Queueing Syst (2010) 65: 365–383
DOI 10.1007/s11134-010-9180-3

Regularly varying tail of the waiting time distribution in M/G/1 retrial queue

Jerim Kim · Jeongsim Kim · Bara Kim

Received: 24 June 2007 / Revised: 27 April 2010 / Published online: 4 June 2010

Abstract We consider an M/G/1 retrial queue where the service time distribution has a regularly varying tail with index $-\beta$, $\beta > 1$. The waiting time distribution is shown to have a regularly varying tail with index $1-\beta$, and the pre-factor is determined explicitly. The result is obtained by comparing the waiting time in the M/G/1 retrial queue with the waiting time in the ordinary M/G/1 queue with random order service policy.

Keywords M/G/1 retrial queue · Regular variation · Waiting time distribution · Stochastic comparison · Random order service

Mathematics Subject Classification (2000) 60K25

1 Introduction

Retrial queues are queueing systems where arriving customers finding the server occupied may retry for service again after a random amount of time. Retrial queues

J. Kim's research was supported by the Korea Research Foundation (KRF) grant funded by the Korea government (MEST) (2009-0076674). B. Kim's research was supported by the Korea Research Foundation (KRF) grant funded by the Korea government (MEST) (2009-0076600).

J. Kim · B. Kim (✉)
Department of Mathematics and Telecommunication Mathematics Research Center, Korea University, 1, Anam-dong, Sungbuk-ku, Seoul, 136-701, Korea
e-mail: bara@korea.ac.kr

J. Kim
e-mail: b1155@hanmail.net

J. Kim
Department of Mathematics Education, Chungbuk National University, 12, Gaeshin-dong, Heungduk-ku, Cheongju, Chungbuk, 361-763, Korea
e-mail: jeongsimkim@chungbuk.ac.kr

have been widely used to model many problems in telephone systems, call centers, and many telecommunication systems. Detailed overviews for retrial queues can be found in the bibliographies [1–3], surveys [8, 17], and books [4, 9].

In this paper, we consider an M/G/1 retrial queueing system where customers arrive from outside according to a Poisson process with intensity λ, and service times B are independent and identically distributed with distribution function F_B. If the server is idle at the time of a customer arrival, the arriving customer begins to be served immediately and leaves the system after service completion. Otherwise, i.e., if the server is busy, the arriving customer joins a retrial group (i.e., forms a source of repeated customers), called an orbit. If an incoming repeated customer from the orbit finds the server idle, this customer is served and leaves the system after service completion. Otherwise, i.e., if the repeated customer finds the server busy, the customer comes back to the orbit immediately and repeats the retrial process. The inter-retrial times of each customer in the orbit are assumed to be exponentially distributed with mean ν^{-1}. The arrival process, the service times, and the retrial times are assumed to be mutually independent. The traffic load ρ is defined as $\rho = \lambda \mathbb{E}B$. We assume that $\rho < 1$ for stability of the system.

Tail behaviors of the queue size and the waiting time distributions in retrial queues began to be investigated recently. Light-tailed behaviors have been studied by Nobel and Tijms [14] and Kim et al. [12, 13]. Nobel and Tijms [14] suggested a light-tailed approximation of the waiting time distribution in the M/G/1 retrial queue when the service time distribution has a finite exponential moment. Kim et al. [12] showed that if the service time distribution has a finite exponential moment then the tail of the queue size distribution is asymptotically given by a geometric function multiplied by a power function in the M/G/1 retrial queue under an additional condition. The result of [12] was generalized to the MAP/G/1 retial queue by Kim et al. [13].

The interest of our work is that we find the heavy-tailed asymptotics for the waiting time distribution in the M/G/1 retrial queue. There are many references for the heavy-tailed asymptotics in ordinary queues. See, for example, [5, 6, 11, 16] and references therein. However, for the heavy-tailed asymptotics in retrial queues, it seems that Shang et al. [15] is the only known result in the open literature. Shang et al. [15] showed that the stationary distribution of the queue length in the M/G/1 retrial queue is subexponential if the stationary distribution of the queue length in the corresponding ordinary M/G/1 queue is subexponential. As a corollary of this property, they proved that the stationary distribution of the queue length has a regularly varying tail if the service time distribution has a regularly varying tail.

The main contribution of this paper is to show that if the service time distribution has a regularly varying tail of index $-\beta$, $\beta > 1$, in the M/G/1 retrial queue, then the waiting time distribution has a regularly varying tail of index $1 - \beta$. More precisely, we prove that if the distribution function F_B of service times satisfies

$$1 - F_B(x) \sim x^{-\beta} L(x) \quad \text{as } x \to \infty$$

with $\beta > 1$ and a slowly varying function L, then the distribution function F_W for the waiting time W of an arbitrary customer satisfies

$$1 - F_W(x) \sim c x^{1-\beta} L(x) \quad \text{as } x \to \infty \tag{1}$$

with a constant $c > 0$ that is given explicitly. Here and subsequently, $f(x) \sim g(x)$ denotes $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Boxma et al. [7] obtained the same result as (1) for the waiting time distribution in the ordinary M/G/1 queue with random order service (ROS) policy. The main result (1) is obtained by comparing the waiting time in the M/G/1 retrial queue with the waiting time in the ordinary M/G/1 queue with ROS policy.

The remainder of this paper is organized as follows: In Sect. 2 we show that if the service time distribution has a regularly varying tail of index $-\beta$, $\beta > 1$, then tails of several first passage time distributions are bounded by a function that is of regular variation with index $-\beta$. In Sect. 3 we present our main result. In Sect. 4 we compare the conditional waiting times between the M/G/1 retrial queue and the ordinary M/G/1 queue with ROS policy. Section 5 is devoted to the derivation of the tail asymptotics stated without proof in Sect. 3.

2 First passage time distributions

We consider the M/G/1 retrial queue where arrival rate is λ and service times have the distribution function F_B. Let us define

$$N(t) = \text{the number of customers in the orbit at time } t,$$

$$C(t) = \begin{cases} 1 & \text{if the server is busy at time } t, \\ 0 & \text{if the server is idle at time } t, \end{cases}$$

$$X(t) = \begin{cases} \text{the elapsed service time of the customer who is in service at time } t \\ \quad \text{if } C(t) = 1, \\ 0 \quad \text{if } C(t) = 0. \end{cases}$$

Then $\{(N(t), C(t), X(t)) : t \geq 0\}$ is a Markov process. We assume that every sample path of the Markov process is right-continuous. Let

$$\begin{aligned} \tau_n &= \inf\bigl\{t > 0 : N(t) = n, C(t) = 1, X(t) = 0\bigr\}, \quad n = 0, 1, 2, \ldots, \\ \sigma_n &= \inf\bigl\{t > 0 : N(t) = n, C(t) = 0\bigr\}, \quad n = 0, 1, 2, \ldots, \\ G_n(x) &= \mathbb{P}\bigl(\sigma_n \leq x \bigm| N(0) = n, C(0) = 1, X(0) = 0\bigr), \quad n = 0, 1, 2, \ldots, \\ H_n(x) &= \mathbb{P}\bigl(\tau_{n-1} \leq x \bigm| N(0) = n, C(0) = 1, X(0) = 0\bigr), \quad n = 1, 2, 3, \ldots; \end{aligned}$$

that is, τ_n is the first time a service starts with n customers in the orbit, σ_n is the first time a service ends with n customers in the orbit, $G_n(x)$ is the distribution function of the time interval that starts when a service starts with n customers in the orbit and ends when a service completes with n customers in the orbit, and $H_n(x)$ is the distribution function of the time interval that starts when a service starts with n customers in the orbit and ends when a service starts with $n - 1$ customers in the orbit. See Fig. 1 for an illustration of a sample path of $\{N(t) : t \geq 0\}$ with σ_n and τ_{n-1}.

