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Stationary analysis of a single queue with remaining service time-dependent arrivals

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Abstract We study a generalization of the $M/G/1$ system (denoted by $rM/G/1$) with independent and identically distributed service times and with an arrival process whose arrival rate $\lambda_0 f(r)$ depends on the remaining service time r of the current customer being served. We derive a natural stability condition and provide a stationary analysis under it both at service completion times (of the queue length process) and in continuous time (of the queue length and the residual service time). In particular, we show that the stationary measure of queue length at service completion times is equal to that of a corresponding $M/G/1$ system. For $f > 0$, we show that the continuous time stationary measure of the $rM/G/1$ system is linked to the $M/G/1$ system via a time change. As opposed to the $M/G/1$ queue, the stationary measure of queue length of the $rM/G/1$ system at service completions differs from its marginal distribution under the continuous time stationary measure. Thus, in general, arrivals of the $rM/G/1$ system do not see time averages. We derive formulas for the average queue length, probability of an empty system and average waiting time under the continuous time stationary measure. We provide examples showing the effect of changing the reshaping function on the average waiting time.

Keywords Residual service time-dependent arrivals · Reshaping function · Queueing systems · Performance evaluation · Piecewise-deterministic processes

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1 Introduction

The goal of the present note is the steady-state analysis of a single-server queueing system with independent and identically distributed (iid) service times and an arrival process whose rate is a function of the remaining service time of the current customer being served, if the server is busy, or a constant λ_0 otherwise. This is a generalization of the $M/G/1$ system. Because the arrival rate is allowed to depend on the remaining service time, we will denote it by the notation "$rM/G/1$." Arrival processes with remaining service time-dependent rates can be used to model systems where customers can directly estimate the remaining service time by observing the amount of work that a server has to treat and use this information to decide whether to join the queue or not. This type of behavior occurs, for example, at checkout queues in supermarkets. A potential application area for $rM/\cdot/\cdot$ systems is call centers [1,7] with inbound and outbound calls. Modern call centers call out customers to connect them with a server even when all servers are busy [17]; the decision to initiate an outbound call can use estimates of the remaining service time of the busy servers. New approaches to call center modeling also allow the control of the arrival process of inbound calls by postponing their routing to an agent or by giving incentives to call back later [13]; such approaches can make use of estimates of the remaining service time of servers. Generalizations of the $rM/G/1$ model may be useful in the analysis of these systems.

Queues with queue-length-dependent and Markov-modulated arrival or service time distributions have been studied in the literature; see, for example, [4,6,12,18]. The only works we are aware of allowing the arrival rate to depend on the remaining service time are [9–11]; these works study the remaining service time process (denoted by $U(t)$ in these works) when the arrival rate and the service rate of the arriving customer depend on U ([10,11] further contain two-state Markov modulation whose transition rates depend on U). The analysis method used in these works is asymptotic approximation as arrival, service and transition rates are scaled by a parameter whose value is sent to ∞. In the current work, we study, within a narrower framework, the joint queue length and remaining service time distribution and our focus is on finding exact solutions.

To simplify exposition, we assume that the iid service times have a density, denoted by $g(\cdot)$. We further comment on this assumption in Sect. 6. The arrival process of customers is Poisson with constant arrival rate λ_0 if the system is empty or $\lambda_0 f(r)$ if the server is busy and the remaining service time of the customer being served is r. In the particular case where $f(r) = 1$ for $r \geq 0$, the system reduces to an $M/G/1$ queue. f can be interpreted in two ways: if $f(r) \in (0, 1)$, $r \in \mathbb{R}_+$, then $f(r)$ can be thought of as the probability that an arriving customer joins the queue after having observed the remaining service time r. f can also be thought of as a control parameter that transforms / reshapes the constant arrival rate λ_0 to optimize system performance. With this interpretation in mind, we will refer to f as the "reshaping function" (the "r" in the abbreviation $rM/G/1$ refers also to "reshaping" of the arrival process). For the latter interpretation, a natural condition on a reshaping function is that it does not change the overall average arrival rate to the system. In Proposition 10 of Sect. 4.2, the average arrival rate to an $rM/G/1$ system is computed to be $\alpha = \frac{\lambda_0}{1-\lambda_0(\bar{\nu}-\nu)}$, where $\nu = \int_0^\infty rg(r)\mathrm{d}r$ is the average service length and $\bar{\nu} = \int_0^\infty F(r)g(r)\mathrm{d}r$ with $F(r) = \int_0^r f(u)\mathrm{d}u$. Thus, under the assumption

$$\nu = \bar{\nu}, \tag{1}$$

the average arrival rate of an $rM/G/1$ system remains λ_0. This assumption will be in force in Sect. 5, where we compare the average waiting times of a range of $rM/G/1$ systems with the same service time distribution and average arrival rate λ_0 but different reshaping functions.

A natural framework for the study of the $rM/G/1$ queue is the piecewise-deterministic processes (PDP) of [5]. Section 2 gives a construction of the $rM/G/1$ process as a piecewise-deterministic Markov process based on this framework. The process is $X_t = (N_t, R_t)$; its first component represents the number of customers (i.e., queue length, including the customer being served) in the system; the second component represents the remaining service time. Section 2.1 gives its generator and Sect. 2.2 derives the dynamics of the embedded random walk $\mathcal{N}$, which is the sequence of queue lengths observed at service completion times; Proposition 2 shows that the dynamics of $\mathcal{N}$ equal those of the embedded random walk (at service completion times) of an $M/G/1$ queue (whose state process is denoted by $\bar{X}$) with constant arrival rate λ_0 and with iid service times $\{\bar{\sigma}_k, k = 1, 2, 3, \ldots\}$, where $\bar{\sigma}_k = F(\sigma_k)$, and $\{\sigma_k\}$ are the iid service times of the original $rM/G/1$ system. The stationary distribution of the $rM/G/1$ system at service completions (and arrivals) follows from this reduction; the details are given in Sect. 3. Proposition 3 derives the stability condition $\rho \doteq \lambda_0\bar{\nu} < 1$, (15) gives the expected stationary queue length at service completions, and (16) gives the stationary moment generating function of the queue length distribution at service completions.

As opposed to $M/G/1$ queues, the stationary distribution of the queue length of an $rM/G/1$ system in continuous time does not equal its stationary distribution at service completions; therefore, for $rM/G/1$ queues, the continuous time stationary distribution and service measures based on it must be computed directly. Section 4 begins with the statement and recursive solution of the balance equation for the stationary distribution of the continuous time process X, which consists essentially of a sequence of linear ordinary differential equations (ODEs) where f serves as an r dependent coefficient. Proposition 8 proves that the solution of the balance equation is indeed the stationary measure of the process X under the stability assumption $\rho < 1$. The proof is based on the PDP framework of [5]. A number of further computations based on the continuous time stationary distribution are given in Sect. 4; in particular, Corollary 2 gives a simple formula for the stationary probability of an empty $rM/G/1$ system in continuous time and Proposition 9 gives a formula for the stationary expected queue length in continuous time. Proposition 10 of Sect. 4.2 gives the average arrival rate for the $rM/G/1$ system, and finally, (64) gives an explicit formula for the average sojourn time of a customer in an $rM/G/1$ system. In general, f may take the value 0 and this may make F noninvertible. For this reason, there is not, in general, a bijective correspondence between the continuous time stationary distribution of the $rM/G/1$ process X and that of the $M/G/1$ process $\bar{X}$. However, for $f > 0$ a bijective correspondence can be established; this is treated in Sect. 4.4.

Section 5 gives two examples showing the impact of reshaping the arrival process on the average waiting time. We observe, as expected, that, for a given average arrival rate, the closer the customers arrive to the end of a service, the shorter will be the average waiting time in the system. Section 6 points out directions for future research.

2 Dynamics of the process

The theory of piecewise-deterministic Markov Processes (PDP) of [5] provides the ideal mathematical framework for the analysis of the $rM/G/1$ queue. For the definition of the process, we will use the PDP definition given in [5, page 57], which uses the following elements (all adopted from [5]): the state space of the process will be

$$E \doteq \bigcup_{k=0}^{\infty} E_k,\ E_0 = B(\mathbf{0},\delta) \subset \mathbb{R}^2, \quad E_k \doteq \{k\} \times \mathbb{R}_+ = \{(k,r), r > 0\}, \quad k \in \{1,2,3,\ldots\},$$

where $\mathbf{0} = (0,0) \in \mathbb{R}^2$ denotes the origin of $\mathbb{R}^2$ and $B(\mathbf{0},\delta)$ denotes an open ball of radius $\delta < 1$; $\mathbf{0}$ represents the empty system (in [5], the letter ζ denotes the second component of $x \in E$; we use r for the same purpose). The $rM/G/1$ process, $X_t = (N_t, R_t) \in E$, $t \geq 0$, will evolve, on each E_k smoothly following the vector field $\mathfrak{X}_k : E_k \mapsto \mathbb{R}^2$ given by

$$\mathfrak{X}_k(x) \doteq \begin{cases} (0,-1), & k > 0, \\ 0, & \text{otherwise}, \end{cases}$$

until it jumps. Let us denote the jump times of X by the sequence $\{T_i, i = 1,2,3,\ldots\}$. The vector field $\mathfrak{X}_k$ defines the following trivial flow:

$$\phi(t,(k,r)) = (k, r-t), \quad k > 0, \qquad \phi(t,\mathbf{0}) = \mathbf{0}; \tag{2}$$

the process X follows this flow in between its jumps:

$$X_t = (N_{T_k}, \phi(t, X_{T_k})) = (N_{T_k}, R_{T_k} - (t - T_k)), T_k < t < T_{k+1}. \tag{3}$$

