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Tail asymptotics for the $M_1, M_2/G_1, G_2/1$ retrial queue with non-preemptive priority

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Abstract
Stochastic networks with complex structures are key modelling tools for many important applications. In this paper, we consider a specific type of network: retrial queueing systems with priority. This type of queueing system is important in various applications, including telecommunication and computer management networks with big data. The system considered here receives two types of customers, of which Type-1 customers (in a queue) have non-pre-emptive priority to receive service over Type-2 customers (in an orbit). For this type of system, we propose an exhaustive version of the stochastic decomposition approach, which is one of the main contributions made in this paper, for the purpose of studying asymptotic behaviour of the tail probability of the number of customers in the steady state for this retrial queue with two types of customers. Under the assumption that the service times of Type-1 customers have a regularly varying tail and the service times of Type-2 customers have a tail lighter than Type-1 customers, we obtain tail asymptotic properties for the numbers of customers in the queue and in the orbit, respectively, conditioning on the server's status, in terms of the exhaustive stochastic decomposition results. These tail asymptotic results are new, which is another main contribution made in this paper. Tail asymptotic properties are very important, not only on their own merits but also often as key tools for approximating performance metrics and constructing numerical algorithms.

Keywords Exhaustive stochastic decomposition approach, Tail asymptotics · Retrial queue · Priority queue · Number of customers · Stationary distribution · Regularly varying distribution

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1 Introduction

Rapid advances in the fields of computer and communication technologies, with fast increasing internet, big data and smartphone applications, have significantly changed every aspect of our life. These accelerated developments have continuously raised new challenges in modelling, performance analysis, system control and optimization, such as those for queueing systems, including priority retrial queues (see, for example, Sutton and Jordan [33], Dimitriou [9], Phung-Duc [32], Walraevens, Claeys and Phung-Duc [34] among others). As a consequence of these challenges, the resulting stochastic models become progressively more complex, due to, for example, dependence structures, dimensions and the size of the data involved. For such networks, non-transformation exact solutions are hardly seen, whereas asymptotic behaviours and properties are among the key candidates that we search for.

In this paper, we consider a single-server retrial queue with two types of customers (Type-1 and Type-2), denoted by $M_1, M_2/G_1, G_2/1$. This model was studied by Falin, Artalejo and Martin in [14]. In this model, customers arrive according to a Poisson process at rate $\lambda > 0$ and with probabilities $q \in (0, 1)$ and $p = 1 - q$ to be Type-1 and Type-2, respectively. In other words, Type-1 and Type-2 customers form two independent Poisson arrival processes with rates $\lambda_1 \equiv \lambda q$ and $\lambda_2 \equiv \lambda p$, respectively. If the server is idle upon the arrival of a Type-1 or Type-2 customer, the customer receives the service immediately and leaves the system after the completion of the service. If an arriving Type-1 customer finds the server busy, it joins the priority queue with an infinite waiting capacity. If a Type-2 customer finds the server busy upon arrival, it enters the orbit and makes retrial attempts later for receiving a service. Each of the Type-2 customers in the orbit repeatedly tries, independently, to receive service according to a Poisson process with a common retrial rate μ until it finds the server idle, and receives its service immediately. Type-1 customers have non-preemptive priority to receive service over Type-2 customers. Thus, as long as the priority queue is not empty, all retrials by Type-2 customers from the orbit are blocked (or failed), and all blocked Type-2 customers return to the orbit with probability one. Each Type-i customer, $i = 1, 2$, has service time T_{β_i}, whose common probability distribution is $F_{\beta_i}(x)$ with $F_{\beta_i}(0) = 0$, and all service times are independent, and are assumed to be independent of the arrivals. T_{β_i} is assumed to have a finite mean $\beta_{i,1}$, $i = 1, 2$, where the second subscript is used to indicate the first moment of the service time. The Laplace–Stieltjes transforms (LST) of the distribution function $F_{\beta_i}(x)$ is denoted by $\beta_i(s)$, $i = 1, 2$. Let $\rho_1 = \lambda_1\beta_{1,1}$, $\rho_2 = \lambda_2\beta_{2,1}$ and $\rho = \rho_1 + \rho_2 = \lambda(q\beta_{1,1} + p\beta_{2,1})$. It follows from [14] that this system is stable if and only if $\rho < 1$. We assume that $\rho < 1$ throughout this paper. For the service times T_{β_i} of Type-i customers, $i = 1, 2$, we make the following assumptions:

A1. *T_{β_1} has tail probability $P\{T_{\beta_1} > t\} \sim t^{-a_1}L(t)$ as $t \to \infty$, where $a_1 > 1$;*
A2. *T_{β_2} has tail probability $P\{T_{\beta_2} > t\} \sim e^{-rt}t^{-a_2}L(t)$ as $t \to \infty$, where $-\infty < a_2 < \infty$ if $r > 0$, or $a_2 > a_1$ if $r = 0$.*

Here, $L(t)$ represents a slowly varying function at ∞ (see Definition A.1).

There are many references in the literature for asymptotic analysis for queues without retrials, for example, Asmussen, Klüppelberg and Sigman [3], Boxma and Denisov [6], and more references can be found in two excellent surveys: Borst et al. [5] and Boxma and Zwart [7]. Concerning retrial queues, we refer readers to the following books or review articles for an updated status of studies and for more references therein: Falin [13], Artalejo and Gómez-Corral [2], Kim and Kim [21] and Phung-Duc [32]. We also mention here the following two references, which are closely related to the study in this paper: Kim, Kim and Ko [23] and Kim, Kim and Kim [22]. Priority retrial queueing systems are a type of very important retrial queues, which find many applications, for example, in computer network management and telecommunication systems. In such systems, there are usually two or more types of customers. A survey of studies on single-server retrial queues with priority calls (or customers), published in 1999, can be found in Choi and Chang [8]. Since then, more publications on priority retrial queues became available, such as Artalejo, Dudin and Klimenok [1], Lee [24], Gómea-Corral [19], Wang [35], Dimitriou [9], Wu and Lian [36], Wu, Wang and Liu [37], Gao [18], Dudin et al. [11], Walraevens, Claeys and Phung-Duc [34] (who, assuming $F_{\beta_1}(x) = F_{\beta_2}(x)$, considered special situations of the model studied in this paper using a different method), among possible others. Readers may refer to [9,37] for more detailed reviews of the above-mentioned studies.

Different from the above-mentioned studies, our focus in this paper is on the heavy-tailed behaviour of stationary (conditional) probabilities. Specifically, under the assumptions that the tail probability of the service time for Type-1 customers is regularly varying and the tail probability of the service time for Type-2 customers is lighter than that for Type-1 customers (see Assumptions A1 and A2), we characterize the tail asymptotic behaviour for the following key system performance metrics:

- PO-0 Conditional tail probability of the number of customers in the orbit given that the server is idle;
- PO-1 Conditional tail probability of the number of customers in the orbit given that the server is serving a Type-1 customer;
- PO-2 Conditional tail probability of the number of customers in the orbit given that the server is serving a Type-2 customer;
- PQ-1 Conditional tail probability of the number of customers in the queue given that the server is serving a Type-1 customer;
- PQ-2 Conditional tail probability of the number of customers in the queue given that the server is serving a Type-2 customer.

We did not include the conditional probability of the number of customers in the queue given that the server is idle, since the queue is obviously empty when the server is idle. The tail asymptotic behaviour is one of the key subjects in applied probability. It is also very useful in approximations and computations, such as providing performance metrics and developing numerical algorithms (see Liu and Zhao [26] for some of its applications).

The main contributions made in this paper are twofold:

(1) The proposal of an exhaustive stochastic decomposition approach for complicated probability generating functions (PGFs), or transformations in general, of a probability distribution. This approach decomposes a complicated PGF into a product of PGFs, or equivalently decomposes a r.v. into a sum of independent r.v.s, called independent components, to which detailed probabilistic interpretations are provided. We refer to this extended version of stochastic decompositions as the exhaustive stochastic decomposition approach. This exhaustive version of the stochastic decomposition is an extension of the idea presented in the literature; for example, see Liu, Wang and Zhao [29]. This paper is one of the sister papers (see Liu, Min and Zhao [25] and Liu and Zhao [27,28]) demonstrating the usefulness of the exhaustive decomposition approach to different types of queueing models.

Stochastic decomposition has been widely used in queueing system analysis. For example, it is well known that for the $M/G/1$ retrial queue, one can stochastically decompose the total number of customers in the system as the independent sum of the total number of customers in the corresponding standard $M/G/1$ queueing system (without retrials) and another random variable (r.v.). The exhaustive version of the stochastic decomposition approach is also to decompose a r.v. into independent components, but in a more systematic way, applied to more complicated PGFs (or transformations), such that detailed probabilistic interpretations for each decomposed component can be provided. We expect that this extended version of decompositions can be applied to other queueing models and used in other applied probability areas.

(2) Tail asymptotic results for the conditional probabilities, specified in PO-i ($i = 0, 1, 2$) and PQ-i ($i = 1, 2$) for the $M_1, M_2/G_1, G_2/1$ priority retrial queueing model. These tail asymptotic results are new. Tail probability behaviour is an important topic, which was not touched in [14]. The stochastic decompositions of these generating functions allow us to analyse tail asymptotic behaviour for each independent component in the decompositions, which lead to our final tail asymptotic results, stated in detail in the following theorem. For the purpose of stating this theorem, let R_{que} be the number of Type-1 customers in the queue, excluding the possible one in the service, let R_{orb} be the number of Type-2 customers in the orbit, and let $I_{\text{ser}} = 0, 1,$ or 2 according to the status of the server: idle, busy with a Type-1 customer, or busy with a Type-2 customer, respectively. To stay consistent with the literature notation, let R_0 be a r.v.whose distribution coincides with the conditional distribution of R_{orb} given that $I_{\text{ser}} = 0$, or

$$R_0 \stackrel{\text{d}}{=} R_{\text{orb}} \mid I_{\text{ser}} = 0,$$

where the symbol "$\stackrel{\text{d}}{=}$" stands for equality in probability distribution. Similarly, for $i = 1, 2$, we define the two-dimensional r.v. (R_{i1}, R_{i2}) as

$$(R_{i1}, R_{i2}) \stackrel{\text{d}}{=} (R_{\text{que}}, R_{\text{orb}}) \mid I_{\text{ser}} = i.$$

We can now state the main results on tail asymptotics.

Theorem 1 *For the $M_1, M_2/G_1, G_2/1$ priority retrial queueing model, studied in [14], under assumptions A1 and A2, we have, as $j \to \infty$,*

(1)

$$P\{R_{\text{orb}} > j | I_{\text{ser}} = 0\} = P\{R_0 > j\} \sim \frac{\lambda_1 \lambda_2^{a_1}}{a_1 \mu (1-\rho)^2 (1-\rho_1)^{a_1-1}} \cdot j^{-a_1} L(j). \tag{1.1}$$

(2)

$$P\{R_{\text{que}} > j | I_{\text{ser}} = 1\} = P\{R_{11} > j\} \sim \frac{\lambda_1^{a_1}}{\rho_1 (1-\rho_1)(a_1-1)} \cdot j^{-a_1+1} L(j); \tag{1.2}$$

$$P\{R_{\text{orb}} > j | I_{\text{ser}} = 1\} = P\{R_{12} > j\} \sim \left[\frac{\rho_2}{1-\rho} + \frac{1}{\rho_1}\right] \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{(a_1-1)(1-\rho_1)^{a_1}} \cdot j^{-a_1+1} L(j); \tag{1.3}$$

$$P\{R_{\text{que}} > j | I_{\text{ser}} = 2\} = P\{R_{21} > j\} \sim \begin{cases} \frac{\lambda_2 \lambda_1^{a_2-1}}{\rho_2 (a_2-1)} \cdot j^{-a_2+1} L(j), & \textit{if } r = 0, \\ \frac{\lambda_2 \lambda_1 (\lambda_1 + r)^{a_2-1}}{\rho_2 r} \cdot \left(\frac{\lambda_1}{\lambda_1 + r}\right)^j j^{-a_2} L(j), & \textit{if } r > 0; \end{cases} \tag{1.4}$$

$$P\{R_{\text{orb}} > j | I_{\text{ser}} = 2\} = P\{R_{22} > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{(1-\rho)(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1} L(j). \tag{1.5}$$

Recalling the assumptions made on the service times, we notice that all above conditional probabilities are also regularly varying with a dominant influence by the service time distribution for Type-1 customers, except for the case $r > 0$, for which the tail is dominated by the service time for Type-2 customers.

