Reflected Brownian motion with drift in a wedge

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Abstract

We study reflecting Brownian motion with drift constrained to a wedge in the plane. Our first set of results provides necessary and sufficient conditions for existence and uniqueness of a solution to the corresponding submartingale problem with drift, and show that its solution possesses the Markov and Feller properties. Next, we study a version of the problem with absorption at the vertex of the wedge. In this case, we provide a condition for existence and uniqueness of a solution to the problem and some results on the probability of the vertex being reached.

1 Introduction

In this paper, we study 2-dimensional Brownian motion with constant drift $\mu \in \mathbb{R}^2$ constrained to a wedge S in $\mathbb{R}^2$. This process may also be referred to as reflected Brownian motion (RBM) with drift in a wedge, and we denote the process itself by Z. For concreteness, we define the wedge in polar coordinates by $\{r \geq 0, 0 \leq \theta \leq \xi\}$ for some $0 < \xi < 2\pi$. Loosely speaking, the behavior of Z may be characterized as follows. In the interior of S, Z behaves as a 2-dimensional Brownian motion. On the other hand, the behavior of Z on the boundary of S is characterized by two reflection

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angles θ_1 and θ_2, depending upon whether the lower boundary ∂S_1 or upper boundary ∂S_2 has been reached. Both $-\pi/2 < \theta_1, \theta_2, < \pi/2$ and the angles are measured from their inward-facing normals, with positive angles corresponding to reflection toward the vertex of the wedge and negative angles away. See figure below for an illustration.

One way to define RBM in a wedge is using a sample-path approach [9, 10, 19, 34] where Z is explicitly characterized as the sum of a 2-dimensional Brownian motion on an arbitrary probability space [22, 24, 31] and a constraining or pushing process which satisfies the specifications related to the directions of reflection given above. This sample-path approach works with or without drift for some but not all parameter regimes of $(\xi, \theta_1, \theta_2)$. It tends not to work in regimes where Z is known not to be a semi-martingale [39] and the pushing process has infinite variation. Recent progress in this direction has however been made [20, 28].

A more probabilistic approach to defining RBM in a wedge was given by Varadhan and Williams [35]. In this case, Z is defined as the solution to a submartingale problem. This approach yields existence and uniqueness results for all parameter regimes but at several points the proofs of [35] rely heavily on the assumption that Z behaves as a standard Brownian motion inside of S. This is not an issue for parameter regimes where the sample-path approach described above may be applied because it is amenable to Brownian motions with drift, and the recent paper [21] demonstrates equivalence between the sample-path and the submartingale approach in such settings. On the other hand, in parameter regimes where the sample-path approach cannot be applied, extending the results of [35] in the direction of allowing Z to behave as a Brownian motion with drift in the interior of S remains an open problem. In this paper, we resolve this open problem for the case of a constant drift. We conjecture that our results could be further generalized for the case when the drift is a bounded function of the current state, but this generalization is beyond the scope of the present paper.

Our primary motivation comes from queueing theory where semi-martingale RBM with drift has long been known to serve as the weak limit of both the properly scaled queue length [5, 17, 18, 30] and workload [4, 6, 26, 41] processes of different queueing systems in heavy-traffic. In such queueing settings, the drift term arises as the result of an imbalance between the input and output processes to the system. The limiting RBM in these cases is often defined using the sample-path approach via the conventional Skorokhod map [19, 33, 36]. More recently, using the extended Skorokhod map [28], RBM with drift which is not a semi-martingale has been proven [29] to be the weak limit of the properly scaled unfinished work process of the generalized processor sharing model in heavy traffic. In this example, the sample-path approach is still employed to define the limiting process with the help of the extended Skorokhod map [28]. We conjecture however that there exist other applied queueing settings where the limiting heavy-traffic process is an RBM with drift which is not a semi-martingale and cannot be rigorously defined via the sample-path approach. One of these settings is the coupled processor model [7, 14]. In such situations, before proving any limit theorems, it is necessary to first establish the existence of RBM with drift through other means such as the submartingale problem.

We also mention that there exists a related stream of literature studying the behavior of reflected Brownian in smooth domains with cusps. The paper [12] appears to be the first to study reflected Brownian motion confined to a cusp in the plane. In this case,

RBM is defined as the solution to a corresponding submartingale problem. Existence and uniqueness results are then proven by conformally mapping RBM in the upper half plane to the cusp and applying a time change. In the follow-up paper [11], it is shown that depending on the geometry of the problem in [12], RBM in a cusp in the plane may or may not turn out to be a semi-martingale. The results in [11] are similar to those in [39] where a wedge instead of a cusp is considered. The authors in [16] use a Dirichlet form approach to construct a diffusion process contained in a d-dimensional Lipschitz domain with cusps. Moreover, conditions are provided under which the constraining process is of bounded variation. The paper [8] considers RBM in a cusp in the plane as the solution to a stochastic differential equation with reflection (SDER). It is proven that in this case there exists a unique weak solution to the corresponding SDER.

The remainder of the paper is organized as follows. Our main results may be found in Sect. 2. In Sect. 2.1, we provide necessary and sufficient conditions for the existence and uniqueness of the solution to the submartingale problem with drift (see Definition 2.1), and show that its solution possesses the strong Markov property and three versions of the Feller property. Next, in Sect. 2.2, we study the submartingale problem with drift absorbed at the vertex of the wedge (see Definition 2.13). We provide results on the existence and uniqueness of the solution to this problem and results on the probability of the absorbed process with drift reaching the vertex of the wedge. Sections 3 through 7 contain the proofs of our main results.

2 Main results

Before stating our main results, we first set up some notation. Let $C_S = C(\mathbb{R}_+, S)$ and, for each $t \geq 0$, let $Z(t) : C_S \rightarrow S$ denote the coordinate map $Z(t)(\omega) = \omega(t)$ for $\omega \in C_S$. Also, let $Z = \{Z(t), t \geq 0\}$ denote the coordinate mapping process on C_S. Let $\mathcal{M}_t = \sigma(Z(s), 0 \leq s \leq t)$ be the underlying natural filtration with terminal σ-algebra $\mathcal{M} = \sigma(Z(s), s \geq 0)$. For each $n \geq 1$ and domain $\Omega \subseteq \mathbb{R}^2$, we denote by $C_b^n(\Omega)$ the set of n times bounded continuously differentiable functions on Ω. We assume that the wedge S is positioned so that one side of it is the positive horizontal half line, and the angle of the wedge is ξ. We define ∂S_1 and ∂S_2 as the two sides of the wedge so that neither includes the vertex, i.e., $\partial S_1 = \{(x, 0) : x > 0\}$ and $\partial S_2 = \{r(\cos \xi, \sin \xi) : r > 0\}$. On the other hand, we define ∂S as the boundary of S, so it includes the vertex. Next (see Fig. 1), we denote by v_1 and v_2 the reflection directions on the boundaries ∂S_1 and ∂S_2, respectively. For convenience, we assume that each v_i is normalized such that $v_i \cdot n_i = 1$, where n_i is the inward-facing normal vector on ∂S_i for $i = 1, 2$. Finally, for $i = 1, 2$, we set the directional derivative operator $D_i = v_i \cdot \nabla$, with ∇ being the gradient operator, the dot is the inner product, and denote by Δ the Laplacian operator.

Fig. 1 RBM in a Wedge

$$\partial S_2$$
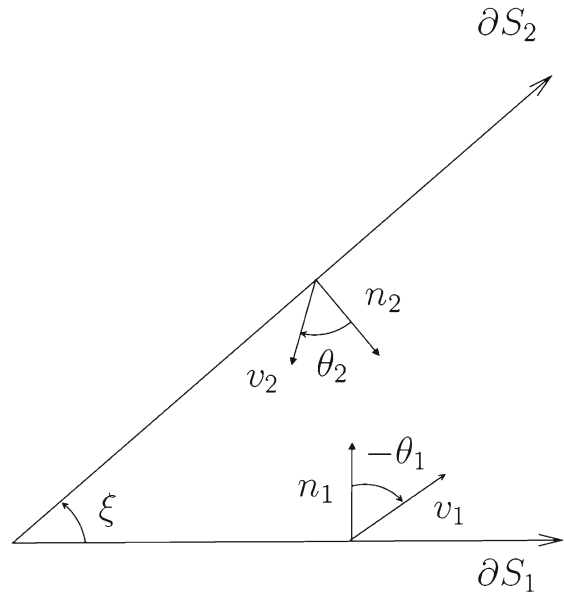
2.1 The submartingale problem with drift

Definition 2.1 (*Submartingale problem with drift*) A family of probability measures $\{\mathbb{P}_\mu^z, z \in S\}$ on $(C_S, \mathcal{M})$ is said to solve the submartingale problem with drift $\mu \in \mathbb{R}^2$ if for each $z \in S$, the following three conditions hold,

1. $\mathbb{P}_\mu^z(Z(0) = z) = 1$;
2. For each $f \in C_b^2(S)$, the process

$$\left\{ f(Z(t)) - \int_0^t \mu \cdot \nabla f(Z(s))ds - \frac{1}{2}\int_0^t \Delta f(Z(s))ds, t \geq 0 \right\}$$

   is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^z)$ whenever f is constant in a neighbourhood of the origin and satisfies $D_i f \geq 0$ on ∂S_i for $i = 1, 2$;
3. $\mathbb{E}_\mu^z\left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}}dt \right] = 0.$

The above definition bears a relationship to the extended Skorokhod problem (ESP) developed in [28]. We shall recall the definition of the ESP below. Let $d(\cdot)$ be a set-valued map from ∂S, the boundary of S, to the class of subsets of $\mathbb{R}^2$ satisfying the following two conditions:

(d1) for any $x \in \partial S$, the image $d(x)$ is a non-empty closed convex cone in $\mathbb{R}^2$ with the vertex being the origin;

(d2) the graph $\{(x, d(x)); \ x \in \partial S\}$ is closed.

For convenience, we extend the definition of $d(\cdot)$ to S by setting $d(x) = \{0\}$ for all $x \in S^\circ$, where S° is the interior of S. For a set $A \subset \mathbb{R}^2$, let co(A) be the closed convex cone generated by A.

Intuitively, the set-valued function $d(\cdot)$ represents all the possible directions of reflection at a given boundary point. If a vector is included in $d(x)$ for some boundary point x, then that vector represents an allowed direction of reflection whenever the reflected process hits the boundary point x. The third item below captures exactly this property. In the case of the present paper we shall always have $d(x) = \{\lambda v_i, \lambda \geq 0\}$ whenever $x \in \partial S_i$ (recall that the vertex is not included in ∂S_i), $i = 1, 2$. However, we shall assume that $d(0) = \mathbb{R}^2$. Despite the fact that we shall use these set functions only in this special case, we recall below the general definition as it appears in [28].

Definition 2.2 (*Extended Skorokhod Problem (ESP)*) A pair of processes $(\phi, \eta) \in C_S \times C(\mathbb{R}_+, \mathbb{R}^2)$ is said to solve the ESP $(S, d(\cdot))$ for $\psi \in C(\mathbb{R}_+, \mathbb{R}^2)$ such that $\psi(0) \in S$ if $\phi(0) = \psi(0)$, and if for all $t \in \mathbb{R}_+$, the following properties hold,

1. $\phi(t) = \psi(t) + \eta(t)$;
2. $\phi(t) \in S$;
3. For every $s \in [0, t]$;

$$\eta(t) - \eta(s) \in \mathrm{co}\left[\cup_{u \in (s,t]} d(\phi(u))\right].$$

Item 2 in the above definition is redundant since we already required that $\phi \in C_S$, but we kept that item as it appears in the original definition in [28].
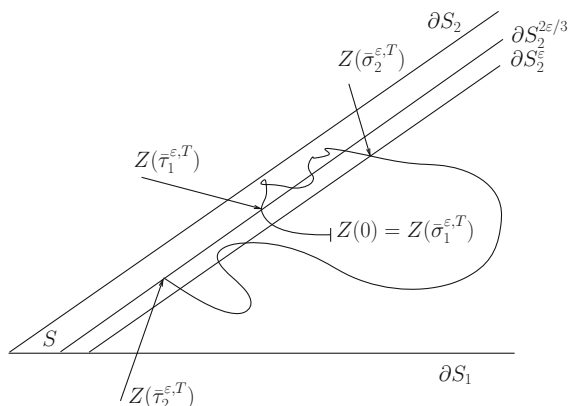
Just like in [35], let

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

The quantity α plays a prominent role. A visual description in a table form of the way various values of α determines the properties of the solution of the submartingale problem without drift is available in [3], Fig. 2.

From here on in this paper we shall always have

$$d(x) = \{\lambda v_i, \lambda \geq 0\} \text{ whenever } x \in \partial S_i, \ i = 1, 2, \tag{1}$$

Fig. 2 The stopping times $\bar{\sigma}_k^{\epsilon,T}$ and $\bar{\tau}_k^{\epsilon,T}$

$$d(x) = \{0\}, \text{ whenever } x \in S^0, \tag{2}$$

and

$$d(0) = \mathbb{R}^2, \tag{3}$$

except in Remark 8.3, where we revert to the general case. In (2) S^0 represents the interior of the wedge S.

Theorem 2.3 *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ there exists a unique solution $\{\mathbb{P}^z_\mu, z \in S\}$ to the submartingale problem with drift. In addition, the following statements hold:*

1. *There exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion with drift μ starting at z under $\mathbb{P}^z_\mu$;*
2. *Setting $Y = Z - X$, the pair (Z, Y) solves the ESP $(S, d(\cdot))$ for X, $\mathbb{P}^z_\mu$-a.s., for each $z \in S$, where the set function $d(\cdot)$ is specified in (1)–(3).*

The above theorem establishes a decomposition

$$Z = X + Y, \tag{4}$$

such that for all $z \in S$ under $\mathbb{P}^z_\mu$ the process X is a standard Brownian motion with drift μ starting at z. In the following two theorems we shall establish several properties of the process Y appearing in the above decomposition. In order to state the first of these two results, we need the definition of the strong p-variation of a function. Let $T > 0$ arbitrary. We call an ordered set $(t_0, t_1, \ldots, t_n)$ a partition of the interval $[0, T]$, if $0 = t_0 < t_1 < \cdots < t_n = T$, for an arbitrary $n \in \mathbb{N}_+$. Let $\pi(T)$ denote the set of all partitions of the interval $[0, T]$. We define the mesh of a partition $\rho = (t_0, \ldots, t_n) \in \pi(T)$ by setting

$$\|\rho\| = \max\{t_i - t_{i-1} : i = 1, \ldots, n\}.$$

Definition 2.4 Let $T > 0$ and $p > 0$. The strong p-variation of a function $f : \mathbb{R}_+ \mapsto \mathbb{R}^k$ on $[0, T]$ is defined by

$$V_p(f, [0, T]) = \sup \left\{ \sum_{t_i \in \rho, \ i \geq 1} \|f(t_i) - f(t_{i-1})\|^p : \rho \in \pi(T) \right\}.$$

Theorem 2.5 *Suppose that $1 < \alpha < 2$. Then for each $p > \alpha$ and $z \in S$,*

$$\mathbb{P}^z_\mu(V_p(Y, [0, T]) < +\infty) = 1, \quad T > 0, \tag{5}$$

and, for each $0 < p \leq \alpha$,

$$\mathbb{P}^0_\mu(V_p(Y, [0, T]) < +\infty) = 0, \quad T > 0. \tag{6}$$

A 2-dimensional continuous process U defined on $(C_s, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^z)$ is said to be of *zero energy* if for each $T > 0$ and each sequence of partitions $(\rho^m) \subset \pi(T)$ such that $\|\rho^m\| \to 0$ as $m \to \infty$ we have

$$\sum_{i=1}^{n(m)} \|U(t_i^m) - U(t_{i-1}^m)\|^2 \xrightarrow{\mathbb{P}_\mu^z} 0 \text{ as } m \to \infty,$$

where $\rho^m = (t_0^m, \ldots, t_{n(m)}^m)$. A process D on $(C_s, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^z)$ is said to be a *Dirichlet process* if it has a decomposition $D = M + U$, where M is a local martingale on the same probability space, and U is a continuous zero-energy process with $U(0) = 0$.