For $A \subset \mathbb{R}^2$, let ∂A denote its boundary in the Euclidian topology. The exit boundary of the process is

$$\Gamma^* \doteq \cup_{k=0}^{\infty} \partial E_k = \partial B(\mathbf{0},\delta) \cup \left(\cup_{k=1}^{\infty} \{(k,0), k > 0\} \right).$$

For $x = (k,r) \in E$, define (following [5, page 57]) $t_*(x) \doteq \inf\{t > 0, \phi_k(t,r) \in \partial E_k\}$, where we use the convention that the infimum of the empty set is empty; $t_*(x)$ is the time when X reaches Γ^* if it does not jump until this happens. By Definitions (2) and (3), X moves with unit speed toward the k-axis on each E_k, $k > 0$, therefore,

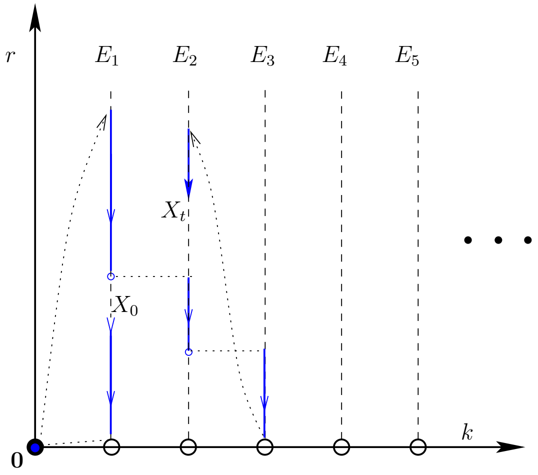


Fig. 1 The state space and a sample path of X

$$t_*(x) = r, x = (k, r) \in E, k > 0.$$

For $k = 0$, the process remains constant $\mathbf{0}$ until an arrival occurs, which implies $t_*(\mathbf{0}) = \infty$. Figure 1 shows an example sample path of X; the horizontal axis is the k-axis, showing the number of customers in the system, and the vertical axis is the r-axis, showing the remaining service time of the current customer in service. Dynamics (3) mean that X travels with unit speed toward the k-axis in between its jumps. Two types of jumps are possible: either an arrival, which is a jump to the right, or a service completion, which is a jump to the left occurring when X hits the k-axis.

The jump dynamics are specified by the rate function $\boldsymbol{\lambda} : E \to \mathbb{R}_+$ and the transition measure Q. For the $rM/G/1$ system, the jump rate function will be

$$\boldsymbol{\lambda}(k, r) \doteq \begin{cases} \lambda_0 f(r), & k > 0, \\ \lambda_0, & k = 0. \end{cases}$$

The transition measure $Q(\cdot, x), x \in E \cup \Gamma^*$, for the $rM/G/1$ system will be as follows: $Q(\cdot, x)$ is the Dirac measure on $(k+1, r)$ for $x = (k, r), k > 0$ and $r > 0$ (represents an arrival to the busy system). For $(k, 0) \in \Gamma^*$, $k > 0$, $Q(\cdot, x)$ is the measure $g(r)\mathrm{d}r$ on E_{k-1} (represents the completion of a service and the start of another, this is exactly when the sample path X hits the k-axis in Fig. 1); $Q(\cdot, \mathbf{0})$ is the measure $g(r)\mathrm{d}r$ on E_1 (represents an arrival to the empty system).

2.1 Generator of X

Let $\mathscr{E}$ denote the σ-algebra of Borel-measurable subsets of E. Let $\{T_n, n = 1, 2, 3, \ldots\}$ denote the jump times of X. For $h : E \times \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$, h measurable, one writes $h \in L_1(X)$ if

$$\mathbb{E}\left[\sum_{i=1}^{\infty}\left|h(X_{T_i}, T_i, \omega)\right|\right] < \infty,$$

and $h \in L_1^{\text{loc}}(X)$ if $h\mathbb{1}_{\{t<\sigma_n\}} \in L_1(X)$ for a sequence of stopping times $\sigma_n \nearrow \infty$. The characterization of the generator of X given in the next paragraph uses these definitions.

The generator of any PDP process is derived explicitly in [5, Theorem (26.14), page 69]; for the $rM/G/1$ process X, it is given by the following operator:

$$\mathfrak{A}h(x) = \begin{cases} -\frac{\mathrm{d}}{\mathrm{d}r}h(x) + \lambda_0 f(r)\left(h(k+1, r) - h(k, r)\right), & x = (k, r), k, r > 0, \\ +\lambda_0 g(r)(h(1, r) - h(\mathbf{0})), & x = \mathbf{0}, \end{cases}$$

where $h \in \mathscr{D}(\mathfrak{A})$, the domain of $\mathfrak{A}$, consisting of measurable functions h on $E \cup \Gamma^*$ satisfying:

1. for each $k > 0$, $h(k, \cdot)$ is absolutely continuous on $\mathbb{R}_+$,
2. $h(k, 0) = \int h(k, r)g(r)\mathrm{d}r$, $k > 0$, and
3. $\mathfrak{B}h \in L_1^{\text{loc}}(X)$, where $\mathfrak{B}h$ is the process $t \mapsto (h(X_0) - h(X_t-))$.

2.2 Embedded random walk at service completion times

Let S_k denote[1] the sequence of service completion times

$$S_1 \doteq \inf\{t, X_{t-} \in \Gamma^*\}, \; S_n \doteq \inf\{t > S_{n-1}, X_{t-} \in \Gamma^*\}, n > 1,$$

and define the process $(\mathcal{N}_n, \mathcal{R}_n) \doteq X_{S_n}$, the state of the system right after service completions. Let

$$\boldsymbol{p}(k, \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, 3, \ldots$$

denote the Poisson distribution with rate λ and define

$$F(r) \doteq \int_0^r f(r-u)\mathrm{d}u = \int_0^r f(u)\mathrm{d}u.$$

[1] In [5], S_k denotes the inter-jump times of the PDP; here they denote the successive times when the process hits the boundary of its state space.

Proposition 1 *The process* $\{(\mathcal{N}_n, \mathcal{R}_n), n = 1, 2, 3, \dots\}$ *is a Markov chain with transition probabilities*

$$P(\mathcal{N}_{n+1} - \mathcal{N}_n = j | \mathcal{N}_n, \mathcal{R}_n) = \begin{cases} \boldsymbol{p}(j+1, F(\mathcal{R}_n)), & \mathcal{N}_n > 0, \\ \boldsymbol{p}(j, F(\mathcal{R}_n)), & \mathcal{N}_n = 0 \end{cases} \tag{4}$$

$$P(\mathcal{R}_{n+1} \in A | \mathcal{N}_{n+1}) = \begin{cases} \delta_0(A), & \mathcal{N}_{n+1} = 0, \\ \int_A g(r)\mathrm{d}r, & \mathcal{N}_{n+1} > 0, \end{cases} \tag{5}$$

where δ_0 *denotes the Dirac measure on* 0.

Proof The definition of the process X (or its strong Markov property) implies that $(\mathcal{N}, \mathcal{R})$ is a Markov chain. The jump distribution Q determines where X jumps after it hits Γ^*: X jumps to $(N_{S_n-} - 1, \sigma_n)$, where σ_n has density g, if $N_{S_n-} > 1$ or it jumps to $\mathbf{0} = (0, 0)$ if $N_{S_n-} = 0$. This gives (5). To compute the conditional density of $\mathcal{N}_{n+1}$ given $(\mathcal{N}_n, \mathcal{R}_n)$ it suffices to compute that of

$$\mathcal{N}_{n+1} - \mathcal{N}_n = (N_{S_{n+1}} - N_{S_{n+1}-}) + (N_{S_{n+1}-} - N_{S_n}). \tag{6}$$

By the strong Markov property of X, the conditional distribution of $N_{S_{n+1}-} - N_{S_n}$ given $(\mathcal{N}_n, \mathcal{R}_n)$ is the same as that of $N_{S_1-} - N_0$ given (N_0, R_0). The two cases $(N_0, R_0) = \mathbf{0}$ and $N_0, R_0 > 0$ are treated separately. Let us start with the latter: conditioned on $(X_0 = (N_0, R_0) = x = (k, r))$, $k > 0$, the dynamics of X imply the following: $S_1 = t_*(x) = r$, and $R_t = r - t$ for $t \in [0, r)$. In the same time interval, the N process is Poisson with time-dependent rate $\lambda_0 f(r - t)$. Therefore, conditioned on $X_0 = (k, r)$, $k > 0$, $N_{S_1-} - N_0$ has Poisson distribution with rate $F(r)$. Furthermore, for $k > 0$, one has $N_{S_1-} > 0$, and therefore, once again by the definition of the jump dynamics of X, $N_{S_1} - N_{S_1-} = -1$ (i.e., the customer whose service has just finished leaves the system). These and (6) imply

$$P(\mathcal{N}_{n+1} - \mathcal{N}_n = j | \mathcal{N}_n, \mathcal{R}_n) = \boldsymbol{p}(j+1, F(\mathcal{R}_n)), \mathcal{N}_n > 0. \tag{7}$$

