969) here and consider instead the minimum proportion of voters whose preferences must be ignored in order to create a voting situation with a specified value of Parameter k. Let $P^{CW}(\infty, IAC_k(\alpha))$ denote the limiting probability that a CW exists when the minimum proportion α of voters must have their preferences removed from a voting situation in order for the reduced voting situation to have a specified Parameter k under the $IAC_k(\alpha)$ assumption. The $IAC_k(\alpha)$ assumption specifies that all voting situations with the given value of α for the specified k are equally likely to be observed. Following earlier discussion, if a voting situation has perfectly single-peaked preferences then some candidate is never ranked as least preferred and therefore $k \leq 4$. However, it is quite possible that a voting situation can exist with $k = 4$ without reflecting preferences that are single-peaked, such as when $n_1 = n_5 = 0$ with all other $n_i > 0$. Changing the value of α in $P^{CW}(\infty, IAC_4(\alpha))$ therefore represents a less restrictive condition in the current study than did the proximity to perfect single-peakedness that is suggested in Niemi's work.

We know that $P^{CW}(\infty, IAC(k))$ behaves as expected as k increases in Table 1. Consideration now changes to whether or not $P^{CW}(\infty, IAC_k(\alpha))$ consistently behaves as we expect when α increases to reflect voting situations that are farther removed from being perfectly represented by a voting situation with a specified Parameter k. We discussed above the logical link to this expectation between Niemi's work and our case of $k = 4$, and we also pursue this analysis for all other values of k. The first step of this analysis is to develop limiting representations for $P^{CW}(\infty, IAC_k(\alpha))$ for each $1 \leq k \leq 5$.

2.1 Limiting representations for $P^{CW}(\infty, IAC_k(\alpha))$

If we define $x_j = n_j/n$ in the limit as $n \to \infty$, a voting situation is described as a 6-tuple $x = (x_1, x_2, \ldots, x_6)$ with nonnegative components such that $\sum_{i=1}^{6} x_i = 1$. Then let $S(\alpha, \infty, k)$ denote the set of all voting situations that require exactly a minimum proportion α of voters to be removed in order to have a reduced voting situation for which there are only $k$remaining preference ranking types. For every voting situation $x \in S(\alpha, \infty, k)$ there must exist a $J \subset \{1, 2, \ldots, 6\}$ with $\#J = 6 - k$ such that

$$\sum_{j \in J} x_j = \alpha, \ \sum_{l \notin J} x_l = 1 - \alpha, \ \text{and } x_j \leq x_l \text{ for all } j \in J \text{ and for all } l \notin J. \tag{1}$$

It follows that $S(\alpha, \infty, k)$ is 4-dimensional, and the space of all possible J is defined to have a 4-dimensional volume $C(\alpha, \infty, k)$, with:

$$C(\alpha, \infty, k) = \binom{6}{6-k} C_J(\alpha, \infty, k). \tag{2}$$

Here, $C_J(\alpha, \infty, k)$ is the 4-dimensional volume of the set of all voting situations that satisfy the set of constraints in (1) for any given $J \subset \{1, 2, \ldots, 6\}$ with $\#J = 6-k$. We note that any of the distinct sets of voting situations that are defined by (1) may overlap, but the dimension of these corresponding intersections is less than four.

For each possible J, let $C'_J(\alpha, \infty, k, A)$, $C'_J(\alpha, \infty, k, B)$ and $C'_J(\alpha, \infty, k, C)$ respectively denote the 4-dimensional volume of the subsets of all voting situations defined in (1) and for which A, B and C are also respectively the CW. The volume of the set of all possible voting situations that are defined by (1) for which a CW exists is then given by $C'_J(\alpha, \infty, k)$, with

$$C'_J(\alpha, \infty, k) = C_J(\alpha, \infty, k, A) + C_J(\alpha, \infty, k, B) + C_J(\alpha, \infty, k, C).$$

The procedure that is used to develop a representation for $C'(\alpha, \infty, k)$ for a specified k as the 4-dimensional volume of the collection of all subsets of voting situations defined by (1) for which a CW exists is outlined in the following steps:

(i) Choose a subset $J_1 \subset \{1, 2, \ldots, 6\}$ with $\#J_1 = 6 - k$.
(ii) Determine the total number λ_1 of isomorphic J that can obtained from J_1 by permuting candidate names, and consider each of these J to be accounted for in later steps.
(iii) Compute the volume $C'_{J_1}(\alpha, \infty, k)$.
(iv) Continue cycling through this process to find $J_2, J_3, \ldots, J_t$ that have not been accounted for by any J_i in previous steps of the process, where t is the total number of equivalent sets of J that have been found by repeating the cycle a total of t times.

Then

$$C'_J(\alpha, \infty, k) = \lambda_1 C'_{J_1}(\alpha, \infty, k) + \lambda_2 C'_{J_2}(\alpha, \infty, k) + \cdots + \lambda_t C'_{J_t}(\alpha, \infty, k). \quad (3)$$

Finally, $P_k^{CW}(\infty, \alpha, IAC)$ is obtained with the IAC assumption for the specified k from (2) and (3) as:

$$P^{CW}(\infty, IAC_k(\alpha)) = \frac{C'_J(\alpha, \infty, k)}{C(\alpha, \infty, k)}. \quad (4)$$

The volumes $C(\alpha, \infty, k)$ and $C'_J(\alpha, \infty, k)$ are obtained by using a MAPLE program coded by one of the authors (see Appendix for details).

2.1.1 The case of $k = 5$

The definitions that are given for α and in (1) lead to the restriction $0 \le \alpha \le \frac{1}{6}$ when $k = 5$, and initial computation leads to

$$C(\alpha, \infty, 5) = \frac{(6\alpha - 1)^4}{4}, \text{ for } 0 \le \alpha \le \frac{1}{6}.$$

To obtain $C'_J(\alpha, \infty, 5)$, the initial J in (i) can be any of the singletons from {1,2,…,6}. Any selection of J will yield equivalent domains for all other singletons in (ii) with $\lambda_1 = 6$ and $t = 1$. Results from (iii) and (3) lead to

$$C'(\alpha, \infty, 5) = \frac{15(6\alpha - 1)^4}{64}, \text{ for } 0 \le \alpha \le \frac{1}{6}\ .$$

It then follows from (4) that

$$P^{CW}(\infty, IAC_5(\alpha)) = \frac{15}{16}, \text{ for } 0 \le \alpha < \frac{1}{6}. \quad (5)$$

Table 2 Equivalent domains of voting situations with $k = 4$

Class (u)	J_u	Collection of equivalent J	λ_u
1	{1,2}	{1,2},{3,5},{4,6}	3
2	{1,3}	{1,3},{2,4},{5,6}	3
3	{1,5}	{1,5},{1,4},{2,3},{2,6},{3,6},{4,5}	6
4	{1,6}	{1,6},{2,5},{3,4}	3

2.1.2 The case of $k = 4$

When $k = 4$, $C(\alpha, \infty, 4)$ is given by:

$$C(\alpha, \infty, 4) = -170\alpha^4 + \frac{245}{2}\alpha^3 - 30\alpha^2 + \frac{5}{2}\alpha, \text{ for } 0 \leq \alpha \leq \frac{1}{5},$$

$$C(\alpha, \infty, 4) = \frac{5}{16}(3\alpha - 1)^4, \text{ for } \frac{1}{5} \leq \alpha \leq \frac{1}{3}.$$

There are $\binom{6}{2} = 15$ possible J from pairs in {1,2,3,4,5,6} that can be partitioned into four distinct classes of equivalent domains of voting situations that are defined by (1). Table 2 below presents theses classes:

The results after computation and algebraic reduction are as follows:

$$P^{CW}(\infty, IAC_4(\alpha)) = \frac{2747\alpha^3 - 1924\alpha^2 + 462\alpha - 38}{40(4\alpha - 1)(17\alpha^2 - 8\alpha + 1)}, \quad \text{for } 0 < \alpha \leq \frac{1}{6}, \tag{6}$$

$$P^{CW}(\infty, IAC_4(\alpha)) = \frac{8241\alpha^4 - 5988\alpha^3 + 1494\alpha^2 - 132\alpha + 1}{120\alpha(4\alpha - 1)(17\alpha^2 - 8\alpha + 1)}, \quad \text{for } \frac{1}{6} \leq \alpha \leq \frac{1}{5},$$

$$P^{CW}(\infty, IAC_4(\alpha)) = \frac{14}{15}, \quad \text{for } \frac{1}{5} \leq \alpha < \frac{1}{3}.$$

2.1.3 The case of $k = 3$

For k = 3, $C(\alpha, \infty, 3)$ is given by:

$$C(\alpha, \infty, 3) = \frac{985}{24}\alpha^4 - \frac{85}{3}\alpha^3 + 5\alpha^2, \quad \text{for } 0 \leq \alpha \leq \frac{1}{4},$$

$$C(\alpha, \infty, 3) = -\frac{2165}{72}\alpha^4 + \frac{385}{9}\alpha^3 - \frac{65}{3}\alpha^2 + \frac{40}{9}\alpha - \frac{5}{18}, \quad \text{for } \frac{1}{4} \leq \alpha \leq \frac{2}{5},$$

$$C(\alpha, \infty, 3) = \frac{5}{6}(2\alpha - 1)^4, \quad \text{for } \frac{2}{5} \leq \alpha \leq \frac{1}{2}.$$

There are $\binom{6}{3} = 20$ possible J triples from {1,2,3,4,5,6} that can be partitioned into four distinct classes of equivalent domains of voting situations that are defined by (1). The partition of equivalent domains is listed below in Table 3.