Theorem 2.6 *Let $1 < \alpha < 2$, and $z \in S$ arbitrary. Then the process Y in decomposition* (4) *is a zero-energy process, and Z is a Dirichlet process.*

Theorem 2.7 *If $1 \le \alpha < 2$, then Z is not a semimartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^z)$, for any $z \in S$.*

Theorem 2.8 *If $\alpha \ge 2$, then for any $\mu \in \mathbb{R}^2$ there is no solution to the submartingale problem with drift.*

Let $\{\mathbb{P}_\mu^z, z \in S\}$ be the solution to the submartingale problem for some $\mu \in \mathbb{R}^2$. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ possesses the strong Markov property if for each stopping time τ and $z \in S$, and each bounded $\mathcal{M}$-measurable function $h : C_S \to \mathbb{R}$ we have that

$$\mathbb{E}_\mu^z[\mathbb{1}_{\{\tau < \infty\}} f(\omega(\cdot + \tau))|\mathcal{M}_\tau] = \mathbb{1}_{\{\tau < \infty\}} \mathbb{E}_\mu^{\omega(\tau)}[f(w(\cdot))], \quad \mathbb{P}_\mu^z\text{-a.s.} \tag{7}$$

Theorem 2.9 *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ the solution to the submartingale problem with drift has the strong Markov property.*

The last subject of this subsection is the Feller property of $\{\mathbb{P}_\mu^z, z \in S\}$. There are various, slightly differing definitions of the Feller property available in the literature. For clarity we list below three definitions.
1. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ has the Feller property if for any $\{z_n, n \ge 1\} \subset S$ converging to $z \in S$, $\mathbb{P}_\mu^{z_n} \Rightarrow \mathbb{P}_\mu^z$ as $n \to \infty$ (see Varadhan and Williams [35]).
2. Let $\hat{C}(S)$ be the set of continuous functions on S vanishing at infinity. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ has the $\hat{C}(S)$-Feller property if for any $f \in \hat{C}(S)$, and $t \ge 0$, the function $z \mapsto \mathbb{E}_\mu^z[f(Z(t))]$ is also in $\hat{C}(S)$.
3. Let $C_b(S)$ be the set of bounded continuous functions on S. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ has the $C_b(S)$-Feller property if for any $f \in C_b(S)$, and $t \ge 0$, the function $z \mapsto \mathbb{E}_\mu^z[f(Z(t))]$ is also in $C_b(S)$.

Remark 2.10 The Feller property obviously implies the $C_b(S)$-Feller property. The $\hat{C}(S)$-Feller property implies the $C_b(S)$-Feller, but the converse is not true (cf. Theorems 1.9 and 1.10 in [2]).

Theorem 2.11 *If $\alpha < 2$, then the solution to the submartingale problem for each $\mu \in \mathbb{R}^2$ has the Feller property.*

Theorem 2.12 *If $\alpha < 2$, then the solution to the submartingale problem for each $\mu \in \mathbb{R}^2$ has the $\hat{C}(S)$-Feller property.*

We note that for the $\mu = 0$ case the Feller property is known ([35], Theorem 3.13). However, the $\hat{C}(S)$-Feller property is new even in the $\mu = 0$ case.

2.2 The absorbed process

Let

$$\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}$$

be the stopping time with respect to $\{\mathcal{M}_t, t \geq 0\}$ representing the first time that Z reaches the vertex of the wedge. Results in this subsection concern the RBM in a wedge up until τ_0. Results of this type were provided in [35] for the drift-less case. In particular, it been shown there that for the drift-less case $\mathbb{P}_0^z(\tau_0 < \infty)$ is equal to zero if $\alpha \leq 0$, and equal to 1 if $\alpha > 0$. We shall begin the study of τ_0 in the presence of a constant drift μ with the following definition.

Definition 2.13 (*The Absorbed Process Problem*) A family of probability measures $\{\mathbb{P}_\mu^{z,0}, z \in S\}$ on $(C_S, \mathcal{M})$ is said to solve the absorbed process problem with drift $\mu \in \mathbb{R}^2$ if for each $z \in S$, the following three conditions hold,

1. $\mathbb{P}_\mu^{z,0}(Z(0) = z) = 1$;
2. The process

$$\left\{ f(Z(t \wedge \tau_0)) - \int_0^{t \wedge \tau_0} \mu \cdot \nabla f(Z(s)) ds - \frac{1}{2} \int_0^{t \wedge \tau_0} \Delta f(Z(s)) ds, t \geq 0 \right\}$$

   is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^{z,0})$, for each $f \in C_b^2(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$;
3. $\mathbb{P}_\mu^{z,0}(Z(t) = 0, \forall t \geq \tau_0) = 1$.

Notice that in the above definition in item 2 the upper limit of the integral is not t but $t \wedge \tau_0$. In other words, in this definition we require that the process specified in item 2 of Definition 2.1 is a submartingale only up to τ_0. Item 3 in the above definition specifies that the reflected process is absorbed at the vertex once the vertex is reached. This definition requires less then Definition 2.1, hence it is possible that the absorbed process problem with drift has a solution for $\alpha \geq 2$, despite the fact stated in Theorem 2.8, namely that the submartingale problem with drift has no solution in that case. This is exactly the subject of the next theorem.

Theorem 2.14 *For each $\mu \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, there exists a unique solution to the absorbed process problem.*

The above theorem is particularly interesting if $\alpha \geq 2$, since Theorem 2.3 does not cover that case. The existence of a solution to the absorbed process problem easily

follows from the existence of a solution to the submartingale problem whenever $\alpha < 2$. However, the uniqueness part of Theorem 2.14 does not follow in an obvious way from Theorem 2.3 even in the $\alpha < 2$ case. Our proof for Theorem 2.14 applies to all $\alpha \in \mathbb{R}$.

Next we state a series of results on the hitting probability of the vertex for the absorbed process in the case of a constant drift.

Theorem 2.15 *If $\alpha \leq 0$, then for each $\mu \in \mathbb{R}^2$ and $z \in S$, $\mathbb{P}_{\mu}^{z,0}(\tau_0 = \infty) = 1$.*

The hitting probability of the vertex is more varied in the case of $\alpha \geq 1$, and before proceeding we must make some observations on the geometry of the wedge. For $n \geq 1$ and a set of vectors $\{a_1, \ldots, a_n\} \subset \mathbb{R}^2$, let $co(a_1, \ldots, a_n)$ denote the closed convex cone generated by $\{a_1, \ldots, a_n\}$. We illustrate two relevant cases for α below.
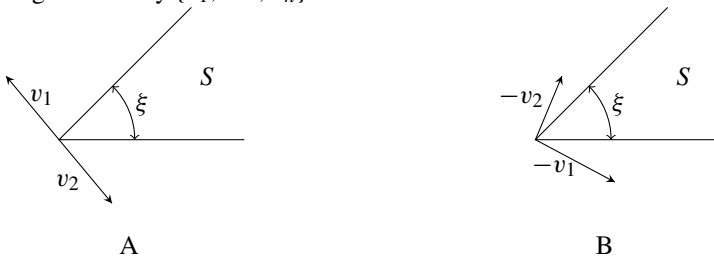
In the above diagrams, case A corresponds to $\alpha = 1$, which occurs if and only if $co(v_1, v_2)$ is a line. Case B corresponds to $\alpha > 1$, which occurs if and only if $co(-v_1, -v_2)$ contains S. In both cases, that is, whenever $\alpha \geq 1$ we have that $co(v_1, v_2) \cap S = \{0\}$. We have $\alpha < 1$ if and only if $co(-v_1, -v_2) \cap S = \{0\}$ and $co(v_1, v_2)$ is not a line. The cases $\alpha \leq 0$, $\alpha \geq 2$ play significant roles in the results in this section, but unfortunately we don't have a simple geometric interpretation for these cases. Intuitively, small value for α means that the combined impact of the reflections on ∂S_1 and ∂S_2 are driving the reflected process more away from the vertex than towards it. Large value for α means that the combined effect of the reflections is driving the reflected process more towards the vertex than away from the vertex. Note also that $\alpha \geq 1$ implies $\xi < \pi$.

Theorem 2.16 *If $\alpha \geq 1$, then*

$$\mathbb{P}_{\mu}^{z,0}(\tau_0 < \infty) > 0 \text{ for each } \mu \in \mathbb{R}^2 \text{ and } z \in S. \tag{8}$$

Moreover, if in addition to the $\alpha \geq 1$ condition we also have that

$$co(v_1, v_2, \mu) \cap S = \{0\}, \tag{9}$$

then for each $z \in S$,

$$\mathbb{P}_{\mu}^{z,0}(\tau_0 < \infty) = 1. \tag{10}$$

We note that in the case of $\alpha > 1$ condition (9) can be cast in an algebraic form. Let R be the 2×2 matrix such that its i-th column vector is v_i, for $i = 1, 2$. If $\alpha > 1$ then condition (9) is equivalent to the requirement the vector $R^{-1}\mu$ has at least one non-negative component.

Remark 2.17 Theorem 2.16 leaves open the possibility that $\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) = 1$ whenever $\alpha \geq 1$. This however is not the case as the following counterexample shows. Let $\alpha \in \mathbb{R}$ be arbitrary and let the drift $\mu \in \mathbb{R}^2$ be given by $\mu = ||\mu||(\cos\eta, \sin\eta) \neq 0$, where $\eta \in (0, \xi)$. Then, it is not hard to show using the proposition below that $\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) < 1$ for each $z \in S\backslash\{0\}$.

Proposition 2.18 *Let S be the 2-d wedge defined above, let S^0 be the interior of S, let B be a 2-d standard Brownian motion with zero drift starting at the origin under a probability measure P, and let $\mu \in \mathbb{R}^2$ given by $\mu = ||\mu||(\cos\eta, \sin\eta) \neq 0$, where $\eta \in (0, \xi)$. Then, if $0 < \xi < \pi$, for each $z \in S^0$,*

$$P(z + B_t + \mu t \in S^0, t \geq 0) > 0.$$

Using the proposition above and Theorem 2.16, we may now deduce that if $\alpha \geq 1$, $\eta \in (0, \xi)$, and $\mu = ||\mu||(\cos\eta, \sin\eta) \neq 0$, we obtain that

$$\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) \in (0, 1) \text{ for each } z \in S \setminus \{0\}.$$

This implies that when $\alpha \geq 1$, hitting the vertex is no longer a 0-1 event for certain values of μ, which contrasts with the drift-less result of [35]. An explicit formula for the hitting probability of the vertex can be found in [15] assuming that S is a quadrant, $\alpha = 1$, and μ points inside of the quadrant (the result of that paper is actually stated in a dimension possibly higher than 2, but we quoted it in the 2-dimensional case, since that is the relevant case in the present paper).

3 Proof of Theorems 2.3, 2.5, 2.6, 2.7, and 2.8

The first two propositions in this section will provide proofs for all statements in Theorem 2.3, except the uniqueness. The proof of these propositions is based on Girsanov's theorem. We know that all these statements hold for $\mu = 0$, and Girsanov's theorem implies that a Brownian motion X with drift μ exists under the transformed probability measures, and (X, Y) solves a particular ESP. Then a result of Kang and Ramanan will imply that the family of transformed probability measures solves the submartingale problem. The exact steps of this plan are below.

Recall that $d(\cdot)$ has been specified in (1)–(3). It is known that in the case of $\mu = 0$ the submartingale problem has a unique solution whenever $\alpha < 2$ (see [35]). In accordance with our notation, that solution will be denoted by $\{\mathbb{P}_0^z, z \in S\}$. We then have the following.

Proposition 3.1 *Let $(C_S, \mathcal{M}, \mathcal{M}_t)$ and Z be defined as in Sect. 2, that is, $Z(t) : C_S \mapsto S$ is the coordinate map $Z(t)(\omega) = \omega(t)$ for $\omega \in C_S$. Then, if $\alpha < 2$,*

1. *There exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion starting at z under $\mathbb{P}_0^z$;*
2. *Setting $Y = Z - X$, the pair (Z, Y) solves the ESP $(S, d(\cdot))$ for X, $\mathbb{P}_0^z$-a.s.*

Proof Let $\alpha < 2$. Then, Condition 1 is immediate from Theorem 2.4 in [23]. It remains to show that for each $z \in S$, $\mathbb{P}_0^z$-a.s., Z and $Y = Z - X$ together solve the ESP (see Definition 2.2 as above) for X with d as defined immediately preceding the statement of the proposition. For $\alpha \in (1, 2)$, this follows by Theorem 2.8 in [23]. We now claim that if $\alpha \leq 1$, (Z, Y) also solves the ESP $(S, d(\cdot))$ for X, $\mathbb{P}^z$-a.s. For any two real numbers $0 < s < t$, if (s, t) belongs to a single excursion from the origin then by a similar proof to the one in part 2 of Theorem 4.2 in [23], one can conclude that item 3 in the definition of the ESP holds. If (s, t) doesn't belong to one excursion, then item 3 is obviously satisfied by $d(0) = \mathbb{R}^2$. □

We are now ready to prove the existence of a solution to the submartingale problem with drift, and some of the properties of the solution we create. In the following Proposition X and Y are the processes whose existence is guaranteed by Proposition 3.1. In particular, for each $z \in S$ the process X is a 2-dimensional Brownian motion with no drift under $\mathbb{P}_0^z$, and $Y = Z - X$.