The argument for the case $X_0 = \mathbf{0}$ is parallel and gives

$$P(\mathcal{N}_{n+1} - \mathcal{N}_n = j | (\mathcal{N}_n, \mathcal{R}_n) = \mathbf{0}) = \boldsymbol{p}(j, F(R_{\tau'_n})), \tag{8}$$

where τ'_n is the first jump time of R after S_n. For $(\mathcal{N}_n, \mathcal{R}_n) = \mathbf{0}$, τ'_n will be a jump from state $\mathbf{0}$ (i.e., an arrival to the empty system) and, by X's definition, $R_{\tau'_n}$'s density, given the whole history of $(\mathcal{N}, \mathcal{R})$, will again be g. This, (7) and (8) imply (4). □

The process $\mathcal{N}$ itself is a Markov chain:

Proposition 2 $\mathcal{N}$ *is a Markov chain with transition matrix*

$$M = \begin{pmatrix} p(0) & p(1) & p(2) & p(3) & p(4) & \cdots \\ p(0) & p(1) & p(2) & p(3) & p(4) & \cdots \\ 0 & p(0) & p(1) & p(2) & p(3) & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{9}$$

where

$$p(j) \doteq \int_0^\infty \boldsymbol{p}(j, \lambda_0 F(r)) g(r) \mathrm{d}r. \tag{10}$$

Proof The conditional distributions (4), (5) and the Markov property of the process $\{(\mathcal{N}_n, \mathcal{R}_n)\}$ imply that $\{\mathcal{N}_n, n = 0, 1, 2, 3, \ldots\}$ is a Markov chain; the distribution of its increments $\Delta\mathcal{N}_n = \mathcal{N}_{n+1} - \mathcal{N}_n$ is

$$P\left(\Delta\mathcal{N}_n = j | \mathcal{N}_n\right) = \mathbb{E}\left[\mathbb{E}\left[\left(\mathbb{1}_{\mathcal{N}_n = j} | \mathcal{N}_n, R_n\right)\right] \mathcal{N}_n\right] = \begin{cases} p(j+1), & \mathcal{N}_n > 0, \\ p(j), & \mathcal{N}_n = 0. \end{cases} \tag{11}$$

This implies that M of (9) is the transition matrix of $\mathcal{N}$. □

We now note the first connection between $rM/G/1$ and $M/G/1$ systems. That $\{\sigma_i\}$ is an iid sequence implies the same for $\bar{\sigma}_i \doteq F(\sigma_i)$. Then, one can write (10) as

$$p(j) = \mathbb{E}[\boldsymbol{p}(j, \lambda_0 \bar{\sigma}_1)], \; j = 0, 1, 2, 3, \ldots$$

and, by [16, Proposition 3.3.2, page 57], these are exactly the transition probabilities of the embedded random walk (at service completion times) of an $M/G/1$ system with constant rate λ_0 and iid service time sequence $\{\bar{\sigma}_i, i = 1, 2, 3, \ldots\}$:

Corollary 1 *The dynamics at service completion times of the $rM/G/1$ system with arrival rate $\lambda_0 f(\cdot)$ and iid service times $\{\sigma_i, i = 1, 2, 3, \ldots\}$ are identical to the dynamics at service completion times of an $M/G/1$ system with constant arrival rate λ_0 and iid service times $\{\bar{\sigma}_i = F(\sigma_i), i = 1, 2, 3, \ldots\}$.*

The next section computes the stationary distribution of $\mathcal{N}$ under a natural stability assumption; before we move on, let us make the following observation:

Remark 1 Let E_t denote the elapsed service time since the beginning of the current service. If we replace the arrival rate $\lambda_0 f(R_t)$ with $\lambda_0 f(E_t)$, conditioned on $\mathcal{R}_n = r$, the number of arrivals between the nth service completion and $(n+1)$st completion will be a Poisson random variable with rate $\lambda_0 \int_0^r f(u)\mathrm{d}u$, i.e., the same as that of the $rM/G/1$ system; therefore, the transition matrix M of the embedded walk $\mathcal{N}$ remains unchanged if we replace the arrival rate $\lambda_0 f(R_t)$ with $\lambda_0 f(E_t)$. This implies that all of our computations concerning $\mathcal{N}$ above and in Sect. 3 below remain unchanged if the arrival rate process is changed from $\lambda_0 f(R_t)$ to $\lambda_0 f(E_t)$.

3 Stationary distribution at service completions or arrival times

A measure q is the stationary measure of $\mathcal{N}$ if and only if it satisfies

$$q = qM. \tag{12}$$

We have seen in Corollary 1 that the dynamics of the $rM/G/1$ system at service completion times are identical to that of the $M/G/1$ system with constant arrival

λ_0 and service times $\{\bar{\sigma}_i = F(\sigma_i)\}$; therefore, (12) is also the balance equation of this $M/G/1$ system at its service completion times. The well-known solution of this system is (see, for example, [16, page 238] or [2, page 281])

$$q(j) = q(0)\bar{p}(j-1) + \sum_{i=1}^{j-1} q(i)\bar{p}(j-i),\ j = 1, 2, 3, \ldots, \tag{13}$$

where $\bar{p}(j) \doteq \sum_{i=j+1}^{\infty} p(j)$. In particular, a (possibly degenerate) invariant distribution always exists and is uniquely defined as soon as $q(0)$ is fixed. By definition, q is nondegenerate if and only if $\sum_{i=1}^{\infty} q(i) < \infty$, i.e., if q is a finite measure on $\mathbb{N}$. [16, Proposition 10.3.1, page 239] gives precisely the condition for this to hold:

Proposition 3 *q of (13) defines a finite measure if and only if $-1 + \sum_n np(n) < 0$, i.e., if*

$$\rho \doteq \lambda_0\bar{\nu} = \lambda_0\mathbb{E}[\bar{\sigma}_i] = \lambda_0\mathbb{E}[F(\sigma_i)] = \lambda_0 \int_0^{\infty} F(r)g(r)\mathrm{d}r < 1. \tag{14}$$

Then, under the stability condition (14), $q(0)$ can be chosen so that $\sum_{i=0}^{\infty} q(i) = 1$. And, with this choice, q will be the unique stationary measure of the process $\mathcal{N}$. To determine the value of $q(0)$ for which q is a proper probability measure, following [16, page 239], one sums both sides of (13) to get

$$\sum_{j=1}^{\infty} q(j) = q(0)\frac{\rho}{1-\rho};$$

then, for $\sum_{i=0}^{\infty} q(i) = 1$, we must have

$$q(0) = 1 - \rho.$$

In the rest of this article, we will take $q(0) = 1 - \rho$ whenever the stability assumption (14) is made. Under these assumptions, $q(0)$ is the stationary limit probability of an empty $rM/G/1$ queue immediately after service completions:

Proposition 4 *The distribution of $\mathcal{N}_n$ converges in total variation norm to q. In particular,*

$$\lim_{n\to\infty} P(\mathcal{N}_n = 0) = q(0) = 1 - \rho = 1 - \lambda\mathbb{E}[F(\sigma_1)] = 1 - \lambda_0 \int_0^{\infty} F(r)g(r)\mathrm{d}r.$$

Proof That q is the stationary distribution of $\mathcal{N}$ follows from (12). $\mathcal{N}$ is strongly aperiodic; by [16, Proposition 10.3.1] it is positive when (14) holds. The convergence in total variation norm follows from these and [16, Theorem 13.3.1]. □

By Corollary 1, all results/computations for the $M/G/1$ queue at service completion times hold for the $rM/G/1$ queue. For example, the expected queue length at service completion times is given by the Pollaczek–Khinchine formula [2, Eq. (5.3), page 281]

$$\mathbb{E}_q[\mathcal{N}_1] = \sum_{k=1}^{\infty} kq(k) = \rho + \frac{\lambda_0^2 \mathbb{E}[F(\sigma_1)^2]}{2(1-\rho)}, \tag{15}$$

where the subscript q of $\mathbb{E}$ denotes that the Markov chain $\mathcal{N}$ is run in its stationary distribution, and the moment generating function of the stationary distribution is [2, (5.8), page 283]

$$\mathbb{E}_q[e^{s\mathcal{N}_1}] = \sum_{k=1}^{\infty} e^{sk} q(k) = \frac{(1-\rho)(1-s)\psi_p(s)}{\psi_p(s) - s}, \tag{16}$$

where ψ_p is the moment generating function of the increments of $\mathcal{N}$:

$$\psi_p(s) = \mathbb{E}\left[e^{\lambda_0(1-s)\bar{\sigma}_1}\right] = \mathbb{E}\left[e^{\lambda_0(1-s)F(\sigma_1)}\right].$$

Stationary distribution at arrival times Let S_n^A be the sequence of arrival times to the system. Then, $(\mathcal{N}_n^A, \mathcal{R}_n^A) = (N_{S_n^A} - 1, R_{S_n^A})$ is the embedded Markov chain of X representing the state of the system just before arrivals. The fact that the queueing process X changes in increments of 1 and -1 exactly at arrival and service completion times implies that, under the stability assumption (14), the process $\mathcal{N}^A$ will also have stationary distribution q, the stationary distribution of $\mathcal{N}$. For details of similar arguments, we refer the reader to [2, Theorem 4.3, page 278] or [8, Sect. 5.3].

