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Analyzing retrial queues by censoring

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Abstract In this paper we analyze the $M/M/c$ retrial queue using the censoring technique. This technique allows us to carry out an asymptotic analysis, which leads to interesting and useful asymptotic results. Based on the asymptotic analysis, we develop two methods for obtaining approximations to the stationary probabilities, from which other performance metrics can be obtained. We demonstrate that the two proposed approximations are good alternatives to existing approximation methods. We expect that the technique used here can be applied to other retrial queueing models.

Keywords Retrial queues · Stationary distribution · Censoring technique · Matrix-product solution · Decay rate · Decay function · Approximations · Algorithms

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1 Introduction

Retrial queues are a type of classical queue with many interesting applications. References on this topic are numerous. The monograph [8] collected many important

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results and references published up to the late 1990s. Readers are also referred to surveys in this area, including [1, 7, 17], and [22]. A more recent list of references can be found in [10] and in the new book [2], which focuses on computational approaches.

It is well known that explicit expressions for the stationary joint distribution $\pi_{n,k}$ of the number of retrial customers in the orbit and the number of busy servers are available for the standard $M/M/1$ and $M/M/2$ retrial queues (see, for example, [8]). However, it was widely believed that an explicit expression for the joint probability distribution $\pi_{i,j}$ when $c > 2$ does not exist (see, for example, p. 25 of [17] and also p. 288 of [8]). When $c > 2$, besides various closed-form properties, several methodologies have been proposed to approximate multiserver retrial queues since the paper [6]. Among them are a finite approach based on truncated models (Sect. 2.4 of [8] and [20]); generalized truncated models (Sect. 2.5 of [8, 19] and [3]); limit theorems, loss models, and interpolation (Sect. 2.8 of [8]); censoring in level-dependent QBD processes [4]; the retrial see time average (RTA) assumption [21]; and the Fredericks–Reisner approximation [9].

Various limit theorems often provide insightful properties, among which are approximations for high-rate retrials and low-rate retrials, respectively, and heavy-traffic approximations (Sect. 2.7 of [8]). Tail asymptotic analysis for the stationary distribution of a retrial queue has been reported recently on an $M/G/1$ retrial queue [15] and its discrete time counterpart model $Geo/G/1$ retrial queue [13] and also on a discrete-time $D\text{-}BMAP/G/1$ retrial queue [14].

In this paper, we use the censoring technique to analyze the $M/M/c$ multiserver retrial queue. Compared to the analysis using other methods, not only does the censoring method lead us to the same explicit expressions for the standard $M/M/1$ and $M/M/2$ retrial queues, but also offers an approach for asymptotic analysis of the stationary tail probabilities of the $M/M/c$ retrial queue for a general value of c. Based on asymptotic results, we propose two alternative approximations for computing the stationary distribution of the $M/M/c$ retrial queue. A numerical comparison to existing computational methods suggests that our algorithm is comparable to the best available methods.

The rest of the paper is organized as follows. In Sect. 2, we provide an analysis of the $M/M/c$ retrial queue based on the censoring technique and obtain some interesting properties about the model. We also demonstrate how the system can be explicitly solved for the cases of $c = 1$ and $c = 2$. A tail asymptotic analysis is done in Sect. 3, in which we obtain the exact decay function for the $M/M/c$ retrial queue as well as other asymptotic properties. Based on asymptotic results and the censored equations, two new approximation methods are proposed in Sect. 4 for computing the joint stationary distribution, which are good alternatives to the best available methods according to a numerical comparison. Concluding remarks are made in the final section.

2 The $M/M/c$ retrial queue

This section serves two purposes: (1) providing a self-contained analysis process for the $M/M/c$ retrial queue in terms of the censoring technique; and (2) obtaining some basic and interesting properties about the retrial queue, some of which will be used throughout the paper. We also demonstrate that based on our analysis, existing explicit expressions for the stationary probability distribution for $c=1$ and $c=2$, respectively, can be easily constructed.

We first recall the description of the $M/M/c$ retrial queueing model. Consider a queueing system with c identical servers in which primary customers arrive according to a Poisson process with rate λ. If at least one server is free upon the arrival of a primary customer, the customer enters service immediately and leaves the system after the service completion. Otherwise, if all servers are busy upon the arrival of a primary customer, the customer joins the orbit and becomes a retrial customer. Each of the retrial customers in the orbit independently repeatedly tries for receiving service according to a Poisson process with rate θ until it finds an idle server upon retrial, and then starts its service immediately and leaves the system after the service completion. For each of the primary or retrial customers, the service time with any of the c servers follows an exponential distribution with common service rate μ. All service times are independent and are also independent of the arrival and the retrial processes.

Let $N(t)$ be the number of retrial customers in the orbit at time t, and let $C(t)$ be the number of busy servers at time t. Then, $(N(t), C(t))$ is a continuous-time Markov chain with the state space

$$S=\big\{(n,k); n=0,1,\dots \text{ and } k=0,1,\dots,c\big\},$$

where n and k are referred to as the level and phase variables, respectively. If the infinitesimal generator Q of the Markov chain is partitioned according to the level, then it is of the type of level-dependent quasi-birth-and-death (QBD):

$$Q=\begin{bmatrix} B_0 & A & & & \\ C_1 & B_1 & A & & \\ & C_2 & B_2 & A & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.1}$$

where

$$B_n=\begin{bmatrix} -(\lambda+n\theta) & \lambda & & & \\ \mu & -(\lambda+\mu+n\theta) & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & (c-1)\mu & -[\lambda+(c-1)\mu+n\theta] & \lambda \\ & & & c\mu & -(\lambda+c\mu) \end{bmatrix},$$

$$A=\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \lambda \end{bmatrix} \quad \text{and} \quad C_n=\begin{bmatrix} 0 & n\theta & & & \\ & 0 & n\theta & & \\ & & \ddots & \ddots & \\ & & & 0 & n\theta \\ & & & & 0 \end{bmatrix}.$$

For the infinitesimal generator Q in (2.1), define $Q_0 = Q$, and for $n = 1, 2, \dots$, define the submatrix Q_n by deleting the first n block rows and columns in Q, that is,

$$Q_n = \begin{bmatrix} B_n & A & & & \\ C_{n+1} & B_{n+1} & A & & \\ & C_{n+2} & B_{n+2} & A & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2.2}$$

By $\widehat{Q}_n = \sum_{k=0}^{\infty} Q_n^k = (I - Q_n)^{-1}$ denote the fundamental matrix of Q_n. Let $\widehat{Q}_n$ be partitioned according to the level, and let the $(1,1)$st block entry of $\widehat{Q}_n$ be denoted by $\widehat{Q}_n(1,1)$. Assume that the system is stable, that is, $\rho = \lambda/(c\mu) < 1$. Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots)$ be the stationary probability vector of the Markov chain partitioned correspondingly, where $\boldsymbol{\pi}_n = (\pi_{n,0}, \pi_{n,1}, \dots, \pi_{n,c})$. Then, according to matrix-analytic theory (for example, referring to p. 260 of [18]), $\boldsymbol{\pi}_n$ has a matrix-product form solution given by

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R_1 R_2 \cdots R_n, \quad n = 1, 2, \dots, \tag{2.3}$$

where $\boldsymbol{\pi}_0$ is the solution (unique up to multiplication by a constant) to

$$\boldsymbol{\pi}_0 (B_0 + R_1 C_1) = 0, \tag{2.4}$$

where

$$R_\ell = A \widehat{Q}_\ell(1,1), \quad \ell = 1, 2, \dots. \tag{2.5}$$

Remark 2.1 The matrices $\{R_\ell\}_{\ell \ge 1}$ have the following probabilistic interpretation (see [5]). The (i,j)th entry $(R_\ell)_{i,j}$ of R_ℓ is the expected sojourn time in the state $(\ell+1, j)$ per unit sojourn in the state (ℓ, i) before returning to level ℓ, given the process started in state (ℓ, i).