Proposition 3.2 *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ the submartingale problem with drift has a solution $\{\mathbb{P}_\mu^z, z \in S\}$, which satisfies the following properties. With X and Y defined in Proposition 3.1, for every $z \in S$ the following hold:*

- *Under $\mathbb{P}_\mu^z$ the process X is a standard Brownian motion with drift μ starting at z;*
- *The pair (Z, Y) solves the ESP $(S, d(\cdot))$ for X, $\mathbb{P}_\mu^z$-s.s.*

Proof Let $\alpha < 2$ and note that by Proposition 3.1 there exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion starting at z under $\mathbb{P}_0^z$. Now let $T \geq 0$ and for each $z \in S$, let $\mathbb{P}_{\mu,T}^z$ be a probability measure on $(C_S, \mathcal{M}, \mathcal{M}_t)$ equivalent (mutually absolutely continuous) to $\mathbb{P}_0^z$ such that under $\mathbb{P}_{\mu,T}^z$, X is a standard Brownian motion with drift μ up to time T, starting at z. In other words, $\{X(t) - \mu t, t \leq T\}$ is a standard (drift-less) $\mathbb{P}_{\mu,T}^z$-Brownian motion starting at z. The measure $\mathbb{P}_{\mu,T}^z$ is defined by

$$\frac{d\mathbb{P}_{\mu,T}^z}{d\mathbb{P}_0^z} = \zeta(T), \tag{11}$$

where $\zeta(T) = \exp\{\mu \cdot (X(T) - X(0)) - \frac{1}{2}|\mu|^2 T\}$.

One can easily show that the family of probability measures $\{\mathbb{P}_{\mu,T}^z, T \in [0, \infty)\}$ is consistent. That is, if $S < T$, then $\mathbb{P}_{\mu,T}^z(A) = \mathbb{P}_{\mu,S}^z(A)$, whenever $A \in \mathcal{M}_S$. From [25], Theorem 4.2 (page 143), it follows that there exists a single probability measure $\mathbb{P}_\mu^z$ such that $\mathbb{P}_\mu^z(A) = \mathbb{P}_{\mu,T}^z(A)$ whenever $A \in \mathcal{M}_T$. Since $\{X(t) - X(0) - \mu t, t \leq T\}$ is a $\mathbb{P}_{\mu,T}^z$-Brownian motion starting at zero for every $T \in [0, \infty)$, it follows that $\{X(t) - X(0) - \mu t, t < \infty\}$ is also a $\mathbb{P}_\mu^z$-Brownian motion starting at zero. Also, (Z, Y) solves the ESP $(S, d(\cdot))$ for X under $\mathbb{P}_\mu^z$-a.s., because by Proposition 3.1 it is true under $\mathbb{P}_0^z$, and the measures $\mathbb{P}_0^z$ and $\mathbb{P}_\mu^z$ constrained to $\mathcal{M}_T$ are mutually absolutely continuous for every $T \in [0, \infty)$. Now let

$$W(t) = X(t) - X(0) - \mu t, \quad t \in [0, \infty),$$

and consider the definition of a weak solution to an SDER (see Definition 2.4 of Kang and Ramanan [21]. The triplet $(C_S, \mathcal{M}, \mathcal{M}_t)$, $\mathbb{P}_\mu^z$, (Z, W) is a weak solution to the SDER with initial condition z associated with $(S, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$, where $b(x) = \mu$ and $\sigma(x) = \mathrm{Id}_{2\times2}$. For the convenience of the reader we recalled the definition of a weak solution of an SDER from [21] in Remark 8.3. Condition 4 in that remark is satisfied by Lemma 4.2 in [37]. Indeed, it follows from that lemma that for every $t \geq 0$ the set $\{s \in [0, t) : Z(s) \in \partial S\}$ has zero Lebesgue measure $\mathbb{P}_0^z$-a.s. Then by the mutual absolute continuity of $\mathbb{P}_\mu^z$ and $\mathbb{P}_0^z$ the same is true under $\mathbb{P}_\mu^z$, and then by taking limits as $t \to \infty$ we get the required property. We note that the "closed graph condition" (see Kang and Ramanan [21], page 5) is satisfied, because we defined $d(0) = \mathbb{R}^2$. From Theorem 2 in [21], it now follows that $\{\mathbb{P}_\mu^z, z \in S\}$ solves the submartingale problem with drift μ. □

We shall use the Lemmas 3.3, 3.4, 3.5, 3.6, and 3.7 for the proof of both the uniqueness part of Theorem 2.3, and for the proof of Theorem 2.8. In these lemmas α may be an arbitrary real number. On the other hand, in these lemmas we start with a probability measure $\mathbb{P}_\mu^z$ that satisfies conditions 1, 2, and 3 of Definition 2.1. This may be surprising, since Theorem 2.8 states that such probability measure does not exist for $\alpha \geq 2$. However, for such α's we use these lemmas to derive a contradiction, thereby proving Theorem 2.8. A simple use of Girsanov's theorem is not sufficient for proving uniqueness. The reason is that assuming that there is another probability measure, Q_μ^z besides P_μ^z that satisfies conditions 1–3 in Definition 2.1, we do not know that Q_μ^z and P_μ^z constrained to $\mathcal{M}_T$ are mutually absolutely continuous for $T \in [0, \infty)$, thus we do not know that X is a Brownian motion with drift μ under Q_μ^z, and do not know that the ESP is satisfied under Q_μ^z either. Therefore the uniqueness proof is a bit more complicated. In the lemma below we shall show that for any solution $\{\mathbb{P}_\mu^z, z \in S\}$ to the submartingale problem satisfies (12). We showed at the end of the proof of Proposition 3.2 that this can be done easily for the solution we created using Girsanov's theorem. But the point is that for the uniqueness proof now we need to show that (12) holds for *any* solution to the submartingale problem.

Lemma 3.3 *Suppose that $\{\mathbb{P}_\mu^z, z \in S\}$ is a solution to the submartingale problem with drift $\mu \in \mathbb{R}^2$. Then, for all $z \in S$,*

$$\mathbb{E}_\mu^z\left[\int_0^\infty \mathbb{1}_{\{Z(t)\in\partial S\}}dt\right] = 0. \tag{12}$$

Proof Let $z \in S$ be arbitrary. In this proof we shall use the Doob–Meyer decomposition for submartingales, which requires that the probability space is augmented. For this reason we denote by $(C_S, \mathcal{F}^z, (\mathcal{F}_t^z))$ the augmentation of the space $(C_S, \mathcal{M}, (\mathcal{M}_t))$ under $\mathbb{P}_\mu^z$, using the concept of augmentation as it is defined in [32], Definition II.67.3. For some technical details on the augmentation of probability spaces see Remark 8.2 in the Appendix. Condition 3 of the submartingale problem gives

$$\mathbb{E}_\mu^z\left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}}dt\right] = 0,$$

for each $z \in S$, thus in order to complete the proof it suffices to prove that

$$\mathbb{E}_\mu^z \left[\int_0^\infty \mathbb{1}_{\{Z(t) \in \partial S_{i,}\}} dt \right] = 0 \text{ for } i = 1, 2. \tag{13}$$

We prove this result for $i = 1$; the result then follows for $i = 2$ by symmetry. The essence of the proof is the following. First we select an area within S that is bounded away from the vertex, and prove that the statement holds while Z is in this area in the following way. By selecting a test function which in this area is simply the projection to the vertical axis we show that $Z_2(t)$ has a decomposition $M(t) + A(t) + \mu_2 t$, where M is a 1-dimensional martingale and A is an increasing process. Then we show that A can increase at t only if $Z_2(t) = 0$, and M is a Brownian motion. These facts make Z_2 a reflected Brownian motion with a drift while Z is in that particular area, hence by the well-known result for a 1-dimensional reflected Brownian motion, the Lebesgue measure of the set of times such process spends on the boundary is zero. Then (13) follows for $i = 1$ by taking limits. Below are the exact steps of this proof.

For each $\varepsilon > 0$, define $S^\varepsilon \subset S$ by $S^\varepsilon = S + (\varepsilon, 0)$, i.e., a wedge with vertex at $(\varepsilon, 0)$ and edges $\partial S_1^\varepsilon = \{(x, 0) \in \mathbb{R}^2, \ x > \varepsilon\}$ and $\partial S_2^\varepsilon = \{(\varepsilon, 0) + \lambda(\cos\xi, \sin\xi), \ \lambda > 0\}$ (recall that ξ is the angle of the wedge S).

Next we shall recursively define the $(\mathcal{F}_t^z)$ stopping times $\bar\sigma_k^{\varepsilon,T}$, $\bar\tau_k^{\varepsilon,T}$ for $k \geq 1$, for every $T > 0$. We define

$$\bar\sigma_1^{\varepsilon,T} = \inf\left\{t \geq 0 : Z(t) \in S^\varepsilon\right\} \wedge T,$$

and

$$\bar\tau_k^{\varepsilon,T} = \inf\left\{t \geq \bar\sigma_k^{\varepsilon,T} : Z(t) \in \partial S_2^{2\varepsilon/3}\right\} \wedge T, \quad k \geq 1,$$

$$\bar\sigma_k^{\varepsilon,T} = \inf\left\{t \geq \bar\tau_{k-1}^{\varepsilon,T} : Z(t) \in \partial S_2^\varepsilon\right\} \wedge T, \quad k \geq 2;$$

Let $Z(t) = (Z_1(t), Z_2(t))$. Let $C > z_2$ be an arbitrary constant. We define the $(\mathcal{F}_t^z)$ stopping time

$$T_C = \inf\left\{t \geq 0 : Z_2(t) \geq C\right\},$$

and in order to simplify the notation, we also introduce the stopping times $\bar\sigma_k = \bar\sigma_k^{\varepsilon,T} \wedge T_C$ and $\bar\tau_k = \bar\tau_k^{\varepsilon,T} \wedge T_C$. Notice that for all $t \leq T$, $t \in [\bar\sigma_k, \bar\tau_k]$ implies that $Z(t) \in S^{2\varepsilon/3}$ and $Z_2(t) \leq C$ (Fig. 2).

Next, as described in the plan above, we would like to study the process Z_2. We shall select a function which maps (z_1, z_2) to z_2 on $S^{2\epsilon/3} \cap \{z_2 \leq C\}$ which is bounded away from the vertex, and make sure that in addition this function satisfies all conditions imposed on f in item 2 of Definition 2.1. Then we use this function to create a submartingale as in item 2 of Definition 2.1.

Let $f_{\varepsilon,C} \in C_b^2(S)$ such that

$$f_{\varepsilon,C}(x, y) = \begin{cases} 0, & \text{if } (x, y) \in S \setminus S^{\varepsilon/3}, \\ y, & \text{if } (x, y) \in S^{2\varepsilon/3}, \ y \leq C. \end{cases} \tag{14}$$

In addition we require that $f_{\varepsilon,C}(x, 0) = 0$ for all $x \geq 0$, and $D_2 f_{\varepsilon,C} \geq 0$ on ∂S_2. It follows from (14) that $D_1 f_{\varepsilon,C} = 0$ on ∂S_1. We show in Lemma 8.1 in the Appendix that such function indeed exists. By the definition of the submartingale problem

$$V_1 = \left\{ V_1(t) = f_{\varepsilon,C}(Z(t)) - \int_0^t \left(\mu \cdot \nabla f_{\varepsilon,C}(Z(s)) - \frac{1}{2}\triangle f_{\varepsilon,C}(Z(s)) \right) ds; \ t \geq 0 \right\} \tag{15}$$

is a regular submartingale under $\mathbb{P}_\mu^z$ on $(\mathcal{F}_t^z)$, thus by Theorem 1.4.14 in [22] it has a unique Doob–Meyer decomposition

$$V_1(t) = M(t) + A(t) \tag{16}$$

where M is a continuous martingale and A is a continuous increasing (not strictly increasing) process. For the definition of regular submartingales see Definition 1.4.12 in [22]. For an arbitrary $k \geq 1$ we have $f_{\varepsilon,C}(Z(t)) = Z_2(t)$, $\mu \cdot \nabla f_{\varepsilon,C}(Z(t)) = \mu_2$, and $\triangle f_{\varepsilon,C}(Z(t)) = 0$ whenever $t \in [\bar{\sigma}_k, \bar{\tau}_k]$, hence by (16) and (15)

$$Z_2(t) = Z_2(\bar{\sigma}_k) + M(t) - M(\bar{\sigma}_k) + A(t) - A(\bar{\sigma}_k) + \mu_2(t - \bar{\sigma}_k) \tag{17}$$

for $t \in [\bar{\sigma}_k, \bar{\tau}_k]$. Next we are going to establish the following two properties. The first is that

$$\int_{\bar{\sigma}_k}^{\bar{\tau}_k} 1_{\{Z_2(t)>0\}} dA(t) = 0, \quad \mathbb{P}_\mu^z-\text{a.s.}, \tag{18}$$

and the second is that

$$\int_{\bar{\sigma}_k}^{\bar{\tau}_k} d\left(\langle M \rangle_t - t\right) = 0, \quad \mathbb{P}_\mu^z-\text{a.s.} \tag{19}$$

We start with proving (18). For any $\delta > 0$ and $k \geq 1$ we define a sequence of $(\mathcal{F}_t^z)$ stopping times

$$\theta_1^\delta = \inf\{t \geq \bar{\sigma}_k : Z_2(t) \geq \delta\} \wedge \bar{\tau}_k, \quad \vartheta_1^\delta = \inf\left\{t \geq \theta_1^\delta : Z_2(t) = \frac{\delta}{2}\right\} \wedge \bar{\tau}_k,$$

$$\theta_n^\delta = \inf\left\{t \geq \vartheta_{n-1}^\delta : Z_2(t) = \delta\right\} \wedge \bar{\tau}_k, \quad n \geq 2,$$

$$\vartheta_n^\delta = \inf\left\{t \geq \theta_n^\delta : Z_2(t) = \frac{\delta}{2}\right\} \wedge \bar{\tau}_k, \quad n \geq 2.$$

Notice that $[\theta_n^\delta, \vartheta_n^\delta]$ is a sub-interval of $[\bar{\sigma}_k, \bar{\tau}_k]$ such that for $t \in [\theta_n^\delta, \vartheta_n^\delta]$ we have $Z_2(t) \geq \delta/2$. All these stopping times are finite because by definition $\bar{\tau}_k \leq T$. Let