Table 3 Equivalent domains of voting situations with $k = 3$

Class (u)	J_u	Collection of equivalent J	λ_u
1	{1,2,3}	{1,2,3},{1,2,4},{1,3,5},{2,4,6},{3,5,6},{4,5,6}	6
2	{1,2,5}	{1,2,5},{1,2,6},{1,4,6},{2,3,5},{3,4,5},{3,4,6}	6
3	{1,3,4}	{1,3,4},{1,3,6},{1,5,6},{2,3,4},{2,4,5},{2,5,6}	6
4	{1,4,5}	{1,4,5},{2,3,6}	2

The results are as follows:

$$P^{CW}(\infty, IAC_3(\alpha)) = \frac{2041\alpha^2 - 1360\alpha + 234}{10(197\alpha^2 - 136\alpha + 24)}, \text{ for } 0 < \alpha \leq \frac{1}{6}, \tag{7}$$

$$P^{CW}(\infty, IAC_3(\alpha)) = \frac{6868\alpha^4 - 4576\alpha^3 + 720\alpha^2 + 24\alpha - 1}{40\alpha^2(197\alpha^2 - 136\alpha + 24)}, \text{ for } \frac{1}{6} \leq \alpha \leq \frac{1}{4},$$

$$P^{CW}(\infty, IAC_3(\alpha)) = \frac{18820\alpha^4 - 25696\alpha^3 + 12624\alpha^2 - 2536\alpha + 157}{40(433\alpha^4 - 616\alpha^3 + 312\alpha^2 - 64\alpha + 4)}, \text{ for } \frac{1}{4} \leq \alpha \leq \frac{1}{3},$$

$$P^{CW}(\infty, IAC_3(\alpha)) = \frac{17848\alpha^4 - 25696\alpha^3 + 13272\alpha^2 - 2824\alpha + 193}{40(433\alpha^4 - 616\alpha^3 + 312\alpha^2 - 64\alpha + 4)}, \text{ for } \frac{1}{3} \leq \alpha \leq \frac{2}{5},$$

$$P^{CW}(\infty, IAC_3(\alpha)) = \frac{149}{160}, \text{ for } \frac{2}{5} \leq \alpha < \frac{1}{2}.$$

2.1.4 The case of $k = 2$

For $k = 2$, $C(\alpha, \infty, 2)$ is given by:

$$C(\alpha, \infty, 2) = -\frac{245}{48}\alpha^4 + \frac{5}{2}\alpha^3, \text{ for } 0 \leq \alpha \leq \frac{1}{3},$$

$$C(\alpha, \infty, 2) = \frac{485}{24}\alpha^4 - \frac{125}{4}\alpha^3 + \frac{135}{8}\alpha^2 - \frac{15}{4}\alpha + \frac{5}{16}, \text{ for } \frac{1}{3} \leq \alpha \leq \frac{1}{2},$$

$$C(\alpha, \infty, 2) = -\frac{955}{24}\alpha^4 + \frac{355}{4}\alpha^3 - \frac{585}{8}\alpha^2 + \frac{105}{4}\alpha - \frac{55}{16}, \text{ for } \frac{1}{2} \leq \alpha \leq \frac{3}{5},$$

$$C(\alpha, \infty, 2) = \frac{5}{16}(3\alpha - 2)^4, \text{ for } \frac{3}{5} \leq \alpha \leq \frac{2}{3}.$$

There are 15 possible $J \subset \{1, 2, 3, 4, 5, 6\}$ with $\#J = 4$ that can be partitioned into four distinct classes of equivalent domains of voting situations defined by (1). The partition of equivalent domains is listed below in Table 4.

The resulting IAC probability representations are as follows:

$$P^{CW}(\infty, IAC_2(\alpha)) = \frac{101\alpha - 48}{2(49\alpha - 24)}, \text{ for } 0 < \alpha \leq \frac{1}{4}, \tag{8}$$

Table 4 Equivalent domains of voting situations with $k = 2$

Class (u)	J_u	Collection of equivalent J	λ_u
1	{1,2,3,5}	{1,2,3,5},{1,2,4,6},{3,4,5,6}	3
2	{1,2,3,4}	{1,2,3,4},{1,3,5,6},{2,4,5,6}	3
3	{1,2,3,6}	{1,2,3,6},{1,2,4,5},{1,3,4,5},{1,4,5,6},{2,3,5,6},{2,3,4,6}	6
4	{1,2,5,6}	{1,2,5,6},{1,3,4,6},{2,3,4,5}	3

$$P^{CW}(\infty, IAC_2(\alpha)) = \frac{148\alpha^4 + 64\alpha^3 - 96\alpha^2 + 16\alpha - 1}{8\alpha^3(49\alpha - 24)}, \text{ for } \frac{1}{4} \leq \alpha \leq \frac{1}{3},$$

$$P^{CW}(\infty, IAC_2(\alpha)) = \frac{6712\alpha^4 - 10256\alpha^3 + 5448\alpha^2 - 1184\alpha + 97}{40(194\alpha^4 - 300\alpha^3 + 162\alpha^2 - 36\alpha + 3)}, \text{ for } \frac{1}{3} \leq \alpha \leq \frac{1}{2},$$

$$P^{CW}(\infty, IAC_2(\alpha)) = \frac{1991\alpha^4 - 4476\alpha^3 + 3726\alpha^2 - 1356\alpha + 181}{5(382\alpha^4 - 852\alpha^3 + 702\alpha^2 - 252\alpha + 33)}, \text{ for } \frac{1}{2} \leq \alpha \leq \frac{3}{5},$$

$$P^{CW}(\infty, IAC_2(\alpha)) = \frac{14}{15}, \text{ for } \frac{3}{5} \leq \alpha < \frac{2}{3}.$$

2.1.5 The case of $k = 1$

For $k = 1$, $C(\alpha, \infty, 1)$ is given by:

$$C(\alpha, \infty, 1) = \frac{\alpha^4}{4}, \text{ for } 0 \leq \alpha \leq \frac{1}{2},$$

$$C(\alpha, \infty, 1) = -\frac{79}{4}\alpha^4 + 40\alpha^3 - 30\alpha^2 + 10\alpha - \frac{5}{4}, \text{ for } 1/2 \leq \alpha \leq 2/3,$$

$$C(\alpha, \infty, 1) = \frac{731}{4}\alpha^4 - 500\alpha^3 + 510\alpha^2 - 230\alpha + \frac{155}{4}, \text{ for } 2/3 \leq \alpha \leq 3/4,$$

$$C(\alpha, \infty, 1) = -\frac{1829}{4}\alpha^4 + 1420\alpha^3 - 1650\alpha^2 + 850\alpha - \frac{655}{4}, \text{ for } 3/4 \leq \alpha \leq 4/5,$$

$$C(\alpha, \infty, 1) = \frac{1}{4}(6\alpha - 5)^4, \text{ for } 4/5 \leq \alpha \leq 5/6.$$

There are six $J \subset \{1, 2, 3, 4, 5, 6\}$ with $\#J = 5$ that all have of equivalent domains of voting situations defined by (1). Calculation and algebraic reduction lead to:

$$P^{CW}(\infty, IAC_1(\alpha)) = 1, \text{ for } 0 < \alpha \leq 1/2, \tag{9}$$

$$P^{CW}(\infty, IAC_1(\alpha)) = \frac{632\alpha^4 - 1328\alpha^3 + 1044\alpha^2 - 368\alpha + 49}{8(79\alpha^4 - 160\alpha^3 + 120\alpha^2 - 40\alpha + 5)}, \text{ for } 1/2 \leq \alpha \leq 2/3,$$