To guide readers to detailed analysis for our main contributions, we provide a step by step presentation in the following sections:

Step 1: Preliminaries. In this step, we present three PGFs, all obtained from [14], one for the tail behaviour in Theorem 1(1), and the other two for Theorem 1(2). We rewrite these generating functions in a preferable form for making decompositions. We also provide probabilistic interpretations for the functions g and h contained in these three PGFs.

Step 2: Stochastic decompositions. In this step, in terms of the exhaustive stochastic decomposition approach, we provide stochastic decompositions for the generating functions (or part of the generating function for the case of the r.v. R_0) provided in Step 1. Equivalently, we show that each of the these functions can be viewed as the generating function for the sum of some well-constructed independent r.v.s, each with a detailed probabilistic interpretation.

Step 3: Tail asymptotics. In this step, we first specify tail asymptotic properties for each decomposed component obtained in Step 2 and then identify the dominant contributions to derive the results stated in Theorem 1.

The rest of the paper is organized as follows: Sect. 2 is our Step 1, in which we express the probability generating functions of interest, obtained in [14], in the forms in favour of making stochastic decompositions, which are our starting point. Section 3 is devoted to Step 2, in which the proposed exhaustive stochastic decompositions are performed on the three key generating functions. Our Step 3, presented in Sect. 4, is to derive the tail asymptotic properties, based on the results obtained in previous sections. Most of the detailed analysis is organized in Appendices B and C following the concluding section, Sect. 5.

2 Preliminaries—step 1

For the $M_1, M_2/G_1, G_2/1$ priority retrial queueing model, according to the definitions of the r.v.s R_0 and (R_{i1}, R_{i2}) for $i = 1, 2$ (given in the Introduction), we have that R_0 has the PGF $R_0(z_2) \stackrel{\text{def}}{=} E(z_2^{R_0}) = E(z_2^{R_{\text{orb}}}|I_{\text{ser}} = 0)$, and (R_{i1}, R_{i2}) has the PGF $R_i(z_1, z_2) \stackrel{\text{def}}{=} E(z_1^{R_{i1}} z_2^{R_{i2}}) = E(z_1^{R_{\text{que}}} z_2^{R_{\text{orb}}}|I_{\text{ser}} = i)$ for $i = 1, 2$. These three generating functions are key probability quantities of the queueing model. In [14], the authors obtained expressions for $R_0(z_2)$, $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$, together with the facts that $P\{I_{\text{ser}} = 0\} = 1 - \rho$, $P\{I_{\text{ser}} = 1\} = \rho_1$ and $P\{I_{\text{ser}} = 2\} = \rho_2$.

In this section, we first rewrite these three generating functions into forms favourable for making stochastic decompositions and also provide probabilistic interpretations for two important functions h and g (in Sects. 2.1.1 and 2.1.2, respectively) contained in the generating functions.

2.1 Generating function $R_0(z_2)$

First, we denote

$$\beta(s) = q\beta_1(s) + p\beta_2(s), \tag{2.1}$$

$$g(z_2) = qh(z_2) + pz_2, \tag{2.2}$$

where the function $h(z_2)$ is uniquely determined by the equation

$$h(z_2) = \beta_1(\lambda - \lambda_1 h(z_2) - \lambda_2 z_2). \tag{2.3}$$

Then, the PGF $R_0(z_2)$, obtained in [14], can be rewritten as

$$R_0(z_2) = \exp\left\{-\psi \int_{z_2}^{1} K(u)\mathrm{d}u\right\}, \tag{2.4}$$

where

$$\psi = \rho\lambda_2/(\mu(1 - \rho)), \tag{2.5}$$

$$K(u) = K_a(u) \cdot K_b(u) \cdot K_c(u), \tag{2.6}$$

with

$$K_a(u) = \frac{1 - \rho_1}{p} \cdot \frac{1 - g(u)}{1 - u}, \tag{2.7}$$

$$K_b(u) = \frac{1}{\rho} \cdot \frac{1 - \beta(\lambda - \lambda g(u))}{1 - g(u)}, \tag{2.8}$$

$$K_c(u) = \frac{1 - \rho}{1 - \rho_1} \cdot \frac{1 - u}{\beta_2(\lambda - \lambda g(u)) - u}. \tag{2.9}$$

$R_0(z_2)$ will be used for us to obtain the tail asymptotic properties in Theorem 1(1). In Step 2, we will show that all $K_a(u)$, $K_b(u)$ and $K_c(u)$ can be viewed as the PGFs of three independent r.v.s., with detailed probabilistic interpretations, and therefore $K(u)$ is also a PGF.

It is worth mentioning that (i) $\beta(s)$ in (2.1) is the LST of the mixed distribution $F_\beta(x) \stackrel{\text{def}}{=} qF_{\beta_1}(x) + pF_{\beta_2}(x)$; and (ii) both $h(z_2)$ in (2.3) and $g(z_2)$ in (2.2) can be regarded as the PGFs of r.v.s, which will be verified in the next subsections.

2.1.1 Probabilistic interpretations of $h(z_2)$

$h(z_2)$ is a function in the expression of function $g(z_2)$ (see (2.2)), which appears in all $K_a(u)$, $K_b(u)$ and $K_c(u)$ (see (2.7), (2.8) and (2.9)). Therefore, as will be seen, the probabilistic interpretations of both $h(z_2)$ and $g(z_2)$ are crucial for the stochastic decomposition of $K(z_2)$.

In this section, we show that $h(z_2)$ is closely related to the busy period T_α of the standard $M/G/1$ queue with arrival rate λ_1 and the service time T_{β_1}. By $F_\alpha(x)$, we denote the probability distribution function of T_α and by $\alpha(s)$ the LST of $F_\alpha(x)$. The following are well-known results about this standard $M/G/1$ queue:

$$\alpha(s) = \beta_1(s + \lambda_1 - \lambda_1\alpha(s)), \tag{2.10}$$

$$\alpha_1 \stackrel{\text{def}}{=} E(T_\alpha) = \beta_{1,1}/(1 - \rho_1). \tag{2.11}$$

Throughout this paper, we use the notation $N_b(t)$ to represent the number of Poisson arrivals with rate b within the time interval $(0, t]$. Now, let us consider $N_{\lambda_2}(T_\alpha)$, the number of arrivals of a Poisson process at rate λ_2 within an independent random time T_α. The PGF of $N_{\lambda_2}(T_\alpha)$ is easily obtained as follows:

$$E\left(z_2^{N_{\lambda_2}(T_\alpha)}\right) = \int_0^\infty \sum_{n=0}^{\infty} z_2^n \frac{(\lambda_2 x)^n}{n!} e^{-\lambda_2 x} \mathrm{d}F_\alpha(x) = \alpha(\lambda_2 - \lambda_2 z_2). \tag{2.12}$$

It then follows from (2.10) that

$$\alpha(\lambda_2 - \lambda_2 z_2) = \beta_1(\lambda - \lambda_1\alpha(\lambda_2 - \lambda_2 z_2) - \lambda_2 z_2). \tag{2.13}$$

By comparing (2.3) and (2.13) and noticing the uniqueness of $h(z_2)$, we immediately have

$$h(z_2) = \alpha(\lambda_2 - \lambda_2 z_2), \tag{2.14}$$

which, together with (2.12), implies that $h(z_2) = E(z_2^{N_{\lambda_2}(T_\alpha)})$ is the PGF of the non-negative integer-valued r.v. $N_{\lambda_2}(T_\alpha)$.

2.1.2 Probabilistic interpretations of $g(z_2)$

For $g(z_2)$ defined in (2.2), we show that it is also the PGF of a non-negative integer-valued r.v., denoted by X_g, i.e., $g(z_2) = E(z_2^{X_g})$. In fact, it follows from (2.2) that

$$X_g \stackrel{\mathrm{d}}{=} \begin{cases} 1, & \text{with probability } p, \\ N_{\lambda_2}(T_\alpha), & \text{with probability } q. \end{cases} \tag{2.15}$$

Also, it is easy to obtain that $E(N_{\lambda_2}(T_\alpha)) = \lambda_2\beta_{1,1}/(1-\rho_1)$ and

$$E(X_g) = p + q\lambda_2\beta_{1,1}/(1-\rho_1) = p/(1-\rho_1). \tag{2.16}$$

2.2 Generating functions $R_i(z_1, z_2)$ for $i = 1, 2$

In this section, we rewrite the generating functions $R_i(z_1, z_2)$, $i = 1, 2$, obtained in [14], in a different form preferred for stochastic decompositions.

To this direction, for $i = 1, 2$, let $F^{(e)}_{\beta_i}(x)$ be the so-called equilibrium distribution of the service distributions $F_{\beta_i}(x)$, which is defined as $F^{(e)}_{\beta_i}(x) \stackrel{\text{def}}{=} \beta^{-1}_{i,1}\int_0^x(1-F_{\beta_i}(t))\mathrm{d}t$. The LST of $F^{(e)}_{\beta_i}(x)$ can be written as $\beta^{(e)}_i(s) = (1-\beta_i(s))/(\beta_{i,1}s)$. Let $T^{(e)}_\beta$ be a r.v. having the distribution $F^{(e)}_\beta(x)$.

Similarly, let $F^{(e)}_\beta(x) \stackrel{\text{def}}{=} (q\beta_{1,1} + p\beta_{2,1})^{-1}\int_0^x(1-F_\beta(t))\mathrm{d}t$ be the equilibrium distribution of the mixed distribution $F_\beta(t)$ (see the last paragraph in Sect. 2.1) of the service times. The LST of $F^{(e)}_\beta(x)$ can be written as $\beta^{(e)}(s) = (1-\beta(s))/((q\beta_{1,1} + p\beta_{2,1})s)$. For $i = 1, 2$, let $T^{(e)}_{\beta_i}$ be a r.v. having the distribution $F^{(e)}_{\beta_i}(x)$. Moreover, let $F^{(e)}_\alpha(x) = \alpha_1^{-1}\int_0^x(1-F_\alpha(t))\mathrm{d}t$ be the equilibrium distribution of $F_\alpha(x)$, which is the busy period function defined in Sect. 2.1.1. The LST of $F^{(e)}_\alpha(x)$ can be written as $\alpha^{(e)}(s) = (1-\alpha(s))/(\alpha_1 s)$. Let $T^{(e)}_\alpha$ be a r.v., having the distribution $F^{(e)}_\alpha(x)$. This notation is introduced here for convenience, but will not be used until Sect. 3.

Then, the PGFs $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$, obtained in [14], can be rewritten as

$$R_1(z_1, z_2) = M_2(z_1, z_2)\cdot M_1(z_1, z_2)\cdot S_{\beta_1}(z_1, z_2)\cdot R_0(z_2), \tag{2.17}$$
$$R_2(z_1, z_2) = S_{\beta_2}(z_1, z_2)\cdot K_a(z_2)\cdot K_c(z_2)\cdot R_0(z_2), \tag{2.18}$$

where $K_a(z_2)$, $K_c(z_2)$ and $R_0(z_2)$ are given in (2.7), (2.9) and (2.4), respectively, and

$$M_1(z_1, z_2) = (1-\rho_1)\cdot\frac{h(z_2)-z_1}{\beta_1(\lambda-\lambda_1 z_1-\lambda_2 z_2)-z_1}, \tag{2.19}$$
$$M_2(z_1, z_2) = \frac{1-\rho}{\lambda_1(1-\rho_1)}\left[\frac{(\lambda-\lambda g(z_2))\,(\beta_2(\lambda-\lambda g(z_2))-\beta_2(\lambda-\lambda_1 z_1-\lambda_2 z_2))}{(\beta_2(\lambda-\lambda g(z_2))-z_2)(h(z_2)-z_1)}+\lambda_1\right], \tag{2.20}$$
$$S_{\beta_i}(z_1, z_2) = \frac{1}{\beta_{i,1}}\cdot\frac{1-\beta_i(\lambda-\lambda_1 z_1-\lambda_2 z_2)}{\lambda-\lambda_1 z_1-\lambda_2 z_2} = \beta^{(e)}_i(\lambda-\lambda q z_1-\lambda p z_2), \quad i = 1, 2. \tag{2.21}$$

$R_1(z_1, z_2)$ and $R_2(z_1, z_2)$ will be used for us to obtain the tail asymptotic properties in Theorem 1(2). In Step 2, we will show that all factors in the expressions of $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$ can be viewed as the PGFs of some r.v.s., with detailed probabilistic interpretations.