Since only the last row in A is nonzero, the same is true for R_ℓ, i.e.,

$$R_\ell = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ r_{\ell,0} & r_{\ell,1} & \cdots & r_{\ell,c} \end{bmatrix}. \tag{2.6}$$

In accordance to Remark 2.1, this structure is a direct outcome of the following interpretation: given that at least one server is idle (or the system starts in a state (ℓ, i) with $0 \le i \le c-1$), the number of customers in the orbit can never increase without making all servers busy first.

Along with (2.3), the special structure (2.6) of R_ℓ leads to the following theorem.

Theorem 2.1 *For the $M/M/c$ retrial queue given in* (2.1), *the stationary probability vector $\boldsymbol{\pi}$ can be expressed as*

$$\boldsymbol{\pi}_n = \pi_{0,c} r_{1,c} r_{2,c} \cdots r_{n-1,c} (r_{n,0}, r_{n,1}, \dots, r_{n,c}), \quad n = 1, 2, \dots, \tag{2.7}$$

and $\boldsymbol{\pi}_0$ is uniquely determined by (2.4) *and the normalizing condition.*

Remark 2.2 The special structure in the $M/M/c$ retrial queue makes the matrix-product solution for $\boldsymbol{\pi}_n$ much simpler, only depending on the largest eigenvalue $r_{\ell,c}$ of R_ℓ for $\ell = 1, 2, \ldots, n-1$ and the last row of R_n.

In the following, we use the censoring technique to obtain properties for the $M/M/c$ retrial queue and to demonstrate how it can lead to an explicit determination of the matrices R_ℓ when $c = 1$ and 2. Let the state space S be partitioned as $S = S_0 \cup S_1$, where both S_0 and S_1 are nonempty, and let Q be partitioned accordingly:

$$Q = \begin{array}{c} \\ S_0 \\ S_1 \end{array} \begin{array}{c} \begin{array}{cc} S_0 & S_1 \end{array} \\ \begin{bmatrix} T & U \\ L & D \end{bmatrix} \end{array}.$$

Then, the censored matrix $Q^{(S_0)}$ with the censoring set S_0 is also an infinitesimal generator given by

$$Q^{(S_0)} = T + U\widehat{D}L, \tag{2.8}$$

where $\widehat{D} = \sum_{k=0}^{\infty} D^k$. Similarly,

$$Q^{(S_1)} = D + L\widehat{T}U, \tag{2.9}$$

where $\widehat{T} = \sum_{k=0}^{\infty} T^k$ (referring to pp. 133–134 of [12] for details). For two subsets S_1 and S_2 of S with $S_1 \subseteq S_2$, we have the property that $Q^{(S_1)} = (Q^{(S_2)})^{(S_1)}$, which means that the censored matrix $Q^{(S_1)}$ can be obtained by censoring the censored matrix $Q^{(S_2)}$ again with the censoring set S_1. For a recurrent Markov chain Q, every censored matrix is again an infinitesimal generator and also recurrent. In this case, the unique invariant measure of the censored matrix (censored Markov chain), up to multiplication by a constant, is the same as the invariant measure of the original Markov chain restricted to the censoring set. If, furthermore, the Markov chain is positive recurrent with the stationary probability vector $\boldsymbol{\pi} = (\pi_k)$, then the stationary probability vector $\boldsymbol{\pi}^{(S_1)} = (\pi_k^{(S_1)})$ of the censored Markov chain with the censoring set S_1 is given by $\pi_k^{(S_1)} = \pi_k / \sum_{j \in S_1} \pi_j$ (for example, referring to [23] for details).

Remark 2.3 According to the above argument, $\boldsymbol{\pi}_0$ is the solution to $\boldsymbol{\pi}_0 Q^{(0)} = 0$ subject to the normalization condition, where

$$Q^{(0)} = B_0 + (A, 0, \ldots)\,\widehat{Q}_1 \begin{pmatrix} C_1 \\ 0 \\ \vdots \end{pmatrix} = B_0 + R_1 C_1 \quad \big(\text{by (2.8) and (2.5)}\big)$$

is the censored Markov chain to level 0, which is the same matrix given in (2.4).

It follows from the skip-free property in both directions in block-sense in (2.1), (2.8), and (2.5) that for $n \geq 1$, the censored matrix $Q^{\leq (n-1)}$ with the censoring set

$L_{\leq(n-1)} = \{(\ell, k); \ell = 0, 1, \dots, n-1 \text{ and } k = 0, 1, \dots c\}$ can be expressed as

$$Q^{\leq(n-1)} = \begin{bmatrix} B_0 & A & & & \\ C_1 & B_1 & A & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n-2} & B_{n-2} & A \\ & & & C_{n-1} & B_{n-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \cdots \\ A & 0 & \cdots \end{bmatrix} \widehat{Q}_n \begin{bmatrix} 0 & 0 & \cdots & C_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} B_0 & A & & & \\ C_1 & B_1 & A & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n-2} & B_{n-2} & A \\ & & & C_{n-1} & B_{n-1} + R_n C_n \end{bmatrix}, \tag{2.10}$$

where

$$R_n C_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & n\theta r_{n,0} & n\theta r_{n,1} & \cdots & n\theta r_{n,c-1} \end{bmatrix}. \tag{2.11}$$

Based on this structure, we immediately have the following property.

Lemma 2.1 (Key Lemma) *For the $M/M/c$ retrial queue, we have*

$$r_{n,0} + r_{n,1} + \cdots + r_{n,c-1} = \frac{\lambda}{n\theta}, \quad n = 1, 2, \dots. \tag{2.12}$$

Proof Since Q is assumed to be recurrent, the censored matrix $Q^{\leq(n-1)}$ is also recurrent. Therefore, each row sum of $Q^{\leq(n-1)}$ is zero. The fact that the sum of the last row is zero leads to (2.12). □

Remark 2.4 This is a key property, which, together with other properties, plays an important role in the asymptotic analysis.

For $n = 0, 1, 2, \dots,$ we further consider the censored matrix $Q^{(n)}$, which is also an infinitesimal generator, obtained by censoring the censored matrix $Q^{\leq n}$ again with the censoring set $L_{(n)} = \{(n, k); k = 0, 1, \dots c\}$:

$$Q^{(n)} = B_n + R_{n+1} C_{n+1} + C_n \widehat{E}(n, n) A, \tag{2.13}$$

where $\widehat{E}(n,n)$ is the (n,n)th block entry of the fundamental matrix $\widehat{E}$ of

$$E = \begin{bmatrix} B_0 & A & & & \\ C_1 & B_1 & A & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n-2} & B_{n-2} & A \\ & & & C_{n-1} & B_{n-1} \end{bmatrix}. \tag{2.14}$$

Due to the special sparse structure of matrix C_n and A, we immediately have

$$Q^{(n)} = B_n + R_{n+1}C_{n+1} + \begin{bmatrix} 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & * \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{2.15}$$

where $*$ stands for a nonzero element. Since $Q^{(n)}$ is an infinitesimal generator, all rows in $Q^{(n)}$ should sum to zero, which, together with (2.11), determines all elements $*$ and leads to

$$Q^{(n)} = \begin{bmatrix} -\sigma_{n,0} & \lambda & & & & & \beta_n \\ \mu & -\sigma_{n,1} & \lambda & & & & \beta_n \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & (c-2)\mu & -\sigma_{n,c-2} & \lambda & \beta_n \\ & & & & (c-1)\mu & -\sigma_{n,c-1} & \lambda+\beta_n \\ 0 & \alpha_{n+1,0} & \cdots & \alpha_{n+1,c-4} & \alpha_{n+1,c-3} & \alpha_{n+1,c-2}+c\mu & \alpha_{n+1,c-1}-\omega_c \end{bmatrix}, \tag{2.16}$$

where

$$\sigma_{n,k} = \lambda + k\mu + n\theta, \quad k = 0, 1, \dots, c-1, \tag{2.17}$$

$$\omega_c = \lambda + c\mu, \tag{2.18}$$

$$\beta_n = n\theta, \tag{2.19}$$

$$\alpha_{n+1,k} = \beta_{n+1} r_{n+1,k}, \quad k = 0, 1, \dots, c-1. \tag{2.20}$$