Clearly,

$$G_n(x) \geq H_n(x), \quad n = 1, 2, 3, \ldots. \tag{2}$$

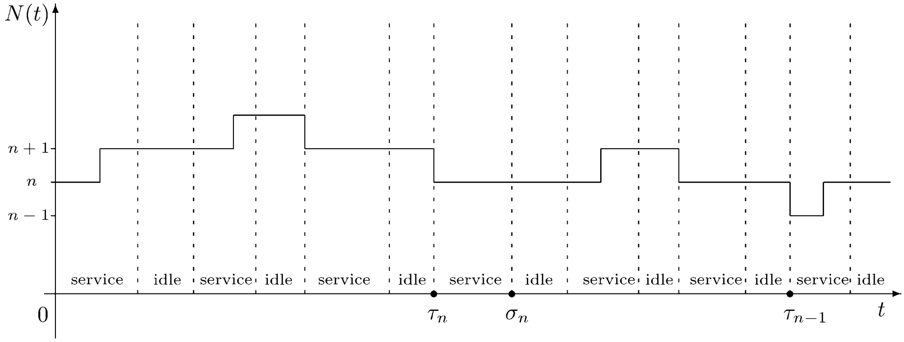

Fig. 1 A sample path of $\{N(t) : t \geq 0\}$

We also have that

$$G_n(x) \leq G_m(x), \quad x \in \mathbb{R}, \text{ if } 0 \leq n \leq m, \tag{3}$$

$$H_n(x) \leq H_m(x), \quad x \in \mathbb{R}, \text{ if } 0 \leq n \leq m, \tag{4}$$

which will be used later. Even though (3) and (4) are intuitively obvious, we give a brief verification of this below.

Suppose that $0 \leq n \leq m$ and $N(0) = n$, $C(0) = 1$, $X(0) = 0$. We introduce a new term, 'k-period'. k-period is the duration that starts when a customer from outside begins service with k customers in the orbit and ends when the server becomes idle with k customers in the orbit. Now we construct a modified system by two procedures. First, whenever a customer arrives and initiates a k-period, we remove the k-period with probability $\frac{m-n}{k+m-n}$. Then for the resulting system, if a service ends with k customers in the orbit, then (i) the next service starts by a customer from the orbit with probability $\frac{(k+m-n)\nu}{(k+m-n)\nu+\lambda}$ and by a customer from outside with probability $\frac{\lambda}{(k+m-n)\nu+\lambda}$, and (ii) the idle time until the next service is exponentially distributed with parameter $\frac{k}{k+m-n}((k+m-n)\nu+\lambda)$. Secondly, during each idle time with k customers in the orbit we shorten the time horizon so that the idle time is exponentially distributed with parameter $(k+m-n)\nu+\lambda$. After the two procedures, the time to the first service completion with n customers in the orbit has the distribution function G_m. Obviously, after the two procedures, the time to the first service completion with n customers in the orbit is smaller than that in the original system, which has the distribution function G_n. Thus (3) is verified. Equation (4) can be shown by a similar argument.

For a distribution function F, the complementary distribution function is denoted by $\overline{F}$, i.e., $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$. In this section, we assume that the service time distribution has a regularly varying tail with index $-\beta$, $\beta > 1$, i.e., $\overline{F}_B(x) \sim x^{-\beta}L(x)$ as $x \to \infty$ with a slowly varying function L. The following proposition asserts that, for all n, $\overline{G}_n(x)$ and $\overline{H}_n(x)$ are bounded by a function that is of regular variation with index $-\beta$. This proposition will be used in the proof of our main result.

Proposition 1 *We have*

$$\overline{G}_n(x) \lesssim x^{-\beta} L(x), \quad n = 0, 1, 2, \ldots,$$

and

$$\overline{H}_n(x) \lesssim x^{-\beta} L(x), \quad n = 1, 2, 3, \ldots,$$

where $f(x) \lesssim g(x)$ *denotes* $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

To prove this, we need a series of lemmas.

Lemma 1 *For* $n \geq 1$,

$$\overline{G}_n(x) \lesssim \overline{F}_B(x) \quad \textit{if and only if} \quad \overline{H}_n(x) \lesssim \overline{F}_B(x).$$

Proof By (2), $\overline{H}_n(x) \lesssim \overline{F}_B(x)$ implies $\overline{G}_n(x) \lesssim \overline{F}_B(x)$. Now we show the converse, that is,

$$\overline{G}_n(x) \lesssim \overline{F}_B(x) \quad \text{implies} \quad \overline{H}_n(x) \lesssim \overline{F}_B(x).$$

Suppose that $\overline{G}_n(x) \lesssim \overline{F}_B(x)$. Letting

$$J_n(x) = \mathbb{P}\big(\tau_{n-1} \leq x \,\big|\, N(0) = n, C(0) = 0\big),$$

we have

$$H_n(x) = G_n * J_n(x), \tag{5}$$

where $*$ denotes the convolution of distributions. We observe that

$$J_n(x) = \frac{n\nu}{n\nu + \lambda} E_{n\nu+\lambda}(x) + \frac{\lambda}{n\nu + \lambda} E_{n\nu+\lambda} * H_n(x), \tag{6}$$

where E_α denotes the exponential distribution function with mean α^{-1}. Substituting (6) into (5) leads to

$$H_n(x) = \frac{n\nu}{n\nu + \lambda} G_n * E_{n\nu+\lambda}(x) + \frac{\lambda}{n\nu + \lambda} G_n * E_{n\nu+\lambda} * H_n(x).$$

This implies

$$H_n(x) = \sum_{k=1}^{\infty} \frac{n\nu}{n\nu + \lambda} \left(\frac{\lambda}{n\nu + \lambda}\right)^{k-1} (G_n * E_{n\nu+\lambda})^{*k}(x),$$

where the superscript $*k$ on the right-hand side denotes the k-fold convolution. Since $\overline{G}_n(x) \lesssim \overline{F}_B(x)$, we have $\overline{G_n * E_{n\nu+\lambda}}(x) \lesssim \overline{F}_B(x)$. By Proposition 2.9 in [16], we obtain $\overline{H}_n(x) \lesssim \overline{F}_B(x)$. □

Now we define

$$A(t) = \text{the number of exogenous arrivals during } (0, t],$$

$$q = 1 - \int_0^\infty e^{-\lambda t}\, dF_B(t),$$
$$\theta = \inf\{t > 0 : C(t) = 0\}.$$

Note that q is the probability that at least one exogenous arrival occurs during a service time. Let

$$G_{n,0}(x) = \mathbb{P}\big(\sigma_n \le x \,\big|\, N(0) = n, C(0) = 1, X(0) = 0, A(\theta) = 0\big),$$
$$K_n(x) = \mathbb{P}\big(\tau_n \le x \,\big|\, N(0) = n, C(0) = 1, X(0) = 0, A(\theta) \ge 1\big),$$

i.e., $G_{n,0}(x)$ is the distribution function of a service time given that there is no exogenous arrival during the service time, and $K_n(x)$ is the distribution function of the time interval between two consecutive epochs at which services start with n customers in the orbit, given that at least one exogenous arrival occurs during the first service time in the interval.