$g_1 \in C_b^2(\mathbb{R})$ be an arbitrary function such that $g_1'(0) = 0$, and $g_1(x) = x$ whenever $x \geq \delta/2$. Relation $g_1'(0) = 0$ implies that $\nabla(g_1 \circ f_{\epsilon,C})$ is zero on the boundary of S, henceMeyerdecomposition

$$V_2 = \left\{ V_2(t) = g_1\left(f_{\varepsilon,C}(Z(t))\right) - \int_0^t \left(\mu \cdot \nabla(g_1 \circ f_{\varepsilon,C})(Z(s)) \right. \right.$$
$$\left. \left. -\frac{1}{2}\triangle(g_1 \circ f_{\varepsilon,C})(Z(s)) \right) ds, \ t \geq 0 \right\}$$

is a martingale with respect to the filtration $(\mathcal{F}_t^z)$ under $\mathbb{P}_\mu^z$. Therefore, $\left\{ V_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right), t \geq 0 \right\}$ is also a martingale with respect to the filtration $\left\{ \mathcal{F}_{(t \vee \theta_n^\delta) \wedge \vartheta_n^\delta}^z, \ t \geq 0 \right\}$ under $\mathbb{P}_\mu^z$. For all $s \in [\theta_n^\delta, \tau_n^\delta]$ we have $g_1(f_{\varepsilon,C}(Z(s))) = Z_2(s)$, $\mu \cdot \nabla(g_1 \circ f_{\varepsilon,C})(Z(s)) = \mu_2$ and $\triangle(g_1 \circ f_{\varepsilon,C})(Z(s)) = 0$, hence for all $t \geq 0$

$$V_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) = V_2(\theta_n^\delta) - Z_2(\theta_n^\delta) + Z_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) - \mu_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta - \theta_n^\delta\right)$$

thus $\left\{ Z_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) - \mu_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right), t \geq 0 \right\}$ is also a martingale with respect to the filtration $\left\{ \mathcal{F}_{(t \vee \theta_n^\delta) \wedge \vartheta_n^\delta}^z, \ t \geq 0 \right\}$ under $\mathbb{P}_\mu^z$. On the other hand, from (17) follows that

$$Z_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) - \mu_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right)$$
$$= Z_2(\theta_n^\delta) - \mu_2\theta_n^\delta + M((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta) - M(\theta_n^\delta) + A((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta) - A(\theta_n^\delta),$$

for all $t \geq 0$. However, the left-hand side in the above identity is a martingale with respect to the filtration $\left\{ \mathcal{F}_{(t \vee \theta_n^\delta) \wedge \vartheta_n^\delta}^z, \ t \geq 0 \right\}$ under $\mathbb{P}_\mu^z$, and so is $M((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta)$ on the right-hand side ($t \geq 0$). Therefore, A must be constant on $[\theta_n^\delta, \tau_n^\delta]$, $\mathbb{P}_\mu^z$-a.s. This holds for all $n \geq 1$, hence

$$\int_{\hat{\sigma}_k}^{\hat{\tau}_k} \sum_{n=1}^\infty 1_{[\theta_n^\delta, \vartheta_n^\delta]}(t) dA(t) = 0, \quad \mathbb{P}_\mu^z\text{-a.s.} \tag{20}$$

If $t \in [\bar{\sigma}_k, \bar{\tau}_k]$ and $Z_2(t) > \delta$, then $Z_2(t) \in [\theta_n^\delta, \vartheta_n^\delta]$ for some $n \geq 1$, hence by (20)

$$\int_{\hat{\sigma}_k}^{\hat{\tau}_k} 1_{\{Z_2(t) > \delta\}}(t) dA(t) = 0, \quad \mathbb{P}_\mu^z\text{-a.s.,}$$

and (18) follows.

Next we are going to show (19). Let $g_2 \in C_b^2(\mathbb{R})$ arbitrary such that $g_2(x) = x^2$ whenever $|x| \leq C$. Then

$$V_3 = \left\{ V_3(t) = g_2\left(f_{\varepsilon,C}(Z(t))\right) - \int_0^t \left(\mu \cdot \nabla(g_2 \circ f_{\varepsilon,C})(Z(s)) \right. \right.$$

$$-\frac{1}{2}\triangle(g_2 \circ f_{\varepsilon,C})(Z(s))\Big) ds, \ t \geq 0\Big\} \tag{21}$$

is a martingale under $\mathbb{P}_\mu^z$ with respect to the filtration $(\mathcal{F}_t^z)$, and for $t \in [\bar\sigma_k, \bar\tau_k]$ we have $g_2\left(f_{\varepsilon,C}(Z(t))\right) = (Z_2(t))^2$, $\mu \cdot \nabla(g_2 \circ f_{\varepsilon,C})(Z(t)) = 2\mu_2 Z_2(t)$ and $\triangle(g_2 \circ f_{\varepsilon,C})(Z(t)) = 2$, hence by Ito's rule applied to $g_2(f_{\varepsilon,C}(Z(t)))$ and by (17)

$$Z_2^2(t) = Z_2^2(\bar\sigma_k) + \int_{\bar\sigma_k}^t 2Z_2(s)dM(s) + \int_{\bar\sigma_k}^t 2\mu_2 Z_2(s)ds + \langle M\rangle_t - \langle M\rangle_{\bar\sigma_k},$$

for $t \in [\bar\sigma_k, \bar\tau_k]$. We note that the $\int_{\bar\sigma_k}^t 2Z_2(s)dA(s)$ term vanished because of (18). From this and from (21) follows that

$$V_3(t) = V_3(\bar\sigma_k) + \int_{\bar\sigma_k}^t 2Z_2(s)dM(s) + \int_{\bar\sigma_k}^t d\left(\langle M\rangle_s - s\right),$$

for $t \in [\bar\sigma_k, \bar\tau_k]$. The process $\{V_3\left((t \vee \bar\sigma_k) \wedge \bar\tau_k\right), \ t \geq 0\}$ is a martingale with respect to the filtration $\left\{\mathcal{F}_{(t\vee\theta_n^\delta)\wedge\vartheta_n^\delta}, \ t \geq 0\right\}$ under $\mathbb{P}_\mu^z$, and can be written by substituting $(t \vee \bar\sigma_k) \wedge \bar\tau_k$ for t in the above identity as

$$V_3(t \vee \bar\sigma_k) \wedge \bar\tau_k) = V_3(\bar\sigma_k) + \int_{\bar\sigma_k}^{t\vee\bar\sigma_k)\wedge\bar\tau_k} 2Z_2(s)dM(s) + \int_{\bar\sigma_k}^{t\vee\bar\sigma_k)\wedge\bar\tau_k} d\left(\langle M\rangle_s - s\right),$$

for all $t \geq 0$. Since the left-hand side is a martingale with respect to the filtration $\left\{\mathcal{F}_{(t\vee\theta_n^\delta)\wedge\vartheta_n^\delta}^z, \ t \geq 0\right\}$ under $\mathbb{P}_\mu^z$, (19) follows. Then by (17), Z_2 is a 1-dimensional Brownian motion with drift μ_2 reflected at zero in $[\bar\sigma_k, \bar\tau_k]$. Therefore,

$$\int_{\bar\sigma_k}^{\bar\tau_k} 1_{\{Z_2(t)=0\}}dt = 0, \quad \mathbb{P}_\mu^z\text{-a.s.}$$

This holds for every $k \geq 1$, hence we also have

$$\sum_{k=1}^\infty \int_{\bar\sigma_k}^{\bar\tau_k} 1_{\{Z_2(t)=0\}}dt = 0, \quad \mathbb{P}_\mu^z\text{-a.s.},$$

and from this

$$\int_0^{T\wedge T_C} 1_{\{Z_1(t)\geq\varepsilon, Z_2(t)=0\}}dt = 0, \quad \mathbb{P}_\mu^z\text{-a.s.}$$

The last identity follows because $t \leq T \wedge T_c$, $Z_1(t) \geq \varepsilon$ and $Z_2(t) = 0$ implies that $t \in [\bar\sigma_k, \bar\tau_k]$ for some $k \geq 1$. The statement of the Lemma now follows by $T, C \uparrow \infty$ and $\varepsilon \downarrow 0$. $\qquad\qquad\square$

Let $\{\mathbb{P}_\mu^z, z \in S\}$ be a solution to the submartingale problem with a drift μ. Next we shall create a process X which is a Brownian motion with drift μ starting at z on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P}_\mu^z)$, for every $z \in S$. We already know that such process exists for the solution that we created in Proposition 3.2. However, for proving the uniqueness of the solution, we need to show the existence of such process X for every solution of the submartingale problem. Such construction has been carried out in [23] and in [21] for the case of zero drift. The generalization to the case of non-zero drift requires only a few obvious changes to the proofs in the case of zero drift, so here we shall only state the results (Lemmas 3.4 and 3.5) without proofs. The corresponding results for the zero drift case are available in [23], Lemma 4.5 and Proposition 4.7.

For each $\delta > 0$, let $S_\delta \subset S$ be the closed set defined in the complex plane by $S_\delta = S + \delta e^{i\xi/2}$. So S_δ is a wedge with vertex at $\delta \left(\cos\left(\xi/2\right), \sin\left(\xi/2\right)\right)$, such that it is included in S and has edges parallel with the respective edges of S.

Set $\tau_0^\delta = 0$, and, for each $k \geq 1$, recursively define

$$\sigma_k^\delta = \inf\{t \geq \tau_{k-1}^\delta : Z(t) \in S_{2\delta}\} \text{ and } \tau_k^\delta = \inf\{t \geq \sigma_k^\delta : Z(t) \in S \backslash S_\delta\}.$$

By Problem 1.2.7 in Karatzas and Shreve [22], σ_k^δ and τ_k^δ are stopping times relative to $\{\mathcal{M}_t, t \geq 0\}$ for every $k \geq 1$.

For each $k \geq 1$ and $\delta > 0$, define the process $\{W_{(k)}^\delta(t), t \geq 0\}$ by setting

$$W_{(k)}^\delta(t) = Z(t \wedge \tau_k^\delta) - Z(t \wedge \sigma_k^\delta) - (t \wedge \tau_k^\delta - t \wedge \sigma_k^\delta)\mu, \quad t \geq 0,$$

and then define the process $\{W^\delta(t), t \geq 0\}$ by setting

$$W^\delta(t) = \sum_{k=1}^{\infty} W_{(k)}^\delta(t), \quad t \geq 0.$$

Lemma 3.4 *For every $\delta > 0$ and $z \in S$ the process W^δ is a square-integrable martingale on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P}_\mu^z)$.*

Lemma 3.5 *There exists a process W on $(C_S, \mathcal{M}, \mathcal{M}_t)$ such that for every $z \in S$ it is a standard 2-dimensional Brownian motion under $\mathbb{P}_\mu^z$ starting at zero, and for every fixed $T > 0$ we have*

$$\mathbb{E}_\mu^z\left[|W(T) - W^\delta(T)|^2\right] \to 0, \quad \text{as } \delta \to 0. \tag{22}$$

Next we shall define the process X by

$$X(t, \omega) = \omega(0) + W(t, \omega) + \mu t, \quad t \geq 0. \tag{23}$$

We define the process Y by

$$Y(t) = Z(t) - X(t), \quad t \geq 0. \tag{24}$$

We shall say that a function $f : \mathbb{R}_+ \mapsto \mathbb{R}^2$ is *flat* on an interval $[s, t] \subset \mathbb{R}_+$, if for every $u \in [s, t]$ we have $f(u) = f(s)$.

Lemma 3.6 *Let $\{\mathbb{P}_\mu^z; z \in S\}$ be an arbitrary solution of the submartingale problem, and let X and Y be the processes defined above. Then the following two statements hold for every $z \in S$:*

1. *Under $\mathbb{P}_\mu^z$ the process X is a standard 2-dimensional Brownian motion on $(C_S, \mathcal{M}, \mathcal{M}_t)$ with drift μ, starting at z;*
2. *for every $n \in \mathbb{N}_+$, and $\delta > 0$, the sample paths of Y are flat on $[\sigma_n^\delta, \tau_n^\delta]$, $\mathbb{P}_\mu^z$-a.s.*

Proof The first statement follows from Lemma 3.5 and property 1 in the definition of the submartingale problem. Next we shall prove the second statement. In this proof we shall first study the properties of the process $\{Z(t) - W^\delta(t) - \mu t, t \geq 0\}$, then take limits to conclude. By the definition of W^δ, the sample paths of

$$\{Z(t) - W^\delta(t) - \mu t, \ t \geq 0\} \text{ are flat on } [\sigma_n^\delta, \tau_n^\delta], \tag{25}$$

for each $\delta > 0$, $n \geq 1$. On the other hand, for every $\delta > 0$, $n \geq 1$ there exists $k \geq 1$ (depending on the sample path) such that $[\sigma_n^\delta, \tau_n^\delta] \subset [\sigma_k^{\delta/2}, \tau_k^{\delta/2}]$. This implies that the sample paths of $\{Z(t) - W^{\delta/2}(t) - \mu t, \ t \geq 0\}$ are also flat on $[\sigma_n^\delta, \tau_n^\delta]$. Iterating this we get that for every $m \geq 1$ the sample paths of $\{Z(t) - W^{\delta/2^m}(t) - \mu t, \ t \geq 0\}$ are also flat on $[\sigma_n^\delta, \tau_n^\delta]$. Comparing this with (25) we conclude that the sample paths of $W^\delta - W^{\delta/2^m}$ are also flat on $[\sigma_n^\delta, \tau_n^\delta]$, thus for every $t \geq 0$

$$\int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dW^{\delta/2^m}(s) = \int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dW^\delta(s).$$

Taking limit as $m \to \infty$ and using (22) we get that

$$\int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dW(s) = \int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dW^\delta(s),$$

$\mathbb{P}_\mu^z$-a.s. This identity and (25) imply that the sample paths of $\{Z(t) - W(t) - \mu t, \ t \geq 0\}$ are flat on $[\sigma_n^\delta, \tau_n^\delta] \cap [0, t]$, $\mathbb{P}_\mu^z$-a.s. Since $t \geq 0$ was arbitrary, this and (23) imply what we wanted to prove. □

In our proof for the uniqueness of the submartingale problem with drift we want to use the known result that such uniqueness holds in the case of $\mu = 0$. The following lemma will be essential in carrying out this plan. It states that two solutions for the submartingale problem with drift yield two solutions for the submartingale problem with no drift using Girsanov-type changes of measures, and in addition, the Radon–Nicodym derivatives associated with these changes of measures are the same.

Lemma 3.7 *Suppose that Q_1 and Q_2 are mutually absolutely continuous probability measures on $\mathcal{M}$, both satisfying properties 1, 2, and 3 of Definition 2.1 with $\mathbb{P}_\mu^z$ replaced by Q_i ($i = 1, 2$). Then there exist probability measures $\tilde{Q}_i$ on $\mathcal{M}$ for $i = 1, 2$,*

such that conditions 1, 2, and 3 of Definition 2.1 are satisfied with P_μ^z replaced by $\tilde{Q}_i$ and μ replaced by the zero vector. Furthermore, for every $T \geq 0$ there exist probability measures $\tilde{Q}_1^T$ and $\tilde{Q}_2^T$ on $\mathcal{M}$ such that for all $T \geq 0$, $A \in \mathcal{M}_T$, and $i = 1, 2$ we have $\tilde{Q}_i^T(A) = \tilde{Q}_i(A)$, $\tilde{Q}_i^T$ and Q_i are mutually absolutely continuous, and

$$\frac{d\tilde{Q}_1^T}{dQ_1} = \frac{d\tilde{Q}_2^T}{dQ_2}. \tag{26}$$

Proof Let Q_1 and Q_2 as above, and let (X^i, Y^i) be as in Lemma 3.6 defined under Q_i. Since X^i is defined by $L^2(Q_i)$ convergence, this implies that $(X^1, Y^1) = (X^2, Y^2)$, which we shall from here on denote by (X, Y). Our main goal in this proof is to write the process $f(Z(t)) - \int_0^t \nabla f(Z(s)) \cdot \mu ds - 1/2 \int_0^t \Delta f(Z(s))ds$ as a sum of a martingale N_i and an increasing process A_i under the measure Q_i for $i = 1, 2$ in such a way that after a Girsanov type change of measure that eliminates the second term, N_i still remains a martingale. The proof of this is quite technical, and the exact steps follow. We shall use the Doob–Meyer decomposition which requires that the probability space satisfies the "usual conditions", and for this purpose we have to augment the probability space $(C_S, \mathcal{M}, (\mathcal{M}_t), Q_i)$, $i = 1, 2$; let this augmentation be $(C_S, \mathcal{F}, (\mathcal{F}_t), Q_i)$. The measures Q_1 and Q_2 are mutually absolutely continuous, hence the filtration $(\mathcal{F}_t)$ and the sigma field $\mathcal{F}$ do not depend on $i = 1, 2$. For technical details concerning the augmentation of a probability space please see Remark 8.2 in the Appendix. In this proof all processes live on the augmented space $(C_S, \mathcal{F}, (\mathcal{F}_t))$, unless specified otherwise.