Lemma 2.2 *For the $M/M/c$ retrial queue, we have*

$$(\pi_{n,0}, \pi_{n,1}, \dots, \pi_{n,c}) Q^{(n)} = 0, \quad n = 0, 1, 2, \dots. \tag{2.21}$$

Proof Since the stationary probabilities $\pi_{n,k}^{(n)}$ of the censored matrix $Q^{(n)}$ with the censoring set $L_{(n)}$ are proportional to the stationary probabilities $\pi_{n,k}$ of the original Markov chain, the conclusion is immediate. □

Lemma 2.3 (Censored Equations) *For the $M/M/c$ retrial queue*, *we have*

$$(r_{n,0}, r_{n,1}, \ldots, r_{n,c})Q^{(n)} = 0, \quad n = 1, 2, \ldots. \tag{2.22}$$

Proof The conclusion is a direct consequence of Theorem 2.1 and Lemma 2.2. □

Corollary 2.1 *For the $M/M/c$ retrial queue*, *we have*

$$(\lambda + n\theta)\pi_{n,0} = \mu\pi_{n,1}, \quad n = 0, 1, 2, \ldots, \tag{2.23}$$

$$(\lambda + n\theta)r_{n,0} = \mu r_{n,1}, \quad n = 1, 2, \ldots. \tag{2.24}$$

Proof The first conclusion can be made from the first equation in $\boldsymbol{\pi}_n Q^{(n)} = 0$, and the second one from the first equation in $(r_{n,0}, r_{n,1}, \ldots, r_{n,c})Q^{(n)} = 0$. □

For the $M/M/c$ retrial queue, for each level n, two independent equations for $r_{n,k}$ are characterized by Lemma 2.1 and Corollary 2.1. Therefore, when $c = 1$, these two independent equations immediately lead to an explicit determination of $r_{n,0}$ and $r_{n,1}$, which determines the stationary distribution as explained in detail in the first example given below. When $c = 2$, we need another independent equation for determining all three $r_{n,k}$ for $k = 0$, 1, and 2 in order to completely determine the stationary distribution. Luckily enough, the censored matrix $Q^{(n)}$ given by (2.16) only depends on $r_{n+1,0}, \ldots, r_{n+1,c-1}$ (not on $r_{n+1,c}$). Therefore, the two independent relationships in Lemma 2.1 and Corollary 2.1 provide an explicit determination of $r_{n+1,0}$ and $r_{n+1,1}$, and then the censored matrix $Q^{(n)}$. The final unknown $r_{n,2}$ can be explicitly determined by solving $(r_{n,0}, r_{n,1}, r_{n,2})Q^{(n)} = 0$ as detailed later in this section. However, when $c \geq 3$, we simply do not have enough independent equations for explicitly determining all $r_{n,k}$ and then the probabilities $\pi_{n,k}$.

To conclude this section, we demonstrate how the explicit solution of $\pi_{n,k}$ can be obtained for $c = 1$ and 2.

The $M/M/1$ retrial queue In this case, according to Lemma 2.1 and Corollary 2.1, for all $n \geq 1$, $r_{n,0}$ and $r_{n,1}$ are given, respectively, as

$$r_{n,0} = \frac{\lambda}{n\theta} \quad \text{and} \quad r_{n,1} = \frac{\lambda + n\theta}{\mu} r_{n,0} = \frac{\lambda(\lambda + n\theta)}{n\theta\mu}.$$

Therefore, all components in Theorem 2.1 are explicitly determined, except $\pi_{0,1}$, which is determined by the normalization condition. We summarize the result in the following corollary.

Corollary 2.2 *For the standard $M/M/1$ retrial queue*, *the stationary distribution is given by*

$$\begin{aligned}\boldsymbol{\pi}_n &= \pi_{0,1} r_{1,1} r_{2,1} \cdots r_{n-1,1}(r_{n,0}, r_{n,1}) \\ &= \pi_{0,1} \frac{(\lambda/\theta\mu)^n}{n!} \left(\prod_{k=1}^{n} (\lambda + k\theta) \right) \left(\frac{\mu}{\lambda + n\theta}, 1 \right), \quad n \geq 1,\end{aligned} \tag{2.25}$$

and

$$\boldsymbol{\pi}_0 = (\pi_{0,0}, \pi_{0,1}) = \pi_{0,1}\left(\frac{\mu}{\lambda}, 1\right), \tag{2.26}$$

where

$$\pi_{0,1} = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\lambda}{\theta}+1}.$$

It is easy to check that the above result is consistent with that given on p. 3 in [8].

The $M/M/2$ retrial queue In this case, according to Lemma 2.1 and Corollary 2.1, for all $n \geq 1$, $r_{n,0}$ and $r_{n,1}$ are explicitly solved as

$$r_{n,0} = \frac{\lambda}{n\theta}\frac{\mu}{\lambda + \mu + n\theta}, \qquad r_{n,1} = \frac{\lambda}{n\theta}\frac{\lambda + n\theta}{\lambda + \mu + n\theta}. \tag{2.27}$$

Hence, the censored matrix $Q^{(n)}$ is explicitly determined by (2.16). By replacing $r_{n,0}$ and $r_{n,1}$ into $(r_{n,0}, r_{n,1}, r_{n,2})\ Q^{(n)} = 0$ and solving its last equation, we can explicitly determine $r_{n,2}$ as

$$\begin{aligned} r_{n,2} &= \frac{\beta_n r_{n,0} + (\lambda + \beta_n) r_{n,1}}{\lambda + 2\mu - (n+1)\theta r_{n+1,1}} \\ &= \frac{\lambda}{n\theta\mu}\frac{\lambda + \mu + (n+1)\theta}{\lambda + \mu + n\theta}\frac{n\theta\mu + (\lambda + n\theta)^2}{3\lambda + 2\mu + 2(n+1)\theta}, \quad n = 1, 2, \ldots. \end{aligned} \tag{2.28}$$

According to Theorem 2.1, to completely explicitly express all probabilities, we still need an expression for $\boldsymbol{\pi}_0$, which can be obtained by solving the following equations:

$$\boldsymbol{\pi}_0 Q^{(0)} = \boldsymbol{\pi}_0 \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda + \mu) & \lambda \\ 0 & 2\mu + \theta r_{1,0} & -(\lambda + 2\mu) + \theta r_{1,1} \end{bmatrix} = 0. \tag{2.29}$$

Specifically,

$$\pi_{0,0} = \frac{\mu}{\lambda}\pi_{0,1} \tag{2.30}$$

$$\pi_{0,1} = \frac{(\lambda + 2\mu) - \theta r_{1,1}}{\lambda}\pi_{0,2} = \frac{\mu}{\lambda}\cdot\frac{3\lambda + 2\mu + 2\theta}{\lambda + \mu + \theta}\pi_{0,2}. \tag{2.31}$$

Finally, $\pi_{0,2}$ is determined by the normalization condition, which leads to a lengthy expression and will not be provided here. The above discussion is summarized into the following corollary.

Corollary 2.3 *For the standard $M/M/2$ retrial queue, the stationary distribution is given by*

$$\begin{aligned}\boldsymbol{\pi}_n &= \pi_{0,2} r_{1,2} r_{2,2} \cdots r_{n-1,2}(r_{n,0}, r_{n,1}, r_{n,2}),\\ &= \frac{\pi_{0,2}(\lambda/\theta\mu)^{n-1}}{(n-1)!}\left(\prod_{k=1}^{n-1}\frac{k\theta\mu+(\lambda+k\theta)^2}{3\lambda+2\mu+2(k+1)\theta}\right)\frac{\lambda+\mu+n\theta}{\lambda+\mu+\theta}(r_{n,0}, r_{n,1}, r_{n,2}),\\ &\quad n \geq 1, \end{aligned} \tag{2.32}$$

$$\boldsymbol{\pi}_0 = \pi_{0,2}\left(\frac{\mu^2}{\lambda^2}\frac{3\lambda+2\mu+2\theta}{\lambda+\mu+\theta}, \frac{\mu}{\lambda}\frac{3\lambda+2\mu+2\theta}{\lambda+\mu+\theta}, 1\right), \tag{2.33}$$

where $r_{n,0}$, $r_{n,1}$, and $r_{n,2}$ are given in (2.27) *and* (2.28), *and $\pi_{0,2}$ is determined according to the normalization condition.*

It is also not difficult to check that the above result is consistent with that on p. 102 in [8].