4 Stationary distribution in continuous time

One of the key properties of $M/G/1$ systems is that their stationary queue length distribution at service completion times is equal to the same distribution under their continuous time stationary measure. We will see in Corollary 2 below that the $rM/G/1$ system does not possess this property; hence, the continuous time stationary measure and related performance measures (such as the average waiting time) for the $rM/G/1$ queue have to be computed separately. This is the goal of the present section. The following verification argument will give us the stationary distribution of X:

1. Derive the balance equation for the stationary distribution,
2. Solve the balance equation,
3. Invoke [5, Proposition (34.7), page 113] to show that the solution is indeed the stationary measure of X (see Proposition 8 below).

For a measure μ on E and $k \in \{1, 2, 3, \ldots\}$, we say that μ has density m on E_k if $\mu(A \cap E_k) = \int_0^\infty 1_A((k, r))m(r)\mathrm{d}r$, for any measurable $A \subset E$. Define

$$\mathscr{M} \doteq \{\mu \text{ is a measure on } E \text{ having density } m(k, \cdot) \text{ on } E_k,\ k = 1, 2, 3, \ldots\}.$$

The balance equation for the stationary distribution is

$$\mathfrak{A}^*(\mu)(x) = 0, x \in E, \tag{17}$$

where $\mathfrak{A}^*$ is the conjugate operator (acting on measures $\mu \in \mathscr{M}$) of the generator operator $\mathfrak{A}$:

$$\mathfrak{A}^*(\mu)(x) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}r} m(k,r) + \lambda_0 f(r)(m(k-1,r) - m(k,r)) + m(k+1,0)g(r), k > 1, r > 0, \\ \frac{\mathrm{d}}{\mathrm{d}r} m(1,r) + \lambda_0 \mu(\mathbf{0}) g(r) + m(2,0)g(r) - \lambda_0 f(r) m(1,r), \qquad r > 0, \\ m(1,0) - \mu(\mathbf{0})\lambda_0. \end{cases} \tag{18}$$

The goal of this section is to show that (up to scaling) there is a unique solution μ^* to the balance equation (17) and this solution is the stationary measure of the continuous time $rM/G/1$ process X. Keep $\mu^*(\mathbf{0}) > 0$ as a free parameter to be fixed below. The third line of (18) gives

$$\begin{aligned} \mathfrak{A}^*(\mu^*)(\mathbf{0}) = \mu^*(\mathbf{0})\lambda_0 - m^*(1,0) = 0, \\ m^*(1,0) = \mu^*(\mathbf{0})\lambda_0. \end{aligned} \tag{19}$$

The last equality and the second line of (18) imply that (17) reduces to the following equation for $m(1,\cdot)$:

$$\frac{\mathrm{d}}{\mathrm{d}r} m(1,r) + g(r)(m^*(1,0) + m^*(2,0)) - \lambda_0 f(r) m(1,r) = 0, r > 0. \tag{20}$$

The classical linear ODE theory implies that the unique solution of (20) vanishing at ∞ is

$$m^*(1,r) = \big(m^*(1,0) + m^*(2,0)\big) \int_r^\infty g(u) e^{(F(r)-F(u))\lambda_0} \mathrm{d}u. \tag{21}$$

Substituting $r = 0$ gives the following formula for $m^*(2,0)$:

$$m^*(1,0) = \big(m^*(1,0) + m^*(2,0)\big)\, p(0) \tag{22}$$

or

$$m^*(2,0) \doteq \frac{1-p(0)}{p(0)} m^*(1,0) > 0, \tag{23}$$

where

$$p(0) = \int_0^\infty g(r) e^{-F(r)\lambda_0} \mathrm{d}r$$

is the 0 increment probability of the embedded Markov chain $\mathcal{N}$, given in (10). That $m^*(2,0) > 0$ implies $m^*(1,\cdot) > 0$. Next derive a second expression for $m^*(2,0)$ by integrating both sides of (21) over $[0,\infty)$:

$$\int_0^\infty m^*(1,r)f(r)\mathrm{d}r = \big(m^*(1,0)+m^*(2,0)\big)\int_0^\infty f(r)\int_r^\infty g(u)e^{(F(r)-F(u))\lambda_0}\mathrm{d}u\mathrm{d}r$$
$$= \big(m^*(1,0)+m^*(2,0)\big)\frac{1}{\lambda_0}(1-p(0)),$$

where we have used Fubini's theorem, $m(1,\cdot) > 0$ and the change of variable $s = F(r)$. Definition (23) of $m^*(2,0)$ implies $m^*(1,0) = \frac{p(0)}{1-p(0)}m^*(2,0)$; substituting this in the last line above gives

$$m^*(2,0) = \lambda_0 \int_0^\infty m^*(1,r)f(r)\mathrm{d}r. \tag{24}$$

Formulas (19), (21) and (23) uniquely determine $m^*(1,\cdot)$ and $m^*(2,0)$ given $\mu^*(\mathbf{0})$. For $k > 1$, (17) uses the first line of (18):

$$\frac{\mathrm{d}}{\mathrm{d}r}m(k,s) + \lambda_0 f(r)(m(k-1,s) - m(k,s)) + m(k+1,0)g(r) = 0, r > 0. \tag{25}$$

The unique solution of this linear equation for $k = 2$ decaying at ∞ is

$$m^*(2,r) = m^*(3,0)\int_r^\infty g(u)e^{(F(r)-F(u))\lambda_0}\mathrm{d}u + \lambda_0\int_r^\infty e^{(F(r)-F(u))\lambda_0}m^*(1,u)f(u)\mathrm{d}u, \tag{26}$$

where $m^*(3,0)$ is yet to be determined. To determine it, set $r = 0$ in the above display to get

$$m^*(3,0) = \frac{1}{p(0)}\left(m^*(2,0) - \lambda_0\int_0^\infty e^{-F(u)\lambda_0}m^*(1,u)f(u)\mathrm{d}u\right). \tag{27}$$

With this, $m^*(2,\cdot)$ and $m^*(3,0)$ are determined uniquely, given $\mu(\mathbf{0})$. (24) and the definition of $m^*(3,0)$ imply $m^*(3,0) > 0$, which in its turn implies $m^*(2,\cdot) > 0$.

Letting $r \to \infty$ in (26) gives $\lim_{r\to\infty} m^*(2,r) = 0$. This and the integration of (25) on $[0,\infty)$ gives

$$-m^*(2,0) + \lambda_0\int_0^\infty m^*(1,r)f(r)\mathrm{d}r - \lambda_0\int_0^\infty m^*(2,r)f(r)\mathrm{d}r + m^*(3,0) = 0.$$

This and (24) now imply a similar equation for $m^*(3,0)$:

$$m^*(3,0) = \lambda_0\int_0^\infty m^*(2,r)f(r)\mathrm{d}r. \tag{28}$$

For $k > 2$, one solves (25) inductively, using $k = 2$ as the base case, to get the following sequence of unique positive solutions of (25) vanishing at ∞:

$$m^*(k+1,0) \doteq \frac{1}{p(0)}\left(m^*(k,0) - \lambda_0 \int_0^\infty e^{-F(u)\lambda_0} m^*(k-1,u) f(u)\mathrm{d}u\right), \quad (29)$$

$$\begin{aligned} m^*(k,r) &\doteq m^*(k+1,0)\int_r^\infty g(u)e^{(F(r)-F(u))\lambda_0}\mathrm{d}u \\ &\quad + \lambda_0 \int_r^\infty e^{(F(r)-F(u))\lambda_0} m^*(k-1,u) f(u)\mathrm{d}u, \end{aligned} \quad (30)$$

$r > 0$, and the solution satisfies

$$m^*(k+1,0) = \lambda_0 \int_0^\infty f(r) m^*(k,r)\mathrm{d}r.$$

The last formulas are the extension of (26) and (27) to $k > 2$. Let us note the foregoing computations as a proposition:

Proposition 5 *Given $\mu^*(\mathbf{0}) > 0$, the balance equation (17) has a unique positive solution μ^* given by (19), (21), (23) for $k = 1$ and (26), recursively, for $k \geq 2$. The solution satisfies*

$$m^*(k+1,0) = \lambda_0 \int_0^\infty m^*(k,r) f(r)\mathrm{d}r \quad (31)$$

for $k \geq 1$.