Next, note that for each $T > 0$ and $\delta > 0$ and $n \geq 1$,

$$\left\{ Z(\tau_n^\delta \wedge t) - Z(\sigma_n^\delta \wedge t), \ t \in [0, T] \right\}$$

is a semimartingale under both Q_1 and Q_2 with respect to the filtration $(\mathcal{F}_t)$. Indeed, it can be written by Lemma 3.6 and by (24) as

$$Z(\tau_n^\delta \wedge t) - Z(\sigma_n^\delta \wedge t) = X(\tau_n^\delta \wedge t) - X(\sigma_n^\delta \wedge t), \quad t \in [0, T].$$

Now let $f \in C_b^2(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$, and f is constant in a neighborhood of the origin. Then, by Itô's rule we have that for $t \in [0, T]$,

$$f(Z(t \wedge \tau_n^\delta)) = f(Z(t \wedge \sigma_n^\delta)) + \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \nabla f(Z(s))dX(s) + \frac{1}{2} \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \Delta f(Z(s))ds, \tag{27}$$

Q_i-a.s., $i = 1, 2$. On the other hand, by condition 2 of Definition 2.1 and by Theorems I.4.10 and I.4.14 in [22], we have for $i = 1, 2$, the unique Doob–Meyer decomposition

$$f(Z(t)) = f(z) + \int_0^t \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2} \int_0^t \Delta f(Z(s))ds + M^i(t) + A^i(t), \ t \leq T, \tag{28}$$

where M^i is a continuous martingale and A^i is a continuous, increasing process on $(C_S, \mathcal{F}, (\mathcal{F}_t), Q_i)$, with $M^i(0) = A^i(0) = 0$. By Proposition 16.32 in Bass [1], we also have that for $T \geq 0$,

$$\mathbb{E}^{Q_i}[||M^i(T)||^2] < \infty, \ i = 1, 2.$$

Let W as in (23), that is, $W(t) = X(t) - z - \mu t$, and for $i = 1, 2$, let $S^i(W)$ be the class of $\mathbb{R}^2$-valued processes on $(C_S, \mathcal{F}, (\mathcal{F}_t))$ such that $U \in S^i(W)$ if it has the form

$$U(t) = \int_0^t G(s) dW(s), \ t \in [0, T],$$

for some 2-dimensional process G such that

$$\mathbb{E}^{Q_i} \left[\int_0^T ||G(s)||^2 ds \right] < \infty.$$

Then, by Theorem IV.36 and Corollary 1 to Theorem IV.37 in [27], there exists a $\mathbb{R}^2$-valued process H^i such that

$$M^i(t) = \int_0^t H^i(s) dW(s) + N^i(t), \ t \in [0, T],$$

where

 (i) N^i is a square-integrable martingale under Q_i,
 (ii) N^i is strongly orthogonal to every member of $S^i(W)$ under Q_i, that is, $N^i U$ is a Q_i-martingale for each $U \in S^i(W)$,
 (iii) $\mathbb{E}^{Q_i} \left[\int_0^T ||H^i(s)||^2 ds \right] < \infty.$

Now, by (28) we have for $t \in [0, T]$,

$$f(Z(t)) = f(z) + \int_0^t \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2} \int_0^t \Delta f(Z(s)) ds$$
$$+ \int_0^t H^i(s) dW(s) + N^i(t) + A^i(t), \ Q_i - \text{a.s.} \qquad (29)$$

The main objective of the rest of the proof is to show that in the above equation $H^i(s)$ can be replaced by $\nabla f(Z(s))$. First we shall show that N_i is flat on $[\sigma_n^\delta, \tau_n^\delta]$, and $H^i(s) = \nabla f(Z(s))$ for $s \in [\sigma_n^\delta, \tau_n^\delta]$. By (29) we have for $t \in [0, T]$,

$$f(Z(t \wedge \tau_n^\delta)) = f(Z(t \wedge \sigma_n^\delta)) + \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2} \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \Delta f(Z(s)) ds$$

$$+ \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} H^i(s) dW(s) + N^i(t \wedge \tau_n^\delta) - N^i(t \wedge \sigma_n^\delta) + A^i(t \wedge \tau_n^\delta) - A^i(t \wedge \sigma_n^\delta),$$

$$(30)$$

Q_i-a.s. Now, for each $i = 1, 2$, we have two Doob–Meyer decompositions of the submartingale

$$\left\{ f(Z(t \wedge \tau_n^\delta)) - f(Z(t \wedge \sigma_n^\delta)) - \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \nabla f(Z(s)) \cdot \mu ds \right.$$
$$\left. - \frac{1}{2} \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \Delta f(Z(s)) ds, \ t \in [0, T] \right\},$$

(27) and (30). Hence, by the uniqueness of the Doob–Meyer decomposition, for each $t \in [0, T]$

$$N^i(t \wedge \tau_n^\delta) - N^i(t \wedge \sigma_n^\delta) + \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} H^i(s) dW(s) = \int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} \nabla f(Z(s)) dW(s),$$

$$(31)$$

Q_i- a.s. N_i is strongly orthogonal to every member of $S^i(W)$, and from [27], Theorem IV.37 follows that $\{N^i(t \wedge \tau_n^\delta) - N^i(t \wedge \sigma_n^\delta), \ t \in [0, T]\}$ is also strongly orthogonal to every member of $S^i(W)$ under Q_i. However, by the above relation it is also a member of $S^i(W)$, hence it follows that $N^i(t \wedge \tau_n^\delta) - N^i(t \wedge \sigma_n^\delta) = 0$ for $t \in [0, T]$. Then by (31) we also have

$$\mathbb{E}^{Q_i} \left[\int_{t \wedge \sigma_n^\delta}^{t \wedge \tau_n^\delta} ||H^i(s) - \nabla f(Z(s))||^2 ds \right] = 0.$$

Next we show that $H^i(s) = \nabla f(Z(s))$ for all $s \leq T$. Indeed,

$$\mathbb{E}^{Q_i} \left[\int_0^t ||H^i(s) - \nabla f(Z(s))||^2 ds \right]$$
$$= \mathbb{E}^{Q_i} \left[\int_0^t \sum_{n=2}^\infty \mathbb{1}_{\left\{ s \in [\tau_{n-1}^\delta, \sigma_n^\delta] \right\}} ||H^i(s) - \nabla f(Z(s))||^2 ds \right]$$
$$\leq \mathbb{E}^{Q_i} \left[\int_0^t \mathbb{1}_{\left\{ Z(s) \in S_{2\delta}^c \right\}} ||H^i(s) - \nabla f(Z(s))||^2 ds \right], \qquad (32)$$

where $S_{2\delta}^c = S \backslash S_{2\delta}$. Moreover, by the dominated convergence theorem,

$$(32) \to \mathbb{E}^{Q_i} \left[\int_0^t \mathbb{1}_{\left\{ Z(s) \in \partial S \right\}} ||H^i(s) - \nabla f(Z(s))||^2 ds \right] = 0 \text{ as } \delta \to 0,$$

where the last identity is by Lemma 3.3. By (29), now follows that for $t \in [0, T]$,

$$f(Z(t)) = f(z) + \int_0^t \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2} \int_0^t \Delta f(Z(s)) ds$$
$$+ \int_0^t \nabla f(Z(s)) dW(s) + N^i(t) + A^i(t), \quad Q_i-\text{a.s.} \qquad (33)$$

Next, for each $t \geq 0$ let

$$\tilde{\zeta}(t) = \exp\left\{ -\mu \cdot (X(t) - z) + \frac{1}{2}||\mu||^2 t \right\},$$

and for each $i = 1, 2$, and $T \geq 0$ define the measure $\tilde{Q}_i^T$ by setting

$$\frac{d\tilde{Q}_i^T}{dQ_i} = \tilde{\zeta}(T). \qquad (34)$$

Then, under $\tilde{Q}_i^T$, $\{X(t), t \in [0, T]\}$ is a Brownian motion (without drift) starting at z, and by (33) we have

$$f(Z(t)) = f(z) + \int_0^t \nabla f(Z(s)) dX(s)$$
$$+ \frac{1}{2} \int_0^t \Delta f(Z(s)) ds + N^i(t) + A^i(t), \quad Q_i-\text{a.s.} \qquad (35)$$

However, note that N^i is also a $\tilde{Q}_i^T$-martingale on $[0, T]$. Indeed, by (34), N^i is a $\tilde{Q}_i^T$-martingale if $N^i\tilde{\zeta}$ is a Q_i-martingale on $[0, T]$. But this follows from the fact that N_i is strongly orthogonal to every member of $S^i(W)$ under Q_i, and by its definition $\tilde{\zeta} - 1 \in S^i(W)$ under Q_i. Just like in the proof of Proposition 3.2, there exists a probability measure $\tilde{Q}_i$ on $\mathcal{M}$ such that $\tilde{Q}_i(A) = \tilde{Q}_i^T(A)$, whenever $A \in \mathcal{M}_T$. Thus, by (35) both $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy property 2 of Definition 2.1 with $\mathbb{P}_\mu^z$ replaced by either $\tilde{Q}_1$ or $\tilde{Q}_2$, and μ replaced by the zero vector. $\tilde{Q}_i^T$ and Q_i are mutually absolutely continuous because of (34) and because $\tilde{\zeta}(T) > 0$, a.s. under Q_i. Relation (26) also follows from (34). Properties 1 and 3 of Definition 2.1 are satisfied if $\mathbb{P}_\mu^z$ is replaced by $\tilde{Q}_i$ because they are satisfied if we replace $\mathbb{P}_\mu^z$ by Q_i, and we already established that $\tilde{Q}_i^T$ and Q_i are mutually absolutely continuous. Property 1 follows immediately from this. Property 3 can be shown by first showing it with the ∞ in the upper limit of the integral replaced by T, then taking $T \to \infty$. $\qquad \square$

Proof of Theorem 2.3 In light of Propositions 3.1 and 3.2 the only missing part is the proof of uniqueness. Let $z \in S$ and suppose that $\mathbb{P}_1^z$ and $\mathbb{P}_2^z$ are two probability measures satisfying conditions 1, 2 and 3 in Definition 2.1 of the submartingale problem with

drift. Let

$$Q_1 = \frac{1}{3}\mathbb{P}_1^z + \frac{2}{3}\mathbb{P}_2^z \quad \text{and} \quad Q_2 = \frac{2}{3}\mathbb{P}_1^z + \frac{1}{3}\mathbb{P}_2^z.$$

Then, one can check that each Q_i also satisfies conditions 1, 2, and 3 in Definition 2.1 of the submartingale problem with drift. In addition, Q_1 and Q_2 are mutually absolutely continuous. In order to complete the proof, it is therefore sufficient to show that $Q_1 \equiv Q_2$. By Lemma 3.7 there exist probability measures $\tilde{Q}_i$, $i = 1, 2$, such that properties 1, 2, and 3 of Definition 2.1 are satisfied with P_μ^z replaced by $\tilde{Q}_i$, and μ replaced by the zero vector. The uniqueness result in Sect. 3.1 of [35] implies $\tilde{Q}_1 = \tilde{Q}_2$. Using the probability measures $\tilde{Q}_i^T$ from the same proposition, we have now that

$$\tilde{Q}_1^T|_{\mathcal{M}_T} = \tilde{Q}_2^T|_{\mathcal{M}_T}.$$

From (26) follows that

$$Q_1|_{\mathcal{M}_T} = Q_2|_{\mathcal{M}_T}.$$

Since T was arbitrary, $Q_1 = Q_2$ follows. □

Next we prove Theorems 2.5, 2.6, and 2.7. These results essentially follow from the facts that we created our solution to the submartingale problem using Girsanov's theorem, and the corresponding results hold for the drift-less case.

Proof of Theorem 2.5 Recall from the proof of Proposition 3.2 that for every $T \geq 0$ there exists the probability measure $\mathbb{P}_{\mu,T}^z$ which is mutually absolutely continuous with respect to $\mathbb{P}_0^z$, and coincides with $\mathbb{P}_\mu^z$ on $\mathcal{M}_T$. By Theorem 2.6 in [23], formulas (5) and (6) hold for $\mu = 0$. Then by the mutual absolute continuity of $\mathbb{P}_0^z$ and $\mathbb{P}_{\mu,T}^z$, (5) and (6) also hold with $\mathbb{P}_\mu^z$ replaced by $\mathbb{P}_{\mu,T}^z$. Since $\mathbb{P}_\mu^z$ and $\mathbb{P}_{\mu,T}^z$ coincide on $\mathcal{M}_T$, both formulas follow. □

Proof of Theorem 2.6 This follows from Theorem 2.4 in [23] using the measure $\mathbb{P}_{\mu,T}^z$ for every $T > 0$, just like in the proof of Theorem 2.5. □

Proof of Theorem 2.7 Suppose that $1 \leq \alpha < 2$, and Z is a continuous semimartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^z)$ for some $z \in S$. Then there exists a decomposition

$$Z(t) = z + M(t) + A(t), \quad t \in [0, \infty), \tag{36}$$

where M is a continuous local martingale and A is a finite variation (FV) process on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^z)$ (see [27], Corollary to Theorem II.31). We know from the proof of Proposition 3.2 that for every $T \in [0, \infty)$ there exists a probability measure $\mathbb{P}_{\mu,T}^z$ on $\mathcal{M}$ which is mutually absolutely continuous with respect to $\mathbb{P}_0^z$, and $\mathbb{P}_{\mu,T}^z(A) = \mathbb{P}_\mu^z(A)$ for all $A \in \mathcal{M}_T$. In addition,

$$\frac{d\mathbb{P}_0^z}{d\mathbb{P}_{\mu,T}^z} = \frac{1}{\zeta(T)}$$

where ζ is defined under (11). We cast (36) in the form

$$Z(t) = z + \tilde{M}(t) + \tilde{A}(t), \quad t \in [0, \infty), \tag{37}$$

where

$$\tilde{M}(t) = M(t) - \int_0^t \zeta(s) d\left[\frac{1}{\zeta}, M\right]_s,$$

$$\tilde{A}(t) = A(t) + \int_0^t \zeta(s) d\left[\frac{1}{\zeta}, M\right]_s.$$

For the definition of the square bracket process we refer to [27], Chapter II, Sect. 6. By the Girsanov–Meyer theorem ([27], Theorem III.31) $\tilde{M}$ is a local martingale on $[0, T]$ and $\tilde{A}$ is a FV process under $\mathbb{P}_0^z$. But this implies that $\tilde{M}$ is a local martingale on $[0, \infty)$ under $\mathbb{P}_0^z$, hence Z must be a semimartingale under $\mathbb{P}_0^z$, which is in contradiction with the result of [39], Theorem 5.   □

The proof of Theorem 2.8 follows from Lemma 3.7, because based on a solution to the submartingale problem with an arbitrary μ, that lemma would guarantee a solution for the drift-less case, but it is known that such solution does not exist for $\alpha \geq 2$.