Remark 2.5 Based on the results obtained in this section, we can also construct a semi-explicit solution for the joint probabilities $\pi_{n,k}$ for $c = 3$ and $c = 4$. By semi-explicit, we mean that the expression of the solution is no longer a simple explicit function of the system parameters. Instead, a probability is expressed in terms of the limit of a sequence of numbers which are explicitly expressed as functions of the system parameters. In the literature, such a semi-explicit solution, but different from ours in its appearance, was derived for $c = 3$ [11, 16]. However, a numerical procedure or algorithm is still needed for computing the probabilities $\pi_{n,k}$.

3 Tail asymptotics

Since there is no explicit solution for the stationary distribution $\pi_{n,k}$ for the $M/M/c$ retrial queue with general c, tail asymptotic analysis becomes more important, aside from its own independent interest. Properties of tail asymptotics can often lead to various performance bounds, approximations, and estimates of errors.

In this section, we first provide an asymptotic analysis on the quantity $r_{n,k}$, based on the Key Lemma and the censored equations, which is an expected taboo sojourn time in state $(n+1, k)$ before returning to level n. Then, based on the above analysis, tail asymptotic results in the joint probabilities $\pi_{n,k}$ as $n \to \infty$ are obtained, including the identification of the exact decay function for $\pi_{n,k}$.

According to Lemma 2.3, the detailed censored equations $(r_{n,0}, r_{n,1}, \ldots, r_{n,c})Q^{(n)} = 0$ for $n = 1, 2, \ldots$ are given by

$$\sigma_{n,0} r_{n,0} - \mu r_{n,1} = 0, \tag{3.1}$$

$$\sigma_{n,1} r_{n,1} - \lambda r_{n,0} - 2\mu r_{n,2} = \beta_{n+1} r_{n+1,0} r_{n,c}, \tag{3.2}$$

$$\sigma_{n,2} r_{n,2} - \lambda r_{n,1} - 3\mu r_{n,3} = \beta_{n+1} r_{n+1,1} r_{n,c}, \tag{3.3}$$

$$\sigma_{n,3}r_{n,3} - \lambda r_{n,2} - 4\mu r_{n,4} = \beta_{n+1}r_{n+1,2}r_{n,c}, \tag{3.4}$$

$$\vdots$$

$$\sigma_{n,c-2}r_{n,c-2} - \lambda r_{n,c-3} - (c-1)\mu r_{n,c-1} = \beta_{n+1}r_{n+1,c-3}r_{n,c}, \tag{3.5}$$

$$\sigma_{n,c-1}r_{n,c-1} - \lambda r_{n,c-2} - c\mu r_{n,c} = \beta_{n+1}r_{n+1,c-2}r_{n,c}, \tag{3.6}$$

$$-\lambda - \lambda r_{n,c-1} + \omega_c r_{n,c} = \beta_{n+1}r_{n+1,c-1}r_{n,c}. \tag{3.7}$$

The following property will be repeatedly used in this section when taking a limit.

Lemma 3.1 *For the $M/M/c$ retrial queue, there exists a positive constant M_0 such that $r_{n,k} \le M_0 < \infty$ for all n and k.*

Proof For $k = 0, 1, \dots, c-1$, the result follows immediately from the Key Lemma, that is, from

$$n(r_{n,0} + r_{n,1} + \dots + r_{n,c-1}) = \frac{\lambda}{\theta}.$$

For $k = c$, it follows from (3.7) and the Key Lemma that for n large,

$$(\lambda + c\mu)r_{n,c} = (n+1)\theta r_{n+1,c-1}r_{n,c} + \lambda + \lambda r_{n,c-1} \le \lambda r_{n,c} + \lambda + \lambda r_{n,c-1},$$

or $c\mu r_{n,c} \le \lambda + \lambda r_{n,c-1}$. This leads to the final result since $r_{n,c-1}$ is bounded. □

Denote by $o(x_n)$ a function of n such that $\lim_{n\to\infty} o(x_n)/x_n = 0$ and by $O(x_n)$ a function of n such that $\lim_{n\to\infty} O(x_n)/x_n = C \ne 0$, where C is a constant. The following is a first-order asymptotic formula for $r_{n,k}$.

Theorem 3.1 (First-order formula) *For $k = 0, 1, 2, \dots, c$,*

$$r_{n,c-k} = \rho\left(\frac{\mu}{n\theta}\right)^k \frac{c!}{(c-k)!} + o\left(\frac{1}{n^k}\right). \tag{3.8}$$

Proof First of all, we show

$$nr_{n,k} \to 0 \quad \text{for } k = 0, 1, \dots, c-2, \tag{3.9}$$

by recursively using (3.1), (3.2), …, (3.5). Specifically, it is clear from (2.12) that we have

$$r_{n,k} \to 0 \quad \text{for } k = 0, 1, \dots, c-1. \tag{3.10}$$

Then, (3.1) implies $nr_{n,0} \to 0$, by which (3.2) implies $nr_{n,1} \to 0$ since $r_{n,c} \le M_0$ for all n; (3.3) implies $nr_{n,2} \to 0$; continue the process until finally (3.5) implies $nr_{n,c-2} \to 0$.

By using the result (3.9) in (2.12), we have

$$nr_{n,c-1} \to \frac{\lambda}{\theta} = \rho\frac{\mu}{\theta}c, \tag{3.11}$$

which is equivalent to

$$r_{n,c-1} = \rho\left(\frac{\mu}{n\theta}\right)c + o\left(\frac{1}{n}\right).$$

Taking the limit, as $n \to \infty$, on both sides of (3.6) gives

$$\lim_{n\to\infty} [n\theta r_{n,c-1} - c\mu r_{n,c}] = 0$$

by using Lemma 3.1, (3.9), and (3.10). Therefore, we have

$$\lim_{n\to\infty} r_{n,c} = \frac{\lambda}{c\mu} = \rho \tag{3.12}$$

by using (3.11), or $r_{n,c} = \rho + o(1)$.

Next, we show

$$n^2 r_{n,k} \to 0 \quad \text{for } k = 0, 1, \ldots, c-3. \tag{3.13}$$

To this end, multiply both sides of the first $c-2$ system equations in (3.1) to (3.7) by n. Taking the limit as $n \to \infty$ in the first equation leads to $n^2 r_{n,0} \to 0$ by noticing the property in (3.9); by which the second equation leads to $n^2 r_{n,1} \to 0$ by using Lemma 3.1 and (3.9); continue this process until the $(c-2)$nd equation leads to $n^2 r_{n,c-3} \to 0$.

Now, consider the limit, as $n \to \infty$, of (3.5) multiplied by n on both sides. Equations (3.9), (3.13), and Lemma 3.1 imply

$$n^2\theta r_{n,c-2} - n(c-1)\mu r_{n,c-1} \to 0,$$

or

$$n^2 r_{n,c-2} \to \rho\left(\frac{\mu}{\theta}\right)^2 c(c-1), \tag{3.14}$$

which is equivalent to $r_{n,c-2} = \rho(\frac{\mu}{n\theta})^2 c(c-1) + o(\frac{1}{n^2})$.

Continue the above procedure by multiplying both sides of the first $c-3$ equations by n^2 and taking the limit as $n \to \infty$ to show that $n^3 r_{n,k} \to 0$ for $k = 0, 1, \ldots, c-4$ and $r_{n,c-3} = \rho(\frac{\mu}{n\theta})^3 c(c-1)(c-2) + o(\frac{1}{n^3}), \ldots,$ until $r_{n,0} = \rho(\frac{\mu}{n\theta})^c c! + o(\frac{1}{n^c})$ is proved. □

Based on the first-order asymptotic result and using the Key Lemma, we can improve the asymptotic formula to the following refined result.