The next proposition links the quantities $m^*(k,0)$ to the stationary distribution of the embedded chain $\mathcal{N}$:

Proposition 6 *Let $\mu^* = (\mu^*(\mathbf{0}), m^*(k,\cdot), k = 1,2,3,\ldots)$ be the unique solution (up to the choice of $\mu^*(\mathbf{0}) > 0$) of the balance equation (17) derived in Proposition 5 above. Then, the measure*

$$m^* \doteq (m^*(1,0), m^*(2,0), m^*(3,0), \cdots)$$

on $\mathbb{N}_+$ is M-invariant, i.e.,

$$m^* M = m^* \quad (32)$$

and

$$m^* = cq \quad (33)$$

for some $c > 0$*, where* q *is the stationary measure given in* (13)*. In particular,*

$$\sum_{k=1}^{\infty} m^*(k,0) < \infty \tag{34}$$

if the stability assumption (14) *holds.*

Proof By Definition (9) of the matrix M, (32) is the following sequence of equations:

$$p(n)(m^*(1,0) + m^*(2,0)) + \sum_{k=2}^{n+1} m^*(k+1,0)p(n+1-k) = m^*(n+1,0), \tag{35}$$

$n = 0, 1, 2, 3, \ldots$ For $n = 0$, (35) reduces to (22), which holds by definition. To prove (35) for $n > 0$, multiply both sides of (20) by $e^{-F(r)\lambda_0}\frac{(F(r)\lambda_0)^n}{n!}$ and integrate from 0 to ∞ to get

$$\begin{aligned} 0 = \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}r} m^*(1,r) e^{-F(r)\lambda_0} \frac{(F(r)\lambda_0)^n}{n!} \mathrm{d}r + (m^*(1,0) + m^*(2,0))p(n) \\ - \lambda_0 \int_0^{\infty} m^*(1,r) e^{-F(r)\lambda_0} \frac{(F(r)\lambda_0)^n}{n!} f(r)\mathrm{d}r. \end{aligned}$$

Integration by parts on the first integral gives

$$0 = (m^*(1,0) + m^*(2,0))p(n) - \lambda_0 \int_0^{\infty} e^{-F(r)\lambda_0} \frac{(F(r)\lambda_0)^{n-1}}{(n-1)!} m^*(1,r) f(r)\mathrm{d}r. \tag{36}$$

For $k = 2, \ldots, n+1$, multiply both sides of (25) by $e^{-F(r)\lambda_0}\frac{(F(r)\lambda_0)^{n+1-k}}{(n+1-k)!}$ and integrate by parts the first term to get

$$\begin{aligned} m^*(k+1,0)p(n+1-k) + \lambda_0 \int_0^{\infty} e^{-F(r)\lambda_0} \frac{(F(r)\lambda_0)^{n+1-k}}{(n+1-k)!} m^*(k-1,r) f(r)\mathrm{d}r \\ - \lambda_0 \int_0^{\infty} e^{-F(r)\lambda_0} \frac{(F(r)\lambda_0)^{n-k}}{(n+1-k)!} m^*(k,r) f(r)\mathrm{d}r = 0. \end{aligned} \tag{37}$$

Summing the last display over k, adding to result (36) and finally noting (29) give (35) for $n > 0$. The Markov chain $\mathcal{N}$ is a constrained random walk on $\mathbb{Z}_+$ with iid increments and hence is obviously irreducible and will therefore have (up to scaling) a unique stationary distribution; (33) follows from this. (34) follows from (33) and Proposition 3. □

Define

$$S(r) \doteq \sum_{k=1}^{\infty} m^*(k, r),$$

whose finiteness under the stability assumption follows from (31) and the previous proposition [see (34)]; (31) also implies

$$\lambda_0 \int_0^{\infty} S(r) f(r) \mathrm{d}r = S(0) - m^*(1, 0) = S(0) - \lambda_0 \mu^*(\mathbf{0}). \tag{38}$$

Remember that $\mu^*(\mathbf{0})$ is still a free parameter. The next proposition computes $\int_0^\infty S(r)\mathrm{d}r$ and $S(0)$ in terms of $\mu^*(\mathbf{0})$ and in terms of the system parameters.

Proposition 7 *Suppose the stability assumption* (14) *holds. Then*

$$S(0) = \lambda_0 \frac{\mu^*(\mathbf{0})}{1-\rho}, \tag{39}$$

and

$$\int_0^{\infty} S(r) \mathrm{d}r = S(0)\nu. \tag{40}$$

Proof Summing the terms of the balance equation gives $S'(r) = -S(0)g(r)$, therefore,

$$S(r) = S(0)G(r), \tag{41}$$

where

$$G(r) \doteq P(\sigma_1 > r) = \int_r^{\infty} g(u) \mathrm{d}u. \tag{42}$$

Integrating both sides of (41) over $[0, \infty)$ gives (40). Next, multiply both sides by $f(r)$ and integrate over $[0, \infty]$:

$$\int_0^{\infty} S(r) f(r) \mathrm{d}r = S(0) \int_0^{\infty} f(r) G(r) \mathrm{d}r = S(0)\bar{\nu},$$

where we have integrated by parts the middle integral. The last display and (38) imply

$$\lambda_0 S(0)\bar{\nu} = S(0) - \lambda_0 \mu^*(\mathbf{0}),$$
$$S(0) = \lambda_0 \mu^*(\mathbf{0}) \frac{1}{1-\rho},$$

which proves (39). □

Let us fix the value for $\mu^*(\mathbf{0})$ to

$$\mu^*(\mathbf{0}) = 1 - \rho = q(0); \tag{43}$$

we will assume (43) whenever the stability assumption (14) is made. This implies, by (39) and (41),

$$S(0) = \lambda_0,\, S(r) = \lambda_0 G(r). \tag{44}$$

A second implication is given in the next lemma.

Lemma 1 *Let $\mu^*(\mathbf{0})$ be fixed as in (43), i.e., we take $\mu^*(\mathbf{0}) = q(0)$. Then*

$$m^* = \lambda_0 q, \tag{45}$$

where $m^* = (m^*(1,0), m^*(2,0), \ldots)$ is as in Proposition 6. In particular,

$$\sum_{k=1}^{\infty} k m^*(k+1,0) = \lambda_0 \sum_{k=1}^{\infty} k q(k) = \lambda_0 \mathbb{E}_q[\mathcal{N}_1]. \tag{46}$$

Proof We know by (33) that $m^* = cq$ for some $c > 0$. Because q is a probability measure, this implies, $c = \sum_{k=1}^{\infty} m^*(k,0) = S(0)$, which equals λ_0 by (44). This proves (45); (46) follows from (45). □

With $\mu^*(\mathbf{0})$ fixed as in (43), the measure μ^* is determined uniquely via Proposition 5. Note that

$$\mu^*(E) = \mu^*(\mathbf{0}) + \int_0^{\infty} S(r)\mathrm{d}r = 1 - \rho + \lambda_0 \nu, \tag{47}$$

where we have used (40), (39) and (43). Thus, in general, with $\mu^*(\mathbf{0})$ fixed as in (43), $\mu^*(E) \neq 1$- to get a proper probability measure, renormalize μ^*:

$$\mu_1^* \doteq \mu^*/\mu^*(E).$$

Proposition 8 below proves that μ_1^* is the unique stationary measure of the $rM/G/1$ process X under the stability assumption (14). The proof will require a subclass of functions in $\mathscr{D}(\mathfrak{A})$ that can separate measures in $\mathscr{M}$. The following lemma identifies such a class.

Lemma 2

$$\mathscr{S} \doteq \{h \in \mathscr{D}(\mathfrak{A}), \sup_{x\in E} |h'(x)| < \infty, \sup_{x \in E} |h(x)| < \infty\}$$

is a separating class of functions for measures in $\mathscr{M}$.

Proof For $\mu_1, \mu_2 \in \mathscr{M}$, $\mu_1 = \mu_2$ if and only if

$$\int_a^b m_{1,k}(r)\mathrm{d}r = \int_a^b m_{2,k}(r)\mathrm{d}r$$

for all $0 < a < b < \infty$ and $k > 0$ ($m_{1,k}$ and $m_{2,k}$ are densities of μ_1 and μ_2 on E_k). Define the standard mollifier

$$\eta(x) \doteq \begin{cases} C_\eta e^{\frac{1}{|x|^1-1}}, & |x| < 1, \\ 0, & |x| > 1, \end{cases}$$

where $C_\eta > 0$ is such that $\int_{-1}^{1} \eta(x)\mathrm{d}x = 1$. For any interval (a, b), $0 < a < b$, define $h_n : E \to [0, 1]$, $1/m < 1/2a$ as follows: for $x = (j, r) \in E$, $j < k$, $h_n(x) = 0$. For $x = (k, r)$

$$h_n(x) = n \int_{-1}^{1} \eta(u/n)\mathbb{1}_{(a,b)}(u + r)\mathrm{d}r,\ j = k. \tag{48}$$

For $j > k$, we proceed recursively:

$$h_n(j, 0) = \int_0^\infty h_n(j-1, r)g(r)\mathrm{d}r, \quad h_n(j, r) = h_n(j, 0)n \int_{nr-1}^{1} \eta(x/n)\mathrm{d}x, \tag{49}$$

where we write $h_n(j, 0)$ instead of $h_n((j, 0))$ to simplify notation. By its definition, $h_n \in \mathscr{S}$ and $\lim_{n\to\infty} h_n(k, r) = \mathbb{1}_{\{(k,r), r\in(a,b)\}}$ almost surely for any measure $\mu \in \mathscr{M}$. This and the bounded convergence theorem imply

$$\lim_{n\to\infty} \int_E h_n(x)\mu(\mathrm{d}x) = \int_a^b m(k, r)\mathrm{d}r.$$

This proves that functions of the form h_n, and therefore the class $\mathscr{S}$ containing them, are a separating class for measures in $\mathscr{M}$.