Lemma 2 *We have*

(a) *For* $n = 0, 1, 2, \ldots,$

$$G_n(x) = (1-q)G_{n,0}(x) + q(K_n * G_n)(x). \tag{7}$$

(b) *For* $n = 0, 1, 2, \ldots,$

$$\overline{G}_{n,0}(x) \le \frac{e^{-\lambda x}}{1-q}\overline{F}_B(x),$$
$$\overline{K}_n(x) \le \frac{1}{q}\overline{H}_{n+1}(x).$$

Proof (a) We decompose $G_n(x)$ as

$$G_n(x) = (1-q)G_{n,0}(x) + q\mathbb{P}\big(\sigma_n \le x \,\big|\, N(0) = n, C(0) = 1, X(0) = 0, A(\theta) \ge 1\big), \tag{8}$$

which can be obtained by conditioning on whether an exogenous arrival occurs during a service time or not. Given $\{N(0) = n, C(0) = 1, X(0) = 0, A(\theta) \ge 1\}$, we have $\tau_n < \sigma_n$, i.e.,

$$\sigma_n = \tau_n + (\sigma_n - \tau_n) \quad \text{with } \sigma_n - \tau_n > 0.$$

Furthermore, given $\{N(0) = n, C(0) = 1, X(0) = 0, A(\theta) \ge 1\}$, we have that

- τ_n and $\sigma_n - \tau_n$ are independent,
- τ_n has the distribution function $K_n(x)$,
- $\sigma_n - \tau_n$ has the distribution function $G_n(x)$.

Therefore,

$$\mathbb{P}\big(\sigma_n \le x \,\big|\, N(0) = n, C(0) = 1, X(0) = 0, A(\theta) \ge 1\big) = K_n * G_n(x). \tag{9}$$

Substituting (9) into (8) leads to (7).

(b) Since $1-q=\mathbb{P}(A(\theta)=0 \mid N(0)=n, C(0)=1, X(0)=0)$, we have

$$(1-q)\overline{G}_{n,0}(x)=\mathbb{P}\big(\theta>x, A(\theta)=0 \,\big|\, N(0)=n, C(0)=1, X(0)=0\big),$$

which is less than or equal to $\mathbb{P}(\theta>x, A(x)=0 \mid N(0)=n, C(0)=1, X(0)=0)$. This proves that

$$(1-q)\overline{G}_{n,0}(x)\le \overline{F}_B(x)e^{-\lambda x}.$$

On the other hand, since $q=\mathbb{P}(A(\theta)\ge 1 \mid N(0)=n, C(0)=1, X(0)=0)$, we have

$$\begin{aligned}
q\overline{K}_n(x)&=\mathbb{P}\big(\tau_n>x, A(\theta)\ge 1 \,\big|\, N(0)=n, C(0)=1, X(0)=0\big)\\
&=\sum_{k=1}^{\infty}\mathbb{P}\big(\tau_n>x, A(\theta)=k \,\big|\, N(0)=n, C(0)=1, X(0)=0\big)\\
&\le\sum_{k=1}^{\infty}\mathbb{P}\big(\tau_n>x, A(\theta)=k \,\big|\, N(0)=n+1, C(0)=1, X(0)=0\big)\\
&\le\mathbb{P}\big(\tau_n>x \,\big|\, N(0)=n+1, C(0)=1, X(0)=0\big)\\
&=\overline{H}_{n+1}(x),
\end{aligned}$$

which completes the proof. □

Now, for $n\ge 1$, we consider an ordinary M/G/1 queue where arrival rate is λ and service times have a distribution function $F_{B^{(n)}}$:

$$F_{B^{(n)}}(x)=\sum_{k=1}^{\infty}\frac{n\nu}{n\nu+\lambda}\left(\frac{\lambda}{n\nu+\lambda}\right)^{k-1}(F_B * E_{n\nu+\lambda})^{*k}(x). \tag{10}$$

We note that $F_{B^{(n)}}$ is the distribution function of $B^{(n)}$ defined as

$$B^{(n)}=\sum_{k=1}^{\mathcal{I}}(B_k+\mathcal{E}_k),$$

where $B_k, \mathcal{E}_k, k=1,2,3,\dots$, and $\mathcal{I}$ are independent random variables with distribution functions

$$\mathbb{P}(B_k\le x)=F_B(x),\quad x\in\mathbb{R}, \tag{11}$$

$$\mathbb{P}(\mathcal{E}_k\le x)=E_{n\nu+\lambda}(x),\quad x\in\mathbb{R}, \tag{12}$$

$$\mathbb{P}(\mathcal{I}=k)=\frac{n\nu}{n\nu+\lambda}\left(\frac{\lambda}{n\nu+\lambda}\right)^{k-1},\quad k=1,2,3,\dots. \tag{13}$$

The mean of $B^{(n)}$ is given by

$$\mathbb{E}B^{(n)} = (\mathbb{E}B_1 + \mathbb{E}\mathcal{E}_1)\mathbb{E}\mathcal{I} = \left(\mathbb{E}B + \frac{1}{n\nu + \lambda}\right)\left(1 + \frac{\lambda}{n\nu}\right).$$

Let

$$\rho^{(n)} \equiv \lambda\mathbb{E}B^{(n)} = \left(\rho + \frac{\lambda}{n\nu + \lambda}\right)\left(1 + \frac{\lambda}{n\nu}\right) \tag{14}$$

denote the offered load in the ordinary M/G/1 queue with the service time distribution function $F_{B^{(n)}}$.

Lemma 3 *Suppose that $\rho^{(n)} < 1$. The distribution function $\mathcal{G}^{(n)}$ of a busy period in the M/G/1 queue satisfies*

$$\overline{\mathcal{G}^{(n)}}(x) \sim \left(1 - \rho^{(n)}\right)^{-\beta-1}\left(1 + \frac{\lambda}{n\nu}\right)x^{-\beta}L(x) \quad \textit{as } x \to \infty.$$

Proof By Proposition 2.9 in [16],

$$\overline{F}_{B^{(n)}}(x) \sim \left(1 + \frac{\lambda}{n\nu}\right)x^{-\beta}L(x) \quad \text{as } x \to \infty. \tag{15}$$

Combining (15) and the main theorem in [10] completes the proof. □

We now prove Proposition 1.