3 Stochastic decompositions—step 2

In this section, we show that all three factors $K_a(u)$, $K_b(u)$ and $K_c(u)$ in the expression of $K(u)$ [see (2.6)] are PGFs. Therefore, the integrand function $K(u)$ in the PGF $R_0(z_2)$ is also a PGF. Furthermore, we show that all factors in the PGFs $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$ [see (2.17) and (2.18)] are PGFs. These decomposition results are needed in Step 3 for deriving the tail asymptotic results stated in Theorem 1.

3.1 Stochastic decomposition of *K*(*u*)

Denote

$$\vartheta = \rho_2/(1-\rho_1) < 1, \tag{3.1}$$

where the inequality follows from the stability condition $\rho = \rho_1 + \rho_2 < 1$.

The decomposition result for $K(u)$ can be stated in the following proposition.

Proposition 3.1 *All factors $K_a(u)$, $K_b(u)$ and $K_c(u)$ in the expression of $K(u)$, given in (2.6), are PGFs. Therefore, $K(u)$ itself is also a PGF. Furthermore, if K is a r.v. having the PGF $K(u)$, and K_a, K_b and K_c are independent r.v.s having the PGFs $K_a(u)$, $K_b(u)$ and $K_c(u)$, respectively, then K can be stochastically decomposed as*

$$K \stackrel{\mathrm{d}}{=} K_a + K_b + K_c, \tag{3.2}$$

where K_a, K_b and K_c can be specifically constructed by

$$K_a \stackrel{\mathrm{d}}{=} \begin{cases} 0, & \text{with probability } 1-\rho_1, \\ N_{\lambda_2}(T_\alpha^{(e)}), & \text{with probability } \rho_1; \end{cases} \tag{3.3}$$

$$K_b \stackrel{\mathrm{d}}{=} N_{\lambda, X_g}(T_\beta^{(e)}); \tag{3.4}$$

and

$$K_c \stackrel{\mathrm{d}}{=} \begin{cases} 0, & \text{with probability } 1-\vartheta, \\ \sum_{i=1}^{J} X_c^{(i)}, & \text{with probability } \vartheta. \end{cases} \tag{3.5}$$

In the above expressions, ϑ is defined by (3.1); $N_{\lambda_2}(T_\alpha^{(e)})$ is the number of arrivals for the Poisson process with rate λ_2 within the time interval $(0, T_\alpha^{(e)}]$; $N_{\lambda, X_g}(T_\beta^{(e)})$ represents the number of the batched Poisson arrivals, with rate λ and batch size X_g, within the time interval $(0, T_\beta^{(e)}]$, where X_g has the PGF $g(z)$ given in (2.2); and $\{X_c^{(i)}\}_{i=1}^\infty$ is a sequence of i.i.d. non-negative integer-valued r.v.s., each with the same PGF $K_a(z) \cdot \beta_2^{(e)}(\lambda - \lambda g(z))$, namely, $X_c^{(i)} \stackrel{\mathrm{d}}{=} K_a + N_{\lambda, X_g}(T_{\beta_2}^{(e)})$, where the two components are assumed to be independent, and $P(J = i) = (1-\vartheta)\vartheta^{i-1}$, $i \geq 1$, independent of $\{X_c^{(i)}\}_{i=1}^\infty$.

We defer the proof of Proposition 3.1 to Appendix B.

3.2 Stochastic decompositions of $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$

It was proved in [14] and in Sect. 3.1, respectively, that $R_0(z_2)$, $K_a(z_2)$ and $K_c(z_2)$ are PGFs. In this section, we show that all four remaining factors $M_i(z_1, z_2)$ and $S_{\beta_i}(z_1, z_2)$, for $i = 1, 2$, in the expressions of $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$ (given in (2.17) and (2.18), respectively) are also PGFs. Compared to the effort for the exhaustive stochastic decomposition for $K(z_2)$, it is more demanding here to get decompositions for $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$.

For stating the decomposition result, we need some preparations. First, for the probabilistic interpretations of the functions $S_{\beta_i}(z_1, z_2)$, $i = 1, 2$, we introduce the concept of splitting.

Definition 3.1 Let N be a non-negative integer-valued r.v., and let $\{X_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. Bernoulli r.v.s, which is independent of N, having the common 0-1 distribution $P\{X_k = 1\} = c$ and $P\{X_k = 0\} = 1 - c$ with $0 < c < 1$. The two-dimensional r.v. $(\sum_{k=1}^{N} X_k, N - \sum_{k=1}^{N} X_k)$, where $\sum_{1}^{0} \equiv 0$, is called an independent $(c, 1 - c)$-splitting of N, denoted by split$(N; c, 1 - c)$.

For probabilistic interpretations of $M_i(z_1, z_2)$, $i = 1, 2$, we introduce the following r.v.s. Let $\{(Y_{n,1}, Y_{n,2})\}_{n=1}^{\infty}$ be a sequence of independent two-dimensional r.v.s, each with the common PGF $qz_1 + pz_2$, and let $\{Z_m\}_{m=1}^{\infty}$ be a sequence of independent r.v.s, each with the common PGF $g(z_2)$. Assume that these two sequences are independent. In terms of the above two sequences, we construct, for $k \geq 1$,

$$(D_{k,1}, D_{k,2}) \stackrel{\mathrm{d}}{=} \sum_{n=1}^{i-1}(Y_{n,1}, Y_{n,2}) + \sum_{m=1}^{k-i}(0, Z_m), \quad \text{with probability } 1/k, \quad i = 1, 2, \ldots, k, \tag{3.6}$$

and then, in terms of $(D_{k,1}, D_{k,2})$, we construct

$$(H_{\beta_1,1}, H_{\beta_1,2}) \stackrel{\mathrm{d}}{=} (D_{k,1}, D_{k,2}) \text{ with probability } (q/\rho_1)kb_{\beta_1,k}, \quad k \geq 1, \tag{3.7}$$

$$(H_{\beta_2,1}, H_{\beta_2,2}) \stackrel{\mathrm{d}}{=} (D_{k,1}, D_{k,2}) \text{ with probability } (p/\rho_2)kb_{\beta_2,k}, \quad k \geq 1, \tag{3.8}$$

where

$$b_{\beta_1,k} = \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \mathrm{d}F_{\beta_1}(t), \quad k \geq 1, \tag{3.9}$$

and

$$b_{\beta_2,k} = \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \mathrm{d}F_{\beta_2}(t), \quad k \geq 1. \tag{3.10}$$

Here, it is worthwhile to mention that $(q/\rho_1) \sum_{k=1}^{\infty} kb_{\beta_1,k} = 1$ (see (B.13) for a proof), and $(p/\rho_2) \sum_{k=1}^{\infty} kb_{\beta_2,k} = 1$ (its proof is similar).

We can now state the decomposition results for $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$.

Proposition 3.2 *Besides* $R_0(z_2)$*,* $K_a(z_2)$ *and* $K_c(z_2)$*, all the other four factors* $M_i(z_1, z_2)$ *and* $S_{\beta_i}(z_1, z_2)$*,* $i = 1, 2$*, in the expressions of* $R_1(z_1, z_2)$ *and* $R_2(z_1, z_2)$*,* *given in* (2.17) *and* (2.18)*, respectively, are also PGFs. Furthermore, for* $i = 1, 2$*, if* (M_{i1}, M_{i2}) *and* $(S_{\beta_i,1}, S_{\beta_i,2})$ *are two-dimensional r.v.s having the PGF* $M_i(z_1, z_2)$ *and* $S_{\beta_i}(z_1, z_2)$*, respectively, then the r.v.s* (R_{i1}, R_{i2}) *for* $i = 1, 2$ *can be stochastically decomposed as*

$$(R_{11}, R_{12}) \stackrel{\mathrm{d}}{=} (M_{21}, M_{22}) + (M_{11}, M_{12}) + (S_{\beta_1,1}, S_{\beta_1,2}) + (0, R_0), \tag{3.11}$$

$$(R_{21}, R_{22}) \stackrel{\mathrm{d}}{=} (S_{\beta_1,1}, S_{\beta_1,2}) + (0, K_a) + (0, K_c) + (0, R_0), \tag{3.12}$$

where, for $i = 1, 2$*,*

$$(S_{\beta_i,1}, S_{\beta_i,2}) \stackrel{\mathrm{d}}{=} \mathrm{split}(N_\lambda(T_{\beta_i}^{(e)}); q, p) \tag{3.13}$$

with $N_\lambda(T_{\beta_i}^{(e)})$ *being the number of Poisson arrivals with rate* λ *within the time interval* $(0, T_{\beta_i}^{(e)}]$*;*

$$(M_{11}, M_{12}) \stackrel{\mathrm{d}}{=} \begin{cases} 0, & \textit{with probability } 1 - \rho_1, \\ \sum_{i=1}^{J} (H_{\beta_1,1}^{(i)}, H_{\beta_1,2}^{(i)}), & \textit{with probability } \rho_1, \end{cases} \tag{3.14}$$

where J *is independent of* $(H_{\beta_1,1}^{(i)}, H_{\beta_1,2}^{(i)})$ $(i \geq 1)$*, having the distribution* $P(J = i) = (1 - \rho_1)\rho_1^{i-1}$*,* $i \geq 1$*;*

$$(M_{21}, M_{22}) \stackrel{\mathrm{d}}{=} \begin{cases} 0, & \textit{with probability } 1 - \vartheta, \\ (H_{\beta_2,1}, H_{\beta_2,2}) + (0, K_a) + (0, K_c), & \textit{with probability } \vartheta; \end{cases} \tag{3.15}$$

and the probabilistic structures for K_a*,* K_c *and* R_0 *have been provided in previous sections.*

We defer the proof of Proposition 3.2 to Appendix B.

4 Tail asymptotics—step 3

In this section, based on the stochastic decomposition results obtained in the previous section, we prove the tail asymptotic results presented in Theorem 1. Before that, we provide some comments on the assumptions in A1 and A2. The service time T_{β_1} has a so-called regularly varying tail at ∞. It is well known that for a distribution F on $(0, \infty)$, if $\overline{F}$ is regularly varying with index $-\sigma$, $\sigma \geq 0$ (see Definition A.1) or $\overline{F} \in \mathcal{R}_{-\sigma}$, then F is subexponential (see Definition A.2) or $F \in \mathcal{S}$ (see, for example Embrechts et al. [12]).

Clearly, under the assumptions in A1 and A2, the service time T_{β_1} of Type-1 customers has a tail probability heavier than the service time T_{β_2} of Type-2 customers. If $r > 0$, T_{β_2} has a light tail, i.e., $E(e^{\varepsilon T_{\beta_2}}) < \infty$ for some $\varepsilon > 0$. If $r = 0$, then T_{β_2} has a regularly varying tail with index $-a_2$. We also notice that since T_α, introduced in

Sect. 2.1.1, is the busy period of the standard $M/G/1$ queue with arrival rate λ_1 and service time T_{β_1}, its asymptotic tail probability is regularly varying according to de Meyer and Teugels [10] (see Lemma A.1).

4.1 Tail asymptotics for R_0—Proof of Theorem 1(1)

In order to obtain the tail asymptotic property for R_0 [or Theorem 1(1)], we first present tail asymptotic properties for the components K_a, K_b and K_c, respectively, and then the property for K. These results are summarized into the following lemma.

Lemma 4.1 *As $j \to \infty$, we have*

(1)

$$P\{K_a > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{(a_1-1)(1-\rho_1)^{a_1}} \cdot j^{-a_1+1} L(j), \tag{4.1}$$

$$P\{K_b > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1} L(j), \tag{4.2}$$

$$P\{K_c > j\} \sim \frac{\vartheta}{1-\vartheta} P\{K_a > j\}; \tag{4.3}$$

(2)

$$P\{K > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho(1-\rho)(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1} L(j). \tag{4.4}$$

Refer to Appendix C for a proof of Lemma 4.1.

