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RESEARCH NOTE

# Ranking by competence using a fuzzy approach

**Abdul Suleman · Fatima Suleman**

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**Abstract** New approaches in economics of education consider the concept of competence to bringing in the multidimensional feature of labour force quality. Competence includes knowledge, skills, behaviours and attitudes. Each individual suits the combination of these elements to perform a task. Thus, different combinations expectedly lead to different task performance inside an occupation. In this paper, we focus on individual heterogeneity in homogeneous job requirements context. In sequel, we establish a theorem on unit simplex that potentially provides a ranking system by competence. The system is designed under a grade of membership framework, with a technique based on fuzzy sets theory. Under certain conditions, the referred theorem allows a mapping of individual multidimensional feature into the interval [0, 1]. So, the greater the value along this interval, the more competent the individual. An empirical study of a banking sector activity illustrates our research.

**Keywords** Heterogeneity · Competence · Fuzzy partition · Convex sets

## 1 Introduction

In transition from school to work, the concept of competence is acknowledged to be important but, in general, it is not observed in the dataset. Although, it is increasingly seen as an appropriate concept to measure the multidimensionality of human capital, it has, however, been restricted to educational background, experience or ability indicators.

A. Suleman (✉)
Department of Quantitative Methods, ISCTE-IUL Lisbon University Institute,
UNIDE Av. Forcas Armadas, 1649-026 Lisbon, Portugal
e-mail: abdul.suleman@iscte.pt

F. Suleman
Department of Political Economy, ISCTE-IUL Lisbon University Institute,
DINAMIA-CET, Lisbon, Portugal
e-mail: fatima.suleman@iscte.pt

The definition of competence and the one of competences' measurement tools are the most challenging problems in the economic analysis of labour force quality (Allen and van der Velden 2005b; van Loo and Semeijn 2004; Suleman and Paul 2007). In modern education perspective, the competences are conceived as composite of knowledge, skills and attitudes, as van Loo and Semeijn (2004) emphasize.

Our goal is to contribute to the wide debate about education as a source of competence according to renewal approach of job matching theory. Therefore, we adopt a holistic and multidimensional perspective of competences. By assumption, this point of view implies individual bundle of competences that is heterogeneous by nature. Explicitly, an individuals' bundle of competences is assumed to be different from another individuals' bundle. We aim to study this heterogeneity using fuzzy sets theory. Thus, our contribution is primarily methodologic.

The assumed holistic perspective implies interdependency in that individual bundle cannot be explained by its components alone. The bundle components are manifest variables indeed.

From the holistic point of view, the interdependency of the components should explain the way individuals select and use knowledge, skills and attitudes to perform a task. To understand this mechanism, we assume that manifest variables are ruled by say latent or not observable variables that must be sought.

To proceed this goal, an appropriate tool is designed under fuzzy sets and convex sets theories. Fuzzy approach allows modelling heterogeneity by grade of membership representation which involves partial rather than crisp membership. To make this approach feasible, we used a fuzzy latent structure based statistical model, known by the acronym GoM—Grade of Membership—(Woodbury and Clive 1974), which is anchored in convex sets theory.

An empirical study focuses on the assessment of competences of clerks in commercial activity of banking industry in Portugal, in 2001. The achieved results permit the employees' characterization by competences and their ranking as well.

As a core result of our study, a latent hierarchical fuzzy structure allowed a statistical representation of each banking employee by a meaningful bundle of competences which is an individualized balance of knowledge, skills, behaviours and attitudes in use. An original theorem on convex sets, first established in Suleman (2006), was then used to potentially build up a ranking system by competence. As result of this theorem, an utility function maps, under certain conditions, each individual bundle into the interval [0, 1]. Thus, greater the value along this interval, more competent the individual.

This paper is set out as follows. Section 2 is devoted to the definition and measurement of competence as generally applied in recent research under job matching. Section 3 provides some theoretical background on convex sets and fuzzy sets theories. Our main theorem is presented there'in. In Sect. 4 the statistical GoM model is reviewed. The empirical study is presented in Sect. 5 where the data and the results are described. Finally, in Sect. 6 the results are summarized and needed research suggested.

## 2 Theoretical framework

The concept of competence is increasingly seen as relevant to study the transition from school to work, but it is not, in general, observed in the dataset. Competence is usually measured by educational background, experience or ability indicators to test the contribution of human capital on earnings inequality.

Recent Netherlands' experience attempts to employ the concept of competence to analyze: (i) the matching between acquired and required knowledge and skills (Allen and van der Velden 2001; Heijke and Ramaekers 1998; Heijke et al. 2002); (ii) the measurement of graduate competences (van Loo and Semeijn 2004); (iii) the labour market success (Semeijn et al. 2005). Moreover, two international surveys were designed to integrate the concept of competence. First, the CHEERS Project (Careers after Higher Education: a European Research Study) (for a general presentation, see Paul et al. 2000) adopts the concept of competence to study the relationship between higher education and employment of Western countries' and Japan' graduates. Second, the REFLEX Project (The Flexible Professional in the Knowledge Society New Demands on Higher Education in Europe) focuses particularly on the matching of acquired and required competences and also on the role of higher education to facilitate the acquisition of such competences (Allen and van der Velden 2005a).

The concept of competence seems more comprehensive than one of the human capital. Thus, we argue that the concept of competence allows to conceive individuals not as stock of human capital but as vectors of different sort of knowledge, skills and attitudes (Léné 1999). Nevertheless, the education constitutes a good indicator of individuals' knowledge. But the question about how education system partakes in the acquisition of competences remains quite unanswered.

Additionally, this concept integrates the interaction between individual competences and job competences—"competence-in-use" (Le Boterf 1998). While human capital considers resources acquired by means of education and training, the competence framework focuses on the effective use of these competence aspects inside jobs.

In their conceptual model, van Loo and Semeijn (2004) provide a definition that undertakes this holistic and multidimensional perspective. In consequence, competences are "composites of individual attributes (knowledge, skills and attitudinal or personal aspects) that represent context-bounded productivity" (van Loo and Semeijn 2004). So, the core question involves the measurement of competences in accordance with this conceptual framework. To fit this framework, the just cited authors adopt a direct measure of competence using self-reports to collect information on the effectively used knowledge, skills, behaviours and attitudes in work context (for a survey of definition and measurement, see Allen and van der Velden 2005b). Alternatively, other subjective measures, as supervisor ratings, are used to gather information on competences in use. Either based on self-reports or supervisor assessments, the collected data are hardly manageable. This is particularly due to usual exhaustive list of well-defined aspects of competences which constitutes the main tool of these kinds of study. Our main contribution to the multidimensional and holistic point of view lies in the way the data is analysed.