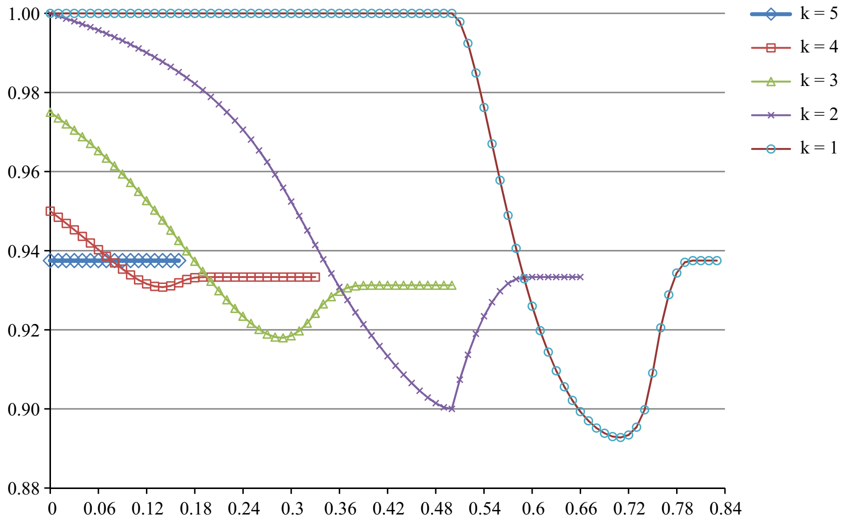

Fig. 2 Graphs of $P^{CW}(\infty, IAC_k(\alpha))$ values for $k = 1, 2, 3, 4, 5$

$$P^{CW}(\infty, IAC_1(\alpha)) = \frac{5200\alpha^4 - 14224\alpha^3 + 14508\alpha^2 - 6544\alpha + 1103}{8(731\alpha^4 - 2000\alpha^3 + 2040\alpha^2 - 920\alpha + 155)},$$
$$\text{for } 2/3 \le \alpha \le 3/4,$$

$$P^{CW}(\infty, IAC_1(\alpha)) = \frac{5(6112\alpha^4 - 19040\alpha^3 + 22200\alpha^2 - 11480\alpha + 2221)}{16(1829\alpha^4 - 5680\alpha^3 + 6600\alpha^2 - 3400\alpha + 655)},$$
$$\text{for } 3/4 \le \alpha \le 4/5,$$

$$P^{CW}(\infty, IAC_1(\alpha)) = \frac{15}{16}, \quad \text{for } \frac{4}{5} \le \alpha < 5/6.$$

2.2 Overall results from $P^{CW}(\infty, IAC_k(\alpha))$ representations

In order to visualize how the $P^{CW}(\infty, IAC_k(\alpha))$ values change as α increases, Fig. 2 shows a graph representation of values that are obtained for $k = 1, 2, 3, 4, 5$ from (9), (8), (7), (6) and (5) respectively.

Figure 2 shows that $P^{CW}(\infty, IAC_5(\alpha))$ is constant over the range of feasible α. The probability $P^{CW}(\infty, IAC_k(\alpha))$ starts out according to expectation when it is maximized at $\alpha = 0$ for each $= 1, 2, 3, 4$; and it then consistently decreases as α increases according to expectations to a minimum probability value at $\alpha_k^{'}$. However, we then find a result contrary to expectation when $P^{CW}(\infty, IAC_k(\alpha))$ consistently increases as α increases to an α value at $\alpha_k^{''}$. Then, $P^{CW}(\infty, IAC_k(\alpha))$ remains at a constant value as α increases over the remainder of its feasible interval. All results are summarized in Table 5.

The use of Parameter k to measure the degree of diversity among voters' preferences is indeed an adequate metric to provide an explanation for the change in the probability that a CW exists in an overall broad context as the value of $k$changes, as seen in Table 1. However, it fails to act consistently according to expectations when changes in the proximity measure α to having Parameter k in a voting situation are considered, as seen in Fig. 2. The connection

Table 5 Summary of change points in $P^{CW}(\infty, IAC_k(\alpha))$ for $k = 1, 2, 3, 4$

k	$P^{CW}(\infty, IAC_k(0))$	$\alpha_k^{'}$	$P^{CW}(\infty, IAC_k(\alpha_k^{'}))$	$\alpha_k^{''}$	$P^{CW}(\infty, IAC_k(\alpha_k^{''}))$
1	1.00000	0.13872	0.89320	0.20000	0.93750
2	1.00000	0.28788	0.90000	0.40000	0.93333
3	0.97500	0.50000	0.91798	0.60000	0.93125
4	0.95000	0.70818	0.93078	0.80000	0.93333

between the case with $k = 4$ and the condition of single-peaked preferences was discussed in detail above, and we now consider this link in more detail.

The connection between the probability that Condorcet's Paradox exists and measures of group mutual coherence was analyzed in Gehrlein (2006). Group mutual coherence refers to the degree to which a group of individuals form their individual preference rankings on candidates according to some rational model, or logically coherent model, for decision making. A candidate is perfectly Weak Positively Unifying if no voter ranks this candidate last, so that no voter would consider the election of this candidate the worst possible outcome. Parameter b is associated with this measure, and it is defined by

$$b = Min\{n_1 + n_3, n_2 + n_4, n_5 + n_6\}. \tag{10}$$

The *Min* function refers to the smallest sum contained in the brackets that follow it. Parameter b measures the proximity of a voting situation to describing a perfectly Weak Positively Unifying candidate by determining the fewest number of ballots on which one of the candidates is ranked last. As noted above, any voting situation for which $b = 0$ in (10) describes voter preferences that are perfectly single-peaked, and single-peakedness certainly represents a model of group preference formation that would be viewed as representing a group of rational individuals who are reflecting group mutual coherence. The ratio $\alpha_b = b/n$ describes the smallest proportion of voters whose preferences would have to be ignored in order for the preferences of the remaining voters to be single-peaked.

Following the logic behind the earlier definition of $P^{CW}(\infty, IAC_k(\alpha))$ above, we let $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ denote the limiting probability that a CW will exist with the assumption $IAC(\alpha_b)$ that all voting situations with limiting Parameter b values given by α_b are equally likely to be observed. A limiting representation for $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ is obtained in Gehrlein (2005) as

$$P_{SP}^{CW}(\infty, IAC(\alpha_b)) = \frac{11\alpha_b^3 - 4\alpha_b^2 - 3\alpha_b + 1}{(1 - 3\alpha_b)(1 - 3\alpha_b^2)}, \text{ for } 0 \le \alpha_b \le 1/4 \tag{11}$$

$$P_{SP}^{CW}(\infty, IAC(\alpha_b)) = \frac{-18\alpha_b^3 + 18\alpha_b^2 - 6\alpha_b + 1}{2\alpha_b(1 - 3\alpha_b^2)}, \text{ for } 1/4 \le \alpha_b < 1/3.$$

The representations in (11) are used to develop the graph for $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ in Fig. 3, where it is clear that $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ continuously decreases as α_b increases over the entire feasible range $0 \le \alpha_b < 1/3$, exactly as intuition suggests.

So, $P^{CW}(\infty, IAC_4(\alpha))$ does not consistently decrease as α increases, while P_{SP}^{CW} $(\infty, IAC(\alpha_b))$ does behave as intuition suggests. This happens regardless of the fact that the existence of single-peaked preferences requires that $k \le 4$, and the conditional probability that $k < 4$ is of measure zero in the subspace of single-peaked voting situations with

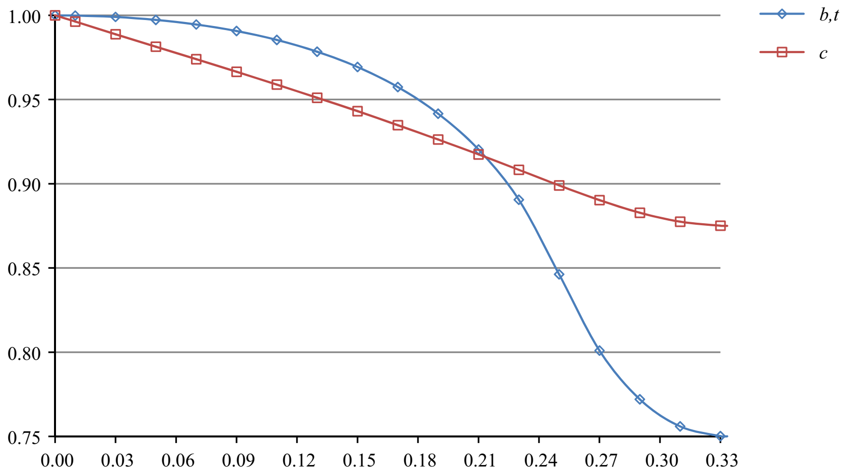

Fig. 3 Graph of $P_{SP}^{CW}(\infty, IAC(\alpha_b))$, $P_{SD}^{CW}(\infty, IAC(\alpha_t))$ and $P_{PC}^{CW}(\infty, IAC(\alpha_c))$

$k \leq 4$, so that both conditions therefore effectively require that $k = 4$. Given that, behavior that is inconsistent with intuition is observed with $P^{CW}(\infty, IAC_4(\alpha))$ and behavior that is consistent with intuition is observed with $P_{SP}^{CW}(\infty, IAC(\alpha_b))$. So, there is some distinct difference between these two conditions. An answer to this dilemma exists in some very interesting work regarding possible 'levels of agreement' among groups of voters.