Proof of Proposition 1 Choose n such that $\rho^{(n)} < 1$. According to Lemma 3, we have

$$\overline{\mathcal{G}^{(n)}}(x) \lesssim x^{-\beta}L(x). \tag{16}$$

By the stochastic comparison of the M/G/1 retrial queue and the ordinary M/G/1 queue with the service time distribution function $F_{B^{(n)}}$, it can be easily shown that

$$\overline{G}_k(x) \le \overline{\mathcal{G}^{(n)}}(x), \quad k \ge n. \tag{17}$$

We have, by (16) and (17),

$$\overline{G}_k(x) \lesssim x^{-\beta}L(x), \quad k \ge n, \tag{18}$$

and by (7),

$$G_k(x) = G_{k,0} * \sum_{i=0}^{\infty}(1-q)q^i K_k^{*i}(x), \quad k = 0, 1, 2, \dots.$$

By Lemma 2(b) and Proposition 2.9 in [16],

$$\overline{G}_k(x) \lesssim x^{-\beta}L(x) \quad \text{if } \overline{H}_{k+1}(x) \lesssim x^{-\beta}L(x),\ k = 0, 1, 2, \dots. \tag{19}$$

The proof is completed by Lemma 1, (18) and (19). □

3 The main result

In this section, we present our main result. The result is proved by comparing the waiting time distribution in the M/G/1 retrial queue with the waiting time distribution in the ordinary M/G/1 ROS queue.

We consider the corresponding ordinary M/G/1 ROS queue where arrival rate is λ and service times have the distribution function F_B. Under the ROS policy, at the completion of a service, the server randomly takes one of the waiting customers into service. Let W_{ROS} denote a generic random variable for the waiting time of an arbitrary customer in the ordinary M/G/1 ROS queue and let $F_{W_{\text{ROS}}}$ be its distribution function.

We state a result of Boxma et al. [7] on the regularly varying tail of the waiting time distribution in the ordinary M/G/1 queue with ROS. We assume that for $\beta > 1$, and a slowly varying function L,

$$\overline{F}_B(x) \sim x^{-\beta} L(x) \quad \text{as } x \to \infty. \tag{20}$$

Lemma 4 [7] *If* (20) *holds for* $\beta > 1$, *and a slowly varying function* L, *then*

$$\overline{F}_{W_{\text{ROS}}}(x) \sim c x^{1-\beta} L(x) \quad \textit{as } x \to \infty,$$

where

$$c = \frac{\rho}{1-\rho} h(\rho, \beta) \frac{1}{\beta - 1} \frac{1}{\mathbb{E}B}, \tag{21}$$

with

$$h(\rho, \beta) = \int_0^1 f(u, \rho, \beta)\, du,$$

$$f(u, \rho, \beta) = \frac{\rho}{1-\rho} \left(\frac{\rho u}{1-\rho} \right)^{\beta - 1} (1-u)^{\frac{1}{1-\rho}} + \left(1 + \frac{\rho u}{1-\rho} \right)^{\beta} (1-u)^{\frac{1}{1-\rho} - 1}.$$

Remark Theorem 4.1 of Boxma et al. [7] provides the result of Lemma 4 for $1 < \beta < 2$. They don't mention the case of $\beta \geq 2$ explicitly. However, we can see that Lemma 4 still holds for $\beta \geq 2$ by following the argument of Sect. 5 in [7]. We confirmed this by personal communication with one of the authors of [7].

We now present our main result. The proof is deferred to Sect. 5.

Theorem 1 *Let* W *be the waiting time of an arbitrary customer in the M/G/*1 *retrial queue. If* (20) *holds for* $\beta > 1$, *and a slowly varying function* L, *then the distribution function* F_W *of* W *satisfies*

$$\overline{F}_W(x) \sim \overline{F}_{W_{\text{ROS}}}(x) \quad \textit{as } x \to \infty,$$

i.e.,

$$\overline{F}_W(x) \sim c x^{1-\beta} L(x) \quad \textit{as } x \to \infty,$$

where c *is given by* (21).

Remark Theorem 1 shows that, if the service time distribution has a regularly varying tail, then the waiting time distribution of the M/G/1 retrial queue has the same tail asymptotics as the waiting time distribution of the ordinary M/G/1 queue with ROS. This is proved in Sect. 5 by using the stochastic comparison of the waiting time in the retrial queue with the waiting time in the ordinary ROS queue.

A similar argument can be applied to the number of customers. More precisely, we can show that, if the service time distribution has a regularly varying tail, then the number of customers in the orbit of the M/G/1 retrial queue has the same tail asymptotics as the queue size in the ordinary M/G/1 queue. For the ordinary M/G/1 queue with arrival rate λ and service time distribution function F_B, it is well known (see, for example, Asmussen et al. [6]) that, if (20) holds for $\beta > 1$, then the stationary queue size $\tilde{N}$ satisfies

$$\mathbb{P}(\tilde{N} > x) \sim \frac{\lambda^\beta}{(\beta-1)(1-\rho)} x^{1-\beta} L(x) \quad \text{as } x \to \infty.$$

Therefore, following a similar argument to the stochastic comparison of this paper, we can conclude that, if (20) holds for $\beta > 1$, then the stationary number of customers in the orbit of the retrial queue satisfies

$$\mathbb{P}(N > x) \sim \frac{\lambda^\beta}{(\beta-1)(1-\rho)} x^{1-\beta} L(x) \quad \text{as } x \to \infty. \tag{22}$$

Here, by a little abuse of notation, N denotes the number of customers in the orbit of the retrial queue at steady state. We note that (22) is consistent with a result of Shang et al. [15].

4 Comparison between the retrial queue and the ordinary queue

In this section, we compare the retrial queue with the ordinary queue. In Sect. 4.1 we compare the conditional waiting time between the M/G/1 retrial queue and the ordinary M/G/1 queue with ROS. In Sect. 4.2 we compare the number of customers between the M/G/1 retrial queue and the ordinary M/G/1 queue.

4.1 Comparison of the conditional waiting time between the retrial queue and the ordinary ROS queue

We consider the M/G/1 retrial queue. When $N(0) \geq 1$, choose an arbitrary customer in the orbit and call it the tagged customer. Let ϕ be the service initiation epoch of the tagged customer. Let

$$\Phi_k(t) = \begin{cases} \mathbb{P}(\phi \leq t \mid N(0) = k, C(0) = 1, X(0) = 0), & k = 1, 2, \dots, \\ U(t), & k = 0, \end{cases} \tag{23}$$

where

$$U(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Now, for each n with $\rho^{(n)} < 1$, where $\rho^{(n)}$ is defined as (14), we consider an ordinary M/G/1 ROS queue where arrival rate is λ and service times have the distribution function $F_{B^{(n)}}$ in (10). Let $N^{(n)}(t)$ be the number of customers in the queue at time t, excluding the one in service, if any. If there is a customer in service at time t, let $X^{(n)}(t)$ be the elapsed service time of the customer in service at that time. If the server is idle at t, $X^{(n)}(t)$ is set to be zero. When $N^{(n)}(0) \geq 1$, choose an arbitrary customer waiting in the queue for service and call it the tagged customer. Let $\phi^{(n)}$ be the service initiation epoch of the tagged customer. Let

$$\Phi_k^{(n)}(t) = \begin{cases} \mathbb{P}(\phi^{(n)} \leq t \mid N^{(n)}(0) = k, X^{(n)}(0) = 0), & k > n, \\ U(t), & k \leq n. \end{cases} \tag{24}$$