As our first goal, we aim to transform the original raw data, gathered in multidimensional individual vectors, into low dimensional vectors to highlight individual characteristics with ease. Thus, at this stage, our focus of interest is the operational side of competences' measure.

As the second goal, we aim to classify individuals by their bundle of competences. Explicitly, we want to develop a new tool to rank individuals in addition to those widely used for performance appraisal. Ideally, this tool should label each individual with a number according to his or her position in a classification system.

Most of performance appraisal methods place heavy emphasis on subjective measures, which represents judgmental assessments based on comparative and absolute procedures (Fisher et al. 1996). Comparative procedures are norm-referenced—ranking (Viswesvaran and Ones 2000), and are used to compare employees directly one against other.

Instead of comparing employees, absolute standards refer to criterion-referenced methods—rating (Viswesvaran and Ones 2000) where performance is measured on the basis of specific dimensions.

Not surprisingly, this presentation of performance appraisal reveals that the methods are, in general, descriptive rather than analytical. Not often, a total score for each employee is computed by summing (or weighted summing) the ratings across all performance aspects. The core outcome is a generic information of each employee anchored on a singular numeric result.

Despite the multidimensional aspects considered, these procedures often ignore the interdependency between measured variables that might affect the final result. In this paper, we alternatively use a latent variables statistical model to filter out the referred interdependency to get insight into real dimensions of competence. The model is based on fuzzy sets theory and it is known by the acronym GoM standing for Grade of Membership. By assumption, the population under study is decomposed into a fuzzy partition which is related to a unit simplex. An original theorem on this convex set leads, under certain conditions, to a competence-based ranking system.

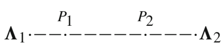
## 3 Mathematical background

The main purpose of this section is to establish a link between convex sets and fuzzy sets theories. The first subsection devotes to those parts of general theory of convex sets that are needed to support the proof of the main theorem. In sequel, some elementary concepts on fuzzy sets theory are provided ending up defining a fuzzy partition which closes the mathematical framework for the competence-based ranking system.

### 3.1 Convex sets

A subset $C \subset \mathbb{R}^J$ is a convex set if and only if $g_1\mathbf{\Lambda}_1 + g_2\mathbf{\Lambda}_2 \in C$, for all $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in C$ and all $g_1, g_2 \in \mathbb{R}$ with $g_1 + g_2 = 1$ and $g_1, g_2 \geq 0$. In sequel, a convex combination of $K$ points of $\mathbb{R}^J$, $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \ldots, \mathbf{\Lambda}_K$, is defined as the linear combination $g_1\mathbf{\Lambda}_1 + g_2\mathbf{\Lambda}_2 + \cdots + g_K\mathbf{\Lambda}_K$ such that the combination coefficients $g_k \geq 0$, $1 \leq k \leq K$, and $\sum_{k=1}^{K} g_k = 1$. It can be shown that a subset $C \subset \mathbb{R}^J$ "is convex if and only if any convex combination of points from $C$ is again in $C$" (Br$\phi$ndsted 1983). A point $\bar{\mathbf{\Lambda}} \in C$ is an extreme point or vertex of $C$ if and only if points $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in C$ do not exist such that $\bar{\mathbf{\Lambda}}$ is a strictly convex combination of $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$. Extreme points connected by an edge are adjacent. A closed edge includes its extreme points while a open edge does not. Given two adjacent extreme points, $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, the path of the edge connecting them can be characterized by their convex combination. This means, a unique number of the interval [0, 1] suffices to position any point in the path.

*Example 1* In the following picture




the convex combination

$$g_1\mathbf{\Lambda}_1 + g_2\mathbf{\Lambda}_2 = (1 - g_2)\,\mathbf{\Lambda}_1 + g_2\mathbf{\Lambda}_2,$$

as $g_1 + g_2 = 1$, represents the path of the edge connecting $\mathbf{\Lambda}_1$ to $\mathbf{\Lambda}_2$. Any point of the path may be referred by its $g_2$ (equivalently $g_1$) coefficient. Thus, we may say $\mathbf{P1} \equiv 0.2$ and $\mathbf{P2} \equiv 0.7$, to mean

$$\mathbf{P}1 = 0.8\boldsymbol{\Lambda}_1 + \mathbf{0.2}\boldsymbol{\Lambda}_2$$

and

$$\mathbf{P}2 = 0.3\boldsymbol{\Lambda}_1 + \mathbf{0.7}\boldsymbol{\Lambda}_2,$$

respectively. If the order relation $\mathbf{Z}_1 \preceq \mathbf{Z}_2$ means "the point $\mathbf{Z}_1$ has lower or equal $g_2$ coefficient than the point $\mathbf{Z}_2$", one is able to write

$$\boldsymbol{\Lambda}_1 \preceq \mathbf{P}1 \preceq \mathbf{P}2 \preceq \boldsymbol{\Lambda}_2.$$

For instance, if $\boldsymbol{\Lambda}_1$ represents the black colour and $\boldsymbol{\Lambda}_2$ the white colour, one could say $\mathbf{P}1$ is darker than $\mathbf{P}2$. Our main result is a generalization of this idea.

A intersection of any family of convex sets in $\mathbb{R}^J$ is again a convex set in this linear space. The smallest convex set containing the set $S \subset \mathbb{R}^J$, i.e., the intersection of all convex sets in $\mathbb{R}^J$ that contain $S$, is called convex hull of $S$. Of interest is the class of bounded convex hulls of non-empty finite set of points of $\mathbb{R}^J$ called polyhedrons or convex polytopes or simply polytopes. Being a convex set, a polytope $\mathbb{P}$, with $K$ extreme points $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_K$, is characterized by the convex combination

$$\mathbb{P} = \left\{ \boldsymbol{\Lambda} : \boldsymbol{\Lambda} = g_1\boldsymbol{\Lambda}_1 + g_2\boldsymbol{\Lambda}_2 + \cdots + g_K\boldsymbol{\Lambda}_K, g_k \geq 0 \wedge \sum_{k=1}^{K} g_k = 1 \right\}, \tag{1}$$