3 Substantive-level agreement and meta-level agreement

To describe this concept, List (2002) suggests that there are different levels of agreement that can exist among voters' preferences. Voters might have a *substantive level agreement* at some elementary level, to the extent that their preferences on candidates tend to reflect some degree of consistency. However, it is further argued that voters might go beyond this elementary level of agreement to achieve a higher degree of *meta-level agreement*, to the extent that the voters can also agree on some 'common dimension' along which the candidates under consideration can be conceptualized. But, while the voters might be largely in agreement as to what this common dimension is and the relative placement of the candidates along it, they might still be in great disagreement as to what the optimal choice is from the candidates that are positioned along that dimension. The concept of a mutual agreement with the positioning of candidates along such a dimension is completely consistent with the notion of single-peaked preferences. List (2002) then goes on to argue that when voters have agreement on such a meta-level, it is significantly more likely to reduce occurrences of paradoxical results like Condorcet's Paradox than is some elementary agreement on a substantive level.

Dryzek and List (2003) extend the same notion, to argue that the existence of significant complexity resulting from multiple relevant issues of consideration might rule out the possibility of universal agreement among voters on any one of the common dimensions for comparing candidates. But, they go on to suggest that the multiple relevant issue dimensions could be coupled with the individual voter's preference rankings on candidates on the issue dimensions to lead to some "intra-dimensional single-peakedness". They also discuss the impact that deliberation and discussion might have on voters' "preference structuration" that would increase the likelihood that any resulting voting situation would be more representative of single-peaked preferences.

List et al. (2013) perform an empirical analysis of the impact that learning and deliberation have on levels of agreement among groups of decision makers. Subjects were formed into groups that were presented with background details from 13 different cases. The individuals were polled to obtain their preferences on issues for the case they were considering both before and after they were given time to have meaningful directed deliberations with other members of their group regarding the relative benefits of issue selection for their case. The level of substantive agreement in each group was measured by $\sum_{i=1}^{k} \gamma_i^2$ for k issues where γ_i is the proportion of voters who most prefer the ith issue, and this measure actually decreased slightly in the study when comparing voter preferences before and after deliberation. However, the proportion of voters in each group that had preferences that were consistent with single-peaked preferences increased as a result of deliberation. This change was strongest for decision situations with lower degrees of salience, such that the issues were not perceived as being as critical to the decision makers' personal interests; and therefore would not have been independently analyzed as thoroughly in advance by the decision makers. The increased proximity to single-peakedness that resulted from learning and deliberation therefore tended to be associated with a decrease in substantive level agreement. So, when the subjects deliberated longer, they tended to disagree more on a substantive level. But, on a meta-level "the more they come to agree about what they are disagreeing about" (List et al. 2013, p. 89).

The conclusion to be reached from these analyses is that there is a definite connection between the degree of voters' understanding of the different 'dimensions' for comparing candidates in an election and their propensity to have single-peaked preferences as a group. The ability of voters to agree mutually on an ordering of candidates along some form of a common dimension leads to a meta-level of agreement that is quite different than simple agreement among preferences on a substantive level. Such a meta-level of agreement among voters then has a significant positive impact on the probability that a CW will exist, compared to a scenario with simple substantive level agreement.

A related general theory was proposed earlier by Grofman and Uhlaner (1985) that is based on the existence of possible voter "meta-preferences" in the formation of voting situations. A meta-preference scenario exists when the voters begin by having preferences for characteristics of broadly defined processes that are then involved in the final determination of their individual preferences on candidates, rather than simply just having preferences on candidates. It is then suggested that the additional structure that would be present in the resulting voting situations, with such mutually agreeable preferences on processes at work the background during the stage of forming preferences on candidates, would lead to more of an overall understanding of the entire decision process. This overall understanding would then lead to more overall stability. Increased stability would presumably minimize the likelihood that events such as Condorcet's Paradox would be observed. While suggesting that such a higher level of mutual understanding of the process that leads to the development of voting situations will lead to increased stability in the voting situations, this study does not stress the importance of having an underlying scenario in which voters mutually agree upon an ordering of candidates along some form of a common dimension.

The impact of the work of List and his coauthors on the observations that we have made to this point is quite evident. When considering $P^{CW}(\infty, IAC_4(\alpha))$, a situation exists that is strongly suggestive of the presence of some underlying substantive level of agreement among voters, since the resulting voting situations indicate that only four of the possible preference rankings are relevant, even with $n \to \infty$. This definitely reflects a scenario that indicates the existence of some degree of elementary rationality in the process by which voters are forming preference rankings on candidates, such that voters clearly do not appear to have been simply randomly selecting an ordering on candidates without thinking about it. This substantive level of agreement is then adequate in an overall sense to provide logically consistent changes in $P^{CW}(\infty, IAC(k))$ as k increases in Table 1, but it does not provide an adequate framework in a more refined sense to result in consistent changes in $P^{CW}(\infty, IAC_4(\alpha))$ as α changes in Fig. 2.

Much more is implied when attention is shifted to the consideration of $P_{SP}^{CW}(\infty, IAC(\alpha_b))$, where it is similarly agreed that only four of the possible preference rankings are relevant. A much stronger meta-level of agreement is suggested in this scenario, because it is further specified that there are only four relevant preference rankings that are feasible in any voting situation *because* some candidate does not tend to be ranked as least preferred by any of the voters. Our measure α therefore acts as a proximity measure in a substantive level of agreement scenario, while α_b acts as measure of proximity in a more strongly structured meta-level of agreement scenario. It is therefore quite reasonable to expect results that are related to α_b to behave more consistently than those related to α. Moreover, there is a much stronger relationship between the decrease in $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ values (1.00 to 0.750) over the range of corresponding α_b in Fig. 3 than we observe in the decrease in $P^{CW}(\infty, IAC_4(\alpha))$ values (0.950 to 0.931) over the range of corresponding α in Fig. 2.

The observation regarding the link between the current study and List's work is further reinforced by considering two other measures of group mutual coherence from Gehrlein (2005).

4 Other measures of group mutual coherence

A candidate is perfectly Weak Negatively Unifying if no voter ranks this candidate as being most preferred, so that all voters are unified in their opposition to having this candidate chosen as the winner. Parameter t measures the proximity of a voting situation to suggesting the existence of such a candidate by counting the minimum number of ballots on which the same candidate is ranked first, with

$$t = Min\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \tag{12}$$

As t decreases in voting situations, a smaller proportion of voters ranks one of the candidates as being the most preferred candidate. A voting situation for which $t = 0$ describes a voting situation in which voters' preferences are perfectly single-troughed or single-dipped, and $k \leq 4$ in such a voting situation. Following the definition of single-peaked preferences, when voters' preferences are single-dipped each voter has a least preferred candidate with minimum utility along some common ordering of candidates. Each voter's utility for candidates must then continuously increase for candidates when moving in either direction left or right from their least preferred candidate in the common ordering. The ratio $\alpha_t = t/n$ that follows from (12) defines the smallest share of voters whose preferences must be ignored for the preferences of the remaining voters to be in complete agreement with single-dipped preferences. Following the definition of $P_{SP}^{CW}(\infty, IAC(\alpha_b))$, let $P_{SD}^{CW}(\infty, IAC(\alpha_t))$ denote the limiting probability that a CW exists when a minimum proportion α_t of voters must be removed from a voting situation to have the remaining voter preferences be perfectly single-dipped. It is proved in Gehrlein (2005) that $P_{SP}^{CW}(\infty, IAC(\alpha_b)) = P_{SD}^{CW}(\infty, IAC(\alpha_t))$ when $\alpha_b = \alpha_t$. So, all discussion above applies to a comparison of how $P_{SD}^{CW}(\infty, IAC(\alpha_t))$ changes with α_t relative to how $P^{CW}(\infty, IAC_4(\alpha))$ changes with α.

A candidate is perfectly Polarizing if no voter ranks this candidate in second place, so that all voters believe that this candidate is either the best or the worst of the three available candidates. Parameter $c$measures the proximity of a voting situation to describing such a candidate, by representing the smallest number of voters in a voting situation who rank the same candidate in second place, with

$$c = Min\left\{n_1 + n_6, n_2 + n_5, n_3 + n_4\right\}. \tag{13}$$

We again have $k \leq 4$ when $c = 0$ for a voting situation, and there is an increase in the number of voters who believe that one of the candidates is either the best or the worst of the three candidates as the value of c decreases. It is important to note that there is no connection between the existence of polarizing candidates and a model with some common ordering of candidates along a dimension, as developed in the models that formed the basis of single-peaked and single-dipped preferences.