Lemma 5 *We have*

(a) *For $k \geq 0$,*

$$\Phi_k(t) \geq \Phi_k^{(n)} * H_1^{*n}(t). \tag{25}$$

(b) *For $k \geq 0$ and $n \geq 0$,*

$$\Phi_{k+n}(t) \geq H_1^{*n} * \left(\frac{n}{k+n} U + \frac{k}{k+n} \Phi_k \right)(t), \tag{26}$$

$$\Phi_{k+n}(t) \geq H_1^{*2n} * \left(\frac{n}{k+n} U + \frac{k}{k+n} \Phi_k^{(n)} \right)(t). \tag{27}$$

Proof (a) Letting

$$\Phi_{k,n}(t) = \begin{cases} \mathbb{P}(\min\{\phi, \tau_n\} \leq t \mid N(0) = k, C(0) = 1, X(0) = 0), & k > n, \\ U(t), & k \leq n, \end{cases}$$

we have

$$\Phi_{k,n}(t) \geq \Phi_k^{(n)}(t). \tag{28}$$

Clearly,

$$\Phi_k(t) \geq \Phi_{k,n} * H_n * H_{n-1} * \cdots * H_1(t). \tag{29}$$

Substituting (28) and $H_m(t) \geq H_1(t)$, $m = 1, 2, \ldots, n$, into (29) yields (25).

(b) We prove (26) by induction on n. If $n = 0$, then (26) is trivial. Suppose that (26) holds for $n = m \geq 0$. Then

$$\begin{aligned} \Phi_{k+m+1}(t) &\geq H_{k+m+1} * \left(\frac{1}{k+m+1} U + \frac{k+m}{k+m+1} \Phi_{k+m} \right)(t) \\ &\geq H_{k+m+1} * \left(\frac{1}{k+m+1} U + \frac{k+m}{k+m+1} H_1^{*m} \right. \\ &\qquad \left. * \left(\frac{m}{k+m} U + \frac{k}{k+m} \Phi_k \right) \right)(t) \\ &\geq H_1^{*(m+1)} * \left(\frac{m+1}{k+m+1} U + \frac{k}{k+m+1} \Phi_k \right)(t). \end{aligned}$$

Thus (26) holds for $n = m + 1$, which completes the proof of (26). Substitution of (25) into (26) yields (27). □

4.2 Comparison of the number of customers in the retrial queue and the ordinary queue

For each n with $\rho^{(n)} < 1$, we consider the embedded Markov chains in the M/G/1 retrial queue and in the ordinary M/G/1 queue with the service time distribution function $F_{B^{(n)}}$. First, we describe the embedded Markov chain in the M/G/1 retrial queue. For $k = 1, 2, 3, \ldots$, let M_k be the number of customers in the orbit immediately after the beginning of the kth service for the M/G/1 retrial queue. We observe that $\{M_k : k = 1, 2, 3, \ldots\}$ is a Markov chain. For an illustration, embedded points are marked with dots in Fig. 2.

Next we describe the embedded Markov chain in the ordinary M/G/1 queue with the service time distribution function $F_{B^{(n)}}$. Recall that the generic service time $B^{(n)}$ is written as

$$B^{(n)} = \sum_{i=1}^{\mathcal{I}} (B_i + \mathcal{E}_i),$$

where $B_i, \mathcal{E}_i$, $i = 1, 2, 3, \ldots$, and $\mathcal{I}$ are independent random variables with distribution functions given by (11)–(13). We call each $B_i + \mathcal{E}_i$ a subservice. Thus a service time in the M/G/1 queue consists of a geometric number of subservices. Furthermore, a subservice consists of two periods, namely B-period and $\mathcal{E}$-period. The lengths of B-period and $\mathcal{E}$-period have distribution functions F_B and $E_{n\nu+\lambda}$, respectively. Figure 3 illustrates the structure of a service time.

For $k = 1, 2, 3, \ldots$, let $M_k^{(n)}$ be the number of customers waiting in the queue for service, excluding the one starting subservice, immediately after the beginning of the kth subservice for the M/G/1 queue with the service time distribution function $F_{B^{(n)}}$. We observe that $\{M_k^{(n)} : k = 1, 2, 3, \ldots\}$ is a Markov chain. For an illustration, embedded points are marked with dots in Fig. 4.

The following lemma provides the relation between the stationary distribution of $\{M_k : k = 1, 2, 3, \ldots\}$ and the stationary distribution of $\{M_k^{(n)} : k = 1, 2, 3, \ldots\}$.

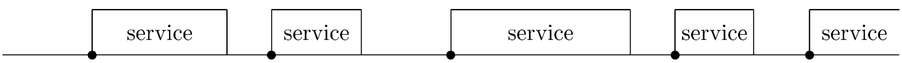

Fig. 2 Embedded points in the M/G/1 retrial queue

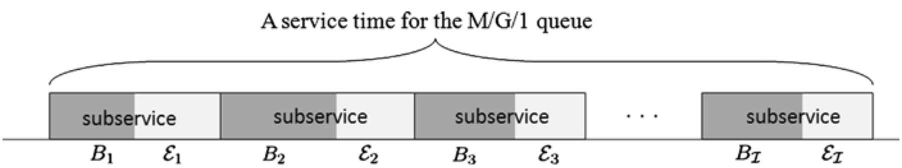

Fig. 3 Structure of a service time for the ordinary M/G/1 queue

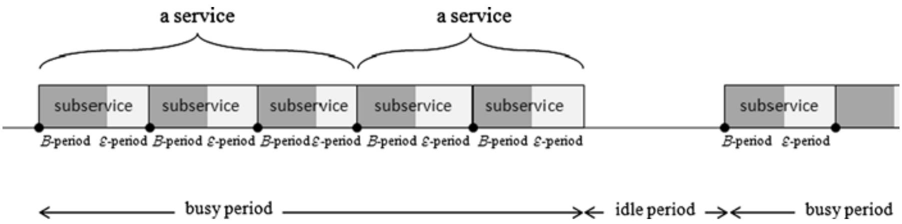

Fig. 4 Embedded points in the ordinary M/G/1 queue with service time distribution function $F_{B^{(n)}}$

Lemma 6 *Let M and $M^{(n)}$ denote random variables having stationary distributions of $\{M_k : k = 1, 2, 3, \dots\}$ and $\{M_k^{(n)} : k = 1, 2, 3, \dots\}$, respectively. Then*

$$(M - n)^+ \le M^{(n)} \quad \textit{in distribution},$$

where $(a)^+ = \max\{a, 0\}$.