It remains to show that $h_n \in \mathscr{D}(\mathfrak{A})$. The following three conditions for this are listed in Sect. 2.1: (1) h_n must be absolutely continuous, and this follows from its definitions (48) and (49); (2) h_n must satisfy $h_n((j, 0)) = \int_0^\infty h_n(j-1, r)g(r)\mathrm{d}r$, and this again holds by definition; and (3) $\mathfrak{B}h_n \in L_1^{loc}(X)$, and this follows from the fact that h_n is bounded. □

Proposition 8 *If the stability assumption* (14) *holds, then* μ_1^* *is the unique stationary measure of the process* X*. In particular, if* X_0 *has distribution* μ_1^* *then* X_t *has the same distribution for all* $t > 0$.

Proof The uniqueness follows from the uniqueness claim of Proposition 5. By [5, Proposition (34.7), page 113], μ_1^* is the stationary distribution of X if

$$\int \mathfrak{A}h(x)\mu_1^*(\mathrm{d}x) = 0 \tag{50}$$

for a class of functions $h \in \mathscr{D}(\mathfrak{A})$ that forms a separating class for measures in $\mathscr{M}$ to which μ_1^* belongs; by Lemma 2 $\mathscr{S} \subset \mathscr{D}(\mathfrak{A})$ is such a class. Thus, to prove the proposition it suffices to prove (50) for $h \in \mathscr{S}$. By definition,

$$\int \mathfrak{A}h(x)\mu_1^*(\mathrm{d}x) = \frac{1}{\mu^*(E)} \int \mathfrak{A}h(x)\mu^*(\mathrm{d}x),$$

and one can directly work with the measure μ^* rather than the normalized μ_1^*. For any $h \in \mathscr{S}$,

$$\int_E \mathfrak{A}h(x)\mu^*(\mathrm{d}x) = \lim_{N\to\infty}\sum_{k=1}^{N}\int_0^\infty \left(-\frac{\mathrm{d}h}{\mathrm{d}r}(k,r) + \lambda_0 f(r)(h(k+1,r)-h(k,r))\right) m^*(k,r)\mathrm{d}r. \tag{51}$$

We begin by an integration by parts:

$$\sum_{k=1}^{N}\int_0^\infty \left(-\frac{\mathrm{d}h}{\mathrm{d}r}(k,r) + \lambda_0 f(r)(h(k+1,r)-h(k,r))\right) m^*(k,r)\mathrm{d}r$$
$$= \sum_{k=1}^{N}\int_0^\infty \left(\frac{\mathrm{d}m^*}{\mathrm{d}r}(k,r) - h(k,0)m^*(k,0) + \lambda_0 f(r)(h(k+1,r)-h(k,r))\right) m^*(k,r)\mathrm{d}r.$$

$h(k,0) = \int_0^\infty g(r)h(k-1,r)\mathrm{d}r$ because $h \in \mathscr{D}(\mathfrak{A})$, therefore,

$$= \sum_{k=1}^{N}\int_0^\infty \left(\frac{\mathrm{d}m^*}{\mathrm{d}r}(k,r) - g(r)h(k-1,r)m^*(k,0) + \lambda_0 f(r)(h(k+1,r)-h(k,r))\right) m^*(k,r)\mathrm{d}r.$$

Rearrange the terms in the sum to factor out the common $h(k,r)$:

$$= \sum_{k=1}^{N}\int_0^\infty \left(\frac{\mathrm{d}m^*}{\mathrm{d}r}(k,r) - g(r)m^*(k+1,0) + \lambda_0 f(r)(m^*(k-1,r)-m^*(k,r))\right) h(k,r)\mathrm{d}r$$
$$+ \int_0^\infty m^*(N,r)f(r)\lambda_0 h(N+1,r)\mathrm{d}r.$$

Now $\mathfrak{A}^*\mu^* = 0$ implies

$$= \int m^*(N,r)f(r)\lambda_0 h(N+1,r)\mathrm{d}r;$$

the last integral goes to 0 with N because $\int m^*(N,r)f(r)\mathrm{d}r \to 0$ and h is bounded. Therefore, the limit on the right-hand side of (51) is 0. This proves (50) and establishes that μ^* is the unique stationary distribution of the process X. □

The last proposition and Proposition 8 give

Corollary 2 *The stationary probability of an empty system in continuous time for a stable* $rM/G/1$ *queue is*

$$\mu_1^*(\mathbf{0}) = \frac{\mu^*(\mathbf{0})}{\mu^*(E)} = \frac{1-\rho}{1-\rho+\lambda_0\nu}.$$

4.1 Expected queue length

As Corollary 2 demonstrates, the probability of a stable $rM/G/1$ being empty under its continuous time stationary distribution does not, in general, equal the same probability under its stationary distribution at service completion or arrival times:

$$P_{\mu_1^*}(N_t = 0) = \mu_1^*(\mathbf{0}) = \frac{1-\rho}{1-\rho+\lambda_0\nu} \neq q(0) = 1-\rho.$$

Thus, in general, the steady-state queue length distribution of a stable $rM/G/1$ system in continuous time differs from the same distribution at service completion and arrival times. The following proposition gives a formula for $\mathbb{E}_{\mu_1^*}[N_1]$ under μ_1^*, the expected queue length under the stationary distribution in continuous time; in general, this quantity will not equal $\mathbb{E}_q[\mathcal{N}_1]$, the expected queue length under the stationary distribution at service completion times.

Proposition 9 *The expected $rM/G/1$ queue length under its continuous time stationary measure equals*

$$\mathbb{E}_{\mu^*}[N_t] = \frac{1}{\mu^*(E)}\left(\lambda_0^2\int_0^\infty\left(\int_0^x uf(u)\mathrm{d}u\right)g(x)\mathrm{d}x + \big((1-\rho)+\mathbb{E}_q[\mathcal{N}_1]\big)\lambda_0\nu\right), \tag{52}$$

where $\mathbb{E}_q[\mathcal{N}_1]$ is the stationary mean queue length at service completion times whose formula is given in (15).

Proof The proof proceeds parallel to that of Proposition 7. Set

$$\varphi(r) \doteq \sum_{k=1}^{\infty} m_k^*(r)k; \tag{53}$$

by definition

$$\mathbb{E}_{\mu_1^*}[N_t] = \mathbb{E}_{\mu^*}[N_t]/\mu^*(E) = \frac{1}{\mu^*(E)}\int_0^\infty \varphi(r)\mathrm{d}r. \tag{54}$$

Let us compute $\int_0^\infty \varphi(r)\mathrm{d}r$. Multiply the first and the second lines of the balance equation (17) by k, $k = 1, 2, 3, 4, \ldots$, and sum over k to get

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r}(r) + \lambda_0 f(r)S(r) + \left(\mu^*(\mathbf{0})\lambda_0 + \sum_{k=1}^{\infty} m_{k+1}^*(0)k\right)g(r) = 0, \tag{55}$$

where, as before,

$$S(r) = \sum_{k=1}^{\infty} m_k^*(r) = S(r) = \lambda_0 G(r);$$

the last equality follows from (44). (43) and (46) simplify the terms in parentheses in (55) to

$$\left(\mu^*(\mathbf{0})\lambda_0 + \sum_{k=1}^{\infty} m^*_{k+1}(0)k\right) = \lambda_0\left((1-\rho) + \mathbb{E}_q[\mathcal{N}_1]\right),$$

where $\mathbb{E}_q[\mathcal{N}_1]$ is the stationary mean queue length at service completions. Then, the unique solution of (55) vanishing at ∞ is

$$\varphi(r) = \lambda_0^2 \int_r^{\infty} f(u)G(u)\mathrm{d}u + \left((1-\rho) + \mathbb{E}_q[\mathcal{N}_1]\right)\lambda_0 G(r).$$

Integrating the last display over r over $[0, \infty]$ yields

$$\begin{aligned}\int_0^{\infty} \varphi(r)\mathrm{d}r &= \lambda_0^2 \int_0^{\infty}\int_r^{\infty} f(u)G(u)\mathrm{d}u\mathrm{d}r + \left((1-\rho) + \mathbb{E}_q[\mathcal{N}_1]\right)\lambda_0\nu \\ &= \lambda_0^2 \int_0^{\infty}\left(\int_0^{x} uf(u)\mathrm{d}u\right)g(x)\mathrm{d}x + \left((1-\rho) + \mathbb{E}_q[\mathcal{N}_1]\right)\lambda_0\nu.\end{aligned}$$