(for details, see Steuer 1986). Although the set of extreme points of $\mathbb{P}$, $\{\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_K\}$, is its unique minimal spanning set, the spanning (1) is not unique, unless $\mathbb{P}$ is a simplex. "Simplices are the only polytopes having a convex basis" (Br$\phi$ndsted 1983). This is equivalent to say that, in general, the mapping $\mathbb{P}$ into so called unit simplex $\mathbf{S}_K$, where

$$\mathbf{S}_K := \left\{ \mathbf{g} : \mathbf{g} = (g_1, g_2, \ldots, g_K) : g_k \geq 0 \wedge \sum_{k=1}^{K} g_k = 1 \right\}, \tag{2}$$

is not one-to-one. In some contexts, one would say $\mathbf{S}_K$ does not necessarily identify $\mathbb{P}$, if this polytope is not a simplex. The problem of identifiability arises in many daily problems and often it is not easy to deal with. In practice, restrictions on $\mathbb{P}$ may lead to its "simplexification" and consequently to its identification. In any case, the extreme points of $\mathbb{P}$ match the ones of $\mathbf{S}_K$, and the points on the edges of the polytope have unique representation in the unit simplex $\mathbf{S}_K$, as shown in Example 1. Woodbury et al. (1997) explore this subject in greater detail.

**Theorem 1** *Let $\mathbf{S}_K$ be the unit simplex with $K$ extreme points, namely $\mathbf{V}_1 = (1, 0, \ldots, 0)$, $\mathbf{V}_2 = (0, 1, \ldots, 0)$, …, and $\mathbf{V}_K = (0, 0, \ldots, 1)$. Consider the strictly decreasing real function $\eta : \mathbb{R}^+ \to [0, 1]$. Then, the function $\rho : \mathbf{S}_K \to \mathbb{R}$ defined by*

$$\rho(\mathbf{g}) \equiv \rho(g_1, g_2, \ldots, g_K) := 1 - \sum_{k=1}^{K-1} \eta(k) \times g_k \tag{3}$$

*is strictly increasing along the edge of adjacent extreme points $\mathbf{V}_m$ and $\mathbf{V}_n$, for $1 \leq m < n \leq K$.*

*Proof* Recall that any point of the unit simplex is written as a convex combination of its extreme points. In particular, the only possible non null coefficients of points in the edge connecting the adjacent extreme points $\mathbf{V}_m$ and $\mathbf{V}_n$ are $g_m$ and $g_n$, respectively. In this case we have $g_m + g_n = 1$, as $\mathbf{g} = (g_1, g_2, \ldots, g_K) \in \mathbf{S}_K$. Thus, $g_m = 1 - g_n$. Then, for points on the referred edge,

$$\rho(\mathbf{g}) = 1 - \eta(m) \times g_m - \eta(n) \times g_n = 1 - \eta(m) \times (1 - g_n) - \eta(n) \times g_n$$
$$\Leftrightarrow \rho(\mathbf{g}) = 1 - \eta(m) + [\eta(m) - \eta(n)] \times g_n. \quad (4)$$

By assumption, $\eta(.)$ is a strictly decreasing function. The quantity under square brackets in (4) is positive, since $m < n$. Moving from $\mathbf{V}_m$ to $\mathbf{V}_n$, along their connecting edge, is equivalent to increase $g_n$ from 0 to 1. Hence, the result.

The achieved result deserves some comments. One may call $\eta(.)$ penalty function because in some sense it "penalizes" the convex combination coefficients, in (3). Although different penalty functions lead to different images $\rho(\mathbf{g})$, the relative order of these images is not, however, affected by the choice of $\eta(.)$. So, in practice any penalty function suffices to order points $\mathbf{g}$ in the edges connecting adjacent vertices. For practical purpose, the harmonic penalty function

$$\eta(k) := \frac{1}{k}, \quad 1 \leq k \leq K, \quad (5)$$

will be used then.

**Corollary 1** *Consider the sequence of $q$ integers, $n_1, n_2, \ldots, n_q$, such that*

$$1 \leq n_1 < n_2 < \cdots < n_q \leq K,$$

*and suppose that the vertices $\mathbf{V}_{n_\alpha}$ and $\mathbf{V}_{n_{\alpha+1}}$, $1 \leq \alpha < q$, are adjacent. Then, the function $\rho(\mathbf{g})$ increases along the path of edges connecting the vertex $\mathbf{V}_{n_1}$ to vertex $\mathbf{V}_{n_q}$, from the lowest value $\rho\left(\mathbf{V}_{n_1}\right)$ to the highest value $\rho\left(\mathbf{V}_{n_q}\right)$.*

### 3.2 Fuzzy sets

The notion of fuzzy set was introduced by Zadeh (1965). In the author's own words, a "fuzzy set is a class of objects with a continuum grades of membership". This set concept is a extension of the one of classical set and it is intended to represent the problems characterized by intrinsic ambiguity (De Luca and Termini 1972).

Formally, a fuzzy set $A$ in $\mathbf{U} \subset \mathbb{R}^J$ is the mapping $g_A : \mathbf{U} \to [0, 1]$. The function $g_A$ is a generalization of the two values characteristic function of classical sets theory. The function $g_A(\mathbf{x})$ is termed grade of membership (GoM) or degree of compatibility of $\mathbf{x}$ with $A$. The values 0 and 1 represent, respectively, the lowest and the highest grades of membership. By this definition, any element $\mathbf{x} \in \mathbf{U}$ is represented by a number between 0 and 1 which is its grade of membership in the fuzzy set $A$. In this perspective, a fuzzy set $A$ in $\mathbf{U}$ may be alternatively defined as the set of ordered pairs (Bellman and Zadeh 1970),

$$A := \{(\mathbf{x}; g_A(\mathbf{x})), \quad \mathbf{x} \in \mathbf{U}\}.$$

If there is $\mathbf{x} \in \mathbf{U}$ such that $g_A(\mathbf{x}) = 1$, the fuzzy set $A$ is said normal. With the definitions of normal fuzzy set together with the one of the fuzzy set itself, we are able to define a fuzzy partition.