Corollary 3.1 (Refined first-order formula)

$$r_{n,c-k} = \rho\frac{c!}{(c-k)!}\left(\frac{\mu}{n\theta}\right)^k + O\left(\frac{1}{n^{k+1}}\right) \quad \textit{for } k = 0, 1, 2, \ldots, c.$$

Proof First, we prove the result for $k = 1$, which is immediate from the Key Lemma and the first-order asymptotic formula:

$$\begin{aligned} r_{n,c-1} &= \lambda/n\theta - [r_{n,c-2} + r_{n,c-3} + \cdots + r_{n,1} + r_{n,0}] \\ &= \lambda/n\theta + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{3.15}$$

Next, we prove the result for $k = 0$. By (3.7), we have

$$\begin{aligned} r_{n,c} &= \frac{\lambda + \lambda r_{n,c-1}}{\omega_c - \beta_{n+1} r_{n+1,c-1}} \\ &= \frac{\lambda + \lambda r_{n,c-1}}{\omega_c - [\lambda + O(\frac{1}{n})]} \\ &= \frac{\lambda}{c\mu} \cdot \frac{1 + r_{n,c-1}}{1 - O(\frac{1}{n})} \\ &= \rho \cdot (1 + r_{n,c-1})\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \rho + O\left(\frac{1}{n}\right). \end{aligned}$$

Now, we prove the result for $k = 2, 3, \ldots, c-1$. By (3.5), we have

$$\begin{aligned} r_{n,c-2} &= \frac{(c-1)\mu r_{n,c-1} + \beta_{n+1} r_{n+1,c-3} r_{n,c} + \lambda r_{n,c-3}}{\sigma_{n,c-2}} \\ &= \frac{(c-1)\mu\lambda/n\theta + O(\frac{1}{n^2})}{\lambda + (c-2)\mu + n\theta} \\ &= \frac{(c-1)\mu\lambda}{(n\theta)^2} \cdot \frac{1 + O(\frac{1}{n})}{1 + O(\frac{1}{n})} \\ &= \rho \frac{c!}{(c-2)!}\left(\frac{\mu}{n\theta}\right)^2 \left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \rho \frac{c!}{(c-2)!}\left(\frac{\mu}{n\theta}\right)^2 + O\left(\frac{1}{n^3}\right), \end{aligned}$$

which completes the proof for $k = 2$. Similarly, the cases of $k = 3, 4, \ldots, c-1$ can be proved.

Finally, we prove the result for $k = c$, which is immediate by using (3.1), or

$$\begin{aligned} r_{n,0} = \frac{\mu}{\lambda + n\theta} r_{n,1} &= \frac{\mu}{n\theta}\left(1 + O\left(\frac{1}{n}\right)\right) r_{n,1} \\ &= \frac{\mu}{n\theta}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\rho c!\left(\frac{\mu}{n\theta}\right)^{c-1} + O\left(\frac{1}{n^c}\right)\right). \end{aligned}$$

□

Based on the kth-order asymptotic result and using the Key Lemma repeatedly, we can improve the asymptotic formula to order $k+1$. However, since the asymptotic expression does not show any unified pattern, we cannot have a single expression for a general k. The proof will soon become too cumbersome. Without providing any details, we list some higher-order asymptotic formulas in the following corollary.

Corollary 3.2 (Higher-order formulas)

$$r_{n,c-2} = \rho\left(\frac{\mu}{n\theta}\right)^2 \frac{c!}{(c-2)!} - \rho\left(\frac{\mu}{n\theta}\right)^3 \frac{c!}{(c-2)!}(2\rho + 2c - 3) + O\left(\frac{1}{n^4}\right),$$

$$r_{n,c-1} = \rho\left(\frac{\mu}{n\theta}\right)\frac{c!}{(c-1)!} - \rho\left(\frac{\mu}{n\theta}\right)^2 \frac{c!}{(c-2)!} + \rho\left(\frac{\mu}{n\theta}\right)^3 \frac{c!}{(c-2)!}(2\rho + c - 1) + O\left(\frac{1}{n^4}\right),$$

and

$$r_{n,c} = \rho + \rho^2 \frac{\mu}{n\theta} + \rho^2(c-1)\frac{\mu}{n^2\theta^2}(\theta + \mu\rho - \mu) + O\left(\frac{1}{n^3}\right).$$

As an immediate consequence of the above discussion, the decay rate of $\pi_{n,k}$ can be found easily.

Corollary 3.3 *For $k = 0, 1, \dots, c$,*

$$\lim_{n\to\infty} \frac{\pi_{n+1,k}}{\pi_{n,k}} = \rho,$$

independent of k.

We now present a more detailed tail asymptotic result for the stationary probability distribution $\pi_{n,k}$ by characterizing the exact decay function for $\pi_{n,k}$. By an exact decay function $h_k(n) > 0$ for $\pi_{n,k}$, we mean that for each k,

$$0 < C_k^{(1)} \le \liminf_{n\to\infty} \frac{\pi_{n,k}}{h_k(n)} \le \limsup_{n\to\infty} \frac{\pi_{n,k}}{h_k(n)} \le C_k^{(2)},$$

where $C_k^{(1)}$ and $C_k^{(2)}$ are constants independent of n. We start with some preliminary results.

Fact 1 *The third-order asymptotic formula for $r_{n,c}$ in Corollary 3.2 can be rewritten as*

$$r_{n,c} = \rho\left[1 + \frac{a}{n} + \frac{b}{n^2} + O\left(\frac{1}{n^3}\right)\right],$$

where $a = \frac{\lambda}{c\theta} > 0$ and $b = a(c-1)[1 - (1-\rho)\mu/\theta]$, which could be negative, positive, or zero.

Fact 2 *This is a standard property of the gamma function* $\Gamma(z)$: *For any complex number* $z \neq 0, -1, -2, \dots,$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)},$$

which can be rewritten as

$$\prod_{j=1}^{n} \frac{j+z}{j} = \frac{(z+1)(z+2)\cdots(z+n)}{n!} \sim \frac{n^z}{z\Gamma(z)} \quad \text{as } n \to \infty.$$

Fact 3 *Let* $\hat{a} > 0$, *and let* $\hat{b}$ *be any real number satisfying* $\hat{b} \neq \hat{a}m - m^2$ *for* $m = 0, 1, 2, \dots$. *Then, we can write*

$$j^2 + \hat{a}j + \hat{b} = (j + \hat{\omega}_1)(j + \hat{\omega}_2), \tag{3.16}$$

where $\hat{\omega}_1, \hat{\omega}_2 \neq 0, -1, -2, \dots$. *Note that* $\hat{\omega}_1$ *and* $\hat{\omega}_2$ *might not be real numbers.*

Fact 4 *For* $\hat{a} > 0$ *and any real number* $\hat{b}$ *satisfying* $\hat{b} \neq \hat{a}m - m^2$ *for* $m = 0, 1, 2, \dots,$ *by Fact 3 we have, as* $n \to \infty$,

$$\begin{aligned}\prod_{j=1}^{n}\left(1 + \frac{\hat{a}}{j} + \frac{\hat{b}}{j^2}\right) &= \prod_{j=1}^{n} \frac{(j+\hat{\omega}_1)}{j} \cdot \frac{(j+\hat{\omega}_2)}{j} \\ &\sim \frac{n^{\hat{\omega}_1}}{\hat{\omega}_1\Gamma(\hat{\omega}_1)} \cdot \frac{n^{\hat{\omega}_2}}{\hat{\omega}_2\Gamma(\hat{\omega}_2)} \quad (\textit{by Fact } 2) \\ &= \frac{n^{\hat{a}}}{\hat{b}\Gamma(\hat{\omega}_1)\Gamma(\hat{\omega}_2)},\end{aligned}$$

where the last equality is due to $\hat{\omega}_1 + \hat{\omega}_2 = \hat{a}$ *and* $\hat{\omega}_1\hat{\omega}_2 = \hat{b} \neq 0$.