Proof of Theorem 2.8 Suppose that $\alpha \geq 2$, let $z \in S$ and suppose that $\mathbb{P}_\mu^z$ is a probability measure on $\mathcal{M}$ satisfying properties 1, 2 and 3 in Definition 2.1 of the submartingale problem with drift. Selecting $Q_1 = Q_2 = \mathbb{P}_\mu^z$ in Lemma 3.7, it follows that there exists a probability measure $\tilde{Q}$ on $\mathcal{M}$ such that properties 1, 2, and 3 of Definition 2.1 are satisfied with $\mathbb{P}_\mu^z$ replaced by $\tilde{Q}$, and μ replaced by the zero vector. However, this is in direct contradiction with Theorem 3.11 in [35].   □

4 Proof of Theorems 2.11 and 2.9

First we shall prove Theorem 2.11. A simple application to Girsanov's theorem does not work for the following reason. Suppose that $(z_n) \subset S$ is a sequence of points such that $z_n \to z$ as $n \to \infty$. In order to prove the Feller property we need to show that $\mathbb{E}_\mu^{z_n}[F] \to \mathbb{E}_\mu^z[F]$ as $n \to \infty$, for every bounded, continuous functional F on C_S. We know that this result holds for the case of $\mu = 0$. However, what we need to show, written under the P_0^z measure is that $\mathbb{E}_0^{z_n}[F\zeta(T)] \to \mathbb{E}_0^z[F\zeta(T)]$ as $n \to \infty$, assuming that $F \in \mathcal{M}_T$ (the Radon–Nicodym derivative ζ is given below (11)). But in order to conclude this last convergence, $\zeta(T)$ needs to be a continuous functional on C_S. The problem is that $\zeta(T)$ is defined in terms of the process X, and we don't know that X depends on Z continuously. Hence instead of using the Girsanov transformation, we shall show that the family $\{P_\mu^z, z \in S\}$ satisfies the Feller property in a different way. The technique of this proof is quite standard. First we show that the family of probability measures $\{\mathbb{P}_\mu^{z_n}\}$ is tight whenever $z_n \to z$, then we show that any limit point satisfies requirements 1–3 of Definition 2.1. Then the Feller property follows from our uniqueness result.

Proposition 4.1 *The family of probability measures* $\{\mathbb{P}_\mu^{z_n}\}$ *is tight for any sequence* $\{z_n, n \geq 1\}$ *in S which converges to some $z \in S$.*

Proof By Theorem 2.4.10 in [22], it is sufficient to show that

$$\lim_{\delta \downarrow 0} \sup_n \mathbb{P}_\mu^{z_n}(\omega : m^T(\omega, \delta) \geq \varepsilon) = 0, \text{ for any } T > 0, \varepsilon > 0. \tag{38}$$

In the above,

$$m^T(\omega, \delta) = \sup_{\substack{|t-s| \leq \delta \\ 0 \leq s, t \leq T}} |\omega(s) - \omega(t)|.$$

Using (11) and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\mathbb{P}_\mu^{z_n}(\omega : m^T(\omega, \delta) \geq \varepsilon) \\
&= \mathbb{E}_0^{z_n}\left[\mathbb{1}_{\{m^T(\omega,\delta) > \varepsilon\}} \exp\left\{\mu \cdot (X(T) - X(0)) - \frac{1}{2}||\mu||^2 T\right\}\right] \\
&\leq \left(\mathbb{E}_0^{z_n}\left[\exp\left\{2\mu \cdot (X(T) - X(0)) - ||\mu||^2 T\right\}\right] \mathbb{P}_0^{z_n}(m^T(\omega, \delta) > \varepsilon)\right)^{1/2} \\
&= \exp\left\{\frac{1}{2}||\mu||^2 T\right\} \left(\mathbb{P}_0^{z_n}(m^T(\omega, \delta) > \varepsilon)\right)^{1/2}. \tag{39}
\end{aligned}
$$

By Theorem 3.13 in [35], $\{\mathbb{P}_0^{z_n}\}$ is tight hence

$$\lim_{\delta \downarrow 0} \sup_n \mathbb{P}_0^{z_n}(\omega : m^T(\omega, \delta) \geq \varepsilon) = 0, \text{ for any } T > 0, \varepsilon > 0,$$

combining with inequality (39), we have (38). □

Proof of Theorem 2.11 Given Proposition 4.1, it only remains to show that any weak limit point $\mathbb{P}_\mu^*$ of the family $\{\mathbb{P}_\mu^{z_n}\}$ is a solution to the submartingale problem starting from z, then by the uniqueness part of Theorem 2.3, $\mathbb{P}_\mu^{z_n} \Rightarrow \mathbb{P}_\mu^z$ as $n \to \infty$.

It is straightforward that $\mathbb{P}_\mu^*$ satisfies condition 1 in Definition 2.1 (the submartingale problem), since for any $k \geq 1$ and the closed set $C_k = \{\omega \in C_S : |\omega(0) - z| \leq \frac{1}{k}\}$, $1 = \lim\sup_n \mathbb{P}_\mu^{z_n}(C_k) \leq \mathbb{P}_\mu^*(C_k)$ hence $\mathbb{P}_\mu^*$ concentrates on $\{\omega \in C_S : \omega(0) = z\}$. The condition 2 is also satisfied, since the submartingale property is preserved under the weak convergence. Now we prove $\mathbb{P}_\mu^*$ satisfies condition 3, we need to show that if $\{z_n, n \geq 1\} \subset S, z \in S$ such that $\lim_{n \to \infty} z_n = z$ and $\mathbb{P}_\mu^{z_n} \Rightarrow \mathbb{P}_\mu^*$, then

$$\mathbb{E}_\mu^*\left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}}dt\right] = 0. \tag{40}$$

Let $\epsilon > 0$ and $t \geq 0$ be arbitrary. Under the local uniform topology the event $\{w \in C_S : |Z(t, \omega)| < \epsilon\}$ is an open set, and $\{w \in C_S : |Z(t, \omega)| \leq \epsilon\}$ is a closed set. By

the Portmanteau Theorem, (11), and the Cauchy–Schwarz inequality

$$
\begin{aligned}
\mathbb{P}_\mu^* \left(|Z(t)| < \epsilon \right) &\leq \liminf_n \mathbb{P}_\mu^{z_n} \left(|Z(t)| < \epsilon \right) = \liminf_n \mathbb{E}_0^{z_n} \left[\zeta(t) 1_{\{|Z(t)| < \epsilon\}} \right] \\
&\leq \liminf_n \left(\mathbb{E}_0^{z_n} \left[(\zeta(t))^2 \right] \mathbb{P}_0^{z_n} \left(|Z(t)| < \epsilon \right) \right)^{1/2} \\
&\leq \exp\left\{ \frac{1}{2} |\mu|^2 t \right\} \limsup_n \left(\mathbb{P}_0^{z_n} \left(|Z(t)| \leq \epsilon \right) \right)^{1/2} \\
&\leq \exp\left\{ \frac{1}{2} |\mu|^2 t \right\} \left(\mathbb{P}_0^z \left(|Z(t)| \leq \epsilon \right) \right)^{1/2}.
\end{aligned}
$$

The process ζ is the Radon-Nykodim derivative defined below (11). The last inequality follows from the Feller property for the drift-less case ([35], Theorem 3.13). We now let $\epsilon \downarrow 0$, and then from property 3 in the definition of the submartingale problem follows that

$$
\mathbb{P}_\mu^* \left(Z(t) = 0 \right) = 0,
$$

for Lebesgue-almost every $t \geq 0$. Identity (40) follows.   □

In the following proof we could have possibly used Girsanov's theorem in the following way. In order to show the strong Markov property, we essentially need to show that $\mathbb{P}_\mu^z[Z(T + t) \in \Gamma | \mathcal{M}_T] = \mathbb{P}_\mu^{Z(T)}(Z(t) \in \Gamma)$, for every Borel-subset Γ of S and every stopping time T. This can be written under the $\mathbb{P}_0^z$ measure as $\mathbb{E}_0^z[\zeta(T + t)/\zeta(T) 1_{Z(T+t) \in \Gamma} | \mathcal{M}_T] = \mathbb{E}_0^{Z(T)}[\zeta(t) 1_{Z(t) \in \Gamma}]$. In order to show that this is true, we would need to prove that $\zeta(T + t)/\zeta(T)$ depends only on $Z(T + \cdot)$, and it depends on $Z(T + \cdot)$ the same way as $\zeta(t)$ depends on $Z(\cdot)$. This is intuitively true, and indeed would be possible to show, but instead we proceed in a simpler fashion. The proof presented below is essentially very similar to the corresponding proofs in [35].

Proof of Theorem 2.9 This follows in the standard way from the uniqueness of the solution to the submartingale problem, using the regular conditional probability measures for $\mathcal{M}$ given $\mathcal{M}_\tau$ under $\mathbb{P}_\mu^z$. In particular, Lemma 3.1, and Corollary 3.3 in [35] remain true in the presence of a drift, without changing a single word in their proofs. Then the strong Markov property follows, again exactly the same way as in [35], Theorem 3.14.   □

5 Proof of Theorem 2.12

In this proof we first show that it is sufficient to prove the $\hat{C}(S)$-Feller property for the drift-less case, using the fact that our solution for the submartingale problem with drift has been derived from the solution for the drift-less case using Girsanov's theorem. The proof for the drift-less case boils down to estimating the probability that $|Z(t)| \leq n$ under P_μ^z.

Proof of Theorem 2.12 It is sufficient to show that the $\hat{C}(S)$-Feller property holds in the case when the drift is zero. Indeed, suppose that for the drift-less case the $\hat{C}(S)$-Feller property holds. By Theorem 1.10 in Bottcher, Schilling, and Wang [2], the $\hat{C}(S)$-Feller property holds for $\{\mathbb{P}^z_\mu, z \in S\}$ if and only if there exists an increasing sequence of bounded sets $B_n \in \mathcal{B}(S)$ with $\cup_{n \geq 1} B_n = S$ such that for every $t > 0$ and $n \geq 1$

$$\lim_{|z| \to \infty} \mathbb{P}^z_\mu(Z(t) \in B_n) = 0.$$

Since we already know that uniqueness holds for the submartingale problem with drift μ, we may assume that $\{\mathbb{P}^z_\mu, z \in S\}$ is exactly the family we created in the existence part of this paper using Girsanov's theorem. By Theorem 1.10 in [2], there exists an increasing sequence of bounded sets $B_n \in \mathcal{B}(S)$ with $\cup_{n \geq 1} B_n = S$ such that for every $t > 0$ and $n \geq 1$,

$$\lim_{|z| \to \infty} \mathbb{P}^z_0(Z(t) \in B_n) = 0,$$

where $\{\mathbb{P}^z_0, z \in S\}$ is the solution of the submartingale problem without drift. By formula (11), using a calculation similar to (39) we have that

$$\mathbb{P}^z_\mu(Z(t) \in B_n) = \mathbb{E}^z_0\left[\zeta(t)\mathbb{1}_{\{Z(t) \in B_n\}}\right] \leq \left(\mathbb{E}^z_0\left[(\zeta(t))^2\right]\mathbb{P}^z(Z(t) \in B_n)\right)^{1/2}$$
$$= \exp\left\{\frac{|\mu|^2}{2}t\right\}\left(\mathbb{P}^z_0(Z(t) \in B_n)\right)^{1/2},$$

which shows that the $\hat{C}(S)$-Feller property holds for $\{\mathbb{P}^z_\mu, z \in S\}$.

In the rest of this proof we shall show that in the case of $\mu = 0$ the $\hat{C}(S)$-Feller property holds. Let X be the process identified in Proposition 3.1. It has been shown in Williams and Varadhan [35] that the Feller property holds, hence the $C_b(S)$-Feller property holds, so by Theorem 1.10 in [2], it is sufficient to show that for every $t > 0$

$$\mathbb{P}^z_0(Z(t) \in B_n) \to 0$$

as $|z| \to \infty$, where $B_n = \{z \in S : |z| \leq n\}$, for $n \geq 1$. Note that

$$\mathbb{P}^z_0(Z(t) \in B_n) = \mathbb{P}^z_0(Z(t) \in B_n, \tau \leq t) + \mathbb{P}^z_0(Z(t) \in B_n, \tau > t),$$

where

$$\tau = \inf\{t \geq 0 : Z(t) \in \partial S\}.$$

We treat the second term first. It is bounded above by

$$\mathbb{P}^z_0(X(t) \in B_n) \leq \mathbb{P}^z_0(|X(t) - z| \geq |z| - n) = \mathbb{P}^0_0(|X(t)| > |z| - n) \to 0,$$

as $|z| \to \infty$, because under $\mathbb{P}_0^0$ the process X is a standard 2-dimensional Brownian motion without drift starting at zero. Next we treat the first term. Let $T_n = \inf\{t \geq 0 : Z(t) \in B_n\}$. Then the first term can be written as a sum of three terms

$$\mathbb{P}_0^z \left(Z(t) \in B_n, \tau \leq t, |Z(\tau)| < \frac{|z|}{2} \right)$$

$$+ \mathbb{P}_0^z \left(Z(t) \in B_n, \tau \leq t, T_n \leq \tau, |Z(\tau)| \geq \frac{|z|}{2} \right)$$

$$+ \mathbb{P}_0^z \left(Z(t) \in B_n, \tau \leq t, T_n > \tau, |Z(\tau)| \geq \frac{|z|}{2} \right)$$

Again, we treat the three terms separately. By Proposition 3.1 we have that $Z(\tau) = X(\tau)$, where X is a standard Brownian motion with zero drift starting at z under $\mathbb{P}_0^z$, thus the first term is bounded above by

$$P P_0^z \left(\tau \leq t, |X(\tau)| < \frac{|z|}{2} \right) \leq \mathbb{P}_0^z \left(\tau \leq t, |X(\tau) - z| > \frac{|z|}{2} \right)$$

$$\leq \mathbb{P}_0^0 \left(\max_{s \leq t} |X(s)| > \frac{|z|}{2} \right) \to 0,$$

as $|z| \to \infty$. The second term is bounded above by

$$\mathbb{P}_0^z(T_n \leq \tau \leq t) \leq \mathbb{P}_0^z(X - z \text{ reaches } B_n - z \text{ by time } t)$$
$$= \mathbb{P}_0^0(X \text{ reaches } B_n - z \text{ by time } t) \to 0$$

as $|z| \to \infty$.