Consider $K \geq 2$ fuzzy sets in $\mathbf{U}$, $A_1, A_2, \ldots, A_K$ associated, respectively, with the grade of membership functions $g_{A_1}, g_{A_2}, \ldots, g_{A_K}$. We say $A_1, A_2, \ldots, A_K$ form a fuzzy partition or a fuzzy $K$-partition of $\mathbf{U}$ if and only if

(i) $A_k$ is normal, for $k = 1, 2, \ldots, K$;

(ii) $\forall \mathbf{x} \in \mathbf{U}, \sum_{k=1}^{K} g_{A_k}(\mathbf{x}) = 1.$

Condition (i) requires that each fuzzy set $A_k$ has at least one crisp element. The Condition (ii) means that all elements of the universe $\mathbf{U}$ are equally included into the partition. In other words, one may say that the partition is formed exhaustively. In this context, the fuzzy set $A_k$ crisp elements will be referred to as prototypes or $k$-prototypes and the fuzzy partition sets $A_1, A_2, \ldots, A_K$ to as typical profiles or extreme profiles (Manton et al. 1994). Of course, typical profiles are characterized by the respective prototypes.

A decomposition of the universe $\mathbf{U}$, by a fuzzy partition, configures a fuzzy structure. Any element $\mathbf{x} \in \mathbf{U}$ is then represented by the vectorial grade of membership function,

$$\mathbf{g}(\mathbf{x}) := \left(g_{A_1}(\mathbf{x}), g_{A_2}(\mathbf{x}), \ldots, g_{A_K}(\mathbf{x})\right). \tag{6}$$

Indeed, the GoM vector $\mathbf{g}(\mathbf{x})$ positions $\mathbf{x}$ in the fuzzy structure. In marketing context, one could refer $\mathbf{g}(\mathbf{x})$ as the positioning of $\mathbf{x}$ in $\mathbf{U}$. In any case, $\mathbf{g}(\mathbf{x})$ is a grade of membership representation of the element $\mathbf{x} \in \mathbf{U}$.

Unlike classical partition, where $K$ elements describe the universe, in a fuzzy partition every element of $\mathbf{U}$ is taken into account. This property positions the fuzzy sets theory as an analytic device to model heterogeneous populations. In this context, the heterogeneity is, however, restricted to elements' grade of membership in $K$ fuzzy sets and it is measured by grade of membership vector as in (6).

Being $g_{A_k}(\mathbf{x}) \geq 0$, $1 \leq k \leq K$, and $\sum_{k=1}^{K} g_{A_k}(\mathbf{x}) = 1$, a fuzzy $K$-partition induces a mapping from $\mathbf{U}$ into the unit simplex $\mathbf{S}_K$. That is, the vectorial membership function $\mathbf{g} : \mathbf{U} \to \mathbf{S}_K$.

When the universe $\mathbf{U}$ is discrete, the elements of $\mathbf{U}$ are frequently referred to as individuals and indexed by an integer. In this case, the $i$th individual GoM vector, $\mathbf{g}_i$, is written as $\mathbf{g}_i := (g_{i1}, g_{i2}, \ldots, g_{iK})$, where the generic coordinate $g_{ik}$ stands for the grade of membership of $i$th individual in the fuzzy set indexed by $k$, namely $A_k$.

In practical problems modelled by a fuzzy $K$-partition, one usually aims to label the fuzzy partition sets $A_1, A_2, \ldots, A_K$ with scientifically acceptable meaning, i.e., a number of coherent characteristics. In this way, $g_{ik}$ is interpreted as the proportion of the fuzzy set $A_k$ characteristics shared by the $i$th individual. If this individual is itself a $k$-prototype, he/she shares integrally the characteristics of the typical profile $A_k$, since in this case $g_{ik} = 1$.

## 4 Statistical model

A pioneer model based on fuzzy partition, known by the acronym GoM, was introduced by Woodbury and Clive (1974). The authors were primarily concerned with modelling discrete heterogenous populations in medical context, characterized “by a large number of discrete conditions, no combination of which occurs with high frequency” (Singer 1989). The model knew further development (e.g. Woodbury et al. (1994, 1997)) and it has been applied in a variety of clinical and social science areas. The book by Manton et al. (1994) is a landmark of its history.

In GoM model the fuzzy partition of the discrete universe under study (population) is related to a latent structure usually unknown to the researcher. The number of partition sets $K$ is fixed a priori. The latent structure configures “convex subsets of a space of probability density functions" (Woodbury et al. 1997) and it is used to explicitly represent heterogeneous populations.

The model uses two sets of latent variables: one relates to $K$ sets of the fuzzy partition, $A_1, A_2, \ldots, A_K$, and the other to individual position in the partition, namely the grade of membership function. They are generically denoted by $\lambda_{kjl}$ and $g_{ik}$, respectively. The former are known as structural parameters and the $g_{ik}$ are called individual coefficients. The structural parameters, gathered in $K$ vectors, are identified with the extreme points of a convex polytope, say $\mathbb{B}$. Individual position in $\mathbb{B}$ is then measured in probabilistic terms by a convex linear combination of these $K$ vectors using $g_{ik}$ as the convex combination coefficients.

The model input consists in $N$ realizations of $J$-dimensional random categorical data vector $\mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{iJ})$, $1 \leq i \leq N$. Often $\mathbf{X}_i$ is referred to as the answer of individual $i$ to $J$ questions. The coordinate $X_{ij}$ provides a response in a finite number of $L_j \geq 2$ categories of $j$th question ($j = 1, 2, \ldots, J$). The $L_j$ categories are assumed exhaustive and mutually exclusive.

Although individuals can be classified by vector responses $\mathbf{X}_i$, in general, this individual record hardly gives meaningful insight about its position in the population. Often the number of questions is too high (e.g. $J = 30$) to be manageable. Furthermore, the assumed holistic nature of competences involves the interdependence of coordinates $X_{ij}$ thus carrying redundant information. This interdependency must be removed somehow and consequently the data reduced.

As we will see soon, grade of membership representation provides data reduction and often leads to a "meaningful interpretation for what otherwise might look like a morass conditions on an individual record" (Singer 1989). In the application below, Sect. 5, $J = 30$ to $K = 3$ reduction is achieved. Before proceeding, some basic model assumptions are considered (for details, see Manton et al. 1994).

**Assumption 1** The $i$th individual vector $\mathbf{g}_i = (g_{i1}, g_{i2}, \ldots, g_{iK})$, $1 \leq i \leq N$, is an independent realization of the random vector $\xi := (\xi_1, \xi_2, \ldots, \xi_K) \in \mathbf{S}_K$, with the distribution function

$$\mathbf{H}(\mathbf{z}) := \Pr(\xi \leq \mathbf{z}).$$

**Assumption 2** Conditional on realized value $\mathbf{g}_i$ for individual $i$, $X_{ij}$ and $X_{ij'}$ $\left(j \neq j'\right)$ are independent random variables.