Remark 3.1 $\Gamma(\hat{\omega}_1)$ and $\Gamma(\hat{\omega}_2)$ are well defined because $\hat{\omega}_1, \hat{\omega}_2 \neq 0, -1, -2, \dots,$ and $\Gamma(\hat{\omega}_1)\Gamma(\hat{\omega}_2)$ is always a nonzero real number even if $\hat{\omega}_1$ and $\hat{\omega}_2$ are not real numbers, because $\hat{\omega}_1$ and $\hat{\omega}_2$ must be complex conjugates if they are not real numbers, which leads to $\Gamma(\hat{\omega}_1)\Gamma(\hat{\omega}_2) = \Gamma(\hat{\omega}_1)\Gamma(\overline{\hat{\omega}_1}) = \Gamma(\hat{\omega}_1)\overline{\Gamma(\hat{\omega}_1)} = |\Gamma(\hat{\omega}_1)|^2 > 0$.

Fact 5 *It follows from Fact 4 that for any integer* $n_0 \geq 1$, *any real number* $\hat{a} > 0$, *and any real number* $\hat{b}$ *satisfying* $\hat{b} \neq \hat{a}m - m^2$ *for* $m = 0, 1, 2, \dots,$ *we have, as* $n \to \infty$,

$$\prod_{j=n_0}^{n}\left(1 + \frac{\hat{a}}{j} + \frac{\hat{b}}{j^2}\right) \sim Cn^{\hat{a}},$$

where C *depends only on* n_0, $\hat{a}$, *and* $\hat{b}$, *but not on* n.

The main tail asymptotic result is proved in the following theorem.

Theorem 3.2 *For the $M/M/c$ retrial queue, an exact decay function $h_k(n)$ is given by*

$$h_k(n) = n^{\frac{\lambda}{c\theta}-(c-k)}\rho^n, \quad n \geq 1,$$

for $k = 0, 1, \dots, c$. That is, for each $k = 0, 1, \dots, c$, there exist two positive constants $C_k^{(1)}$ and $C_k^{(2)}$, independent of n, such that

$$C_k^{(1)} h_k(n) \leq \pi_{n,k} \leq C_k^{(2)} h_k(n).$$

Proof For the number b defined in Fact 1, we can always find two numbers $b^{(1)}$ and $b^{(2)}$ satisfying $b^{(1)} \leq b \leq b^{(2)}$ and $b^{(1)}, b^{(2)} \neq am - m^2$ for $m = 0, 1, 2, \dots,$ where a is also defined in Fact 1. Fact 1 implies that there exists a positive integer N_0 such that

$$0 < \rho\left(1 + \frac{a}{n} + \frac{b^{(1)}}{n^2}\right) < r_{n,c} < \rho\left(1 + \frac{a}{n} + \frac{b^{(2)}}{n^2}\right) \quad \text{for all } n > N_0.$$

Hence, for all $n > N_0$,

$$\begin{aligned} 0 &< \rho^{n-N_0} \prod_{j=N_0+1}^{n} \left(1 + \frac{a}{j} + \frac{b^{(1)}}{j^2}\right) < r_{N_0+1,c} r_{N_0+2,c} \cdots r_{n,c} \\ &< \rho^{n-N_0} \prod_{j=N_0+1}^{n} \left(1 + \frac{a}{j} + \frac{b^{(2)}}{j^2}\right), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} 0 &< C^{(0)}\rho^n \prod_{j=N_0+1}^{n} \left(1 + \frac{a}{j} + \frac{b^{(1)}}{j^2}\right) < r_{1,c} r_{2,c} \cdots r_{n,c} \\ &< C^{(0)}\rho^n \prod_{j=N_0+1}^{n} \left(1 + \frac{a}{j} + \frac{b^{(2)}}{j^2}\right), \end{aligned} \tag{3.17}$$

where $C^{(0)} = \rho^{-N_0} \prod_{j=1}^{N_0} r_{j,c}$. Note that $C^{(0)}$ is a positive constant independent of n.

By Fact 5, we have a lower bound and an upper bound for $r_{1,c} r_{2,c} \cdots r_{n,c}$, given, respectively, by

$$C^{(0)}\rho^n \prod_{j=N_0+1}^{n} \left(1 + \frac{a}{j} + \frac{b^{(1)}}{j^2}\right) \sim C^{(1)} \cdot n^a \rho^n = C^{(1)} \cdot n^{\frac{\lambda}{c\theta}} \rho^n \tag{3.18}$$

and

$$C^{(0)}\rho^n \prod_{j=N_0+1}^{n} \left(1 + \frac{a}{j} + \frac{b^{(2)}}{j^2}\right) \sim C^{(2)} \cdot n^a \rho^n = C^{(2)} \cdot n^{\frac{\lambda}{c\theta}} \rho^n, \tag{3.19}$$

where $C^{(1)}$ and $C^{(2)}$ are positive constants, independent of n. The rest of the proof follows from Theorem 2.1 and the refined first-order asymptotic formula. □

Remark 3.2 The determination of the constants $C_k^{(1)}$ and $C_k^{(2)}$ is possible for $c=1$ and $c=2$. In fact, we have

$$C_k^{(1)} = C_k^{(2)} = C_k = \frac{\pi_{0,1}}{(\lambda/\theta)\Gamma(\lambda/\theta)}\left(\frac{\mu}{\theta}\right)^{c-k} \quad \text{when } c=1,$$

$$C_k^{(1)} = C_k^{(2)} = C_k = \frac{\pi_{0,2}}{\lambda/\theta+\mu/\theta+1}\cdot\frac{\Gamma(3\lambda/2\theta+\mu/\theta+2)}{\Gamma(w_1+1)\Gamma(w_2+1)}\left(\frac{\mu}{\theta}\right)^{c-k}\frac{c!}{k!}$$
$$\text{when } c=2,$$

where

$$w_1, w_2 = \frac{(2\lambda/\theta+\mu/\theta)\mp\sqrt{4(\lambda/\theta)(\mu/\theta)+(\mu/\theta)^2}}{2} > 0.$$

Proof of Remark 3.2 For the $M/M/1$ retrial queue, by the explicit expression of $r_{n,1}$, we have

$$\pi_{n,1} = \pi_{0,1}r_{1,1}r_{2,1}\cdots r_{n,1} = \pi_{0,1}\prod_{j=1}^{n}\frac{\lambda(\lambda+j\theta)}{j\theta\mu} = \pi_{0,1}\rho^n\prod_{j=1}^{n}\frac{\lambda/\theta+j}{j}$$
$$\sim \frac{\pi_{0,1}}{(\lambda/\theta)\Gamma(\lambda/\theta)}\cdot n^{\lambda/\theta}\rho^n \quad \text{(by Fact 2).}$$

For the $M/M/2$ retrial queue, by the explicit expression of $r_{n,2}$, we have

$$\pi_{n,2} = \pi_{0,2}r_{1,2}r_{2,2}\cdots r_{n,2} = \pi_{0,2}\prod_{j=1}^{n}\left(\frac{\bar\lambda}{j\bar\mu}\,\frac{\bar\lambda+\bar\mu+1+j}{\bar\lambda+\bar\mu+j}\,\frac{j\bar\mu+(\bar\lambda+j)^2}{3\bar\lambda+2\bar\mu+2+2j}\right)$$
$$= \pi_{0,2}\rho^n\prod_{j=1}^{n}\left(\frac{1}{j}\,\frac{\bar\lambda+\bar\mu+1+j}{\bar\lambda+\bar\mu+j}\,\frac{(w_1+j)(w_2+j)}{3\bar\lambda/2+\bar\mu+1+j}\right),$$

where we use $\bar\lambda = \lambda/\theta$ and $\bar\mu = \mu/\theta$ for simplicity of notation. Therefore,

$$\pi_{n,2} = \pi_{0,2}\rho^n\left(\prod_{j=1}^{n}\frac{\bar\lambda+\bar\mu+1+j}{\bar\lambda+\bar\mu+j}\right)\left(\prod_{j=1}^{n}\frac{w_1+j}{j}\right)\left(\prod_{j=1}^{n}\frac{w_2+j}{3\bar\lambda/2+\bar\mu+1+j}\right)$$
$$\sim C_2\cdot\rho^n n^{1+w_1+w_2-(3\bar\lambda/2+\bar\mu+1)} \quad \text{(by Fact 2)}$$
$$= C_2\cdot n^{\frac{\lambda}{2\theta}}\rho^n \quad (\text{since } w_1+w_2 = 2\bar\lambda+\bar\mu).$$

The rest of the proof follows from Theorem 2.1 and the refined first-order asymptotic formula. □

4 Approximations

Based on the tail asymptotic results obtained in the previous section, we propose two approximation methods for computing the joint stationary distribution $\pi_{n,k}$. By comparison with existing methods, we conclude that both the proposed approximations are comparable to the best existing ones.