Proof Suppose that $M_1 = 0$ and $M_1^{(n)} = 0$. Then induction on k shows that, for $k = 1, 2, 3, \dots,$

$$(M_k - n)^+ \le M_k^{(n)} \quad \text{in distribution.}$$

Letting $k \to \infty$ completes the proof. □

5 Proof of main result

In this section, we prove Theorem 1, which means $\overline{F}_W(x) \sim \overline{F}_{W_{\mathrm{ROS}}}(x) \sim cx^{1-\beta} L(x)$ as $x \to \infty$. Since $W \ge W_{\mathrm{ROS}}$ in distribution, we have

$$\overline{F}_W(x) \ge \overline{F}_{W_{\mathrm{ROS}}}(x) \quad \text{for all } x \in \mathbb{R}.$$

This and Lemma 4 yield

$$\liminf_{x \to \infty} \frac{\overline{F}_W(x)}{x^{1-\beta} L(x)} \ge c.$$

Therefore, Theorem 1 is proved if we show

$$\limsup_{x \to \infty} \frac{\overline{F}_W(x)}{x^{1-\beta} L(x)} \le c. \tag{30}$$

We break the proof of (30) into 4 steps.

Step 1 *Define distribution functions A_k, $k = 0, 1, 2, \dots,$ by*

$$A_k = \frac{\nu}{(k+1)\nu + \lambda} U + \frac{k\nu}{(k+1)\nu + \lambda} \Phi_k + \frac{\lambda}{(k+1)\nu + \lambda} \Phi_{k+1},$$

where Φ_k is given by (23). *Let Ψ be a distribution function defined as*

$$\Psi(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\int_0^x \mathbb{P}(M=k)a_l(y)A_{k+l}(x-y)\,dy,$$

where

$$a_l(y) = \frac{1}{\mathbb{E}B}\int_y^{\infty} e^{-\lambda t}\frac{(\lambda t)^l}{l!}\,dF_B(t). \tag{31}$$

Then

$$\overline{F}_W(x) \le \rho\overline{\Psi * E_{\nu+\lambda}}(x), \quad x > 0. \tag{32}$$

Proof We consider the M/G/1 retrial queue. We choose an arbitrary customer who arrives at the queue and call it the tagged customer. Let

$$I = \begin{cases} 1 & \text{if the tagged customer arrives while the server is busy,} \\ 0 & \text{otherwise.} \end{cases}$$

By the 'Poisson arrivals see time averages' (PASTA) property,

$$\mathbb{P}(I=1) = \rho. \tag{33}$$

When $I = 1$, let us define the following epochs; see Fig. 5:

$t_* =$ the arrival epoch of the tagged customer,

$t_1 =$ the beginning epoch of the service period during which the tagged customer arrives,

$t_2 =$ the end epoch of the service period during which the tagged customer arrives,

$t_3 =$ the beginning epoch of the next service after time t_2,

$t_4 =$ the service initiation epoch of the tagged customer.

When $I = 1$, let $\mathcal{A}$ denote the number of exogenous arrivals during the time interval (t_1, t_2) excluding the tagged customer. Given $I = 1$, $N(t_1)$ and $(\mathcal{A}, t_2 - t_*)$ are

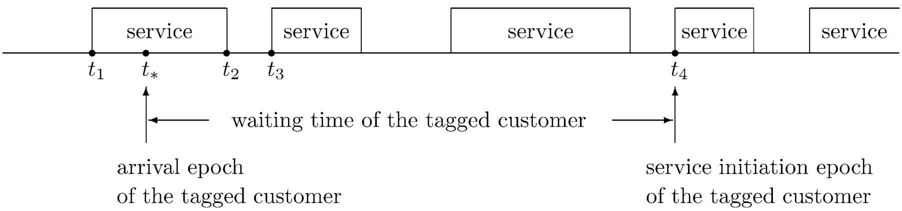

Fig. 5 Arrival epoch and service initiation epoch of the tagged customer in the M/G/1 retrial queue

independent. Further $N(t_1)$ has the same distribution as M. Therefore,

$$\mathbb{P}\big(N(t_1)=k, \mathcal{A}=l,\ t_2-t_*\le y \,\big|\, I=1\big)=\mathbb{P}(M=k)\mathbb{P}(\mathcal{A}=l, t_2-t_*\le y \mid I=1). \tag{34}$$

Given $I=1$, the joint distribution of $\mathcal{A}$ and t_2-t_* is given by

$$\frac{d}{dy}\mathbb{P}(\mathcal{A}=l,\ t_2-t_*\le y \mid I=1)=a_l(y), \quad l=0,1,2,\dots,\ y\ge 0, \tag{35}$$

where $a_l(y)$ is defined as (31). By (33), (34) and (35), we have

$$\frac{d}{dy}\mathbb{P}\big(I=1, N(t_1)=k, \mathcal{A}=l, t_2-t_*\le y\big)=\rho\mathbb{P}(M=k)a_l(y),$$
$$l=0,1,2,\dots, y\ge 0. \tag{36}$$

If $I=1,\ N(t_1)=k,\ \mathcal{A}=l$ and $t_2-t_*=y$, then $N(t_2)=k+l+1$; the $k+l+1$ customers in the orbit at time t_2 consists of the tagged customer and the other $k+l$ customers. Hence, given $\{I=1,\ N(t_1)=k, \mathcal{A}=l, t_2-t_*=y\}$, t_3-t_2 and t_4-t_3 have distribution functions $E_{(k+l+1)\nu+\lambda}$ and A_{k+l}, respectively. Furthermore, given $\{I=1,\ N(t_1)=k, \mathcal{A}=l, t_2-t_*=y\}$, t_3-t_2 and t_4-t_3 are independent. Therefore,

$$\begin{aligned}\mathbb{P}\big(t_4-t_2\le x \,\big|\, I=1, N(t_1)=k, \mathcal{A}=l,\ t_2-t_*=y\big) &= E_{(k+l+1)\nu+\lambda} * A_{k+l}(x)\\ &\ge E_{\nu+\lambda} * A_{k+l}(x).\end{aligned} \tag{37}$$

By (36) and (37), the complementary distribution function $\overline{F}_W$ of the waiting time in the retrial queue satisfies the following: For $x>0$,

$$\begin{aligned}\overline{F}_W(x)&=\mathbb{P}(I=1, t_4-t_*>x)\\ &\le \rho\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\mathbb{P}(M=k)a_l(y)\overline{E_{\nu+\lambda} * A_{k+l}}(x-y)\,dy\\ &=\rho\overline{E_{\nu+\lambda} * \Psi}(x),\end{aligned}$$

which means (32). □

Step 2 *For each n with $\rho^{(n)}<1$, let*

$$A_k^{(n)}=\frac{n\nu}{n\nu+\lambda}\frac{1}{k+1}U+\frac{n\nu}{n\nu+\lambda}\frac{k}{k+1}\Phi_k^{(n)}+\frac{\lambda}{n\nu+\lambda}\Phi_{k+1}^{(n)}, \quad k=0,1,2,\dots,$$

where $\Phi_k^{(n)}$ is given by (24). Define a distribution function $\Psi^{(n)}$ as

$$\Psi^{(n)}(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{x}\mathbb{P}\big(M^{(n)}=k\big)a_l(y)A_{k+l}^{(n)}(x-y)\,dy.$$