For analyzing the third term we define the stopping time $T_n^\tau = \inf\{t \geq \tau : Z(t) \in B_n\}$. The third term is bounded above by

$$\mathbb{P}_0^z \left(\tau < T_n \leq t, |Z(\tau)| > \frac{|z|}{2} \right) \leq \mathbb{P}_0^z \left(\tau < T_n \leq \tau + t, |Z(\tau)| > \frac{|z|}{2} \right)$$

$$\leq \mathbb{P}_0^z \left(T_n^\tau \leq t, |Z(\tau)| > \frac{|z|}{2} \right).$$

By the strong Markov property this can be written as

$$\int_{\partial S \cap B_{|z|/2}^c} \mathbb{P}_0^x(T_n \leq t) \mathbb{P}_0^z(Z(\tau) \in dx).$$

By the scaling property (Lemma 2.1 in [40]), the process $\{Z(t), t \geq 0\}$ under $\mathbb{P}_0^x$ induces the same measure on $\mathcal{M}$ as $\{|x|Z(t/|x|^2), t \geq 0\}$ induces under $\mathbb{P}_0^{x/|x|}$, for every non-zero $x \in S$. Then the above expression can be written as

$$\int_{\partial S \cap B_{|z|/2}^c} \mathbb{P}_0^{x/|x|} \left(|x|^2 T_{n/|x|} \leq t \right) \mathbb{P}_0^z(Z(\tau) \in dx)$$

$$= \int_{\partial S_1 \cap B_{|z|/2}^c} \mathbb{P}_0^{u_1}\left(|x|^2 T_{n/|x|} \le t\right) \mathbb{P}_0^z(Z(\tau) \in dx)$$

$$+ \int_{\partial S_2 \cap B_{|z|/2}^c} \mathbb{P}_0^{u_2}\left(|x|^2 T_{n/|x|} \le t\right) \mathbb{P}_0^z(Z(\tau) \in dx),$$

where u_1 and u_2 are the unit vectors $u_1 = (1, 0)$, and $u_2 = (\cos\xi, \sin\xi)$. By symmetry it is sufficient to show that the first term converges to 0 as $|z| \to \infty$. If $|z|/2 > 2n$, then it is bounded above by

$$\sup_{|x| > |z|/2} \mathbb{P}_0^{u_1}(|x|^2 T_{1/2} \le t) = \mathbb{P}_0^{u_1}\left(\frac{|z|^2}{4} T_{1/2} \le t\right) \to 0,$$

as $|z| \to \infty$, which completes the proof of the proposition. $\qquad\square$

6 Proof of Theorem 2.14

Before proving the existence part of Theorem 2.14, we must first establish some preliminary results. Let $\{B_t, t \ge 0\}$ be the coordinate mapping process on $C(\mathbb{R}_+, \mathbb{R}^2)$, whose natural filtration is given by $\mathcal{W}_t = \sigma(B_s, 0 \le s \le t)$ for $t \ge 0$, and let $\mathcal{W} = \sigma(B_s, s \ge 0)$. Recall that v_i is the reflection direction on ∂S_i for $i = 1, 2$, and let R be the 2×2 matrix defined by $R_{ij} =$ the i-th component of v_j. The following result is adapted from Theorem 3.1 in [38]. This result has two parts. The first part is deterministic, and it states that the Skhorohod problem in S for any continuous function has a unique solution up to the first time the vertex is reached. In the second part some purely technical results are stated when the space of continuous functions is considered with its natural filtration and terminal sigma field.

Proposition 6.1 *For any $w \in C(\mathbb{R}_+, \mathbb{R}^2)$ with $w(0) \in S$, there exists a unique triple (ϕ, η, T_0), where $\phi \in C_S$, $\eta \in C(\mathbb{R}_+, [0, +\infty]^2)$ and $T_0 : C(\mathbb{R}_+, \mathbb{R}^2) \to [0, +\infty]$, satisfying the following four conditions,*

1. *$\phi(t) = w(t) + R\eta(t)$ for each $t \in [0, T_0)$;*
2. *$\phi(t) \ne 0$ for all $t < T_0$ and $\phi(t) = 0$ for all $t \ge T_0$;*
3. *For $j = 1, 2$, $\eta(0) = 0$ and $\eta_j(\cdot)$ is non-decreasing and finite for $t \in [0, T_0)$;*
4. *For $j = 1, 2$, η_j only increases when $\phi(t)$ is on $\partial S_j \backslash \{0\}$.*

Furthermore, we have the following two properties

(i) *T_0 is a stopping time on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}, \mathcal{W}_t)$;*
(ii) *Define the map $\Gamma : C(\mathbb{R}_+, \mathbb{R}^2) \mapsto C_S$ such that $\Gamma(w) = \phi$. Then, $\tau_0 \circ \Gamma = T_0$, the map $\Gamma_t \equiv \Gamma(\cdot)(t)$ is $\mathcal{W}_t$-measurable and Γ is $\mathcal{W}/\mathcal{M}$-measurable.*

The main idea of the existence proof is the following. We start with a measure $\hat{\mathbb{P}}_0^z$ under which the coordinate mapping process is a standard drift-less Brownian motion, then transform this measure into $\hat{\mathbb{P}}_\mu^z$ in such way that under the transformed measure it becomes a Brownian motion with drift μ. Then we prove that the reflected process

from the above Proposition induces a measure under $\hat{\mathbb{P}}_\mu^z$ that satisfies the conditions of Definition 2.13. The exact details are below.

For each $z \in S$, let $\hat{\mathbb{P}}_0^z$ be the unique measure on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}, \mathcal{W}_t)$ under which $\{B_t, t \geq 0\}$ is a standard Brownian motion starting at z (the subscript 0 indicates that the Brownian motion has zero drift under $\hat{\mathbb{P}}_0^z$). Next, for each $\mu \in \mathbb{R}^2$ and $T \geq 0$, define the measure $\hat{\mathbb{P}}_{\mu,T}^z$ on $\mathcal{W}_T$ by

$$\frac{d\hat{\mathbb{P}}_{\mu,T}^z}{d\hat{\mathbb{P}}_0^z} = \exp\left(\mu(B_T - z) - \frac{1}{2}||\mu||^2 T\right),$$

and define also

$$\hat{B}_t = B_t - \mu t,$$

and note that $\hat{B}$ is a standard 2-dimensional Brownian motion starting at z under $\hat{\mathbb{P}}_{\mu,T}^z$ on $[0, T]$. Then, by Theorem 4.2 in [25] there exists a measure $\hat{\mathbb{P}}_\mu^z$ on $\mathcal{W}$ which coincides with $\hat{\mathbb{P}}_{\mu,T}^z$ on $\mathcal{W}_T$ for all $T \geq 0$. For every $z \in S$ and $\mu \in \mathbb{R}^2$ the process B is a Brownian motion with drift μ starting at z under the probability measure $\hat{\mathbb{P}}_\mu^z$. Moreover, since by Proposition 6.1, Γ is a measurable map from $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}, \mathcal{W}_t)$ to $(C_S, \mathcal{M}, \mathcal{M}_t)$, we may introduce our candidate solution $\mathbb{P}_\mu^{z,0}$: the measure induced on $\mathcal{M}$ by the mapping Γ under $\hat{\mathbb{P}}_\mu^z$, i.e.,

$$\mathbb{P}_\mu^{z,0}(A) \equiv \hat{\mathbb{P}}_\mu^z(\Gamma^{-1}(A)), \text{ for each } A \in \mathcal{M}. \tag{41}$$

We next prove that the family of measures $\{\mathbb{P}_\mu^{z,0}, z \in S\}$ is a solution to the absorbed process problem of Definition 2.13. Since conditions 1 and 3 of Definition 2.13 are trivially satisfied by $\{\mathbb{P}_\mu^{z,0}, z \in S\}$, it remains to prove that condition 2 is satisfied as well. This is achieved in Lemma 6.2.

Lemma 6.2 *Let the family of measures $\{\mathbb{P}_\mu^{z,0}, z \in S\}$ be defined as in (41) and let Z be the coordinate-mapping process on $(C_S, \mathcal{M}, \mathcal{M}_t)$. Then, the process*

$$\left\{f(Z(t \wedge \tau_0)) - \int_0^{t \wedge \tau_0} \mu \cdot \nabla f(Z(s))ds - \frac{1}{2}\int_0^{t \wedge \tau_0} \Delta f(Z(s))ds, t \geq 0\right\} \tag{42}$$

is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^{z,0})$, for each $f \in C_b^2(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$.

Proof For each $w \in C(\mathbb{R}_+, \mathbb{R}^2)$, let $\phi(w) = \Gamma(w)$ and note that by Proposition 6.1 we may write $\phi(t) = w(t) + R\eta(t)$ for all $t \geq 0$. Now, on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}_t, \mathcal{W})$, we consider the process $\{\phi(t), t \geq 0\}$ with $\phi(t) = \Gamma_t(w)$ for any $w \in C(\mathbb{R}_+, \mathbb{R}^2)$. Recall that the coordinate mapping process $\{B_t, t \geq 0\}$ is a Brownian motion on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}_t, \mathcal{W})$ under $\hat{\mathbb{P}}^z$, and $\{B_t - \mu t, t \geq 0\}$ is a Brownian motion on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}_t, \mathcal{W})$ under $\hat{\mathbb{P}}_\mu^z$. Notice, by Theorem 1 in [39], that $R\eta(t)$ is of finite

variation on $[0, t \wedge T_0]$ for any $t \geq 0$, and $w(t) = B(w)(t)$, we get that $\{\phi(t), t \geq 0\}$ is a semimartingale under $\hat{\mathbb{P}}^z$. On the other hand, the Girsanov transform keeps the semimartingale property, so $\{\phi(t), t \geq 0\}$ is also a semimartingale under $\hat{\mathbb{P}}^z_\mu$. Hence, for each $f \in C^2_b(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$, we use Itô's formula under $\hat{\mathbb{P}}^z_\mu$ and get

$$
\begin{aligned}
&f(\phi(t \wedge T_0)) - f(\phi(0)) \\
&= \sum_{i=1}^{2} \int_0^{t \wedge T_0} \frac{\partial f}{\partial x_i}(\phi(s)) d(w_i(s) - \mu_i s) + \int_0^{t \wedge T_0} \mu \cdot \nabla f(\phi(s)) ds \\
&\quad + \int_0^{t \wedge T_0} (D_1 f(\phi(s)), D_2 f(\phi(s))) \cdot d\eta(s) + \frac{1}{2} \int_0^{t \wedge T_0} \Delta f(\phi(s)) ds.
\end{aligned}
$$

Since we have by Proposition 6.1 that for $i = 1, 2$,

$$
d\eta_i(s) = \mathbb{1}_{\{\phi(s) \in \partial S_i \setminus \{0\}\}} d\eta_i(s), \quad s \geq 0,
$$

and by the assumption on f that for $i = 1, 2$,

$$
D_i f(\phi(s)) \mathbb{1}_{\{\phi(s) \in \partial S_i\}} \geq 0, \quad s \geq 0,
$$

it follows that the process

$$
\left\{ \int_0^{t \wedge T_0} (D_1 f(\phi(s)), D_2 f(\phi(s))) \cdot d\eta(s), t \geq 0 \right\}
$$

is increasing. On the other hand since $\{B_t - \mu t, t \geq 0\}$ is a Brownian motion under $\hat{\mathbb{P}}^z_\mu$, the process

$$
\left\{ \sum_{i=1}^{2} \int_0^{t \wedge T_0} \frac{\partial f}{\partial x_i}(\phi(s)) d(w_i(s) - \mu_i s), t \geq 0 \right\}
$$

is a martingale under $\hat{\mathbb{P}}^z_\mu$, so

$$
\begin{aligned}
&f(\phi(t \wedge T_0)) - \int_0^{t \wedge T_0} \mu \cdot \nabla f(\phi(s)) ds - \frac{1}{2} \int_0^{t \wedge T_0} \Delta f(\phi(s)) ds \\
&= f(\phi(0)) + \sum_{i=1}^{2} \int_0^{t \wedge T_0} \frac{\partial f}{\partial x_i}(\phi(s)) d(w_i(s) - \mu_i s) \\
&\quad + \int_0^{t \wedge T_0} (D_1 f(\phi(s)), D_2 f(\phi(s))) \cdot d\eta(s)
\end{aligned}
$$

is a submartingale under $\hat{\mathbb{P}}^z_\mu$. It follows from (41) that the process under (42) is also a submartingale under the induced measure $\bar{\mathbb{P}}^z_\mu$. $\qquad\square$

Proof of the existence part of Theorem 2.14 The existence of a solution to the absorbed process problem follows from Lemma 6.2.   □

Proof of the uniqueness part of Theorem 2.14 The proof of the uniqueness of the solution to the absorbed process problem is very similar to that of the solution of the submartingale problem, hence in this section we shall state the appropriate lemmas and indicate the necessary changes in order to adapt the proofs in Sect. 3 to the absorbed process problem.

Lemma 6.3 *Let $\alpha \in \mathbb{R}$ arbitrary, and suppose that $\{\mathbb{P}_{\mu}^{z,0}, z \in S\}$ is a solution to the absorbed process problem with drift $\mu \in \mathbb{R}^2$. Then, for all $z \in S$,*

$$\mathbb{E}_{\mu}^{z,0}\left[\int_0^{\tau_0} \mathbb{1}_{\{Z(t) \in \partial S\}} dt \right] = 0.$$

Proof The proof is almost identical to that of Lemma 3.3 with the modification that all processes must be stopped at τ_0.   □

Lemma 6.4 *Let $\alpha \in \mathbb{R}$ be arbitrary. Suppose that $\{\mathbb{P}_{\mu}^{z,0}, z \in S\}$ is a solution to the absorbed process problem with drift $\mu \in \mathbb{R}^2$. Then there exists a process X on $(C_S, \mathcal{M}, \mathcal{M}_t)$ such that for all $z \in S$, X is a Brownian motion with drift μ under $\mathbb{P}_{\mu}^{z,0}$ starting at z and stopped at τ_0. In addition, $Y = Z - X$ is flat on $[\sigma_n^{\delta}, \tau_n^{\delta}]$.*

Proof The proof is very similar to the proofs of Lemmas 3.4, 3.5, and 3.6, with the difference that all processes must be stopped at τ_0.   □

Proof of the uniqueness part of Theorem 2.14 The proof is basically a copy of the proof of the uniqueness of the solution to the submartingale problem (Lemma 3.7 and the proof of Theorem 2.3). The necessary changes in order to adapt that proof to the present situation are the following:

- All processes must be stopped at τ_0;
- The constraint that the test function f is constant in a neighborhood of the vertex must be erased (above (27) in the adapted proof);
- At the end of the proof, instead of using the uniqueness of the solution to the submartingale problem with no drift, we must use the uniqueness of the solution to the absorbed process problem with no drift ([35], Theorem 2.1).