**Assumption 3** The probability of a response $l$ for the $j$th question by a $k$-prototype is $\lambda_{kjl}$.

**Assumption 4** The probability of a response $l$ for the $j$th question by individual $i$, conditional on $\mathbf{g}_i$, is

$$p_{ijl} \equiv \Pr\left(X_{ij} = l|\mathbf{g}_i\right) := \sum_{k=1}^{K} g_{ik}\lambda_{kjl}. \tag{7}$$

By Assumption 1, $\mathbf{g}_i$ is implicit realization of individual $i$ and occurs simultaneously with the observation of $\mathbf{X}_i$. The Assumption 2 is common to latent variables models and it is referred to as conditional independence. This means "that for any given manifest variable $\left[X_{ij}\right]$, the other observed variables provide no information on that variable beyond information provided by $\mathbf{g}_i$" (Haberman 1979). Assumption 3 makes the fuzzy partition prototypes feasible and consequently the fuzzy partition itself. In GoM model, the fuzzy $K$-partition is then identified by the $K$ vectors

$$\mathbf{\Lambda}_k := \left(\lambda_{k11}, \ldots, \lambda_{k1L_1}, \lambda_{k21}, \ldots, \lambda_{k2L_2}, \ldots, \lambda_{kJ1}, \ldots, \lambda_{kJL_J}\right), \quad k = 1, 2, \ldots, K. \tag{8}$$

Being probabilities, $\lambda_{kjl}$ satisfy the following two conditions

$$\lambda_{kjl} \geq 0 \quad \text{and} \quad \sum_{l=1}^{L_j} \lambda_{kjl} = 1, \quad \text{for each } k \text{ and } j.$$

Finally, Assumption 4 is the GoM model itself. In words of Manton et al. (1994) it "bridges the definition of the fuzzy partition with a parametric expression for the probability of a specific discrete response". The individual $i$ is then characterized by the probabilistic profile $\mathbf{p}_i := \left(p_{i11}, \ldots, p_{i1L_1}, p_{i21}, \ldots, p_{i2L_2}, \ldots, p_{iJ1}, \ldots, p_{iJL_J}\right)$, where each coordinate $p_{ijl}$ is obtained through the relation (7).

Woodbury et al. (1997) show that the parametric space of $\mathbf{p}_i$ is a convex polytope, $\mathbb{B}$, and $\mathbf{\Lambda}_k$, $1 \leq k \leq K$, are the extreme points of $\mathbb{B}$, i.e., the structure of this polytope. In this perspective, one can write $\mathbf{p}_i$ as a convex combination of the unique minimal spanning set of $\mathbb{B}$, namely $\{\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \ldots, \mathbf{\Lambda}_K\}$, as $\mathbf{p}_i = \sum_{k=1}^{K} g_{ik}\mathbf{\Lambda}_k$. Thus, $\mathbf{g}_i = (g_{i1}, g_{i2}, \ldots, g_{iK}) \in \mathbf{S}_K$ defines, in probability terms, the barycentric coordinates of individual $i$ in $\mathbb{B}$.

As referred earlier, the representation of $\mathbf{p}_i$ by $\mathbf{g}_i$ is not unique, unless the polytope $\mathbb{B}$ is a simplex. The same authors consider restrictions on $\mathbb{B}$ that lead to the identification of $\mathbf{p}_i$. This issue is not an easy matter and deserves further research.

The likelihood of a $N$-dimensional random sample, based on a fuzzy $K$-partition, can be written as

$$\mathbf{L}_K^* := \prod_{i=1}^{N} \int_{\mathbf{S}_K} \prod_{j=1}^{J} \prod_{l=1}^{L_j} \left( \sum_{k=1}^{K} g_{ik}\lambda_{kjl} \right)^{y_{ijl}} d\mathbf{H}, \tag{9}$$

where $\mathbf{H}(.)$ is the distribution function of implicit individual random vector, as defined in Assumption 1 and $y_{ijl}$ are indicator variables with unit value if $x_{ij} = l$ and zero otherwise.

In general, the model parameters are not uniquely determined by the observed random variables, unless restrictions on the parametric space are considered. This phenomenon is known as lack of parameters, statistical or distribution identifiability (Paulino and Pereira 1994).

Woodbury et al. (1994, 1997) show that one can only expect statistical identifiability of the likelihood (9) if the parametric space is restricted to structural parameters $\lambda_{kjl}$ and moments up to order $J$ of $\mathbf{H}(.)$. Additionally, a sufficient condition is provided for the parameters be uniquely related to the data, which includes, among others, the relation

$$K < \frac{J}{2}. \tag{10}$$

This condition leads GoM model as a data reduction device thus providing a reduction to less than half the initial dimension $J$.

As a consequence of restrictions to grant parameters identifiability, involving moments of individual coefficients $g_{ik}$ distribution and not the coefficient itself, their estimates, say $\hat{g}_{ik}$, must be used with care. Neverthless, additional result in Woodbury et al. (1994) allows the usage of $\hat{g}_{ik}$ to estimate unbiased moments of $\mathbf{H}(.)$. So, in practice, the conditional to $\mathbf{g}_i$ likelihood function

$$\mathbf{L}_K := \prod_{i=1}^{N} \prod_{j=1}^{J} \prod_{l=1}^{L_j} \left( \sum_{k=1}^{K} g_{ik}\lambda_{kjl} \right)^{y_{ijl}} \tag{11}$$

is therefore preferred and, of course, it is more useful for the purpose of our work.

Estimation strategy to-date is based on maximum likelihood method using $\mathbf{L}_K$. The maximization process is evaluated in two steps. In the first one, both sets of parameters are estimated using only the set of variables considered potentially important components of the extreme profiles; these variables are called internal. In the second step, only additional structural parameters are estimated using all remaining variables entered into analysis: these are external variables (Singer 1989). Often external variables are demographic characteristics of individuals. The choice of variable type is subjective and mostly context dependent.

Due to excessive number of parameters involved, the calculation of the sample information matrix is not an easy task. Additional difficulty may arise in applications because the information matrix is expected to be metastable (i.e. it has multiple singularities) (Woodbury et al. 1978).