Proof of Theorem 1(1) By (2.4), the PGF $R_0(z)$ is expressed in terms of the PGF $K(z)$. Therefore, the tail probability of R_0 is determined by the tail probability of K. The following asymptotic result is a straightforward application of Theorem 5.1 in [25]:

$$P\{R_0 > j\} \sim \frac{a_1-1}{a_1}\psi \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho(1-\rho)(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1} L(j), \tag{4.5}$$

where ψ is given in (2.5).

4.2 Tail asymptotics for R_{ik} ($i, k = 1, 2$)—Proof of Theorem 1(2)

In order to obtain tail asymptotic properties for R_{ik}, $i, k = 1, 2$, based on the stochastic decompositions for R_{ik} in (3.11) and (3.12), we first present tail asymptotic properties for the components $S_{\beta_i,k}$ and M_{ik} in (3.13), (3.14) and (3.15). These results are summarized into the following lemma.

Lemma 4.2 *As $j \to \infty$, we have*

(1)

$$P\{S_{\beta_1,1} > j\} \sim \frac{\lambda_1^{a_1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j), \tag{4.6}$$

$$P\{S_{\beta_1,2} > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j), \tag{4.7}$$

$$P\{S_{\beta_2,1} > j\} \sim \begin{cases} \frac{\lambda_2 \lambda_1^{a_2-1}}{\rho_2(a_2-1)} \cdot j^{-a_2+1} L(j), & \textit{if } r = 0, \\ \frac{\lambda_2 \lambda_1 (\lambda_1+r)^{a_2-1}}{\rho_2 r} \cdot \left(\frac{\lambda_1}{\lambda_1+r}\right)^j j^{-a_2} L(j), & \textit{if } r > 0, \end{cases} \tag{4.8}$$

$$P\{S_{\beta_2,2} > j\} \sim \begin{cases} \frac{\lambda_2^{a_2}}{\rho_2(a_2-1)} \cdot j^{-a_2+1} L(j), & \textit{if } r = 0, \\ \frac{\lambda_2^2 (\lambda_2+r)^{a_2-1}}{\rho_2 r} \cdot \left(\frac{\lambda_2}{\lambda_2+r}\right)^j j^{-a_2} L(j), & \textit{if } r > 0; \end{cases} \tag{4.9}$$

(2)

$$P\{M_{11} > j\} \sim \frac{\rho_1}{1-\rho_1} P\{S_{\beta_1,1} > j\}, \tag{4.10}$$

$$P\{M_{12} > j\} \sim \left[\frac{1}{(1-\rho_1)^{a_1}} - 1\right] \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j), \tag{4.11}$$

$$P\{M_{21} > j\} = \vartheta P\{S_{\beta_2,1} > j\}, \tag{4.12}$$

$$P\{M_{22} > j\} \sim \vartheta \frac{\lambda_1 \lambda_2^{a_1-1}}{(1-\rho)(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1} L(j). \tag{4.13}$$

Refer to Appendix C for a proof of Lemma 4.2.

Now, we are in the position to prove Theorem 1(2). Note that the tail asymptotic properties for K_a, K_c and R_0 have already been derived earlier. Based on Lemma 4.2 and by identifying dominant contributions made from the tail probabilities $P\{M_{ik} > j\}$, $P\{S_{\beta_i,k} > j\}$, $P\{K_a + K_c > j\}$ and $P\{R_0 > j\}$ to $P\{R_{ik} > j\}$, Theorem 1(2) can be proved as follows.

Proof of Theorem 1(2) Recall (3.11). By (4.12), (4.8) and (1.1), M_{21} and R_0 have tail probabilities lighter than $j^{-a_1+1}L(j)$, and by (4.10), (4.13), (4.6) and (4.7), M_{11}, M_{22}, $S_{\beta_1,1}$ and $S_{\beta_1,2}$ have regularly varying tails with index $-a_1+1$. By (3.11) and applying Lemma A.5, we obtain

$$\begin{aligned} P\{R_{11} > j\} &= P\{M_{21} + M_{11} + S_{\beta_1,1} > j\} \\ &\sim P\{M_{11} + S_{\beta_1,1} > j\} \sim \frac{1}{1-\rho_1} P\{S_{\beta_1,1} > j\} \quad \text{(refer to (4.6))}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} P\{R_{12} > j\} &= P\{M_{22} + M_{12} + S_{\beta_1,2} + R_0 > j\} \\ &\sim P\{M_{22} + M_{12} + S_{\beta_1,2} > j\} \quad \text{(refer to (4.13), (4.11) and (4.7))}. \end{aligned} \tag{4.15}$$

By a similar argument, it follows from (3.12) that

$$P\{R_{21} > j\} = P\{S_{\beta_2,1} > j\} \quad \text{(refer to (4.8))}, \tag{4.16}$$
$$P\{R_{22} > j\} = P\{S_{\beta_2,2} + K_a + K_c + R_0 > j\}$$
$$\sim P\{K_a + K_c > j\} \quad \text{(refer to (4.1) and (4.3))}, \tag{4.17}$$

where we have used the fact, by (4.9), that $S_{\beta_2,2}$ has a tail probability lighter than $j^{-a_1+1}L(j)$.

Recall the definition of R_{ik}, $i, k = 1, 2$, given in Sect. 1. We know that $P\{R_{\text{que}} > j | I_{\text{ser}} = i\} = P\{R_{i1} > j\}$ and $P\{R_{\text{orb}} > j | I_{\text{ser}} = i\} = P\{R_{i2} > j\}$, $i = 1, 2$. This completes the proof of Theorem 1(2). □

5 Concluding remarks

In this paper, we considered a priority retrial queueing model, which was first considered in [14]. This is a relatively complex queueing system with two types of arrivals, priority and non-priority, joining a regular queue and a retrial orbit, respectively, and served by a singer server. This model was considered in [14], and expressions of the generating functions for some key probability measures were obtained by the authors. However, tail probability behaviour, which is important on its own merits and also in applications, of these key probability measures was not touched in [14], which is the focus of this paper. We proposed an exhaustive stochastic decomposition approach to decompose a complicated generating function (or a transformation) into independent components with clear probabilistic interpretations. For each component in the decomposition, we provide tail asymptotic results for the associated probabilities, with which we successfully obtained tail asymptotic results (Theorem 1) for probabilities of the number of customers in the queue or in the orbit, under various conditions.

The proposed exhaustive stochastic decomposition approach is a generalization of the standard decomposition, frequently used in probability and also in queueing applications. We expect that this decomposition approach can be used for other queueing systems and in other areas of applied probability. In fact, this paper is one of the sister papers (see, for example, [25,27,28]) in exploring its potential usefulness and importance.

It is of interest to continue the research in this direction. One of the natural extensions is to apply this exhaustive stochastic decomposition approach to other queueing systems. Another interesting problem is to explore its potentials in the analysis of statistical queueing networks, defined through big data (see, for example, [33]).

To conclude the paper, we would like to provide intuition on the results in Theorem 1(2). First, let us recall a well-known result for the standard $M/G/1$ queue: If the service time is regularly varying with index $-a_1$, then the stationary queue length is also regularly varying, but with index $-a_1+1$. Such a conclusion can be made through a distributional Little's law (see, for example [3]). For the model studied in this paper, the condition $I_{\text{ser}} = 1$ means that the server is serving a Type-1 customer. Under this condition, both types of customers have to wait, customers of Type-1 in the queue and customers of Type-2 in the orbit. Therefore, both $R_{\text{que}}|I_{\text{ser}} = 1$ and $R_{\text{orb}}|I_{\text{ser}} = 1$ have an asymptotic tail of the form $Const \cdot j^{-a_1+1}L(j)$ (given in (1.2) and (1.3), respectively), due to the regularly varying assumption for the service time of Type-1 customers in Assumption A1. Here, and throughout the appendices, $Const$ stands for a (but not necessarily the same) constant. On the other hand, the condition $I_{\text{ser}} = 2$ means that the server is serving a Type-2 (lower-priority) customer, which implies that no Type-1 customers were waiting in the queue at the beginning of service of this Type-2 customer. In other words, $I_{\text{ser}} = 2$ implies that all Type-1 customers in the queue must be those who arrived after the beginning of the service time of this Type-2 customer. Therefore, $R_{\text{que}}|I_{\text{ser}} = 2$ has an asymptotic tail of the form given in (1.4), determined by the service time assumption (in Assumption A2) of Type-2 customers. However, $R_{\text{orb}}|I_{\text{ser}} = 2$ still has an asymptotic tail of the form $Const \cdot j^{-a_1+1}L(j)$ (see (1.5), determined by the assumption on Type-1 customer's service time), since the customers in the orbit could be those who arrived during the service times of Type-2 customers, and/or during the service times of Type-1 customers who were served before the current Type-2 customer in service, due to the priority discipline.

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Appendix A: Definitions and useful results from the literature

Definition A.1 (for example, see Bingham, Goldie and Teugels [4]) A measurable function $U : (0, \infty) \to (0, \infty)$ is regularly varying at ∞ with index $\sigma \in (-\infty, \infty)$ (written $U \in \mathcal{R}_\sigma$) iff $\lim_{t\to\infty} U(xt)/U(t) = x^\sigma$ for all $x > 0$. If $\sigma = 0$ we call U slowly varying, i.e., $\lim_{t\to\infty} U(xt)/U(t) = 1$ for all $x > 0$.

For a distribution function F, denote $\bar{F} \overset{\text{def}}{=} 1 - F$ for the remainder of the paper.

Definition A.2 (for example, see Foss, Korshunov and Zachary [17]) A distribution F on $(0, \infty)$ belongs to the class of subexponential distribution (written $F \in \mathcal{S}$) if $\lim_{t\to\infty} \overline{F^{*2}}(t)/\overline{F}(t) = 2$, where $\overline{F} = 1 - F$ and F^{*2} denotes the second convolution of F.

Lemma A.1 *(de Meyer and Teugels [10]) Under Assumption A1,*

$$P\{T_\alpha > t\} \sim \frac{1}{(1-\rho_1)^{a_1+1}} \cdot t^{-a_1}L(t) \quad \text{as } t \to \infty. \tag{A.1}$$

The result (A.1) is straightforward due to the main theorem in [10].

Lemma A.2 *(pp.580–581 in [12]) Let N be a r.v. with $P\{N = k\} = (1-\sigma)\sigma^{k-1}$, $0 < \sigma < 1$, $k \geq 1$, and $\{Y_k\}_{k=1}^\infty$ be a sequence of non-negative, i.i.d. r.v.s having a common subexponential distribution F. Define $S_n = \sum_{k=1}^n Y_k$. Then, $P\{S_N > t\} \sim (1 - F(t))/(1-\sigma)$ as $t \to \infty$.*

Lemma A.3 *(Proposition 3.1 in [3], or Theorem 3.1 in [16]) Let $N_\lambda(t)$ be a Poisson process with rate λ and let T be a positive r.v. with distribution F, which is independent of $N_\lambda(t)$. If $\bar{F}(t) = P\{T > t\}$ is heavier than $e^{-\sqrt{t}}$ as $t \to \infty$, then $P(N_\lambda(T) > j) \sim P\{T > j/\lambda\}$ as $j \to \infty$.*

Lemma A.3 holds for any distribution F with a regularly varying tail because it is heavier than $e^{-\sqrt{t}}$ as $t \to \infty$.

Lemma A.4 *(p.181 in [20]) Let $N_\lambda(t)$ be a Poisson process with rate λ and let T be a positive r.v. with distribution F, which is independent of $N_\lambda(t)$. If $\bar{F}(t) = P\{T > t\} \sim e^{-wt}t^{-h}L(t)$ as $t \to \infty$ for $w > 0$ and $-\infty < h < \infty$, then*

$$P(N_\lambda(T) > j) \sim \lambda(\lambda + w)^{h-1}\left(\frac{\lambda}{\lambda + w}\right)^j j^{-h}L(j), \quad j \to \infty.$$

Lemma A.5 *(p.48 in [17]) Let F, F_1 and F_2 be distribution functions. Suppose that $F \in \mathcal{S}$. If $\bar{F}_i(t)/\bar{F}(t) \to c_i$ as $t \to \infty$ for some $c_i \geq 0$, $i = 1, 2$, then $\overline{F_1 * F_2}(t)/\bar{F}(t) \to c_1 + c_2$ as $t \to \infty$, where the notation $F_1 * F_2$ stands for the convolution of F_1 and F_2.*

Lemma A.6 *(pp.162–163 in [20]) Let N be a discrete non-negative integer-valued r.v. with mean value μ_N, and let $\{Y_k\}_{k=1}^{\infty}$ be a sequence of non-negative i.i.d. r.v.s with mean value μ_Y. Define $S_0 \equiv 0$ and $S_n = \sum_{k=1}^n Y_k$. If $P\{Y_k > x\} \sim c_Y x^{-h}L(x)$ as $x \to \infty$ and $P\{N > m\} \sim c_N m^{-h}L(m)$ as $m \to \infty$, where $h > 1$, $c_Y \geq 0$ and $c_N \geq 0$, then $P\{S_N > x\} \sim (c_N\mu_Y^h + \mu_N c_Y)x^{-h}L(x)$ as $x \to \infty$.*

Remark A.1 It is a convention that in Lemma A.6, $c_Y = 0$ and $c_N = 0$ means that $\lim_{x\to\infty} P\{Y_k > x\}/(x^{-h}L(x)) = 0$ and $\lim_{m\to\infty} P\{N > m\}/(m^{-h}L(m)) = 0$, respectively.