The ratio $\alpha_c = c/n$ that follows from (13) is therefore a measure of the proximity of a voting situation to representing a scenario in which voters have completely polarized preferences. Gehrlein (2005) obtains a limiting representation $P_{PC}^{CW}(\infty, IAC(\alpha_c))$ for the probability that a CW exists when a minimum proportion α_c of voters must be removed from a voting situation to have the remaining voter preferences reflect the existence of a perfectly polarizing candidate, with

$$P_{PC}^{CW}(\infty, IAC(\alpha_c)) = \frac{139\alpha_c^3 - 28\alpha_c^2 - 54\alpha_c + 16}{16(1-3\alpha_c)(1-3\alpha_c^2)}, \text{ for } 0 \leq \alpha_c \leq 1/4 \tag{14}$$

$$P_{PC}^{CW}(\infty, IAC(\alpha_c)) = \frac{39\alpha_c^3 - 63\alpha_c^2 + 29\alpha_c - 1}{16\alpha_c(1-3\alpha_c^2)}, \text{ for } 1/4 \leq \alpha_c < 1/3.$$

The representation on (14) was used to compute $P_{PC}^{CW}(\infty, IAC(\alpha_c))$ for the graph in Fig. 3, which shows some interesting results. First, $P_{PC}^{CW}(\infty, IAC(\alpha_c))$ does consistently decreases according to intuition as α_c increases over its range. Figure 3 also shows a significantly weaker relationship between the decrease in $P_{PC}^{CW}(\infty, IAC(\alpha_c))$ values (1.000 to 0.875) over its range of corresponding α_c than we observe in the decrease of $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ values (1.000 to 0.750) over the range of corresponding α_b. There are definite reasons to explain why the strength of the relationship between $P_{PC}^{CW}(\infty, IAC(\alpha_c))$ and α_c is weaker than the strength of the relationship between $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ and α_b.

5 The difference between parameters *b*, *t* and *c*

The development of the model with a perfect Weak Positively Unifying Candidate, where there is increased agreement that some candidate is not the least preferred as Parameters b decreases, suggests that increases in the level of *concordance* or similarity among voters' preferences should exist with decreases in Parameter b. In the same way, the model with a perfect Weak Negatively Unifying Candidate, where there is increased agreement that some candidate is not the most preferred as Parameters t decreases, also suggests that the level of concordance among voters preferences will increase as Parameter t decreases. On the other hand, our model with a Weak Polarizing candidate has a very different nature, and suggests that voters' preferences should become more *antagonistic* or divergent as Parameter c decreases. It is not suggested that holding an election in such a scenario is a desirable situation, but such situations do exist and one would not describe the voters as acting irrationally while forming their preferences in such situations. So, when the phrase mutually coherent preferences is used, it does not necessarily imply that voters' preferences are mutually concordant or in agreement. It means that voters' preferences are being formed by an underlying process that implies that voters are generally acting in a manner that can be exemplified by a model of understandable behavior. It is of interest to see how levels of concordance and antagonism actually change as Parameters b,t and c change.

The classical measure of Concordance from basic statistical analysis is given by Kendall's Coefficient of Concordance (Kendall and Smith 1939), which reduces in our case to $H_1(\boldsymbol{n})$, with

$$H_1(\boldsymbol{n}) = \frac{(n_5+n_6-n_1-n_2)^2+(n_2+n_4-n_3-n_5)^2+(n_1+n_3-n_4-n_6)^2}{2n^2}.$$

To determine how $H_1(\boldsymbol{n})$ changes as k changes for Parameters b, t and c; we start by developing a representation for the conditional expected value $E\left[H_1(\boldsymbol{n})|n, IAC_{SP}(m)\right]$ of $H_1(\boldsymbol{n})$ given the assumption of $IAC_{SP}(m)$ for n voters with a specified value of Parameter b equal to m. This process starts by partitioning the space of all possible $\boldsymbol{n}$ into three subspaces for each of Parameters b, t and c.

The three subspaces for which Parameter b has a specified value of m is defined by using n_{ij} to denote the sum $n_i + n_j$ from Fig. 1:

$$Subspace_1^b:\ n_{56} = m,\ n_{13} \geq m \text{ and } n_{24} \geq m$$

$$Subspace_2^b:\ n_{56} \geq m+1,\ n_{13} = m \text{ and } n_{24} \geq m$$

$$Subspace_3^b:\ n_{56} \geq m+1,\ n_{13} \geq m+1 \text{ and } n_{24} = m.$$

In order to accumulate the sum of all values, $S_i^b(m, F(\boldsymbol{n}))$, of a function $F(\boldsymbol{n})$ over all $\boldsymbol{n}$ in Subspace $i$when Parameter b has a specified m, we use:

For $Subspace_1^b$:

$$S_1^b(m, F(\boldsymbol{n})) = \sum\nolimits_{n_{13}=m}^{n-2m} \sum\nolimits_{n_3=0}^{n_{13}} \sum\nolimits_{n_4=0}^{n-m-n_{13}} \sum\nolimits_{n_6=0}^{m} F(\boldsymbol{n}),\quad with\ m \leq \frac{n}{3}. \tag{15}$$

To obtain $F(\boldsymbol{n})$ from summation indexes in $Subspace_1^b$:

$$n_3 = n_3 \quad n_1 = n_{13} - n_3 \quad n_6 = n_6 \quad n_5 = m - n_6$$

$$n_4 = n_4 \quad n_2 = n - m - n_{13} - n_4$$

For $Subspace_2^b$:

$$S_2^b(m, F(\boldsymbol{n})) = \sum\nolimits_{n_{56}=m+1}^{n-2m} \sum\nolimits_{n_5=0}^{n_{56}} \sum\nolimits_{n_1=0}^{m} \sum\nolimits_{n_2=0}^{n-m-n_{56}} F(\boldsymbol{n}),\ with\, m \leq \frac{n-1}{3}. \tag{16}$$

To obtain $F(\boldsymbol{n})$ from summation indexes in $Subspace_2^b$:

$$n_5 = n_5\ n_6 = n_{56} - n_5\ n_1 = n_1\ n_3 = m - n_1$$

$$n_2 = n_2\ n_4 = n - m - n_{56} - n_2$$

For $Subspace_3^b$:

$$S_3^b(m, F(\boldsymbol{n})) = \sum\nolimits_{n_{56}=m+1}^{n-2m-1} \sum\nolimits_{n_5=0}^{n_{56}} \sum\nolimits_{n_4=0}^{m} \sum\nolimits_{n_3=0}^{n-m-n_{56}} F(\boldsymbol{n}),\quad with\ m \leq \frac{n-2}{3}. \tag{17}$$

To obtain $F(\boldsymbol{n})$ from summation indexes in $Subspace_3^b$:

$$n_5 = n_5 \quad n_6 = n_{56} - n_5 \quad n_4 = n_4 \quad n_2 = m - n_4$$

$$n_3 = n_3 \quad n_1 = n - m - n_{56} - n_3.$$

In order to obtain a count of the total number of possible voting situations for which Parameter b has a value of m, we use $F(\boldsymbol{n}) = 1$ in (15), (16) and (17), and after reduction:

$$\begin{aligned} S_1^b(m,1) &= \frac{(n+1-3m)(m+1)\left[(n+1)(n+5)-3m(m+1)\right]}{6} \\ S_2^b(m,1) &= \frac{(n-3m)(m+1)\left[(n+1)(n+5)-3m(m+2)\right]}{6} \\ S_3^b(m,1) &= \frac{(n-1-3m)(m+1)\left[(n+1)(n+6)-3m(m+3)\right]}{6}. \end{aligned} \quad (18)$$

By using all of this with the well-known fact that the total number of voting situations for n voters is given by $\left[\prod_{i=1}^{5}(n+i)\right]/120$, we obtain a representation for the probability, $P^b(n,m)$, that Parameter b has a value equal to m for n voters with IAC:

$$P^b(n,m) = \frac{\sum_{i=1}^{3} S_i^b(m,1)}{\left[\prod_{i=1}^{5}(n+i)\right]/120} = \frac{60(n-3m)(m+1)\left[(n+1)(n+5)-3m(m+2)\right]}{(n+1)(n+2)(n+3)(n+4)(n+5)},$$
$$for\ m \leq \frac{n-1}{3}. \quad (19)$$