The measurement of model fit is another critical issue in the context of GoM model, due to boundary parameters. Often goodness-of-fit is assessed by a likelihood ratio test, comparing $\mathbf{L}_K$ to the so-called independence model $\mathbf{L}_1$, using the test statistic $\chi^2 := -2\ln(\mathbf{L}_K/\mathbf{L}_1)$. Under null hypothesis (i.e. when $\mathbf{L}_1$ is the true model), this statistic is approximated to a qui-square distribution with the number of degrees of freedom $\upsilon$ equals the number of estimated parameters. Due to excessive number of parameters involved, approximation of $\chi^2$ to Normal distribution, provided by the statistic $Z_F := \sqrt{2\chi^2} - \sqrt{2\upsilon - 1}$ is used alternatively. The empirical value $z_F > z_{(1-\alpha)}$ leads to the rejection of the null hypothesis, where $z_{(1-\alpha)}$ is the $(1-\alpha) \times 100\%$ quantile of standard Normal distribution.

Having obtained a good fit with $\mathbf{L}_K$, the main concern is the characterization of the fuzzy $K$-partition through the estimates of structural parameters $\lambda_{kjl}$. In this regard, Marini et al. (1996) suggest $(1+\delta)$ times the sample frequency criterion, where $\delta \in [0, 1]$. The prevalence of the category $l$ of the $j$th variable on the extreme profiles is then measured by the relative value of $\hat{\lambda}_{kjl}$, the maximum likelihood estimate of $\lambda_{kjl}$. The variable-category pair $(j, l)$ is said to contribute substantively to discriminate typical profiles whenever $\hat{\lambda}_{kjl}$ is, at least, $(1+\delta)$ greater than the corresponding sample frequency, say $\hat{f}_{jl}$. The referred authors used $\delta = 0.40$ in their empirical work.

## 5 Empirical study

In order to assess individual competence, we focused on a single occupation and gathered information on supervisor perception about how he/she uses knowledge, skills, behaviours and attitudes to perform a task. Therefore, in our study, we assume individual heterogeneity in homogeneous job requirements.

To proceed this purpose, we used data from an original survey, conducted in the banking sector during 2001, designed to study the valuating competences. The purpose of the data collection was to assess banking employees' competences and subsequently to test the influence of assessed competences on earnings and incentives (Suleman and Paul 2005) and to determine sources of sorts of competences (Suleman and Paul 2007). Several aspects of competences were then collected either by assessment or by gathering general information on other attributes (Table 2 in Appendix).

The competences were then defined as acquired bundle of knowledge, skills, behaviours and attitudes effectively used by employees to perform successfully a task or to solve a problem. Dataset and results are available upon request.

**Table 1** Data structure of commercial activity of banking sector

| Variable $(j)$ | Levels $(L_j)$ | Description | Type |
|---|---|---|---|
| 1–3 | 5 | Knowledge | I |
| 4–9 | 5 | Behaviours and attitudes toward others | I |
| 10–21 | 5 | Behaviours and attitudes toward organization | I |
| 22–30 | 5 | Cognitive and technical skills | I |
| 31 | 3 | Bank | E |
| 32 | 6 | Age | E |
| 33 | 3 | Education level | E |
| 34 | 5 | Seniority | E |
| 35 | 2 | Gender | E |
| 36 | 2 | Occupation: with or without Portfolio of clients | E |

I: internal; E: external

## 5.1 Data structure

From initial 600 individuals, seven were dropped out due to their non-standard education system. The final sample of $N = 593$ clerks of the banking sector, charged with commercial tasks, in three financial corporations integrates information on age, gender, level and domain of education, tenure; job characteristics: name and corresponding position; supervisors' rating on performance and competences (knowledge, skills and attitudes); and compensation schemes (salary grade, base pay, other fixed compensation schemes, bonuses, profit sharing and stock-options). The competence list was built up from banking industry oriented researches and it was checked with human resources managers and couple of supervisors of banking agencies.

For this particular study, a subset of variables was used according to Table 1. In this table the first column indicates variable number and the second column indicates the number of categories or levels in each variable. The fourth column signals the variable type, I for internal and E for external, that is useful for GoM model application below. In total $J = 36$ were considered: $J^+ = 30$ internal and $J^* = 6$ external. In Appendix, Table 2 gives information on internal variables in detail.

## 5.2 Model application

The internal variables were used to set up typical competence profiles of banking employees charged with commercial activity. On the other side, external variables, in number of six, hopefully provide refinements on these profiles by associating useful information on demographic and occupational characteristics.

Following the knowledge of the long tradition in economic of education to understand the sort of skills related to education, we have choosen the education attainment as the pivotal variable. This variable has three levels, specifically: low secondary, secondary and higher. Put in words of GoM model, the number of the fuzzy partition sets was fixed to $K = 3$. This is equivalent to say that the fuzzy structure of banking commercial activity is, by assumption, supported by three extreme profiles.

An empirical measure of model goodness of fit leads to the value $z_F = 111.03$, much higher than $z_{(0.95)} = 1.645$, the 95% quantile of the standard normal distribution. So, the model based on a fuzzy 3-partition seems reasonably acceptable.

### 5.2.1 *Typical profiles*

In this study, we set subjectively $\delta = 0.25$, i.e., 1.25 times criterion is adopted to signal prevalent conditions. This is equivalent to say that the variable-category pair $(j, l)$ is assumed to contribute substantively for the extreme profile $k$ if

$$\hat{\lambda}_{kjl} \geq 1.25 \times \hat{f}_{jl}, \tag{12}$$

where $\hat{f}_{jl}$ is the sample frequency on that variable–category pair. Analysing the achieved results, one realizes that lower levels competences are assigned to one extreme profile while medium levels are assigned to other extreme profile and higher levels to the remaining extreme profile. Thus, the emerged typical profiles may be labelled as Low, Medium and High, respectively. This condition defines a fuzzy structure that is hierarchical.

Put it in mathematical terms, if the order relation $\preceq$ between extreme profiles stands for "has lower or equal competences than", one may fairly write Low $\preceq$ Medium $\preceq$ High. This important finding claims for other individual characteristics (e.g. age, gender, tenure) and on demand-side variables (e.g. bank, occupation) that possibly induce performance. This information can be provided by $J^* = 6$ external variables, for which the same 1.25 criterion was used.