The following two criteria are from Feller (see p.441 in [15]), which are often used to verify that a function is completely monotone.

Criterion A.1 If $\vartheta_1(\cdot)$ and $\vartheta_2(\cdot)$ are completely monotone, so is their product $\vartheta_1(\cdot)\vartheta_2(\cdot)$.

Criterion A.2 If $\vartheta_3(\cdot)$ is completely monotone and $\vartheta_4(\cdot)$ is a positive function with a completely monotone derivative $\vartheta_4'(\cdot)$, then $\vartheta_3(\vartheta_4(\cdot))$ is completely monotone.

To prove Lemma C.2, let us list some notations and results which will be used. Let $F(x)$ be any distribution on $[0, \infty)$ with the LST $\phi(s)$. We denote the nth moment of $F(x)$ by ϕ_n, $n \geq 0$. It is well known that $\phi_n < \infty$ iff

$$\phi(s) = \sum_{k=0}^{n} \frac{\phi_k}{k!}(-s)^k + o(s^n), \quad n \geq 0. \tag{A.2}$$

Based on (A.2), we introduce the notation $\phi_n(s)$ and $\widehat{\phi}_n(s)$, defined by

$$\phi_n(s) \stackrel{\text{def}}{=} (-1)^{n+1}\left\{\phi(s) - \sum_{k=0}^{n} \frac{\phi_k}{k!}(-s)^k\right\}, \quad n \geq 0, \tag{A.3}$$

$$\widehat{\phi}_n(s) \stackrel{\text{def}}{=} \phi_n(s)/s^{n+1}, \quad n \geq 0. \tag{A.4}$$

Lemma A.7 *(pp.333–334 in [4]) Assume that $n < d < n+1$, $n \in \{0, 1, 2, \ldots\}$, then the following two statements are equivalent:*

$$1 - F(t) \sim t^{-d} L(t), \quad t \to \infty;$$

and

$$\phi_n(s) \sim \frac{\Gamma(d-n)\Gamma(n+1-d)}{\Gamma(d)} s^d L(1/s), \quad s \downarrow 0.$$

Lemma A.8 *(Lemma 3.3 in [27]) Assume that $n \in \{1, 2, \ldots\}$, then the following two statements are equivalent:*

$$1 - F(t) \sim t^{-n} L(t), \quad t \to \infty; \tag{A.5}$$

and

$$\lim_{s \downarrow 0} \frac{\widehat{\phi}_{n-1}(xs) - \widehat{\phi}_{n-1}(s)}{L(1/s)/(n-1)!} = -\log x, \quad \textit{for all } x > 0. \tag{A.6}$$

In [27], Lemma A.8 was proved by applying Karamata's theorem, the monotone density theorem and Theorem 3.9.1 presented in [4] (see, p.27, p.39 and pp.172–173, respectively).

Appendix B: Proofs for results in step 2

Proof of Proposition 3.1

By (2.14) and the definition of $\alpha^{(e)}(s)$, we can write $(1 - h(u))/(1 - u) = \lambda_2\alpha_1 \cdot \alpha^{(e)}(\lambda_2 - \lambda_2 u)$, from which, and by (2.7), (2.2) and (2.11), we have,

$$K_a(u) = \frac{1-\rho_1}{p}\left[q \cdot \frac{1-h(u)}{1-u} + p\right] = \rho_1\alpha^{(e)}(\lambda_2 - \lambda_2 u) + 1 - \rho_1, \tag{B.1}$$

which leads to the stochastic decomposition given in (3.3) for the r.v. K_a.

It follows from (2.8) that

$$K_b(z) = \beta^{(e)}(\lambda - \lambda g(z)) = \int_0^\infty \sum_{n=0}^{\infty} (g(z))^n \frac{(\lambda x)^n}{n!} e^{-\lambda x} \mathrm{d}F_\beta^{(e)}(x) = E\left(z^{N_{\lambda, X_g}(T_\beta^{(e)})}\right), \tag{B.2}$$

which leads to the stochastic decomposition given in (3.4) for the r.v. K_b.

It follows from (2.9) that

$$\begin{aligned} K_c(u) &= (1-\vartheta)\left[1-\frac{1-g(u)}{1-u}\cdot\frac{1-\beta_2(\lambda-\lambda g(u))}{1-g(u)}\right]^{-1} \\ &= \frac{1-\vartheta}{1-\vartheta\cdot K_a(u)\cdot\beta_2^{(e)}(\lambda-\lambda g(u))} \\ &= 1-\vartheta+\vartheta\cdot\sum_{i=1}^{\infty}(1-\vartheta)\vartheta^{i-1}\big(K_a(u)\cdot\beta_2^{(e)}(\lambda-\lambda g(u))\big)^i, \end{aligned} \tag{B.3}$$

which leads to the stochastic decomposition given in (3.5) for the r.v. K_c.

Finally, (3.2) follows immediately from (2.6).

Proof of Proposition 3.2

We divide the proof of Proposition 3.2 into three parts.

$S_{\beta_i}(z_1, z_2)$ are PGFs with detailed probabilistic interpretations

Following Definition 3.1, for the split $(N; c, 1-c)$ it is easy to see that $(\sum_{k=1}^{N} X_k, N-\sum_{k=1}^{N} X_k)$ has the PGF

$$\begin{aligned} E\big(z_1^{\sum_{k=1}^N X_k} z_2^{N-\sum_{k=1}^N X_k}\big) &= \sum_{n=0}^{\infty}\left[\prod_{k=1}^{n} E(z_1^{X_k} z_2^{1-X_k})\right] P\{N=n\} \\ &= \sum_{n=0}^{\infty}(cz_1+(1-c)z_2)^n P\{N=n\}, \end{aligned} \tag{B.4}$$

where $\prod_1^0 \equiv 1$.

Recalling (2.21), we can write, for $i=1,2$,

$$S_{\beta_i}(z_1,z_2) = \beta_i^{(e)}(\lambda-\lambda_1 z_1-\lambda_2 z_2) = \int_0^{\infty}\sum_{n=0}^{\infty}(qz_1+pz_2)^n((\lambda x)^n/n!)e^{-\lambda x}\mathrm{d}F_{\beta_i}^{(e)}(x), \tag{B.5}$$

which leads to (3.13) by setting $N = N_\lambda(T_{\beta_i}^{(e)})$ and $c = q$ in (B.4).

$M_1(z_1, z_2)$ is a PGF with a detailed probabilistic interpretation

It follows from (2.19) that

$$M_1(z_1,z_2) = \frac{1-\rho_1}{1-\rho_1 H_{\beta_1}(z_1,z_2)} = \sum_{n=0}^{\infty}(1-\rho_1)\rho_1^n\left(H_{\beta_1}(z_1,z_2)\right)^n, \tag{B.6}$$

where

$$H_{\beta_1}(z_1, z_2) = \frac{1}{\rho_1} \cdot \frac{\beta_1(\lambda - \lambda_1 z_1 - \lambda_2 z_2) - h(z_2)}{z_1 - h(z_2)} \tag{B.7}$$

$$= \frac{1}{\rho_1} \cdot \frac{\beta_1(\lambda - \lambda_1 z_1 - \lambda_2 z_2) - \beta_1(\lambda - \lambda_1 h(z_2) - \lambda_2 z_2)}{z_1 - h(z_2)} \quad \text{(by (2.3))}. \tag{B.8}$$

Clearly, by (B.6), (M_{11}, M_{12}) can be regarded as a random sum of two-dimensional r.v.s. provided that $H_{\beta_1}(z_1, z_2)$ is the PGF of a two-dimensional r.v. To verify this, we rewrite (B.8) as a power series. Note that

$$\beta_1(\lambda - \lambda q z_1 - \lambda p z_2) = \int_0^\infty \sum_{k=0}^\infty \frac{(\lambda(q z_1 + p z_2) t)^k}{k!} \cdot e^{-\lambda t} \mathrm{d}F_{\beta_1}(t)$$

$$= \beta_1(\lambda) + \sum_{k=1}^\infty b_{\beta_1,k} (q z_1 + p z_2)^k, \tag{B.9}$$

where $b_{\beta_1,k}$ is given in (3.9). By (2.3) and (B.9),

$$h(z_2) = \beta_1(\lambda - \lambda q h(z_2) - \lambda p z_2) = \beta_1(\lambda) + \sum_{k=1}^\infty b_{\beta_1,k} (q h(z_2) + p z_2)^k. \tag{B.10}$$

Substituting (B.9) and (B.10) into the numerator of the right-hand side of (B.8), we obtain

$$H_{\beta_1}(z_1, z_2) = \frac{1}{\rho_1} \cdot \sum_{k=1}^\infty b_{\beta_1,k} \left(\frac{(q z_1 + p z_2)^k - (q h(z_2) + p z_2)^k}{z_1 - h(z_2)} \right)$$

$$= \frac{q}{\rho_1} \cdot \sum_{k=1}^\infty b_{\beta_1,k} \sum_{i=1}^k (q z_1 + p z_2)^{i-1} (q h(z_2) + p z_2)^{k-i}$$

$$= \frac{q}{\rho_1} \cdot \sum_{k=1}^\infty k b_{\beta_1,k} \cdot D_k(z_1, z_2), \tag{B.11}$$

where

$$D_k(z_1, z_2) = \frac{1}{k} \sum_{i=1}^k (q z_1 + p z_2)^{i-1} (q h(z_2) + p z_2)^{k-i}. \tag{B.12}$$

Note that $q h(z_2) + p z_2 = g(z_2)$ and $q z_1 + p z_2$ are the PGFs of (one or two-dimensional) r.v.s. It follows from (B.12) that for $k \geq 1$, $D_k(z_1, z_2)$ is the PGF of a two-dimensional r.v. $(D_{k,1}, D_{k,2})$, which can be constructed according to (3.6).

In addition,

$$\sum_{k=1}^{\infty} k b_{\beta_1,k} = \sum_{k=1}^{\infty} \int_0^{\infty} \frac{(\lambda t)^k}{(k-1)!} e^{-\lambda t} \mathrm{d}F_{\beta_1}(t) = \lambda \int_0^{\infty} t \mathrm{d}F_{\beta_1}(t) = \rho_1/q. \quad \text{(B.13)}$$

Namely, $(q/\rho_1)\sum_{k=1}^{\infty} k b_{\beta_1,k} = 1$, which, together with (B.11), implies that $H_{\beta_1}(z_1, z_2)$ is the PGF of a two-dimensional r.v. $(H_{\beta_1,1}, H_{\beta_1,2})$, which can be constructed according to (3.7).

By (B.6), the two-dimensional r.v. (M_{11}, M_{12}) can be constructed according to (3.14), which is a random sum of i.i.d. two-dimensional r.v.s $(H^{(i)}_{\beta_1,1}, H^{(i)}_{\beta_1,2})$, $i \geq 1$, each with the same PGF $H_{\beta_1}(z_1, z_2)$.