Several approximation methods are available for the joint stationary probabilities $\pi_{n,k}$ in the literature as mentioned in the introduction. Approximations based on generalized truncated models are among the best (see, for example, [2]). To improve the overall quality in an approximation the key is to find a better approximation for the tail probabilities. Our methods are based on tail asymptotic expressions for $r_{n,k}$ when $n > N$ is large and on the solution of the censored equations, in which $r_{N+1,k}$ are replaced by the proposed asymptotic solution.

There are three steps to compute the stationary probabilities. In the first step, we propose an approximation for $r_{N+j,k}$ based on the first-order asymptotic formula; in the second step, we compute all $r_{n,k}$ for $n = N, N-1, \dots, 1$ and $k = c, c-1, \dots, 0$ based on the censored equations; and in the third step, we complete the computations of the stationary probabilities.

4.1 Approximations to $r_{N+j,k}$

Let N be the truncation size. We propose two approximations for $r_{N+j,k}$, both based on the first-order asymptotic formula and censored equation (3.7). For this purpose, we rewrite the refined first-order formula as

$$\begin{aligned} r_{N+j,c-k} &= \rho \frac{c!}{(c-k)!}\left(\frac{\mu}{N\theta}\right)^k \left(\frac{N}{N+j}\right)^k + O\left(\frac{1}{N^{k+1}}\right) \\ &= r_{N,c-k}\left(\frac{N}{N+j}\right)^k + O\left(\frac{1}{N^{k+1}}\right), \quad j = 1, 2, \dots; k = 0, 1, \dots, c. \end{aligned}$$

It follows that

$$r_{N+j,c-k} \sim r_{N,c-k}\left(\frac{N}{N+j}\right)^k \quad \text{as } N \to \infty. \tag{4.1}$$

It is also true that

$$r_{N+j,c-k} \sim r_{N,c-k}\frac{N}{N+j} \quad \text{as } N \to \infty. \tag{4.2}$$

By the Key Lemma, we can rewrite censored equation (3.7) as

$$r_{N+j,c} = \frac{\lambda + \lambda r_{N+j,c-1}}{c\mu + \beta_{N+j+1}(r_{N+j+1,c-2} + r_{N+j+1,c-3} + \cdots + r_{N+j+1,0})}, \quad j \geq 1. \tag{4.3}$$

Approximation I: Based on (4.2) and (4.3), for $j \geq 1$, we propose the following approximation $\tilde{r}_{N+j,k}$ for $k = 0, 1, \ldots, c$:

$$\tilde{r}_{N+j,c-k} = \tilde{r}_{N,c-k}\frac{N}{N+j} \quad \text{for } k = 1, 2, \ldots, c, \tag{4.4}$$

and

$$\begin{aligned} \tilde{r}_{N+j,c} &= \frac{\lambda + \lambda \tilde{r}_{N+j,c-1}}{c\mu + \beta_{N+j+1}(\tilde{r}_{N+j+1,c-2} + \tilde{r}_{N+j+1,c-3} + \cdots + \tilde{r}_{N+j+1,0})} \\ &= \frac{\lambda(1 + \frac{N}{N+j}\tilde{r}_{N,c-1})}{c\mu + \beta_N(\tilde{r}_{N,c-2} + \tilde{r}_{N,c-3} + \cdots + \tilde{r}_{N,0})}, \end{aligned} \tag{4.5}$$

where $\tilde{r}_{N,k}$ will be computed in the next subsection.

Remark 4.1 We notice that for every $j > 0$, the approximations $\tilde{r}_{N+j,k}$ for $k = 0, 1, \ldots, c-1$ satisfy (2.12) in the Key Lemma.

Approximation II: This approximation is based on (4.1) and (4.3): for $j \geq 1$,

$$\tilde{r}_{N+j,c-k} = \tilde{r}_{N,c-k}\left(\frac{N}{N+j}\right)^k \quad \text{for } k = 1, 2, \ldots, c, \tag{4.6}$$

and

$$\tilde{r}_{N+j,c} = \frac{\lambda + \lambda \tilde{r}_{N+j,c-1}}{c\mu + \beta_{N+j+1}(\tilde{r}_{N+j+1,c-2} + \tilde{r}_{N+j+1,c-3} + \cdots + \tilde{r}_{N+j+1,0})}, \tag{4.7}$$

where $\tilde{r}_{N+j,c-1}$ and $\tilde{r}_{N+j+1,k}$ for $k = 0, 1, \ldots, c-2$ are given in (4.6), and $\tilde{r}_{N,k}$ will be computed in the next subsection.

4.2 Approximations to $r_{n,k}$ for $n \leq N$

Based on the two proposed approximations, in this subsection, we develop two respective algorithms for computing approximations $\tilde{r}_{n,k}$ to $r_{n,k}$ for $n = N, N-1, \ldots, 1$ and $k = c, c-1, \ldots, 0$. We provide details for using Approximation I, and details for using Approximation II can be easily obtained by replacing approximation formulas (4.4) and (4.5) by (4.6) and (4.7), respectively.

The approximation to $r_{n,k}$ for $n \leq N$ is the solution to the censored equations, in which $r_{N+1,k}$ are replaced by the approximations $\tilde{r}_{N+1,k}$. Specifically, we first rewrite the censored equations as

$$r_{n,c} = \frac{\lambda + \lambda r_{n,c-1}}{c\mu + \beta_{n+1}(r_{n+1,c-2} + r_{n+1,c-3} + \cdots + r_{n+1,0})},$$

$$
\begin{aligned}
r_{n,c-k} &= \frac{(c-k+1)\mu r_{n,c-k+1} + \beta_{n+1} r_{n+1,c-k-1} r_{n,c} + \lambda r_{n,c-k-1}}{\sigma_{n,c-k}}, \\
&\quad k = 1, 2, \ldots, c-1, \\
r_{n,0} &= \frac{\mu r_{n,1}}{\sigma_{n,0}},
\end{aligned}
$$

where the Key Lemma was used for obtaining the first equation. To solve the above nonlinear system numerically, we will use the following direct iterative formula:

$$
r_{n,c}^{\text{new}} = \frac{\lambda + \lambda r_{n,c-1}^{\text{last}}}{c\mu + \beta_{n+1}(r_{n+1,c-2}^{\text{last}} + r_{n+1,c-3}^{\text{last}} + \cdots + r_{n+1,0}^{\text{last}})}, \tag{4.8}
$$

$$
\begin{aligned}
r_{n,c-k}^{\text{new}} &= \frac{(c-k+1)\mu r_{n,c-k+1}^{\text{last}} + \beta_{n+1} r_{n+1,c-k-1}^{\text{last}} r_{n,c}^{\text{last}} + \lambda r_{n,c-k-1}^{\text{last}}}{\sigma_{n,c-k}}, \\
&\quad k = 1, 2, \ldots, c-1,
\end{aligned} \tag{4.9}
$$

$$
r_{n,0}^{\text{new}} = \frac{\mu r_{n,1}^{\text{last}}}{\sigma_{n,0}}. \tag{4.10}
$$