The complementary characteristics of typical profiles, provided by external variables, show that education is positively related to competence acquisition. Higher education levels lead to higher competence. But, contrary to other human capital prediction, the same does not hold for the seniority and age, being the later one a proxy of experience. Indeed, tenure and age do not lead typical profiles to higher competence, as education does. Besides these supply-side characteristics, either man or woman can accommodate any profile as well, despite some higher tendency for male in Low typical profile ($\hat{\lambda}_{1,33,2} = 72.39\%$ against an observed frequency of 51.96%). Furthermore, the prevalence of customer portfolio in High typical profile is found much higher than the expected in the population as whole, $\hat{\lambda}_{3,36,3} = 52.57\%$ against the sample frequency of 33.05%. Almost surely, i.e., $\hat{\lambda}_{1,36,1} = 100\%$, the Low extreme profile has no assigned portfolio. Despite the apparent tendency for possessing customer portfolio ($\hat{\lambda}_{2,36,1} = 75.61\%$ against $\hat{f}_{36,1} = 66.95\%$), this condition is not prevalent in Medium typical profile.

Under 1.25 criterion, High typical profile may be characterized as young, graduated, male or female and, with high probability, charged of customers portfolio. On the opposite side, Low typical profile is characterized as old, least graduated, most probably male and allocated to more generic tasks. Medium typical profile positions between these two extremes. Finally, concerning organizational filiation, one can realize that no typical profile is affected to any particular bank. This fact may signal transferable nature of competences within industry rather than firm specificity.

### 5.2.2 *Individual heterogeneity*

The next stage of our analysis concerns employees' distribution on the fuzzy structure defined by typical profiles, as above-described. A fuzzy 3-partition may be analyzed graphically through the unit simplex $\mathbf{S}_3$ (Fig. 1), whose vertices, the points (1, 0, 0), (0, 1, 0) and (0, 0, 1),

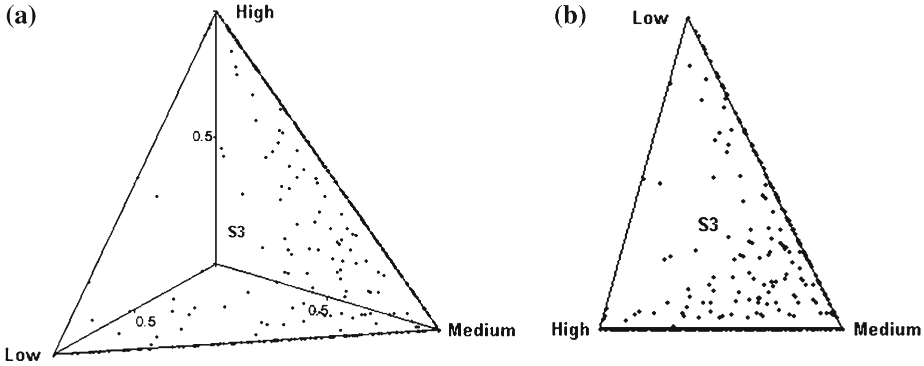



**Fig. 1** Fuzzy structure of banking commercial activity: (**a**) $\mathbf{S}_3$ in 3-D image and (**b**) $\mathbf{S}_3$ projection on the paper plane. The majority of employees position on Low-Medium and Medium-High edges. These particular employees share two extreme profiles only. So, they meet the conditions on Theorem 1 and may be ordered by competence

are assigned to the prototypes of Low, Medium and High typical profiles, respectively. The estimated position in $\mathbf{S}_3$ of $i$th individual is achieved by its estimated grade of membership representation $\hat{\mathbf{g}}_i = (\hat{g}_{i1}, \hat{g}_{i2}, \hat{g}_{i3})$. In this vector, the coordinate $\hat{g}_{ik}$ is the estimated rate of characteristics of extreme profile $k$, $1 \leq k \leq 3$, shared by him or her.

Intuitively, one would expect individuals fairly spread in $\mathbf{S}_3$, sharing differently the characteristics of typical profiles. The achieved results are, however, rather counter-intuitive.

Indeed, by depicting $\hat{\mathbf{g}}_i$ in $\mathbf{S}_3$ (Fig. 1), we found 76% of individuals lying in the edges connecting Low to Medium and this profile to High, vertices included. Additionally, 1% lyes in the open edge Low-High. This means, more than three quarter share only two extreme profiles characteristics. These individuals (77%) fully match the conditions on Theorem 1 and may be ordered by competence. However, individuals in the path Low-Medium-High are not comparable with ones in the path Low-High concerning job performance.

Indeed, the gradual nature of GoM vectors $\mathbf{g}_i$, transferring progressively weight from $g_{i1}$ to $g_{i2}$, in Low-Medium edge, and from $g_{i2}$ to $g_{i3}$, in Medium-High edge, leads individuals in theses edges with symmetrical job performance. The same does not hold for the individuals in Low-High edge where higher performance in some competences acts as a compensation of lower performance in others.

Thus, the observed distribution (Fig. 1) motivates a particular decomposition of $\mathbf{S}_3$ into two classes: one, called unsymmetrical, contains the open edge Low-High; the other contains the closed edges Low-Medium and Medium-High and also a vicinity of these edges, in terms defined below in (13), and we call it symmetrical.

Assume two extreme profiles dominate individual $i$ if he or she positions in a vicinity of 0.90 of these profiles. Formally, we write 0.90 dominance criterion as

$$g_{i1} + g_{i2} \geq 0.90 \ \vee \ g_{i2} + g_{i3} \geq 0.90, \tag{13}$$

which is intended to define "almost full membership" in the path Low-Medium-High. The conditions in (13) fix the symmetrical class which accommodates 93% of individuals. Let us consider this class first.

If conditions on Theorem 1 are relaxed to points in 0.90 vicinity of the edges, as defined in (13), the theorem result could be applied to all members of the symmetrical class. In this way, to every individual may be attached a number to reasonably represent its position in the fuzzy structure. So, greater the number, more competent the individual.

Explicitly, for the harmonic penalty function $\eta(k) := \frac{1}{k}$, and noting that the path Low-Medium-High is competence increasing, the function

$$\rho_{\mathbf{c}}\left(\hat{\mathbf{g}}_i\right) := 1 - \sum_{k=1}^{2} \frac{1}{k} \times \hat{g}_{ik} = 1 - \left(\hat{g}_{i1} + \frac{1}{2} \times \hat{g}_{i2}\right) \tag{14}$$

may be used as an utility function to map individual $i$ position estimate in $\mathbf{S}_3$ into the interval [0, 1]. However, we must state clear that the distribution of the statistic $\rho_{\mathbf{c}}(.)$ in (14) is not known so far. In this way, it is not possible to have any notion of the precision of its particular numeric value. So, the results must be used cautiously.