$M_2(z_1, z_2)$ is a PGF with a detailed probabilistic interpretation

Let

$$H_{\beta_2}(z_1, z_2) = \frac{p}{q\rho_2} \cdot \frac{\beta_2(\lambda - \lambda_1 z_1 - \lambda_2 z_2) - \beta_2(\lambda - \lambda_1 h(z_2) - \lambda_2 z_2)}{z_1 - h(z_2)}. \quad \text{(B.14)}$$

Using (B.14), (2.7) and (2.9), we can rewrite (2.20) as follows:

$$M_2(z_1, z_2) = \vartheta \cdot H_{\beta_2}(z_1, z_2) \cdot K_a(z_2) \cdot K_c(z_2) + 1 - \vartheta. \quad \text{(B.15)}$$

Similarly to (B.11), we can derive (details omitted) from (B.14) that

$$H_{\beta_2}(z_1, z_2) = \frac{p}{\rho_2} \cdot \sum_{k=1}^{\infty} k b_{\beta_2,k} \cdot D_k(z_1, z_2), \quad \text{(B.16)}$$

with $b_{\beta_2,k}$ given in (3.10).

Similarly to (B.13), we can verify that $(p/\rho_2)\sum_{k=1}^{\infty} k b_{\beta_2,k} = 1$, which, together with (B.16), implies that $H_{\beta_2}(z_1, z_2)$ is the PGF of a two-dimensional r.v. $(H_{\beta_2,1}, H_{\beta_2,2})$, which can be constructed according to (3.8).

By (B.15), the two-dimensional r.v. (M_{21}, M_{22}) can be constructed according to (3.15).

Appendix C: Proofs of Lemmas 4.1 and 4.2 in step 3

In this section, we prove Lemma 4.1 and Lemma 4.2. Before proceeding, we provide the following facts, which will be used in our proof multiple times.

Applying Karamata's theorem (for example, p.28 in [4]), and using Assumption A1 and Lemma A.1, respectively, gives, as $t \to \infty$,

$$P\{T^{(e)}_{\beta_1} > t\} \sim \frac{\lambda_1}{\rho_1(a_1 - 1)} \cdot t^{-a_1+1} L(t), \quad \text{(C.1)}$$

and

$$P\{T_{\alpha}^{(e)} > t\} \sim \frac{1}{\alpha_1(a_1-1)(1-\rho_1)^{a_1+1}} \cdot t^{-a_1+1}L(t). \tag{C.2}$$

Applying Proposition 8.5 (p.181 in [20]) to the density $\bar{F}_{\beta_2}(t)/\beta_{2,1}$ and using Assumption A2, give, as $t \to \infty$,

$$P\{T_{\beta_2}^{(e)} > t\} \sim \begin{cases} \frac{\lambda_2}{\rho_2(a_2-1)} \cdot t^{-a_2+1}L(t), & \text{if } r=0, \\ \frac{\lambda_2}{\rho_2 r} \cdot e^{-rt}t^{-a_2}L(t), & \text{if } r>0. \end{cases} \tag{C.3}$$

Furthermore, since $F_\beta(x) = qF_{\beta_1}(x) + pF_{\beta_2}(x)$ and based on Assumptions A1 and A2, we have $P\{T_\beta > t\} = qP\{T_{\beta_1} > t\} + pP\{T_{\beta_2} > t\} \sim qt^{-a_1}L(t)$ as $t \to \infty$, from which Karamata's theorem implies that

$$P\{T_{\beta}^{(e)} > t\} \sim \frac{\lambda_1}{\rho(a_1-1)} \cdot t^{-a_1+1}L(t). \tag{C.4}$$

Proof of Lemma 4.1

Recall (2.4), which relates the PGF of K to the PGF of R_0. With this relationship, we first study the tail probability for K, which can be regarded as the sum of independent r.v.s K_a, K_b and K_c (refer to (3.2)). By (3.3), (C.2) and applying Lemma A.3, we have

$$P\{K_a > j\} = \rho_1 P\{N_{\lambda_2}(T_{\alpha}^{(e)}) > j\} \sim \frac{\lambda_1\lambda_2^{a_1-1}}{(a_1-1)(1-\rho_1)^{a_1}} \cdot j^{-a_1+1}L(j), \quad j \to \infty. \tag{C.5}$$

Recall (3.4), $K_b = N_{\lambda,X_g}(T_\beta^{(e)}) \stackrel{\text{d}}{=} \sum_{i=1}^{N_\lambda(T_\beta^{(e)})} X_g^{(i)}$, where $X_g^{(i)}$ has the common distribution X_g. By (2.15), and then applying Lemma A.3 and using Lemma A.1, we know that

$$P\{X_g > j\} \sim qP\{N_{\lambda_2}(T_\alpha) > j\} \sim q\lambda_2^{a_1}(1-\rho_1)^{-a_1-1} \cdot j^{-a_1}L(j).$$

Similarly, applying Lemma A.3 and using (C.4), we have

$$P\{N_\lambda(T_\beta^{(e)}) > j\} \sim \frac{q\lambda^{a_1-1}}{(q\beta_{1,1}+p\beta_{2,1})(a_1-1)} \cdot j^{-a_1+1}L(j),$$

based on which, by (2.16) and applying Lemma A.6, we have,

$$P\{K_b > j\} \sim \frac{\lambda_1\lambda_2^{a_1-1}}{\rho(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1}L(j), \quad j \to \infty. \tag{C.6}$$

Next, we study $P\{K_c > j\}$. By (3.5), we know that $P\{K_c > j\} = \vartheta P\{\sum_{i=1}^{J} X_c^{(i)} > j\}$, where $P(J=i) = (1-\vartheta)\vartheta^{i-1}$, $i \geq 1$, and $X_c^{(i)}$ has the same distribution as that for $X_c = K_a + N_{\lambda,X_g}(T_{\beta_2}^{(e)})$. Note that $N_{\lambda,X_g}(T_{\beta_2}^{(e)}) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N_\lambda(T_{\beta_2}^{(e)})} X_g^{(i)}$, where $X_g^{(i)}$ has the common tail probability $P\{X_g > j\} \sim Const \cdot j^{-a_1}L(j)$ and $P\{N_\lambda(T_{\beta_2}^{(e)}) > j\} \sim Const \cdot j^{-a_2+1}L(j)$. Therefore, by applying Lemma A.6 (and noticing that $a_2 > a_1$ if $r = 0$ in Assumption A2), we have

$$P\{N_{\lambda,X_g}(T_{\beta_2}^{(e)}) > j\} \sim Const \cdot \max(j^{-a_2+1}L(j), j^{-a_1}L(j)) = o(1) \cdot j^{-a_1+1}L(j). \tag{C.7}$$

By (C.5), (C.7), applying Lemma A.2 and Lemma A.5, we have, as $j \to \infty$,

$$P\{K_c > j\} \sim \frac{\vartheta}{1-\vartheta} P\{X_c > j\} = \frac{\vartheta}{1-\vartheta} P\{K_a + N_{\lambda,X_g}(T_{\beta_2}^{(e)}) > j\} \sim \frac{\vartheta}{1-\vartheta} P\{K_a > j\}, \tag{C.8}$$

which, together with (C.5), (C.6) and (3.2), leads to

$$P\{K_a + K_c > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{(1-\rho)(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1}L(j), \quad j \to \infty, \tag{C.9}$$

$$P\{K > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho(1-\rho)(a_1-1)(1-\rho_1)^{a_1-1}} \cdot j^{-a_1+1}L(j), \quad j \to \infty. \tag{C.10}$$

Proof of Lemma 4.2

By Proposition 3.2, $S_{\beta_1,1} \stackrel{\mathrm{d}}{=} N_{\lambda_1}(T_{\beta_1}^{(e)})$, $S_{\beta_1,2} \stackrel{\mathrm{d}}{=} N_{\lambda_2}(T_{\beta_1}^{(e)})$, $S_{\beta_2,1} \stackrel{\mathrm{d}}{=} N_{\lambda_1}(T_{\beta_2}^{(e)})$ and $S_{\beta_2,2} \stackrel{\mathrm{d}}{=} N_{\lambda_2}(T_{\beta_2}^{(e)})$. By (C.1) and applying Lemma A.3, we obtain

$$P\{S_{\beta_1,1} > j\} \sim \frac{\lambda_1^{a_1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1}L(j), \tag{C.11}$$

$$P\{S_{\beta_1,2} > j\} \sim \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1}L(j). \tag{C.12}$$

By (C.3) and applying Lemma A.3 and Lemma A.4, we obtain

$$P\{S_{\beta_2,1} > j\} \sim \begin{cases} \frac{\lambda_2 \lambda_1^{a_2-1}}{\rho_2(a_2-1)} \cdot j^{-a_2+1}L(j), & \text{if } r = 0, \\ \frac{\lambda_2 \lambda_1 (\lambda_1+r)^{a_2-1}}{\rho_2 r} \cdot \left(\frac{\lambda_1}{\lambda_1+r}\right)^j j^{-a_2}L(j), & \text{if } r > 0, \end{cases} \tag{C.13}$$

$$P\{S_{\beta_2,2} > j\} \sim \begin{cases} \frac{\lambda_2^{a_2}}{\rho_2(a_2-1)} \cdot j^{-a_2+1}L(j), & \text{if } r = 0, \\ \frac{\lambda_2^2 (\lambda_2+r)^{a_2-1}}{\rho_2 r} \cdot \left(\frac{\lambda_2}{\lambda_2+r}\right)^j j^{-a_2}L(j), & \text{if } r > 0. \end{cases} \tag{C.14}$$

Next, we will study the asymptotic tail probabilities of the r.v.s $M_{ik},\ i,k=1,2$. By Proposition 3.2, we know that

$$P\{M_{1k} > j\} = \rho_1 P\Big\{\sum_{i=1}^{J} H^{(i)}_{\beta_1,k} > j\Big\}, \quad k=1,2, \tag{C.15}$$

$$P\{M_{21} > j\} = \vartheta P\{H_{\beta_2,1} > j\}, \tag{C.16}$$

$$P\{M_{22} > j\} = \vartheta P\{H_{\beta_2,2} + K_a + K_c > j\}. \tag{C.17}$$

To proceed further, we need to study the tail probabilities of the r.v.s $H_{\beta_i,k}$ for $i,k=1,2$.

Tail asymptotics for $H_{\beta_1,1}$ and $H_{\beta_2,1}$

Taking $z_2 \to 1$ in (B.8) and (B.14), we can write

$$E(z_1^{H_{\beta_1,1}}) = H_{\beta_1}(z_1,1) = \frac{1}{\rho_1}\cdot\frac{\beta_1(\lambda_1-\lambda_1 z_1)-1}{z_1-1} = \beta_1^{(e)}(\lambda_1-\lambda_1 z_1), \tag{C.18}$$

$$E(z_1^{H_{\beta_2,1}}) = H_{\beta_2}(z_1,1) = \frac{p}{q\rho_2}\cdot\frac{\beta_2(\lambda_1-\lambda_1 z_1)-1}{z_1-1} = \beta_2^{(e)}(\lambda_1-\lambda_1 z_1). \tag{C.19}$$

Therefore, $H_{\beta_i,1} \overset{\mathrm{d}}{=} N_{\lambda_1}(T^{(e)}_{\beta_i}) \overset{\mathrm{d}}{=} S_{\beta_i,1}$, $i=1,2$, and

$$P\{H_{\beta_i,1} > j\} = P\{S_{\beta_i,1} > j\}, \quad i=1,2, \tag{C.20}$$

whose asymptotic tails are presented in (C.11) and (C.13), respectively.

Tail asymptotics for $H_{\beta_1,2}$

Unlike for the other r.v.s discussed earlier, more effort is required for the asymptotic tail behaviour for $H_{\beta_1,2}$, which will be presented in Proposition C.1. Before doing that, we first present a nice bound on the tail probability of $H_{\beta_1,2}$, which is very suggestive for an intuitive understanding of the tail property for $H_{\beta_1,2}$.

Taking $z_1 \to 1$ in (B.12) and (B.11), we have,

$$E(z_2^{D_{k,2}}) = D_k(1,z_2) = \frac{1}{k}\sum_{i=1}^{k}(q+pz_2)^{i-1}(qh(z_2)+pz_2)^{k-i}, \tag{C.21}$$

$$E(z_2^{H_{\beta_1,2}}) = H_{\beta_1}(1,z_2) = \frac{q}{\rho_1}\cdot\sum_{k=1}^{\infty} k b_{\beta_1,k}\cdot D_k(1,z_2). \tag{C.22}$$

It follows from (C.21) that, for $k \geq 1$,

$$D_{k,2} \stackrel{\text{d}}{=} \sum_{n=1}^{i-1} Y_n + \sum_{n=1}^{k-i} Z_n \quad \text{with probability } 1/k \quad \text{for } i = 1, 2, \ldots, k,$$

where $\{Y_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ are sequences of independent r.v.s, which are independent of each other, with Y_n and Z_n having PGFs $q + pz_2$ and $qh(z_2) + pz_2$, respectively.