Let $W^{(n)}$ be the waiting time of an arbitrary customer in the ordinary M/G/1 ROS queue with the service time distribution function $F_{B^{(n)}}$. Then

$$\overline{F}_{W^{(n)}}(x)\ge\rho\overline{\Psi^{(n)}}(x). \tag{38}$$

Proof We consider the ordinary M/G/1 ROS queue with the service time distribution function $F_{B^{(n)}}$. We choose an arbitrary customer who arrives at the queue and call it the tagged customer. Recall the structure of a service in Fig. 3. Let

$$I^{(n)} = \begin{cases} 1 & \text{if the tagged customer arrives in a } B\text{-period},\\ 2 & \text{if the tagged customer arrives in a } \mathcal{E}\text{-period},\\ 0 & \text{otherwise.}\end{cases}$$

By the PASTA property, we have

$$\mathbb{P}\big(I^{(n)} = 1\big) = \rho; \qquad \mathbb{P}\big(I^{(n)} = 2\big) = \rho^{(n)} - \rho; \qquad \mathbb{P}\big(I^{(n)} = 0\big) = 1 - \rho^{(n)}. \tag{39}$$

When $I^{(n)} = 1$, let us define the following epochs; see Fig. 6:

$t_*^{(n)}$ = the arrival epoch of the tagged customer,

$t_1^{(n)}$ = the beginning epoch of the B-period during which the tagged customer arrives,

$t_2^{(n)}$ = the end epoch of the B-period during which the tagged customer arrives,

$t_3^{(n)}$ = the beginning epoch of the next subservice after time $t_2^{(n)}$,

$t_4^{(n)}$ = the service initiation epoch of the tagged customer.

When $I^{(n)} = 1$, let $\mathcal{A}^{(n)}$ denote the number of exogenous arrivals during the time interval $(t_1^{(n)}, t_2^{(n)})$ excluding the tagged customer. Given $I^{(n)} = 1$, $N^{(n)}(t_1^{(n)})$ and $(\mathcal{A}^{(n)}, t_2^{(n)} - t_*^{(n)})$ are independent. Furthermore, $N^{(n)}(t_1^{(n)})$ has the same distribution as $M^{(n)}$. Therefore,

$$\begin{aligned}&\mathbb{P}\big(N^{(n)}\big(t_1^{(n)}\big) = k, \mathcal{A}^{(n)} = l,\ t_2^{(n)} - t_*^{(n)} \le y \,\big|\, I^{(n)} = 1\big)\\ &\quad= \mathbb{P}\big(M^{(n)} = k\big)\mathbb{P}\big(\mathcal{A}^{(n)} = l,\ t_2^{(n)} - t_*^{(n)} \le y \,\big|\, I^{(n)} = 1\big).\end{aligned} \tag{40}$$

Given $I^{(n)} = 1$, the joint distribution of $\mathcal{A}^{(n)}$ and $t_2^{(n)} - t_*^{(n)}$ is given by

$$\frac{d}{dy}\mathbb{P}\big(\mathcal{A}^{(n)} = l,\ t_2^{(n)} - t_*^{(n)} \le y \,\big|\, I^{(n)} = 1\big) = a_l(y), \quad l = 0, 1, 2, \ldots,\ y \ge 0. \tag{41}$$

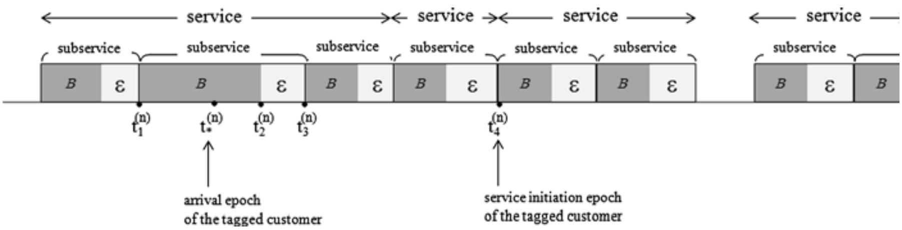

Fig. 6 Arrival epoch and service initiation epoch of the tagged customer in the ordinary M/G/1 ROS queue with service time distribution $F_{B^{(n)}}$

By (39), (40) and (41), we have

$$\frac{d}{dy}\mathbb{P}\big(I^{(n)}=1,\ N^{(n)}\big(t_1^{(n)}\big)=k, \mathcal{A}^{(n)}=l,\ t_2^{(n)}-t_*^{(n)}\le y\big)=\rho\mathbb{P}\big(M^{(n)}=k\big)a_l(y). \tag{42}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\big(t_4^{(n)}-t_*^{(n)}\le x \,\big|\, I^{(n)}=1, N^{(n)}\big(t_1^{(n)}\big)=k, \mathcal{A}^{(n)}=l,\ t_2^{(n)}-t_*^{(n)}=y\big)\\
&\quad=\mathbb{P}\big(t_4^{(n)}-t_2^{(n)}\le x-y \,\big|\, I^{(n)}=1, N^{(n)}\big(t_1^{(n)}\big)=k, \mathcal{A}^{(n)}=l,\ t_2^{(n)}-t_*^{(n)}=y\big)\\
&\quad\le\mathbb{P}\big(t_4^{(n)}-t_3^{(n)}\le x-y \,\big|\, I^{(n)}=1, N^{(n)}\big(t_1^{(n)}\big)=k, \mathcal{A}^{(n)}=l, t_2^{(n)}-t_*^{(n)}=y\big)\\
&\quad=\mathbb{P}\big(t_4^{(n)}-t_3^{(n)}\le x-y \,\big|\, I^{(n)}=1, N^{(n)}\big(t_2^{(n)}\big)=k+l+1\big)\\
&\quad\le\mathbb{P}\big(t_4^{(n)}-t_3^{(n)}\le x-y \,\big|\, I^{(n)}=1, N^{(n)}\big(t_3^{(n)}-\big)=k+l+1\big),
\end{aligned} \tag{43}$$

where the first inequality follows from $t_4^{(n)}-t_2^{(n)}\ge t_4^{(n)}-t_3^{(n)}$ and the last inequality follows from $N^{(n)}(t_2^{(n)})\le N^{(n)}(t_3^{(n)}-)$. When $I^{(n)}=1$ and $N^{(n)}(t_3^{(n)}-)=k+l+1$, the tagged customer and the other $k+l$ customers are waiting for service immediately before time $t_3^{(n)}$. Therefore, when $I^{(n)}=1$ and $N^{(n)}(t_3^{(n)}-)=k+l+1$, we have the following at time $t_3^{(n)}$:

- a B-period begins without service completion with probability $\frac{\lambda}{n\nu+\lambda}$,
- a service is completed and the tagged customer starts service with probability $\frac{n\nu}{n\nu+\lambda}\frac{1}{k+l+1}$,
- a service is completed and a customer among the other $k+l$ customers starts service with probability $\frac{n\nu}{n\nu+\lambda}\frac{k+l}{k+l+1}$.