This and (54) give (52). □

4.2 Average arrival rate

The random variable

$$\mathcal{A}_n \doteq \mathcal{N}_{n+1} - \mathcal{N}_n + 1$$

represents the number of arrivals to the $rM/G/1$ system between the nth and $(n+1)$st service completions. It follows from (58) that its conditional distribution given $\mathcal{N}_n$ is

$$P(\mathcal{A}_n = j|\mathcal{N}_n) = \begin{cases} p(j), & \mathcal{N}_n > 0, \\ p(j+1), & \mathcal{N}_n = 0. \end{cases} \tag{56}$$

It follows from the Markov property of $\mathcal{N}$ that $(\mathcal{N}, \mathcal{A})$ is a Markov chain and is stationary whenever $\mathcal{N}$ is, with stationary distribution

$$P(\mathcal{A}_\infty = j, \mathcal{N}_\infty = k) = \begin{cases} p(j)q(k), & k > 0, \\ p(j+1)q(0), & k = 0. \end{cases}$$

Then, by the ergodic theorem for stable Markov chains,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{n=1}^{\infty} \mathcal{A}_n &= q(0)\left(1 + \sum_{j=1}^{\infty} jp(j)\right) + (1-q(0))\left(\sum_{j=1}^{\infty} jp(j)\right) \\ &= q(0) + \rho = 1 - \rho + \rho = 1. \end{aligned} \tag{57}$$

Define the interservice time

$$\tau_n \doteq S_{n+1} - S_n;$$

similar to the sequence $\mathcal{A}_n$, the distribution of τ_n is completely determined by $\mathcal{N}$ with the following conditional distribution:

$$P(\tau_n > t | \mathcal{N}_n) = \begin{cases} G(t), & \mathcal{N}_n > 0, \\ \int_0^\infty \lambda_0 e^{-\lambda_0 s} G((t-s)^+) \mathrm{d}s, & \mathcal{N}_n = 0, \end{cases} \tag{58}$$

where the second distribution is the convolution of g and the exponential distribution with rate λ_0 (this is the distribution of the sum of a service time and the first arrival time to the system). The process $(\mathcal{N}, \tau)$ is stable whenever $\mathcal{N}$ is, with stationary distribution

$$P(\tau_\infty > t, \mathcal{N}_\infty = k) = \begin{cases} G(t) q(k), & k > 0, \\ \left(\int_0^\infty \lambda_0 e^{-\lambda_0 s} G((t-s)^+) \mathrm{d}s\right) q(0), & k = 0. \end{cases}$$

The law of large numbers for Markov chains [16, Theorem 17.0.1, page 422] implies

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{n=1}^{\infty} \tau_n = \lim_{n\to\infty} \frac{1}{n} S_n &= q(0)\left(\frac{1}{\lambda_0} + \nu\right) + (1 - q(0))\nu \\ &= q(0)\frac{1}{\lambda_0} + \nu = (1-\rho)\frac{1}{\lambda_0} + \nu. \end{aligned} \tag{59}$$

Proposition 10 *Let A_t denote the number of arrivals to an $rM/G/1$ queue up to time t. Then, the ergodic average arrival rate to the $rM/G/1$ system equals*

$$\lim_{t\to\infty} \frac{A(t)}{t} = \alpha \doteq \frac{\lambda_0}{1-\rho+\lambda_0\nu} = \frac{\lambda_0}{\mu^*(E)}. \tag{60}$$

Proof

$$\lim_{n\to\infty} \frac{A_{T_n}}{T_n} = \lim_{n\to\infty} \frac{A_{T_n}/n}{T_n/n} = \frac{\lambda_0}{1-\rho+\lambda_0\nu}, \tag{61}$$

which follows from (57) and (59). For any other sequence $t_m \nearrow \infty$, we know that there exists a sequence n_m with $T_{n_m} < t_m < T_{n_m+1}$. The Borel Cantelli Lemma and the fact that $\mathcal{A}_n$ has finite moments independent of n imply

$$\lim_{n\to\infty} \frac{\mathcal{A}_n}{T_n} = 0. \tag{62}$$

It follows from the monotonicity of T_n and A_n that

$$\frac{A_{T_{n_m}}}{T_{n_m+1}} \leq \frac{A_{t_m}}{t_m} \leq \frac{A_{T_{n_m+1}}}{T_{n_m}}.$$

This, (62) and (61) imply (60). □

4.3 Average sojourn and waiting time

Let ς_k be the sojourn time (the total amount of time spent) of the kth customer arriving to the system. Little's law is the following statement:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varsigma_k = \frac{\lim_{t\to\infty} N_t/t}{\lim_{t\to\infty} A_t/t}. \tag{63}$$

The classical proof of this result outlined in [14] depends on the distribution of X only to the following extent: that N represents the number of customers in a single-server queueing system and that the ergodic limits related to N and A exist; the existence of the ergodic limits follows from the stationarity of N (see, for example, [3, Theorem 1.6.4, page 50]) and Proposition 10 above. Therefore, the classical proof requires no change for the current setup. For the $rM/G/1$ system, (63) and Proposition 10 give the following formula for the average sojourn time:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varsigma_k &= \frac{\mathbb{E}_{\mu_1^*}[N_t]}{\alpha} \\ &= \lambda_0 \int_0^\infty \left(\int_0^x u f(u) \mathrm{d}u \right) g(x) \mathrm{d}x + \big((1-\rho) + \mathbb{E}_q[\mathcal{N}_1] \big) \nu. \end{aligned} \tag{64}$$

This gives the following formula for the average waiting time:

$$\begin{aligned}\omega &\doteq \lambda_0 \int_0^\infty \left(\int_0^x u f(u) \mathrm{d}u \right) g(x) \mathrm{d}x + \big((1-\rho) + \mathbb{E}_q[\mathcal{N}_1] \big) \nu - \nu \\ &= \lambda_0 \int_0^\infty \left(\int_0^x u f(u) \mathrm{d}u \right) g(x) \mathrm{d}x + \big(\mathbb{E}_q[\mathcal{N}_1] - \rho \big) \nu, \end{aligned} \tag{65}$$

where $\mathbb{E}_q[\mathcal{N}_1]$ can be computed with formula (15).

4.4 Connection to $M/G/1$ queue in continuous time

By Corollary 1, we know that the embedded random walk $\mathcal{N}$ at service completions of the $rM/G/1$ queue has identical dynamics to that of an $M/G/1$ queue with constant rate λ_0 and sequence of service times $\{F(\sigma_1), F(\sigma_2), \ldots\}$—which implies that these systems have the same stationary measures at service completions. Then, a natural question is whether there is a similar correspondence between the continuous time stationary distributions. When f takes the value 0 over a nonzero interval, its integral F becomes not invertible. Because of this, in general, the continuous time stationary distribution of the $M/G/1$ system cannot completely be mapped to that of the $rM/G/1$ system (remember that σ_i has density g; when $f = 0$ is allowed, $F(\sigma_i)$ may have no density and $F(\sigma_i)$ may have compact support even when σ takes values in all of $\mathbb{R}_+$). However, for $f > 0$ an exact mapping between the stationary measures is possible; the details follow.

Assuming $f > 0$ implies $F(r) = \int_0^r f(u)\mathrm{d}u$ is strictly increasing. Let H denote its inverse function; that F is differentiable implies the same for H and the inverse function has the derivative

$$\frac{\mathrm{d}H}{\mathrm{d}s}(s) = \frac{1}{f(H(s))}. \tag{66}$$

Define

$$\bar{g}(s) \doteq g(H(s))\frac{\mathrm{d}H}{\mathrm{d}s}(s) = \frac{g(H(s))}{f(H(s))}, s > 0.$$

For $f > 0$, the change of variable formula of calculus implies that $F(\sigma_i)$ has density $\bar{g}$. The same formula allows one to rewrite the operator $\mathfrak{A}^*$ defining the balance equations of the $rM/G/1$ system in the $s = F(r)$ variable thus:

$$\bar{\mathfrak{A}}^*(\mu)(x) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}m(k,s) + \lambda_0(m(k-1,s) - m(k,s)) + m(k+1,0)\bar{g}(s), k > 1, s > 0, \\ \frac{\mathrm{d}}{\mathrm{d}s}m(1,s) + \bar{g}(s)\lambda_0\mu(\mathbf{0}) + \bar{g}(s)m(2,0) - \lambda_0 m(1,s), \qquad s > 0, \\ \mu(\mathbf{0})\lambda_0 - m(1,0), \end{cases} \tag{67}$$

$\mu \in \mathscr{M}$. The equation

$$\bar{\mathfrak{A}}^*(\mu) = 0 \tag{68}$$

is the balance equation of the $M/G/1$ system with rate λ_0 and service density $\bar{g}$. Let us denote the continuous time process representing this $M/G/1$ system by $\bar{X}$ (which can be written in the PDP framework employed in Sect. 2). The relation between the solution of (68) and the solution of the balance equation (17) is given in the following proposition.