The function $\rho_{\mathbf{c}}(.)$ assumes the values 0 and 1 to identify the least and the most competent individuals, respectively. The value 0 corresponds to prototypes of Low extreme profile and the value 1 to ones of High extreme profile. The prototypes of Medium profile are worth 0.5. Individuals in the edge connecting Low to Medium are rated, by $\rho_{\mathbf{c}}(.)$, in the interval [0.0, 0.5] while individual lying in the edge connecting Medium to High are rated in the interval [0.5, 1.0]. The symmetrical class fits almost perfectly the Corollary 1, indeed.

In a slightly different perspective, although fully matching Theorem 1 conditions, the utility function $\rho_{\mathbf{c}}(.)$ orders individuals in the unsymmetrical class as well. These individuals are attached a rank in the interval (0, 1). But, one can hardly compare the unsymmetrical class to symmetrical one in this particular job context as noted earlier.

Regardless some *outliers*, the function $\rho_{\mathbf{c}}(.)$ was applied to all individuals. Noting that the utility function (14) depends only on $g_{i1}$ and $g_{i2}$, individual position in $\mathbf{S}_3$ can be depicted in a peculiar 2-D image, as in Fig. 2. In this figure, the origin is the projection of the point (0.5, 0.5, 0.0). Two straight lines must be taken into account. The lower one corresponds to ranks of individuals in Low-Medium edge and the upper one to individuals in Medium-High edge. As known, these edges are characterized by vectors $(g_{i1}, g_{i2}, 0)$ and $(0, g_{i2}, g_{i3})$, respectively.

Starting from the Low extreme profile, where the utility function $\rho_{\mathbf{c}}(.)$ is worth 0, the $g_{i1}$ decreases while $g_{i2}$ increases by the same amount. The intersection point $\rho_{\mathbf{c}}(.) = 0.25$ is the rank of individual represented by the vector (0.5, 0.5, 0.0). The lower line ends at the point (0, 1, 0) which represents the prototypes of the Medium extreme profile and it is worth the rank 0.5. Similar argument applies to upper straight line.

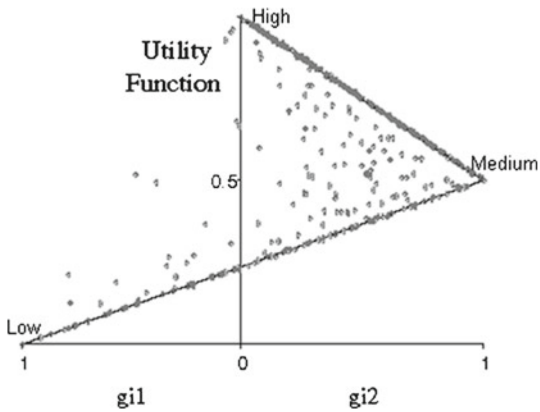



**Fig. 2** A projected image of competence function. The lower straight line corresponds to Low-Medium edge individuals rank. The intersection point corresponds to the vector (0.5, 0.5, 0.0) which is ranked 0.25. The end point of this line corresponds to Medium extreme profile. The *upper straight line* refers to Medium-High edge. Individuals in this line are increasingly rated from 0.5 to 1. This later value is the High extreme profile prototypes rank

The unsymmetrical class individuals with $g_{i1} > 0.5$ position in the lower straight line while the remaining ($g_{i1} < 0.5$) position in the upper straight line. All estimated values $\rho_{\mathbf{c}}\left(\hat{\mathbf{g}}_i\right)$ are depicted in grey.

Figure 2 shows additionally a higher concentration of individuals in Medium-High edge vicinity. Its estimated rate rounds 70%. This signals the demand for higher graduated young people which are expected to have appropriate needed skills.

## 6 Conclusion

Fuzzy sets theory and particularly fuzzy partitions are increasingly being used in classification problems as an alternative method to usual clustering techniques (Tolley and Manton 1992). These later widely used techniques aim to classify individuals into crisply defined sets. In contrast, fuzzy partitions allow individuals' representation by GoM vectors that define their position in the universe under study.

Using a statistical GoM model to analyse an original multivariate categorical dataset on competence assessment of $N = 593$ banking employees, one realizes that the estimated fuzzy structure of commercial activity of banking industry is hierarchical. This major finding leads to a particular use of Theorem 1, potentially allowing employees to be ordered by competences. Moreover, the estimated bundle of competences distribution in the fuzzy structure shows that within individuals, bundles are almost symmetrical or homogeneous, despite the prevalence of between individuals heterogeneity. Scarce room for expertise was found.

Unlike other techniques used on researches under job matching, GoM model seems to be more appropriate to describe individual heterogeneity, as it explicitly incorporates individual representation. This distinguish feature of the model seems to be unexplored in economics of education so far. Despite the small sample size of the available data, we hope this paper shed light on its potentiality to understand individual bundle of competences to design incisive human resources policies.

However, it must be emphasized that in GoM model many points are opened to debate. The issue of identifiability remains far from resolved. Nevertheless, our research top priority is the distribution of the statistic $\rho_{\mathbf{c}}\left(\hat{\mathbf{g}}_i\right)$. And this is not an easy task, we guess.

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## Appendix

Competences rating inquiry

See Table 2.

**Table 2** List of competences used in the survey in commercial activity of banking industry

| Class of competence | Specification |
|---|---|
| Knowledge | General technical knowledge |
| | Specific technical knowledge |
| | Foreign languages |

**Table 2** continued

| Class of competence | Specification |
|---|---|
| Behaviours and attitudes toward others | Relationship with colleagues |
| | Team working |
| | Communication skills |
| | Willingness to help others |
| | Negotiation skills |
| | Persuasion skills |
| Behaviours and attitudes toward organization | Perseverance and goal-oriented attitudes |
| | Client-oriented attitudes |
| | Autonomy |
| | Responsibility |
| | Adaptability |
| | Innovative attitudes |
| | Favourable learning attitudes |
| | Proactive attitudes toward learning |
| | Following rules and procedures |
| | Cooperation (with organizational goals) |
| | Working time flexibility |
| | Punctuality |
| Cognitive and technical skills | Work planning |
| | Computer skills |
| | Analytical skills |
| | Ability to select and process information |
| | Problem solving |
| | Learning ability |
| | Ability to transfer knowledge and experience |
| | Ability to understand the banking specificities |
| | Ability to understand corporation strategy |

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