Thus

$$\mathbb{P}\big(t_4^{(n)}-t_3^{(n)}\le x \,\big|\, I^{(n)}=1, N^{(n)}\big(t_3^{(n)}-\big)=k+l+1\big)=A_{k+l}^{(n)}(x). \tag{44}$$

Substituting (44) into (43) leads to

$$\mathbb{P}\big(t_4^{(n)}-t_*^{(n)}\le x \,\big|\, I^{(n)}=1, N^{(n)}\big(t_1^{(n)}\big)=k, \mathcal{A}^{(n)}=l, t_2^{(n)}-t_*^{(n)}=y\big)\le A_{k+l}^{(n)}(x-y). \tag{45}$$

By (42) and (45),

$$\begin{aligned}
\overline{F}_{W^{(n)}}(x)&\ge\mathbb{P}\big(I^{(n)}=1, t_4^{(n)}-t_*^{(n)}>x\big)\\
&\ge\rho\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\mathbb{P}\big(M^{(n)}=k\big)a_l(y)\overline{A_{k+l}^{(n)}}(x-y)\,dy\\
&=\rho\overline{\Psi^{(n)}}(x),
\end{aligned}$$

which means (38). □

Step 3 *For each n with* $\rho^{(n)} < 1$,

$$A_{k+n}(x) \geq A_k^{(n)} * H_1^{*2n}(x), \quad k = 0, 1, 2, \dots. \tag{46}$$

Proof By (27), we have

$$\begin{aligned} A_{k+n}(x) \geq \bigg[& \left(1 - \frac{k\nu}{(k+n+1)\nu + \lambda} - \frac{\lambda}{(k+n+1)\nu + \lambda} \frac{k+1}{k+n+1}\right) U \\ & + \frac{k\nu}{(k+n+1)\nu + \lambda} \Phi_k^{(n)} \\ & + \frac{\lambda}{(k+n+1)\nu + \lambda} \frac{k+1}{k+n+1} \Phi_{k+1}^{(n)} \bigg] * H_1^{*2n}(x). \end{aligned} \tag{47}$$

By a rather tedious calculation, it can be verified that

$$\frac{\lambda}{(k+n+1)\nu + \lambda} \frac{k+1}{k+n+1} \leq \frac{\lambda}{n\nu + \lambda},$$

$$\frac{k\nu}{(k+n+1)\nu + \lambda} + \frac{\lambda}{(k+n+1)\nu + \lambda} \frac{k+1}{k+n+1} \leq \frac{n\nu}{n\nu + \lambda} \frac{k}{k+1} + \frac{\lambda}{n\nu + \lambda}.$$

According to these equations, (47) leads to

$$A_{k+n}(x) \geq \left(\frac{n\nu}{n\nu + \lambda} \frac{1}{k+1} U + \frac{n\nu}{n\nu + \lambda} \frac{k}{k+1} \Phi_k^{(n)} + \frac{\lambda}{n\nu + \lambda} \Phi_{k+1}^{(n)} \right) * H_1^{*2n}(x),$$

which means (46). □

Step 4 *The assertion* (30) *holds.*

Proof For $x \geq 0$,

$$\begin{aligned} \Psi(x) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_0^x \mathbb{P}(M = k) a_l(y) A_{k+l}(x-y)\, dy \\ &\geq \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_0^x \mathbb{P}\big((M-n)^+ = k\big) a_l(y) A_{k+l+n}(x-y)\, dy. \end{aligned}$$

According to Lemma 6 and (46), the above equation leads to

$$\begin{aligned} \Psi(x) &\geq \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_0^x \mathbb{P}\big(M^{(n)} = k\big) a_l(y)\ A_{k+l}^{(n)} * H_1^{*2n}(x-y)\, dy \\ &= \Psi^{(n)} * H_1^{*2n}(x). \end{aligned} \tag{48}$$

According to (32), (48) and (38),

$$\overline{F}_W(x) \leq \overline{F_{W^{(n)}} * E_{\nu+\lambda} * H_1^{*2n}}(x). \tag{49}$$

Lemma 4 together with (15) yields

$$\overline{F}_{W^{(n)}}(x) \sim c_n x^{1-\beta} L(x) \quad \text{as } x \to \infty, \tag{50}$$

where

$$c_n = \left(1 + \frac{\lambda}{n\nu}\right) \frac{\rho^{(n)}}{1-\rho^{(n)}} h\left(\rho^{(n)}, \beta\right) \frac{1}{\beta - 1} \frac{1}{\mathbb{E}B^{(n)}}.$$

By (50) and Proposition 1, we have

$$\overline{F_{W^{(n)}} * E_{\nu+\lambda} * H_1^{*2n}}(x) \sim c_n x^{1-\beta} L(x) \quad \text{as } x \to \infty,$$

which together with (49) leads to

$$\limsup_{x\to\infty} \frac{\overline{F}_W(x)}{x^{1-\beta} L(x)} \le c_n. \tag{51}$$

Finally, we obtain (30) by letting $n \to \infty$ in (51). □

Acknowledgement We thank an anonymous referee for valuable comments and helpful suggestions.

References

1. Artalejo, J.R.: A classified bibliography of research on retrial queues: Progress in 1990–1999. Top **7**, 187–211 (1999)
2. Artalejo, J.R.: Accessible bibliography on retrial queues. Math. Comput. Model. **30**, 1–6 (1999)
3. Artalejo, J.R.: Accessible bibliography on retrial queues: Progress in 2000–2009. Math. Comput. Model. **51**, 1071–1081 (2010)
4. Artalejo, J.R., Gomez-Corral, A.: Retrial Queueing Systems. Springer, Berlin (2008)
5. Asmussen, S.: Applied Probability and Queues, 2nd edn. Springer, Berlin (2003)
6. Asmussen, S., Klüppelberg, C., Sigman, K.: Sampling at subexponential times, with queueing applications. Stoch. Process. Appl. **79**, 265–286 (1999)
7. Boxma, O.J., Foss, S.G., Lasgouttes, J.-M., Queija, R.: Waiting time asymptotics in the single server queue with service in random order. Queueing Syst. **46**, 35–73 (2004)
8. Falin, G.I.: A survey of retrial queues. Queueing Syst. **7**, 127–168 (1990)
9. Falin, G.I., Templeton, J.G.C.: Retrial Queues. Chapman & Hall, London (1997)
10. De Meyer, A., Teugels, J.L.: On the asymptotic behaviour of the distributions of the busy period and service time in M/G/1. J. Appl. Probab. **17**, 802–813 (1980)
11. Kim, B., Lee, J., Wee, I.-S.: Tail asymptotics for the fundamental period in the MAP/G/1 queue. Queueing Syst. **57**, 1–18 (2007)
12. Kim, J., Kim, B., Ko, S.-S.: Tail asymptotics for the queue size distribution in an M/G/1 retrial queue. J. Appl. Probab. **44**, 1111–1118 (2007)
13. Kim, B., Kim, J., Kim, J.: Tail asymptotics for the queue size distribution in the MAP/G/1 retrial queue. Queueing Syst. (to appear). doi:10.1007/s11134-010-9179-9
14. Nobel, R.D., Tijms, H.C.: Waiting-time probabilities in the M/G/1 retrial queue. Stat. Neerl. **60**, 73–78 (2006)
15. Shang, W., Liu, L., Li, Q.: Tail asymptotics for the queue length in an M/G/1 retrial queue. Queueing Syst. **52**, 193–198 (2006)
16. Sigman, K.: Appendix: A primer on heavy-tailed distributions. Queueing Syst. **33**, 261–275 (1999)
17. Yang, T., Templeton, J.G.C.: A survey on retrial queues. Queueing Syst. **2**, 201–233 (1982)