The same procedure that was used above can be used to partition the space of possible voting situations for which Parameter c has a specified value of m:

$$Subspace_1^c\colon\ n_{34} = m,\ n_{16} \geq m \text{ and } n_{25} \geq m$$

$$Subspace_2^c\colon\ n_{34} \geq m+1,\ n_{16} = m \text{ and } n_{25} \geq m$$

$$Subspace_3^c\colon\ n_{34} \geq m+1,\ n_{16} \geq m+1 \text{ and } n_{25} = m.$$

As above, the sum of all values of a function $F(\boldsymbol{n})$ over all $\boldsymbol{n}$ in Subspace i when Parameter c has a specified m, is denoted $S_i^c(m, F(\boldsymbol{n}))$, with:

For $Subspace_1^c$:

$$S_1^c(m, F(\boldsymbol{n})) = \sum_{n_{16}=m}^{n-2m} \sum_{n_1=0}^{n_{16}} \sum_{n_2=0}^{n-m-n_{16}} \sum_{n_3=0}^{m} F(\boldsymbol{n}), \quad with\ m \leq \frac{n}{3}. \quad (20)$$

To obtain $F(\boldsymbol{n})$ from summation indexes in $Subspace_1^c$:

$$n_1 = n_1 \quad n_6 = n_{16} - n_1 \quad n_3 = n_3 \quad n_4 = m - n_3$$

$$n_2 = n_2 \quad n_5 = n - m - n_{16} - n_2$$

For $Subspace_2^c$:

$$S_2^c(m, F(\boldsymbol{n})) = \sum_{n_{34}=m+1}^{n-2m} \sum_{n_4=0}^{n_{34}} \sum_{n_1=0}^{m} \sum_{n_2=0}^{n-m-n_{34}} F(\boldsymbol{n}), \quad with\ m \leq \frac{n-1}{3}. \quad (21)$$

To obtain $F(\boldsymbol{n})$ from summation indexes in $Subspace_2^c$:

$$n_4 = n_4 \quad n_3 = n_{34} - n_4 \quad n_1 = n_1 \quad n_6 = m - n_1$$
$$n_2 = n_2 \quad n_5 = n - m - n_{34} - n_2$$

For $Subspace_3^c$:

$$S_3^c(m, F(\boldsymbol{n})) = \sum\nolimits_{n_{34}=m+1}^{n-2m-1} \sum\nolimits_{n_4=0}^{n_{34}} \sum\nolimits_{n_2=0}^{m} \sum\nolimits_{n_1=0}^{n-m-n_{34}} F(\boldsymbol{n}), \quad with\ m \leq \frac{n-2}{3}. \tag{22}$$

To obtain $F(\boldsymbol{n})$ from summation indexes in $Subspace_3^c$:

$$n_4 = n_4 \quad n_3 = n_{34} - n_4 \quad n_2 = n_2 \quad n_5 = m - n_2$$

$$n_1 = n_1 \quad n_6 = n - m - n_{34} - n_1.$$

After calculations are performed, we find $P^c(n, m) = P^b(n, m)$. This result verifies the findings in Gehrlein (2005), but it is reproduced here since the IAC-space partitioning process that is used for Parameter c will be very useful in later discussion. Simple symmetry arguments can be applied to show that $P^t(n, m) = P^b(n, m)$.

A representation for $E[H_1(\boldsymbol{n})|n, IAC_{SP}(m)]$ is then obtained by following the logic that led to (19) with (15), (16) and (17), and

$$E[H_1(\boldsymbol{n})|n, IAC_{SP}(m)] = \frac{\sum_{i=1}^{3} S_i^b(m, H_1(\boldsymbol{n}))}{\sum_{i=1}^{3} S_i^b(m, 1)}, \quad for\ m \leq \frac{n-1}{3}.$$

The resulting representation was obtained, but it is very complex, and it was verified by computer enumeration for several n. We set $k = \alpha_k n$ and then consider the limiting case as $n \to \infty$ to obtain $E[H_1(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)]$, with

$$E[H_1(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)] = \frac{1044\alpha_b^4 - 957\alpha_b^3 + 121\alpha_b^2 + 87\alpha_b - 21}{60(3\alpha_b^2 - 1)}, \text{ for } 0 \leq \alpha_b < 1/3. \tag{23}$$

Based on the definition of $H_1(\boldsymbol{n})$ it follows obviously from Fig. 1 that

$$E[H_1(\boldsymbol{n})|\infty, IAC_{SD}(\alpha_t)] = E[H_1(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)].$$

We also obtain a representation for $E[H_1(\boldsymbol{n})|n, IAC_{PC}(m)]$ with Parameter c for finite n by using the $S_i^c(m, H_1(\boldsymbol{n}))$ for $1 \leq i \leq 3$ from (20), (21) and (22), with

$$E[H_1(\boldsymbol{n})|n, IAC_{PC}(m)] = \frac{\sum_{i=1}^{3} S_i^c(m, H_1(\boldsymbol{n}))}{\sum_{i=1}^{3} S_i^c(m, 1)}, \quad for\ m \leq \frac{n-1}{3}.$$

The resulting representation is very complex, but it was verified by computer enumeration. When attention is restricted to the limiting case, we find

$$E[H_1(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)] = \frac{72\alpha_c^4 - 66\alpha_c^3 + 13\alpha_c^2 + 6\alpha_c - 3}{15(3\alpha_c^2 - 1)}, \text{ for } 0 \leq \alpha_c < 1/3. \tag{24}$$

Computed values of $E[H_1(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)]$ and $E[H_1(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)]$ are obtained from (23) and (24) respectively for each $\alpha_b, \alpha_c = 0(.03).33$ and the results are shown in graphical form in Fig. 4.

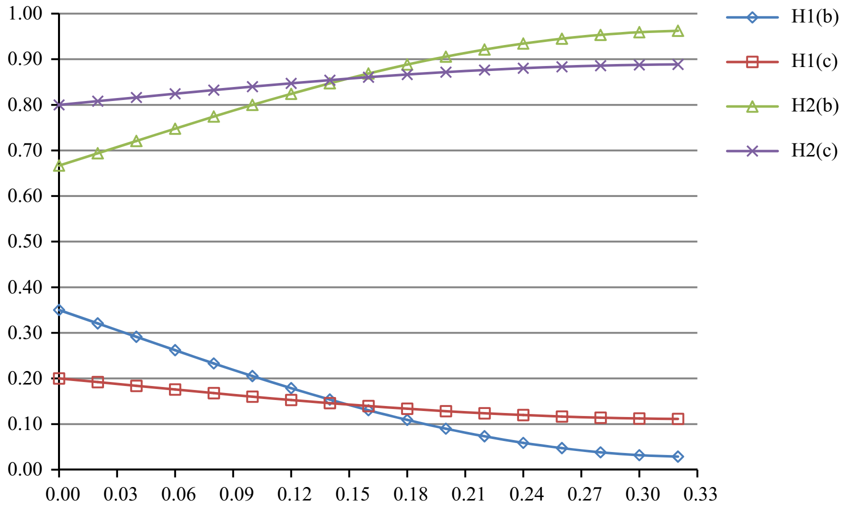

Fig. 4 Graphs of $E\left[H_i(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)\right]$ and $E\left[H_i(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)\right]$ for $i = 1, 2$

It is clearly observed in Fig. 4 that there is a consistent decrease in the expected value of $H_1(\boldsymbol{n})$ as each of α_b, α_t and α_c increases. However, the relationship between the expected value of $H_1(\boldsymbol{n})$ and α_b, and therefore α_t, is much stronger than the relationship between the expected value of $H_1(\boldsymbol{n})$ and α_c.

As noted above, the opposite effect of concordance of voters' preferences is the degree of *antagonism* that exists among voters' preferences, and Kuga and Nagatani (1974) conducted an analysis of the relationship between the degree of antagonism that is present in voting situation and the probability that a CW exists. Their measure of antagonism is given by $H_2(\boldsymbol{n})$, with

$$H_2(\boldsymbol{n}) = \frac{4\left[(n_1+n_2+n_4)(n_3+n_5+n_6)+(n_1+n_2+n_3)(n_4+n_5+n_6)+(n_1+n_3+n_5)(n_2+n_4+n_6)\right]}{3(n-1)(n+1)},$$

for odd n.

It was proved in their analysis in the limit as $n \to \infty$ with IAC that a negative correlation exists between $H_2(\boldsymbol{n})$ and the probability that a CW exists.