The iterative method is extremely efficient in numerical computations since we can set the initial values $r_{n,k}^{\text{init}}$ for $r_{n,k}$ according to asymptotic properties as, for $n = N, N-1, \ldots, 1$,

$$
\begin{aligned}
r_{n,c}^{\text{init}} &= \rho, \\
r_{n,c-1}^{\text{init}} &= \frac{\lambda}{\beta_n}, \\
r_{n,c-k}^{\text{init}} &= 0 \quad \text{for } k = 2, 3, \ldots, c.
\end{aligned}
$$

The detailed algorithm is given below. Let

$$
\begin{aligned}
r_{n,k}^{\text{old}} = r_{n,k}^{\text{last}} = r_{n,k}^{\text{init}} &\quad \text{for } n = N, N-1, \ldots, 1 \text{ and } k = c, c-1, \ldots, 0, \\
r_{N+1,k}^{\text{old}} = r_{N+1,k}^{\text{last}} = \frac{N}{N+1} \cdot r_{N,k}^{\text{last}} &\quad \text{for } k = c, c-1, \ldots, 0,
\end{aligned}
$$

where the last equation has the same argument as in Approximation I.

For each fixed n (starting with $n = N$), the following algorithm computes all $\tilde{r}_{n,k}$ for $k = c, c-1, \ldots, 0$.

For $n = N$,

Iteration: If $n > 0$, then do the following; otherwise stop (all $\tilde{r}_{n,k}$ have been computed).

(1) Compute $r_{n,c}^{\text{new}}$ according to (4.8) and update $r_{n,c}^{\text{last}} = r_{n,c}^{\text{new}}$;
(2) For $k = 1, 2, \ldots, c-1$, compute $r_{n,c-k}^{\text{new}}$ according to (4.9) and update $r_{n,c-k}^{\text{last}} = r_{n,c-k}^{\text{new}}$;
(3) Compute $r_{n,0}^{\text{new}}$ according to (4.10) and update $r_{n,0}^{\text{last}} = r_{n,0}^{\text{new}}$;

(4) If

$$\max_{0\le k\le c}\left|r_{n,k}^{\text{new}}/r_{n,k}^{\text{old}}-1\right|<10^{-8},$$

then set $\tilde{r}_{n,k}=r_{n,k}^{\text{new}}$ for $0\le k\le c$. Let $n=n-1$ and go back to Iteration. Otherwise, set $r_{n,k}^{\text{old}}=r_{n,k}^{\text{last}}$ for $k=c,c-1,\dots,0$ and go back to (1).

We have now computed all approximations for $\tilde{r}_{n,k}$ for $n=N,N-1,\dots,1$ and $k=c,c-1,\dots,0$. For all experiments conducted, the maximal number of iterations for a relative error less than 10^{-8} is only 19.

4.3 Computations of stationary probabilities

In this section, we complete all computations for obtaining a distribution $\tilde{\boldsymbol{\pi}}$ which is an approximation to the true distribution $\boldsymbol{\pi}$. The computations are divided into the following steps.

Step 1. Substitute $\tilde{r}_{1,0},\tilde{r}_{1,1},\dots,\tilde{r}_{1,c-1}$ into (2.4) to have

$$\boldsymbol{p}_0(B_0+\tilde{R}_1C_1)=\mathbf{0},$$

where $\boldsymbol{p}_0=(p_{0,0},p_{0,1},\dots,p_{0,c})$ and

$$\tilde{R}_1C_1=\begin{bmatrix}0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ 0&0&0&\cdots&0\\ 0&\theta\tilde{r}_{1,0}&\theta\tilde{r}_{1,1}&\cdots&\theta\tilde{r}_{1,c-1}\end{bmatrix}.$$

Solve the system of equations satisfying $\boldsymbol{p}_0\boldsymbol{e}=1$ (to control all components $p_{0,k}<1$ for avoiding computational exploration).

Step 2. For $n=1,2,\dots,N$, compute $\boldsymbol{p}_n=p_{n-1,c}(\tilde{r}_{n,0},\tilde{r}_{n,1},\dots,\tilde{r}_{n,c})$ recursively.

Step 3. Normalize $\boldsymbol{p}_n$ for $n=1,2,\dots,N$ to avoid computational exploration to obtain $\boldsymbol{q}_n$ as follows:

$$\varpi=(\boldsymbol{p}_0+\boldsymbol{p}_1+\cdots+\boldsymbol{p}_N)\boldsymbol{e},$$
$$\boldsymbol{q}_n=\boldsymbol{p}_n/\varpi,\quad n=0,1,2,\dots,N.$$

Step 4. For $j=1,2,\dots$, let $\boldsymbol{q}_{N+j}=(q_{N+j,0},q_{N+j,1},\dots,q_{N+j,c})$ be computed according to $\boldsymbol{q}_{N+j}=q_{N+j-1,c}(\tilde{r}_{N+j,c},\tilde{r}_{N+j,c},\dots,\tilde{r}_{N+j,c})$. Practically, we compute $\boldsymbol{q}_{N+j}$ for $j=1,2,\dots,N_0$ such that $\boldsymbol{q}_{N+N_0}\boldsymbol{e}$ is significantly small.

Step 5. Perform the final normalization to obtain $\tilde{\boldsymbol{\pi}}_n$ for $n\ge 0$:

$$\tilde{\boldsymbol{\pi}}_n=\frac{\boldsymbol{q}_n}{(1+\kappa)},\quad n\ge 0,\tag{4.11}$$

where

$$\kappa=\sum_{n=1}^{N+N_0}\sum_{k=0}^{c}q_{n,k}.$$

Table 1 Truncation size when Approximation I

	$\theta = 0.01$	$\theta = 0.1$	$\theta = 1$	$\theta = 10$	$\theta = 100$
$\rho = 0.1$	2(2)	2(1)	2(1)	2(1)	2(2)
$\rho = 0.3$	10(2)	5(1)	5(1)	3(1)	2(2)
$\rho = 0.6$	97(3)	25(3)	13(2)	9(3)	5(2)
$\rho = 0.8$	397(2)	80(3)	32(3)	19(3)	12(3)

Table 2 Truncation size when Approximation II

	$\theta = 0.01$	$\theta = 0.1$	$\theta = 1$	$\theta = 10$	$\theta = 100$
$\rho = 0.1$	2(2)	2(1)	2(1)	2(1)	2(2)
$\rho = 0.3$	11(2)	5(1)	4(1)	3(1)	2(2)
$\rho = 0.6$	101(3)	24(3)	11(1)	7(2)	4(2)
$\rho = 0.8$	400(3)	74(2)	14(1)	14(2)	7(2)

4.4 Numerical analysis

We compared the two approximations proposed in this section to those available in the literature (see, for example, Sect. 3.4 in [2]). This comparison suggests that both of our approximations are comparable to the best available ones in the literature in the sense that the truncation size from using our approximations is similar to the size by using generalized truncation methods, which are considered to be among the best available methods for approximations. We provide the required minimal truncation size for approximations such that the relative error, compared to the true value of the blocking probability, is smaller than 10^{-4}.

The number in the parentheses provided in Tables 1 and 2 is the ranking of our approximation method compared to three other methods, which were compared in Table 3.10 of [2]. A reading of 1 means that the size of truncations required by using our method is the smallest compared to the other three methods, 2 means that our method is the second best, etc. The comparison suggests that both our methods are good alternatives to the best existing methods.

A comparison of the mean number of customers in the orbit confirms our conclusion, for which details are not provided here.

5 Concluding remarks

In this paper, the $M/M/c$ retrial queue is considered in terms of censoring technique. We focused on the tail asymptotic analysis, which is an important aspect and has not been done in the past. We also demonstrated that among other possible applications, the tail asymptotic results can be used to develop approximation methods for the stationary joint distribution for the $M/M/c$ retrial queue model. We expect that this method can be applied to other retrial queues, such as retrial queues with impatient customers, discrete time retrial queues, etc.

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