Proposition 11 *Assume $f > 0$. Let μ^* be the solution of (17) given in Proposition 5. Then $\bar{\mu}^* \in \mathscr{M}$, defined by $\bar{\mu}^*(\mathbf{0}) \doteq \bar{\mu}^*(\mathbf{0})$ and by the densities $\bar{m}^*(k,s) \doteq m^*(k, H(s))$ on E_k, $k = 1, 2, 3, \ldots$, solves (68) and does so uniquely up to the choice of $\bar{\mu}^*(\mathbf{0})$. Furthermore, if the stability condition (14) holds and $\bar{\mu}^*(\mathbf{0})$ is set to $1 - \rho$, we have $\bar{\mu}^*(E) = 1$ and $\bar{\mu}^*$ is the unique continuous time stationary measure of $\bar{X}$.*

Proof $\mathfrak{A}^*(\mu^*) = 0 \Rightarrow \bar{\mathfrak{A}}^*(\bar{\mu}^*) = 0$ follows from the chain rule. The uniqueness claim follows from the linearity of (68). That $\bar{\mu}^*(E) = 1$ under Assumption (14) and $\bar{\mu}^*(\mathbf{0}) = 1 - \rho$ follows from the following observation:

$$\bar{\mu}^*(E_k) = \int_0^\infty \bar{m}^*(k,s)\mathrm{d}s = \int_0^\infty m^*(k,r)f(r)\mathrm{d}r = q(k), k > 0;$$

the first equality follows from the change of variable $r = H(s)$, and the last equality follows from (31) and (45). That $\bar{\mu}^*$ is the stationary measure of $\bar{X}$ is proved exactly as in the proof of Proposition 8. □

5 Illustration

Let us now observe the consequences of the results derived in the previous section over two examples. Figure 2a shows the average waiting times for three $rM/G/1$ systems with uniformly distributed service time on the interval [0, 1], as a function of the arrival rate λ_0. We consider three cases for the reshaping function f, increasing, constant and decreasing in r: $f(r) = \frac{3}{4}(2-2r)\mathbb{1}_{(0,1)}(r)$, $f(r) = \mathbb{1}_{(0,1)}(r)$ and $f(r) = 3r\mathbb{1}_{(0,1)}(r)$ (the constant case corresponds to the $M/U/1$ queue). All of these reshaping functions f satisfy (1); therefore, they all have the same average arrival rate $\alpha = \lambda_0$, utilization $\rho = \lambda_0/2$ and empty system probability $\mu^*(\mathbf{0}) = q(0) = 1 - \rho$. Moreover, for all of these reshaping functions f and the assumed system parameters, formula (65) for the average waiting time has a simple explicit expression: for $f(r) = \frac{3}{4}(2-2r)\mathbb{1}_{(0,1)}(r)$,

$$\omega = \omega_1 = \frac{\lambda_0}{8} + \frac{3\lambda_0^2}{40(1-\rho)},$$

for $f(r) = \mathbb{1}_{(0,1)}(r)$ (this is the $M/U/1$ case),

$$\omega = \omega_2 = \frac{\lambda_0}{6} + \frac{\lambda_0^2}{12(1-\rho)},$$

and for $f(r) = 3r\mathbb{1}_{(0,1)}(r)$,

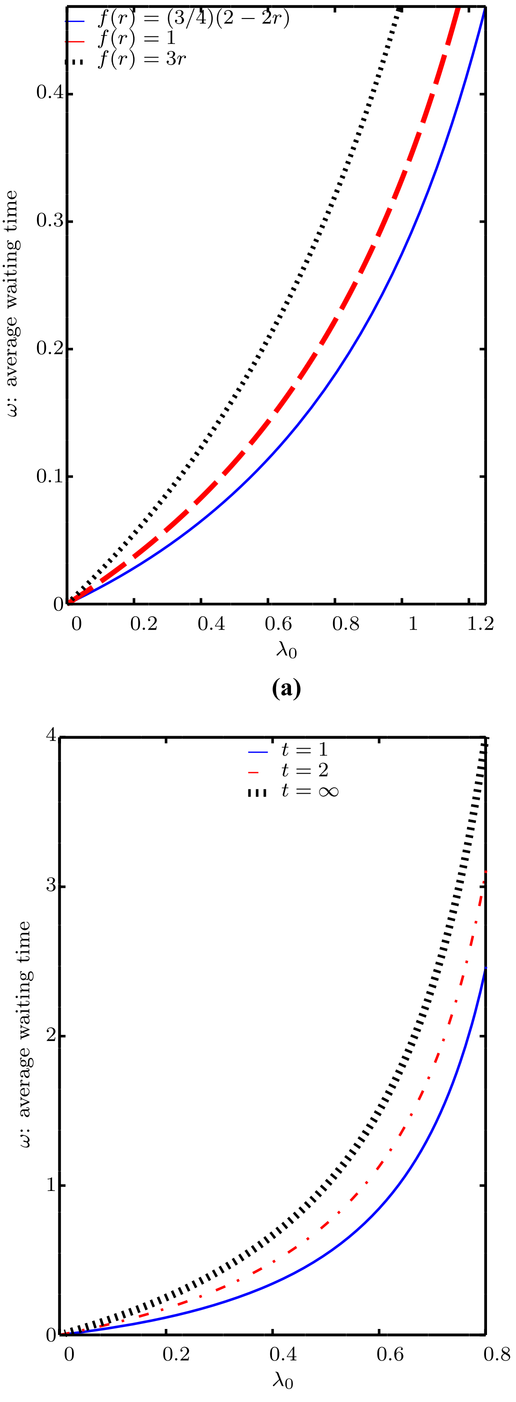
$$\omega = \omega_3 = \frac{\lambda_0}{4} + \frac{9\lambda_0^2}{80(1-\rho)}.$$

We note $\omega_1 < \omega_2 < \omega_3$ for all λ_0 such that $\rho < 1$: i.e., pushing arrivals toward service completions (while keeping the average arrival rate constant) reduces average waiting times. Figure 2a shows the graphs of these functions as λ_0 varies.

Let us now consider an example in which the service time is exponentially distributed with mean $\nu = 1$ and the reshaping function $f(r) = (1-e^{-t})^{-1}\mathbb{1}_{0\leq r\leq t}$; this function restricts arrivals to the last t units of time of service and it satisfies (1). The average waiting time ω of (65) reduces for this case to

$$\omega(t,\lambda_0) = \lambda_0 \frac{1-e^{-t}(t+1)}{1-e^{-t}}\left(1 + \frac{\lambda_0^2}{(1-e^{-t})(1-\rho)}\right);$$

Fig. 2 Average waiting time as a function of the average arrival rate for different reshaping functions. **a** $g(r) = \mathbb{1}_{(0,1)}(r)$, **b** $g(r) = e^{-r}$, $f(r) = \frac{1}{1-e^{-t}}\mathbb{1}_{(0,t)}(r)$



(a)

(b)

ω is increasing in t, i.e., once again, concentrating arrivals near service completions (while keeping the average arrival rate constant) reduces the average waiting time. Figure 2b shows the graph of $\omega(t,\cdot)$ for $t = 1, 2$ and $t = \infty$ (the last corresponds to an $M/M/1$ queue).

We conclude this section with the following observation from our second example: set $t = 3\nu = 3$ in the last example, i.e., we restrict arrivals to the interval $[0, 3\nu]$, where ν is the mean service time. For $\lambda_0 = 0.7$, the system's utilization is $\rho = 0.7$ and the corresponding average waiting time turns out to be $\omega(3, 0.7) = 1.6041$; the same waiting time for the same parameter values but without reshaping is $\omega(\infty, 0.7) = 1.84$. Thus, this not so heavy reshaping reduces average waiting time by 13%.

6 Conclusion

Let us comment briefly on possible future research. We have assumed that the service time distribution has a density g. The analysis at service completions does not depend on this assumption, and the results of Sects. 2.2 and 3 continue to hold without change when σ_i does not have a density. The analysis of Sect. 4 does make use of the assumption that σ_i has a density, but the resulting performance measure formulas (average queue length, probability of an empty system, average waiting time) remain meaningful even when σ_i does not have a density and one expects these results to hold under general service distributions. One simple method of extending our analysis to the general case would be, first, a smooth approximation of the given service distribution, and then taking weak limits. The details of such an argument could be given in future work. The special case of a deterministic constant service time case can be directly handled by appropriate modifications of the balance equation and our arguments based on it.

A natural question is the convergence of the distribution of X_t to the stationary distribution μ^*. As one of the referees pointed out, one way to establish this with precise rates of convergence would be to apply the approach of [15] based on coupling (at the first hitting time to $\mathbf{0}$) and monotonicity arguments. Future research could attempt to give details of this.

In many situations, one may only have an estimate of the remaining service time (rather than the ability to directly observe it, as assumed in the current work). One possible future work is the modeling and analysis of such a setup. We think that, given the possible applications in call centers, another natural direction is the treatment of many servers. Instead of allowing the rate to depend directly on the remaining service times of all of the servers, a possibility is to allow it to depend on a function of them (for example, their minimum or an estimate of it). Finally, it may also be of interest to apply the approach used in the present article to models where the arrival and service rates depend on the queue length as well as the remaining service time.

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