Following the discussion above, we now consider the relationship between the conditional expected value of $H_2(\boldsymbol{n})$ and specified values of Parameters b, t and c. The resulting representation for $E\left[H_2(\boldsymbol{n})|n, IAC_{SP}(m)\right]$ and $E\left[H_2(\boldsymbol{n})|n, IAC_{SP}(m)\right]$ are also very complicated, but they have been verified by enumeration for several values of n. The limiting representations are given by

$$E\left[H_2(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)\right] = \frac{2(72\alpha_b^4 - 66\alpha_b^3 - 5\alpha_b^2 + 6\alpha_b + 3)}{9(1-3\alpha_b^2)}, \text{ for } 0 \leq \alpha_b < 1/3. \tag{25}$$

$$E\left[H_2(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)\right] = \frac{2(36\alpha_c^4 - 33\alpha_c^3 - 16\alpha_c^2 + 3\alpha_c + 6)}{15(1-3\alpha_c^2)}, \text{ for } 0 \leq \alpha_c < 1/3. \tag{26}$$

We note that $E[H_2(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)] + E[H_1(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)] = 1$, but it is also found that this result only holds in the limit as $n \to \infty$.

Computed values of $E[H_2(\boldsymbol{n})|\infty, IAC_{SP}(\alpha_b)]$ and $E[H_2(\boldsymbol{n})|\infty, IAC_{PC}(\alpha_c)]$ are obtained from (25) and (26) respectively for each $\alpha_b, \alpha_c = 0(.03).33$ and the numerical results are displayed graphically in Fig. 4, where it is clearly observed that there is a consistent increase in the expected value of $H_2(\boldsymbol{n})$ as each of α_b, α_t and α_c increases. As we observed above for $H_1(\boldsymbol{n})$, the relationship between the expected value of $H_2(\boldsymbol{n})$ and α_b, and therefore α_t, is much stronger than the relationship between the expected value of $H_2(\boldsymbol{n})$ and α_c.

We observe that both Parameters b and t are much more strongly correlated to measures of both expected concordance and expected antagonism than is Parameter c. It was mentioned above that Kuga and Nagatani (1974) proved that there is a negative correlation between $H_2(\boldsymbol{n})$ and the probability that a CW exists as $n \to \infty$ with IAC. The results of Fig. 4 tell us that there is a much stronger relationship between both Parameters b and t to $H_2(\boldsymbol{n})$ than we observe between Parameter c and $H_2(\boldsymbol{n})$. So, the much stronger relationship that is observed between $P_{SP}^{CW}(\infty, IAC(\alpha_b))$ $\left[P_{SD}^{CW}(\infty, IAC(\alpha_t))\right]$ and α_b $[\alpha_t]$ than that observed between $P_{PC}^{CW}(\infty, IAC(\alpha_c))$ and α_c in Fig. 3 is to be quite expected. A negative correlation between antagonism and the probability that a CW exists logically corresponds to a positive correlation between concordance and the probability that a CW exists. So, the same arguments can be made on the basis of considering changes in expected concordance values of $H_1(\boldsymbol{n})$ in Fig. 4, compared to the probabilities that a CW exists in Fig. 3.

6 Conclusion

Our results are displaying three different levels of agreement, which have three corresponding levels of impact on the probability that a CW exists, and all are completely consistent with the theory proposed by List and his coauthors. A substantive level of agreement is found with the basic assumption that only four of the preference rankings are considered to be feasible preference rankings in voting situations. This assumption provides an adequate level of some elementary agreement among voters' preferences to induce $P^{CW}(\infty, IAC(k))$ to consistently change in an expected fashion as k increases, but it then fails to provide sufficient structure to voters' preferences to result in the observation of consistent changes in $P^{CW}(\infty, IAC_4(\alpha))$ as α changes as a proximity measure.

The models based on Parameters b and t provide significantly more structure to voters' preferences to yield an underlying common ordering of candidates along some dimension, as implied by single-peakedness and single-dippedness respectively. These two models completely conform to the requirements for meta-level agreement, and they produce the strongest relationship between the associated parameter values and the probability that a CW exists.

We find that the model based on Parameter c falls between these two extreme cases. It does reflect a model in which voters can be viewed as forming preferences on a common rational basis. This model provides an adequate logical structure to voters' preferences to yield a consistent relationship between changes in Parameter c and the probability that a CW exists. However, it falls short of requiring the existence of a common ordering of candidates along a common dimension, as suggested for a complete meta-level agreement in the work of List and his coauthors; and we therefore do not observe as strong a relationship between Parameter c and the probability that a CW exists as we do with the models based on Parameters b and t.

The results that have been obtained in this current study are obviously applicable only to the case of three-candidate elections. It is generally accepted that increasing the number of candidates will significantly increase the likelihood that any paradoxical voting outcomes might be observed. While the type of analysis that is presented here is not easily extended to the consideration of more than three candidates, it is very tempting to speculate that conditions approaching single-peaked preferences or single-dipped preferences would have an even more dramatic impact on the probability that a CW exists for more than three candidates than we have just observed.

Appendix: Volume computation

Given $J \subset \{1, 2, \ldots, 6\}$ with $\#J = 6 - k$, let $S_J(\alpha, \infty, k)$ be the set of all voting situations that satisfy the set of constraints in (1). One way to evaluate the 4-dimensional volume $C_J(\alpha, \infty, k)$ of $S_J(\alpha, \infty, k)$ proceeds as follows:

Step 1 *Rewrite all the constraints that defined $S_J(\alpha, \infty, k)$in terms of a 4-component vector y*. To achieve this, choose $s \in J$ and $t \notin J$; then rewrite each constraints in (1) taking into account that $x_s = \alpha + x_s - \sum_{j \in J} x_j$ and $x_t = 1 - \alpha + x_t - \sum_{j \notin J} x_j$. The new set of constraints denoted by $S_J(y, \alpha, \infty, k)$ now depends only on the 4-component vector $y = (x_j)_{j \neq s,t}$ and each constraint can be put in the standard form $c_j y \leq \alpha_j,\ 1 \leq j \leq p$ where c_j is a 4-component vector, α_j is an affine function of the parameter α and p is the total number of such constraints.

Step 2 *Find the set V(J,y) of all vertices of $S_J(y, \alpha, \infty, k)$***.** To do this, consider any subset $\{j_1, j_2, j_3, j_4\}$ of $\{1, 2, \ldots, p\}$ such that $c_{j_1},\ c_{j_2},\ c_{j_3}$ and c_{j_4} are linearly independent and let v be the unique solution to $c_{j_1} y = \alpha_{j_1},\ c_{j_2} y = \alpha_{j_2},\ c_{j_3} y = \alpha_{j_3}$ and $c_{j_4} y = \alpha_{j_4}$. Note that v may not be a vertex of $S_J(y, \alpha, \infty, k)$ since there may still exist some other constraints $c_j y = \alpha_j,\ j \neq j_1, j_2, j_3, j_4$ that should be satisfied by v. Thus v is called a potential vertex and is affected a validity domain that consists in the set

$$R_v = \{c_j v \leq \alpha_j,\ \text{for all } j \neq j_1, j_2, j_3, j_4\}$$

Since each α_j is an affine function of α, each R_v is either empty or a union of some real ranges on $\alpha \in [0, 1]$. Let I_v be the collection of all $r \in [0, 1]$ such that r is a bound of an interval in the validity domain of v. Collecting all bounds r from all validity domains over potential vertices in a single finite subset $I = \{r_1, r_2, \ldots, r_q\}$ of [0,1] with $r_j < r_{j+1}$, the set $V(J, y)$ of all vertices of $S_J(y, \alpha, \infty, k)$ for $r_j < \alpha < r_{j+1}$ is the set of all potential vertices the validity domains of which contain the interval (r_j, r_{j+1}).

Step 3 *Find a triangulation of $S_J(y, \alpha, \infty, k)$to derive the volume.* Note that each facet F_j of $S_J(y, \alpha, \infty, k)$ corresponds to at least one constraint $c_j y \leq \alpha_j$ from the definition of $S_J(y, \alpha, \infty, k)$. Each vertex can then be attached to the subset of facets it belongs to. Choosing a vertex, said v^1, from $V(J, y)$,a dissection of $S_J(y, \alpha, \infty, k)$ is obtained by considering all pyramids $v^1 F_j$ with apex v^1 and bases F_j such that v^1 is out of F_j. This is in fact the initial step of the well known Cohen and Hickey algorithm of triangulating a polytope (Cohen and Hickey 1979). This operation is then applied recursively to find a triangulation of $S_J(y, \alpha, \infty, k)$ into simplices, each containing five points that are affine independent. Finally the volume of $S_J(y, \alpha, \infty, k)$ is the sum of the volumes of each simplex obtained in its triangulation using the following formula of the 4-dimensional volume of a simplex $\Delta(a_0, a_1, a_2, a_3, a_4, a_5)$:

$$vol(\Delta(a_0, a_1, a_2, a_3, a_4, a_5)) = \frac{|\det(a_1 - a_0, a_2 - a_0, a_3 - a_0, a_4 - a_0, a_5 - a_0)|}{4!}. \quad (27)$$