We say that Y is stochastically smaller than Z, written as $Y \leq_{st} Z$, if $P\{Y > t\} \leq P\{Z > t\}$ for all t. It is easy to see that $Y_{n_1} \leq_{st} Z_{n_2}$ for all $n_1, n_2 \geq 1$. Define

$$D_{k,2}^L \stackrel{\text{d}}{=} \sum_{n=1}^{k-1} Y_n \quad \text{and} \quad D_{k,2}^U \stackrel{\text{d}}{=} \sum_{n=1}^{k-1} Z_n.$$

Then, by Theorem 1.2.17 (p.7 in [31]),

$$D_{k,2}^L \leq_{st} D_{k,2} \leq_{st} D_{k,2}^U. \tag{C.23}$$

Furthermore, it follows from (C.22) that $H_{\beta_1,2} \stackrel{\text{d}}{=} D_{k,2}$, with probability $(q/\rho_1)kb_{\beta_1,k}$, for $k \geq 1$.

Now, define the r.v.s $H_{\beta_1,2}^L$ and $H_{\beta_1,2}^U$ as follows:

$$H_{\beta_1,2}^L \stackrel{\text{d}}{=} D_{k,2}^L \quad \text{with probability } (q/\rho_1)kb_{\beta_1,k} \text{ for } k \geq 1,$$
$$H_{\beta_1,2}^U \stackrel{\text{d}}{=} D_{k,2}^U \quad \text{with probability } (q/\rho_1)kb_{\beta_1,k} \text{ for } k \geq 1.$$

Then, by (C.23),

$$H_{\beta_1,2}^L \leq_{st} H_{\beta_1,2} \leq_{st} H_{\beta_1,2}^U. \tag{C.24}$$

Note that $H_{\beta_1,2}^L$ and $H_{\beta_1,2}^U$ have the following PGFs:

$$E(z_2^{H_{\beta_1,2}^L}) = \frac{q}{\rho_1} \cdot \sum_{k=1}^{\infty} kb_{\beta_1,k} \cdot E(z_2^{D_{k,2}^L}) = \frac{q}{\rho_1} \cdot \sum_{k=1}^{\infty} kb_{\beta_1,k} \cdot (q + pz_2)^{k-1}, \tag{C.25}$$

$$E(z_2^{H_{\beta_1,2}^U}) = \frac{q}{\rho_1} \cdot \sum_{k=1}^{\infty} kb_{\beta_1,k} \cdot E(z_2^{D_{k,2}^U}) = \frac{q}{\rho_1} \cdot \sum_{k=1}^{\infty} kb_{\beta_1,k} \cdot (qh(z_2) + pz_2)^{k-1}. \tag{C.26}$$

Next, we will study the asymptotic behaviour of $P\{H_{\beta_1,2}^L > j\}$ and $P\{H_{\beta_1,2}^U > j\}$, respectively. Let N be a r.v. with probability distribution $P\{N = k\} = (q/\rho_1)kb_{\beta_1,k}$, $k \geq 1$. Therefore, (C.25) and (C.26) can be written as

$$H_{\beta_1,2}^L \stackrel{\text{d}}{=} \sum_{k=1}^{N-1} Y_k \quad \text{and} \quad H_{\beta_1,2}^U \stackrel{\text{d}}{=} \sum_{k=1}^{N-1} Z_k,$$

where N is independent of both Z_k and Y_k, $k \geq 1$.

Then, it is immediately clear that

$$P\{N > m\} = (q/\rho_1) \sum_{k=m+1}^{\infty} k b_{\beta_1,k} = (q/\rho_1) \left[m\overline{b}_{\beta_1,m+1} + \sum_{k=m+1}^{\infty} \overline{b}_{\beta_1,k} \right], \quad \text{(C.27)}$$

where $\overline{b}_{\beta_1,k} = \sum_{n=k}^{\infty} b_{\beta_1,n}$.

Using the definition of $b_{\beta_1,n}$ in (3.9), and applying Lemma A.3, we know that $\overline{b}_{\beta_1,k} = P\{N_\lambda(T_{\beta_1}) > k-1\} \sim \lambda^{a_1} k^{-a_1} L(k)$ as $k \to \infty$, which, together with Proposition 1.5.10 in [4], implies that

$$P\{N > m\} \sim \frac{a_1 q \lambda^{a_1}}{\rho_1(a_1 - 1)} m^{-a_1+1} L(m) \quad as\ m \to \infty. \quad \text{(C.28)}$$

Recall the following three facts: (i) Y_k is a $0-1$ r.v., which implies that $P\{Y_k > j\} \to 0$ as $j \to \infty$; (ii) Z_k has the same probability distribution as X_g defined in (2.15), which implies that $P\{Z_k > j\} = P\{X_g > j\} \sim Const \cdot j^{-a_1} L(j)$ as $j \to \infty$; and (iii) $E(Y_k) = p$ and $E(Z_k) = E(X_g) = p/(1-\rho_1) < \infty$, given in (2.16). Then, by Lemma A.6, we know that

$$P\{H^L_{\beta_1,2} > j\} \sim a_1 \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j), \quad \text{(C.29)}$$

$$P\{H^U_{\beta_1,2} > j\} \sim \frac{a_1}{(1-\rho_1)^{a_1-1}} \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j). \quad \text{(C.30)}$$

Remark C.1 It follows from (C.24) that $P\{H^L_{\beta_1,2} > j\} \leq P\{H_{\beta_1,2} > j\} \leq P\{H^U_{\beta_1,2} > j\}$, whereas the asymptotic properties of $P\{H^L_{\beta_1,2} > j\}$ and $P\{H^U_{\beta_1,2} > j\}$ are given in (C.29) and (C.30), respectively. This suggests that $P\{H_{\beta_1,2} > j\} \sim c \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j)$ as $j \to \infty$ for some constant $c \in \left(a_1, a_1/(1-\rho_1)^{a_1-1}\right)$. In Proposition C.1, we will verify that this assertion is true.

Proposition C.1 *As* $j \to \infty$,

$$P\{H_{\beta_1,2} > j\} \sim \frac{1-\rho_1}{\rho_1} \left[\frac{1}{(1-\rho_1)^{a_1}} - 1 \right] \cdot \frac{\lambda_1 \lambda_2^{a_1-1}}{\rho_1(a_1-1)} \cdot j^{-a_1+1} L(j). \quad \text{(C.31)}$$

To prove this proposition, we need the following two lemmas (Lemma C.1 and Lemma C.2). Setting $z_1 = 1$ in (B.7) and noting $h(z_2) = \alpha(\lambda_2 - \lambda_2 z_2)$, we obtain

$$E(z_2^{H_{\beta_1,2}}) = H_{\beta_1}(1, z_2) = \frac{1}{\rho_1} \cdot \frac{\beta_1(\lambda_2 - \lambda_2 z_2) - \alpha(\lambda_2 - \lambda_2 z_2)}{1 - \alpha(\lambda_2 - \lambda_2 z_2)} = \gamma(\lambda_2 - \lambda_2 z_2), \quad \text{(C.32)}$$

where

$$\gamma(s) = \frac{1}{\rho_1} \cdot \frac{\beta_1(s) - \alpha(s)}{1 - \alpha(s)}. \tag{C.33}$$

Lemma C.1 *$\gamma(s)$ is the LST of a probability distribution on* $[0, \infty)$.

Proof By Theorem 1 in [15] (see p.439), the lemma is true iff $\gamma(0) = 1$ and $\gamma(s)$ is completely monotone, i.e., $\gamma(s)$ possesses derivatives of all orders such that $(-1)^n \frac{d^n}{ds^n}\gamma(s) \geq 0$ for $s > 0, n \geq 0$. It is easy to check by (C.33) that $\tau(0) = 1$. Next, we only need to verify that $\gamma(s)$ is completely monotone. By (C.33) and (2.10) and using Taylor expansion, we have

$$\begin{aligned}
\gamma(s) &= \frac{1}{\rho_1} \cdot \frac{\beta_1\big(s + \lambda_1 - \lambda_1\big) - \beta_1(s + \lambda_1 - \lambda_1\alpha(s))}{1 - \alpha(s)} \\
&= \frac{1}{\rho_1} \cdot \frac{1}{1 - \alpha(s)} \left[\sum_{n=0}^{\infty} \frac{\beta_1^{(n)}(s + \lambda_1)}{n!}(-\lambda_1)^n - \sum_{n=0}^{\infty} \frac{\beta_1^{(n)}(s + \lambda_1)}{n!}(-\lambda_1\alpha(s))^n \right] \\
&= \frac{1}{\rho_1} \cdot \sum_{n=1}^{\infty} \frac{(-\lambda_1)^n \beta_1^{(n)}(s + \lambda_1)}{n!} \Big(1 + \alpha(s) + \cdots + (\alpha(s))^{n-1}\Big),
\end{aligned} \tag{C.34}$$

where $\beta_1^{(n)}(\cdot)$ represents the nth derivative of $\beta_1(\cdot)$.

It is easy to check that, for $n \geq 1$, both $(-1)^n \beta_1^{(n)}(s + \lambda_1)$ and $1 + \alpha(s) + \cdots + (\alpha(s))^{n-1}$ are completely monotone, and so is their product (by Criterion A.1 in Appendix A). Therefore, $\gamma(s)$ is completely monotone. □

Remark C.2 Let T_γ be a r.v. whose the probability distribution has the LST $\gamma(s)$. Then, the expression $Ez_2^{H_{\beta_{1,2}}} = \gamma(\lambda_2 - \lambda_2 z_2)$ in (C.32) implies that $H_{\beta_{1,2}} \stackrel{\text{d}}{=} N_{\lambda_2}(T_\gamma)$.

Lemma C.2 *As $t \to \infty$,*

$$P\{T_\gamma > t\} \sim \frac{1 - \rho_1}{\rho_1} \left[\frac{1}{(1 - \rho_1)^{a_1}} - 1 \right] \frac{\lambda_1}{\rho_1(a_1 - 1)} \cdot t^{-a_1+1} L(t). \tag{C.35}$$

Proof First, let us rewrite (C.33) as

$$\gamma(s) = \frac{1}{\rho_1} - \frac{1}{\rho_1} \cdot \frac{1 - \beta_1(s)}{s} \cdot \frac{s}{1 - \alpha(s)}. \tag{C.36}$$

In the following, we divide the proof of Lemma C.2 into two parts, depending on whether a_1 (> 1) is an integer or not.
Case 1: Non-integer $a_1 > 1$. Suppose that $n < a_1 < n + 1$, $n \in \{1, 2, \ldots\}$. Since $P\{T_{\beta_1} > t\} \sim t^{-a_1} L(t)$ and $(1 - \rho_1)^{a_1+1} P\{T_\alpha > t\} \sim t^{-a_1} L(t)$, we know that $\beta_{1,n} < \infty$, $\beta_{1,n+1} = \infty$, $\alpha_n < \infty$ and $\alpha_{n+1} = \infty$. Define $\beta_{1,n}(s)$ and $\alpha_n(s)$ in a

manner similar to that in (A.3). Therefore,

$$\frac{1-\beta_1(s)}{s} = \beta_{1,1} + \sum_{k=2}^{n} \frac{\beta_{1,k}}{k!}(-s)^{k-1} + (-1)^n \frac{\beta_{1,n}(s)}{s}, \tag{C.37}$$

$$\frac{1-\alpha(s)}{s} = \alpha_1 + \sum_{k=2}^{n} \frac{\alpha_k}{k!}(-s)^{k-1} + (-1)^n \frac{\alpha_n(s)}{s}. \tag{C.38}$$

By Lemma A.7,

$$(1-\rho_1)^{a_1+1}\alpha_n(s) \sim \beta_n(s) \sim \frac{\Gamma(a_1-n)\Gamma(n+1-a_1)}{\Gamma(a_1)} s^{a_1} L(1/s), \quad s \downarrow 0. \tag{C.39}$$

Furthermore, it follows from (C.38) that

$$\begin{aligned} \frac{s}{1-\alpha(s)} &= \frac{1/\alpha_1}{1 + (1/\alpha_1)\sum_{k=2}^{n} \frac{\alpha_k}{k!}(-s)^{k-1} + (-1)^n (1/\alpha_1)\frac{\alpha_n(s)}{s}} \\ &= \frac{1}{\alpha_1} - \sum_{k=1}^{n-1} u_k s^k - (-1)^n \frac{\alpha_n(s)}{\alpha_1^2 s} + O(s^n), \end{aligned} \tag{C.40}$$

where $u_1, u_2, \ldots, u_{n-1}$ are constants. By (C.36), (C.37) and (C.40), we have

$$\gamma(s) = 1 + \sum_{k=1}^{n-1} e_k s^k + (-1)^n \cdot \frac{1}{\rho_1\alpha_1}\left[\frac{\beta_{1,1}}{\alpha_1} \cdot \frac{\alpha_n(s)}{s} - \frac{\beta_{1,n}(s)}{s}\right] + O(s^n), \tag{C.41}$$

where $e_1, e_2, \ldots, e_{n-1}$ are constants. Based on the above, we define $\gamma_{n-1}(s)$ in a manner similar to that in (A.3). Applying (C.39), we have

$$\begin{aligned} \gamma_{n-1}(s) &\sim \frac{1}{\rho_1\alpha_1}\left[\frac{\beta_{1,1}}{\alpha_1} \cdot \frac{\alpha_n(s)}{s} - \frac{\beta_{1,n}(s)}{s}\right] \\ &\sim \frac{\lambda_1}{\rho_1} \cdot \frac{1-\rho_1}{\rho_1}\left[\frac{1}{(1-\rho_1)^{a_1}} - 1\right] \cdot \frac{\Gamma(a_1-n)\Gamma(n+1-a_1)}{(a_1-1)\Gamma(a_1-1)} s^{a_1-1} L(1/s). \end{aligned} \tag{C.42}$$

Then, making use of Lemma A.7, we complete the proof of Lemma C.2 for non-integer $a_1 > 1$.