                                                                 □

7 Proof of Theorems 2.15, 2.16, and Proposition 2.18

We need to recall Proposition 6.1. According to this proposition, for every $w \in C(\mathbb{R}_+, \mathbb{R}^2)$ with $w(0) \in S$ there exists a triple (ϕ, η, T_0) such that items 1–4 and (i),(ii) hold. Since B is the coordinate mapping process on $C(\mathbb{R}_+, \mathbb{R}^2)$, we may replace w in item 1 with B, and write

$$\phi(t) = B(t) + R\eta(t), \quad t \in [0, T_0).$$

We also know that B is a 2-dimensional Brownian motion with drift μ starting at z under $\hat{\mathbb{P}}_\mu^z$ for every $z \in S$, hence we can write

$$\phi(t) = z + W(t) + \mu t + R\eta(t), \quad t \in [0, T_0), \tag{43}$$

where W is a standard 2-dimensional Brownian motion starting at zero under $\hat{\mathbb{P}}_\mu^z$. The measure $\mathbb{P}_\mu^{z,0}$ was defined as the measure induced by Γ on $\mathcal{M}$ under $\hat{\mathbb{P}}_\mu^z$. From this and from $\tau_0 \circ \Gamma = T_0$ follows that

$$\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) = \hat{\mathbb{P}}_\mu^z(T_0 < \infty). \tag{44}$$

We shall use (44) repeatedly in the coming proofs. The proof of Theorem 2.15 hinges on the fact that it is known to be true in the drift-less case.

Proof of Theorem 2.15 By Theorem 2.2 in [35], we have that $\mathbb{P}_0^{z,0}(\tau_0 = \infty) = 1$, thus by (44) also $\hat{\mathbb{P}}_0^z(T_0 = \infty) = 1$. For every $n \in \mathbb{N}_+$ the measures $\hat{\mathbb{P}}_0^z$ and $\hat{\mathbb{P}}_\mu^z$ are mutually absolutely continuous on $\mathcal{W}_n$, so $\hat{\mathbb{P}}_0^z(T_0 < n) = 0$ implies $\hat{\mathbb{P}}_\mu^z(T_0 < n) = 0$. Then $\hat{\mathbb{P}}_\mu^z(T_0 = \infty) = 1$ follows, and this and (44) give the required result. $\qquad\square$

The first statement of Theorem 2.16 could have been shown by a simple application of Girsanov's theorem. However, the appearance of the drift in the secont statement is essential, that is, there is no corresponding fact for the $\mu = 0$ case, and in the present proof below the two parts form an organic unit, thus we elected not to use Girsanov's theorem even in the first part of the proof.

Proof of Theorem 2.16 First we are going to show (8). By the $\alpha \geq 1$ condition there exists a vector $b \in \mathbb{R}^2$ such that $b \cdot z < 0$ for all $z \in S, z \neq 0$, and $b \cdot v_i \geq 0$ for $i = 1, 2$. Indeed, if $\alpha \geq 1$ then $\mathrm{co}(-v_1, -v_2)$ is either a line containing S within one side, or it is a wedge with angle less than π containing S. In either case the existence of such a vector follows. Then, by identity (43), for each $z \in S$,

$$0 \geq b \cdot \phi(t) = b \cdot z + b \cdot W_t + b \cdot v_1 \eta_1(t) + b \cdot v_2 \eta_2(t) + b \cdot \mu t$$
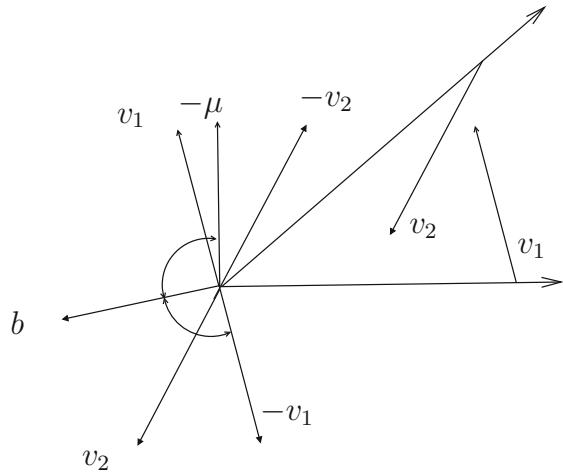$$\geq b \cdot z + b \cdot W_t + b \cdot \mu t,$$

for $t < T_0$, $\hat{\mathbb{P}}_\mu^z$-a.s., and so

$$\hat{\mathbb{P}}_\mu^z(0 \geq b \cdot z + b \cdot W_t + b \cdot \mu t, \ t < T_0) = 1.$$

Therefore,

$$\hat{\mathbb{P}}_\mu^z(0 \geq b \cdot z + b \cdot W_t + b \cdot \mu t, \ t < \infty)$$
$$\geq \hat{\mathbb{P}}_\mu^z(\{0 \geq b \cdot z + b \cdot W_t + b \cdot \mu t, t < T_0\} \cap \{T_0 = \infty\})$$
$$= \hat{\mathbb{P}}_\mu^z(T_0 = \infty).$$

This implies

$$\hat{\mathbb{P}}_\mu^z(-b \cdot z \geq b \cdot W_t + b \cdot \mu t, t < \infty) \geq \hat{\mathbb{P}}_\mu^z(T_0 = \infty). \tag{45}$$

However, $\hat{\mathbb{P}}_\mu^z(-b \cdot z \geq b \cdot W_t + b \cdot \mu t, \ t < \infty) < 1$, and together with (44) this proves the result. Suppose now that (9) also holds, in addition to $\alpha \geq 1$. Then $\text{co}(v_1, v_2, \mu)$ is either a wedge with angle less than π, or a half-space, or a line. Then the same is true for $\text{co}(-v_1, -v_2, -\mu)$, and if it is a wedge or a half-space then it contains S, and if it is a line then it contains S in one side. In all cases we can select b so that in addition to $b \cdot z < 0$ for all $z \in S$, $z \neq 0$, and $b \cdot v_i \geq 0$ for $i = 1, 2$, we also have $b \cdot \mu \geq 0$. Figure 3 illustrates the case when $\text{co}(-v_1, -v_2, -\mu)$ is a wedge containing S. In that case we select b such that it points into the complement of the wedge $\text{co}(-v_1, -v_2, -\mu)$, and bisects the angle of the complement of $\text{co}(-v_1, -v_2, -\mu)$. Once such b has been selected, we have that $\hat{\mathbb{P}}_\mu^z(-b \cdot z \geq b \cdot W_t + b \cdot \mu t, \ t < \infty) = 0$, so (45) and (44) imply (10). $\qquad\qquad\square$

Proof of Proposition 2.18 Suppose first that $0 < \xi \leq \pi/2$. Let $z \in S^0$ and $\mu \in \mathbb{R}^2$ be given by $\mu = ||\mu||(\cos \eta, \sin \eta) \neq 0$, where $\eta \in (0, \xi)$. Next, set

$$X_t = z + B_t + \mu t, \ t \geq 0, \tag{46}$$

where B_t is a standard 2-d Brownian motion.

Now translate the origin of the coordinate axes to z and then rotate the axis in a counterclockwise direction by the angle η. By the translational and rotational invariance of Brownian motion, in these new coordinates the process X may be written as

$$\hat{X}_t = \hat{B}_t + |\mu|\hat{e}_1 t, \ t \geq 0,$$

where $\hat{B}$ is a standard Brownian motion and $\hat{e}_1 = (1, 0)$. Next, in the new coordinate system denote by $\mathcal{L}_1$ the line corresponding ∂S_1, and by $\mathcal{L}_2$ the line corresponding to ∂S_2. Then, in the new coordinate system the interior of S may be expressed as

$$S^0 = \{\hat{z} \in \mathbb{R}^2 : \mathcal{L}_1(\hat{z}_1) < \hat{z}_2 < \mathcal{L}_2(\hat{z}_1)\},$$

where $\mathcal{L}_i(\hat{z}_1)$ is a coordinate uniquely determined by the relation $(\hat{z}_1, \mathcal{L}_i(\hat{z}_1)) \in \mathcal{L}_i$, for $i = 1, 2$. Hence, in order to complete the proof for the case of $0 < \xi \leq \pi/2$ it suffices to show that

$$P(\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1), t \geq 0) > 0.$$

First note that since $0 < \eta < \xi \leq \pi/2$, we may write

$$\mathcal{L}_1(\hat{z}_1) = -a - b\hat{z}_1 \quad \text{and} \quad \mathcal{L}_2(\hat{z}_1) = c + d\hat{z}_1,$$

for $a, b, c, d > 0$. Hence,

$$
\begin{aligned}
&\{\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1), t \geq 0\} \\
&= \{-a - b|\mu|t - b\hat{B}_t^1 < \hat{B}_t^2 < c + d|\mu|t + d\hat{B}_t^1, t \geq 0\}.
\end{aligned}
\tag{47}
$$

From (47), it now follows after some algebra that

$$
\begin{aligned}
&\{\hat{B}_t^1 > \max(-a/2b, -c/2d) - (\|\mu\|/2)t, t \geq 0\} \\
&\cap \{-(a/2) - (b/2)\|\mu\|t < \hat{B}_t^2 < (c/2) + (d/2)\|\mu\|t, t \geq 0\} \\
&\subseteq \{\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1), t \geq 0\}.
\end{aligned}
$$

Next, by the independence of $\hat{B}^1$ and $\hat{B}^2$, we have that

$$
\begin{aligned}
&P(\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1), t \geq 0) \\
&\geq P(\hat{B}_t^1 > \max(-a/2b, -c/2d) - (\|\mu\|/2)t, t \geq 0) \\
&\quad \times P(-(a/2) - (b/2)\|\mu\|t < \hat{B}_t^2 < (c/2) + (d/2)\|\mu\|t, t \geq 0).
\end{aligned}
$$

However, since $a, b, c, d, \|\mu\| > 0$, it follows by (4.3) of Doob [13] that the two probabilities on the right-hand side above are greater than zero. This completes the proof for the case of $0 < \xi \leq \pi/2$.

Now suppose that $\pi/2 < \xi < \pi$ and $z \in S^0$ and $0 < \eta < \xi$. In this case we show that there exists a wedge $\bar{S} \subset S$ such that $z \in \bar{S}^0$ and $P(X_t \in \bar{S}^0, t \geq 0) > 0$, which is sufficient to complete the proof. First, suppose that $0 < \eta < \pi/2$. In this case the wedge $\bar{S}$ can be defined in the usual way by setting $\xi = \pi/2$ and placing the vertex of $\bar{S}$ at a point $\bar{z} \in S^0$ such that $\bar{z} < z$. The results above then yield the desired result. Next, suppose $\pi/2 \leq \eta < \xi$. Then, set $\bar{S} = v + \{r \geq 0, \pi - (\xi + \eta)/2 \leq \theta \leq (\xi + \eta)/2\}$, where the vertex $v = (v_1, v_2) \in S^0$ is such that $v_1 = z_1$ and $v_2 < z_2$. The results above again yield the desired result. $\quad\square$

Appendix

Lemma 8.1 *There exists a function $f_{\varepsilon,C} \in C_b^2(S)$ satisfying*

$$f_{\varepsilon,C}(x, y) = \begin{cases} 0, & \text{if } (x, y) \in S \setminus S^{\varepsilon/3}, \\ y, & \text{if } (x, y) \in S^{2\varepsilon/3}, \; y \leq C, \end{cases}$$

such that in addition $f_{\varepsilon,C}(x, 0) = 0$ for all $x \geq 0$, and $D_i f_{\varepsilon,C} \geq 0$ on ∂S_i.

Proof Let $h_1 \in C_b^2(\mathcal{R})$ such that $h_1(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$h_1(x) = \begin{cases} 0, & \text{if } x \leq \varepsilon/3, \\ 1, & \text{if } x \geq 2\varepsilon/3. \end{cases}$$

Let $h_2 \in C_b^2(\mathcal{R})$ such that $h_2(y) = y$ if $y \leq C$. Then

$$f_{\varepsilon,C} = h_1 \left(x - \frac{y}{\tan \xi} \right) h_2(y) \quad (x, y) \in S$$

satisfies the requirements of the lemma. Note that for any $\delta > 0$ we have $(x, y) \in S^{\delta}$ if and only if $x - y/\tan \xi \geq \delta$. Using this fact repeatedly one can verify the above statement by straightforward calculation. $\qquad\square$

Remark 8.2 There are slightly different definitions for the term "augmented filtration" in the literature; we use this term as defined in [32], Definition II.67.3. Let $\mathbb{P}$ be an arbitrary probability measure on $\mathcal{M}$, and let $(C_S, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be the augmentation of the probability space $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P})$, in the above sense. It is known that right-continuous martingales (submartingales) on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P})$ are also right-continuous martingales (submartingales) on $(C_S, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ (Lemma II.67.10 in [32]). The probability measure $\mathbb{P}$ in the second probability space is the extension of $\mathbb{P}$ from $\mathcal{M}$ to $\mathcal{F}$, without changing the notation. Also, it follows from the martingale characterization of Brownian motion (Theorem 3.3.16 in [22]) that a Brownian motion on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P})$ is also a Brownian motion on $(C_S, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Remark 8.3 For the convenience of the reader in this remark we shall recall the notion of the Stochastic Differential Equation with Reflection (SDER) from Definition 2.4 of Kang and Ramanan [21]. In that paper the definition appears for arbitrary dimension and for a more general domain, but here we specify it to our case for the domain given by S in 2 dimensions. Let $d : S \to \mathbb{R}^2$ be a set-valued map from ∂S to the class of subsets of $\mathbb{R}^2$ satisfying conditions (d1) and (d2) in Sect. 2.1, then extend it to the interior of S by defining $d(x) = \{0\}$ if x lies in the interior of S. The other inputs of the SDER are the drift and dispersion coefficients $b : S \mapsto \mathbb{R}^2$ and $\sigma : S \mapsto \mathbb{R}^{2 \times 2}$. A weak solution to the SDER consists of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, a family of probability measures $\{\mathbb{P}^z, z \in S\}$, and a pair of continuous adapted processes (Z, W) (each Z and W 2-dimensional), satisfying the following conditions for each $z \in S$:

1. W is a standard 2-dimensional Brownian motion on the above filtered probability space starting at zero under $\mathbb{P}^z$;
2. $\int_0^t |b(Z(s))| ds + \int_0^t |\sigma(Z(s))|^2 ds < \infty$, for all $t \in [0, \infty)$, $\mathbb{P}^z$-a.s.;
3. Defining the 2-dimensional processes X and Y as

$$X(t) = z + \int_0^t b(Z(s)) ds + \int_0^t \sigma(Z(s)) dW(s), \quad t \in [0, \infty) \qquad (48)$$

and $Y = Z - X$, the couple (Z, Y) solves the ESP for X associated with $(S, d(\cdot))$, almost surely under $\mathbb{P}^z$;
4. The set $\{t \in [0, \infty) : Z(t) \in \partial S\}$ has zero Lebesgue measure, $\mathbb{P}^z$-a.s.

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