Table 6 Vertices of $S_J(y, \alpha, \infty, 4)$ and their repartition on facets

Vertices	Components	Facets for the vertex
v^1	$(\alpha, \alpha, \alpha, 0)$	3,4,5,10
v^2	$(\alpha, \alpha, \alpha, \alpha)$	6,7,8,9
v^3	$(\alpha, \alpha, 1-4\alpha, 0)$	2,3,4,10
v^4	$(\alpha, \alpha, 1-4\alpha, , \alpha)$	1,6,7,9
v^5	$(\alpha, 1-4\alpha, \alpha, 0)$	2,3,5,10
v^6	$(\alpha, 1-4\alpha, \alpha, \alpha)$	1,6,8,9
v^7	$\left(\frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}\right)$	3,4,5,6,7,8
v^8	$\left(\frac{\alpha}{2}, \frac{\alpha}{2}, 1-\frac{5\alpha}{2}, \frac{\alpha}{2}\right)$	1,2,3,4,6,7
v^9	$\left(\frac{\alpha}{2}, 1-\frac{5\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}\right)$	1,2,3,5,6,8
v^{10}	$(1-4\alpha, \alpha, 0)$	2,4,5,10
v^{11}	$(1-4\alpha, \alpha, \alpha)$	1,7,8,9
v^{12}	$\left(1-\frac{5\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}\right)$	1,2,4,5,7,8

For illustration, we now apply this method to evaluate the 4-dimensional volume of $S_J(\alpha, \infty, 4)$ with $J = \{6, 5\}$. By setting $x_6 = \alpha - x_5$, $x_1 = 1 - \alpha - x_2 - x_3 - x_4$ and $y = (x_2, x_3, x_4, x_5)$, $S_J(\alpha, \infty, 4)$ is represented by set $S_J(y, \alpha, \infty, 4)$ of all 4-component vectors y such that:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} y \leq \begin{pmatrix} 1-\alpha \\ 1-2\alpha \\ -\alpha \\ -\alpha \\ -\alpha \\ 0 \\ 0 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \tag{28}$$

By solving all possible combinations of four equations extracted from the list of constraints in (28), the collection of all validity domains shows that the set of vertices is stable for α in [0,1/5] and [1/5,1/3] respectively. Moreover, the set of constraints in (28) is not feasible for α >1/3 and is of dimension lower than 4 for α =0 or α =1/3. For α in [0,1/5], the list of vertices is provided in Table 6.

A facet F_j of $S_J(y, \alpha, \infty, 4)$ corresponds to the constraint $c_j y = \alpha_j$ obtained by saturating the jthconstraint in (28). In Table 6, the presence of j in the column "Facets for the vertex" means that the corresponding vertex lays on F_j. Starting with v^{12} as the initial vertex, we obtain a triangulation of $S_J(y, \alpha, \infty, 4)$ into eight simplexes as shown in Fig. 5.

In Fig. 5, each simplex in the triangulation corresponds to a terminal node and consists in the set of the five vertices linked to that node. The 4-dimensional volume of $\mathrm{S_J(y}, \alpha, \infty, 4)$ is then derived by performing (27) for each of the eight simplexes obtained:

$$vol(S_J(y, \alpha, \infty, 4)) = \frac{1}{6}\alpha(1-4\alpha)(17\alpha^2 - 8\alpha + 1) \text{ for } 0 < \alpha < \frac{1}{5}.$$

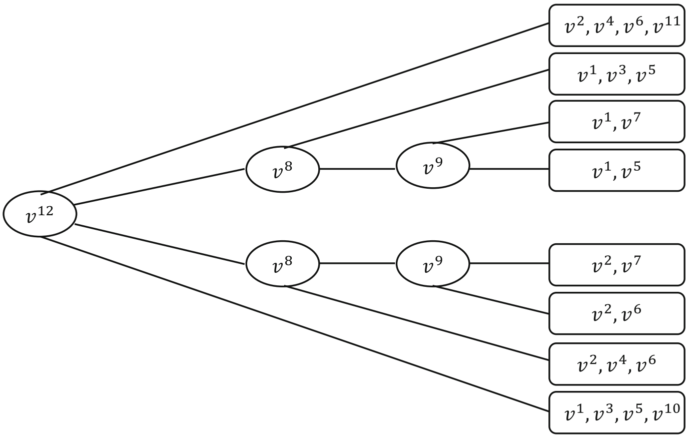

Fig. 5 A triangulation of $S_J(y, \alpha, \infty, 4)$

Since there are fifteen possible J $\subset \{1, 2, \ldots, 6\}$ with #J = 2, then for $0 < \alpha < \frac{1}{5}$

$$C(\alpha, \infty, 4) = 15vol(S_J(y, \alpha, \infty, 4))$$
$$= -170\alpha^4 + \frac{245}{2}\alpha^3 - 30\alpha^2 + \frac{5}{2}\alpha.$$

The same technique applies for α in [1/5,1/3], and subsequently for all the other sets of voting situations considered in the paper. To overcome the difficulty due to a huge number of potential vertices and multiple validity domains for distinct value k = 5,4,3,2,1, we have built a program using MAPLE codes for computerized evaluations.[2]

It is worth noticing that all the results we have obtained have been checked by using an alternative technique based on Barvinok algorithm (see e.g. Lepelley et al. 2008). In this case, we rewrite each set of constraints in terms of m and n_j, $j = 1, 2, \ldots, 6$ instead of $\alpha = m/n$ and $x_j = n_j/n$, where m denotes the minimum number of voters we should remove from a voting situation in order for the reduced voting situation to have $k$remaining preference ranking types. We then derive, via Barvinok algorithm, the quasi polynomial representing the total number of integer points that satisfy the set of constraints as a function of m and n. The corresponding volume is the coefficient of the leading term in n when m is replaced by αn in that representation.

References

Arrow, K.J.: Social choice and individual values, 2nd edn. Yale University Press, New Haven (1963)

Black, D.: The theory of committees and elections. Cambridge University Press, Cambridge (1958)

Cohen, J., Hickey, T.: Two algorithms for determining volumes of convex polyhedral. J. ACM **26**, 401–414 (1979)

de Condorcet, M.: An essay on the application of probability theory to plurality decision-making (1785). In: McLean, I., Hewitt, F. (eds.) Condorcet: Foundations of Social Choice and Political Theory, pp. 120–130. Edward Elgar Press, Hants (1994)

Dryzek, J.S., List, C.: Social choice theory and deliberative democracy: a reconciliation. Br. J. Polit. Sci. **33**, 1–28 (2003)

[2] This program is available from the authors upon request.

Felsenthal, D.S., Maoz, Z., Rapoport, A.: The Condorcet-efficiency of sophisticated voting under the plurality and approval procedures. Behav. Sci. **35**, 24–36 (1990)
Gehrlein, W.V.: Probabilities of election outcomes with two parameters: the relative impact of unifying and polarizing candidates. Rev. Econ. Des. **9**, 317–336 (2005)
Gehrlein, W.V.: Condorcet's paradox. Springer Publishing, Berlin (2006)
Gehrlein, W.V., Moyouwou, I., Lepelley, D.: The impact of voters' preference diversity on the probability of some electoral outcomes. Math. Soc. Sci. **66**, 352–365 (2013)
Gehrlein, W.V., Plassmann, F.: A comparison of theoretical and empirical evaluations of the Borda Compromise. Soc. Choice Welfare **43**, 747–772 (2014)
Grofman, B., Uhlaner, C.: Metapreferences and reasons for stability in social choice: thoughts on broadening and clarifying the debate. Theor. Decis. **19**, 31–50 (1985)
Kendall, M.G., Smith, B.B.: The problem of m rankings. Ann. Math. Stat. **10**, 275–287 (1939)
Kuga, K., Nagatani, H.: Voter antagonism and the paradox of voting. Econometrica **42**, 1045–1067 (1974)
Lepelley, D., Louichi, A., Smaoui, H.: On Ehrhart polynomials and probability calculations in voting theory. Soc. Choice Welfare **30**, 363–383 (2008)
List, C.: Two concepts of agreement. Good Soc. **11**, 72–79 (2002)
List, C., Luskin, R.C., Fishkin, J.S., McLean, I.: Deliberation, single-peakedness, and the possibility of meaningful democracy: evidence from deliberative polls. J. Polit. **75**, 80–95 (2013)
Niemi, R.G.: Majority decision-making under partial unidimensionality. Am. Polit. Sci. Rev. **63**, 488–497 (1969)
Nurmi, H.: Voting paradoxes and how to deal with them. Springer, Berlin (1999)
Vickery, W.: Utility, strategy and social decision rules. Q. J. Econ. **74**, 507–535 (1960)
Ward, B.: Majority voting and alternative forms of public enterprise. In: Margolis, J. (ed.) Public economy of urban communities, pp. 112–126. Johns Hopkins Press, Baltimore (1965)