Case 2: Integer $a_1 > 1$. Suppose that $a_1 = n \in \{2, 3, \ldots\}$. Since $P\{T_{\beta_1} > t\} \sim t^{-n}L(t)$ and $(1-\rho_1)^{n+1}P\{T_\alpha > t\} \sim t^{-n}L(t)$, we know that $\alpha_{n-1} < \infty$ and $\beta_{1,n-1} < \infty$, but whether α_n or $\beta_{1,n}$ is finite or not remains uncertain. This uncertainty is essentially determined by whether $\int_x^\infty t^{-1}L(t)\mathrm{d}t$ is convergent or not.

Define $\widehat{\beta}_{1,n-1}(s)$ and $\widehat{\alpha}_{n-1}(s)$ in a way similar to that in (A.4). Then,

$$\frac{1-\beta_1(s)}{s} = \beta_{1,1} + \sum_{k=2}^{n-1} \frac{\beta_{1,k}}{k!}(-s)^{k-1} + (-s)^{n-1}\widehat{\beta}_{1,n-1}(s), \tag{C.43}$$

$$\frac{1-\alpha(s)}{s} = \alpha_1 + \sum_{k=2}^{n-1} \frac{\alpha_k}{k!}(-s)^{k-1} + (-s)^{n-1}\widehat{\alpha}_{n-1}(s). \tag{C.44}$$

By Lemma A.8, we obtain, for $x > 0$,

$$\begin{aligned}&(1-\rho_1)^{n+1}\widehat{\alpha}_{n-1}(xs) - (1-\rho_1)^{n+1}\widehat{\alpha}_{n-1}(s)\\ &\sim \widehat{\beta}_{1,n-1}(xs) - \widehat{\beta}_{1,n-1}(s) \sim -(\log x)L(1/s)/(n-1)! \quad \text{as } s \downarrow 0.\end{aligned} \tag{C.45}$$

Furthermore, it follows from (C.44) that

$$\begin{aligned}\frac{s}{1-\alpha(s)} &= \frac{1/\alpha_1}{1 + (1/\alpha_1)\sum_{k=2}^{n-1} \frac{\alpha_k}{k!}(-s)^{k-1} + (-s)^{n-1}(1/\alpha_1)\widehat{\alpha}_{n-1}(s)}\\ &= \frac{1}{\alpha_1} - \sum_{k=1}^{n-1} u'_k s^k - (-s)^{n-1}\frac{\widehat{\alpha}_{n-1}(s)}{\alpha_1^2} + O(s^n),\end{aligned} \tag{C.46}$$

where $u'_1, u'_2, \ldots, u'_{n-1}$ are constants. By (C.36), (C.43) and (C.46), we have

$$\gamma(s) = 1 + \sum_{k=1}^{n-1} e'_k s^k + (-1)^n s^{n-1} \cdot \frac{1}{\rho_1\alpha_1}\left[\frac{\beta_{1,1}}{\alpha_1} \cdot \widehat{\alpha}_{n-1}(s) - \widehat{\beta}_{1,n-1}(s)\right] + O(s^n), \tag{C.47}$$

where $e'_1, e'_2, \ldots, e'_{n-1}$ are constants. Based on this, we define $\widehat{\gamma}_{n-2}(s)$ in a way similar to that in (A.4). Then,

$$\widehat{\gamma}_{n-2}(s) = (-1)^n e'_{n-1} + \frac{1}{\rho_1\alpha_1}\left[\frac{\beta_{1,1}}{\alpha_1} \cdot \widehat{\alpha}_{n-1}(s) - \widehat{\beta}_{1,n-1}(s)\right] + O(s). \tag{C.48}$$

It follows from (C.47) and (C.45) that

$$\begin{aligned}\lim_{s\downarrow 0} \frac{\widehat{\gamma}_{n-2}(xs) - \widehat{\gamma}_{n-2}(s)}{L(1/s)/(n-2)!} &= \frac{1}{\rho_1\alpha_1}\left[\frac{\beta_{1,1}}{\alpha_1} \cdot \lim_{s\downarrow 0} \frac{\widehat{\alpha}_{n-1}(xs) - \widehat{\alpha}_{n-1}(s)}{(n-1)L(1/s)/(n-1)!} - \lim_{s\downarrow 0} \frac{\widehat{\beta}_{1,n-1}(xs) - \widehat{\beta}_{1,n-1}(s)}{(n-1)L(1/s)/(n-1)!}\right]\\ &= \frac{\lambda_1}{\rho_1}\frac{1-\rho_1}{\rho_1}\left[\frac{1}{(1-\rho_1)^n} - 1\right] \cdot \left(-\frac{1}{n-1}\log x\right).\end{aligned} \tag{C.49}$$

Applying Lemma A.8, we complete the proof of Lemma C.2 for integer $a_1 = n \in \{2, 3, \ldots\}$. □

Proof of Proposition C.1 It follows directly from Remark C.2, Lemma C.2 and Lemma A.3. □

Referring to Remark C.1, we know from (C.31) that $c=(1-\rho_1)\rho_1^{-1}\left[1/(1-\rho_1)^{a_1}-1\right]$. Now let us confirm that $a_1 < c < a_1/(1-\rho_1)^{a_1-1}$, which is equivalent to checking that $a_1\rho_1(1-\rho_1)^{a_1-1}+(1-\rho_1)^{a_1} < 1$ and $a_1\rho_1+(1-\rho_1)^{a_1} > 1$. This is true because $a_1\rho_1(1-\rho_1)^{a_1-1}+(1-\rho_1)^{a_1}$ is decreasing in $\rho_1 \in (0,1)$ and $a_1\rho_1+(1-\rho_1)^{a_1}$ is increasing in $\rho_1 \in (0,1)$.

Tail asymptotics for $H_{\beta_2,2}$

As we shall see in the next subsection, our main results do not require a detailed asymptotic expression for $P\{H_{\beta_2,2} > j\}$. It is enough to verify that it is $o(1)\cdot j^{-a_1+1}L(j)$ as $j \to \infty$.

Taking $z_1 \to 1$ in (B.16), we have

$$E(z_2^{H_{\beta_2,2}}) = H_{\beta_2}(1,z_2) = \frac{p}{\rho_2}\cdot\sum_{k=1}^{\infty} kb_{\beta_2,k}\cdot D_k(1,z_2). \tag{C.50}$$

It follows from (C.50) that $H_{\beta_2,2} \stackrel{\text{d}}{=} D_{k,2}$, with probability $(p/\rho_2)kb_{\beta_2,k}$, for $k \ge 1$. Define the r.v. $H_{\beta_2,2}^U \stackrel{\text{d}}{=} D_{k,2}^U$, with probability $(p/\rho_2)kb_{\beta_2,k}$, for $k \ge 1$. Then, by (C.23), we have, $H_{\beta_2,2} \le_{st} H_{\beta_2,2}^U$. Note that $H_{\beta_2,2}^U$ has the PGF

$$E(z_2^{H_{\beta_2,2}^U}) = \frac{p}{\rho_2}\cdot\sum_{k=1}^{\infty} kb_{\beta_2,k}\cdot E(z_2^{D_{k,2}^U}) = \frac{p}{\rho_2}\cdot\sum_{k=1}^{\infty} kb_{\beta_2,k}\cdot (qh(z_2)+pz_2)^{k-1}. \tag{C.51}$$

Let N^* be a r.v. with probability distribution $P\{N^*=k\} = (p/\rho_2)kb_{\beta_2,k}$, $k \ge 1$. Therefore, (C.51) implies $H_{\beta_2,2}^U \stackrel{\text{d}}{=} \sum_{k=1}^{N^*-1} Z_k$, where N^* is independent of Z_k, $k \ge 1$. Similar to (C.27), we can write

$$P\{N^* > m\} = (p/\rho_2)\left[m\overline{b}_{\beta_2,m+1} + \sum_{k=m+1}^{\infty}\overline{b}_{\beta_2,k}\right], \tag{C.52}$$

where $\overline{b}_{\beta_2,k} = \sum_{n=k}^{\infty} b_{\beta_2,n}$. By the definition of $b_{\beta_2,n}$ given in (3.10) and applying Lemma A.3 and Lemma A.4, we know that $\overline{b}_{\beta_2,k} = P\{N_\lambda(T_{\beta_2}) > k-1\} = O(1)\cdot k^{-a_2}L(k)$ as $k \to \infty$. Furthermore, by (C.52) and applying Proposition 1.5.10 in [4], we have

$$P\{N^* > m\} = O(1)\cdot m^{-a_2+1}L(m) \quad as\ m \to \infty. \tag{C.53}$$

As pointed out in Sect. 2, $P\{Z_k > j\} \sim Const\cdot j^{-a_1}L(j)$ as $j \to \infty$. By Lemma A.6, we know $P\{H_{\beta_2,2}^U > j\} = O(1)\cdot \max\left(j^{-a_2+1}L(j), j^{-a_1}L(j)\right)$ as

$j \to \infty$. Since $P\{H_{\beta_2,2} > j\} \le P\{H^U_{\beta_2,2} > j\}$ and $a_2 > a_1$, we have

$$P\{H_{\beta_2,2} > j\} = O(1) \cdot \max\left(j^{-a_2+1}L(j), j^{-a_1}L(j)\right) = o(1) \cdot j^{-a_1+1}L(j). \tag{C.54}$$

After the above preparations, we now return to the proof of the tail asymptotic properties for $M_{ik},\ i, k = 1, 2$.

By (C.15) and applying Lemma A.2, together with (C.20), we have

$$P\{M_{11} > j\} \sim \frac{\rho_1}{1-\rho_1} P\{H_{\beta_1,1} > j\} = \frac{\rho_1}{1-\rho_1} P\{S_{\beta_1,1} > j\} \quad \text{(refer to (C.11))}, \tag{C.55}$$

$$P\{M_{12} > j\} \sim \frac{\rho_1}{1-\rho_1} P\{H_{\beta_1,2} > j\} \quad \text{(refer to (C.31))}. \tag{C.56}$$

Immediately, from (C.16) and (C.20),

$$P\{M_{21} > j\} = \vartheta P\{H_{\beta_2,1} > j\} = \vartheta P\{S_{\beta_2,1} > j\} \quad \text{(refer to (C.13))}. \tag{C.57}$$

By (C.17) and applying Lemma A.5, together with (C.54),

$$P\{M_{22} > j\} = \vartheta P\{H_{\beta_2,2} + K_a + K_c > j\} \sim \vartheta P\{K_a + K_c > j\} \quad \text{(refer to (C.9))}. \tag{C.58}$$

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