

M/M/∞ queues in semi-Markovian random environment

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Received: 29 January 2007 / Revised: 11 March 2008 / Published online: 30 April 2008
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Abstract In this paper we investigate an M/M/∞ queue whose parameters depend on an external random environment that we assume to be a semi-Markovian process with finite state space. For this model we show a recursive formula that allows to compute all the factorial moments for the number of customers in the system in steady state. The used technique is based on the calculation of the raw moments of the measure of a bidimensional random set. Finally the case when the random environment has only two states is deeper analyzed. We obtain an explicit formula to compute the above mentioned factorial moments when at least one of the two states has sojourn time exponentially distributed.

Keywords M/M/∞ queues · Random environment · Factorial moments

Mathematics Subject Classification (2000) 60K25 · 60K37 · 60D05

1 Introduction

The M/M/∞ queue is one of the simplest model in queueing theory. This is due to the joint situation to have a memory-less arrival process and an infinite set of servers that allows customers to behave independently from each other. This suddenly stops to be true after introducing some correlation between customers. In this paper we achieve that by introducing an independent random environment that modulates the system

Part of this research took place while the author was still post-doc at EURANDOM, Eindhoven, The Netherlands. The work was supported by the Spanish Ministry of Education and Science by the Grant MTM2007-63140.

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parameters, i.e. the arrival rate and the server speeds. Queues with variable service and arrival speeds arise naturally in practice and therefore many classical works can be found. Most of the results deal with the single server queue, see for example [10, 13, 14] and references therein. Neuts [8] analyzed the M/M/1 queue as well as the M/M/C queue in random environment by using the matrix-geometric approach while Takine and Sengupta [15] looked at the infinite server queue when only the arrival process was subject to a Markovian modulation. The infinite server queue in random environment has then been studied by Keilson and Servi [7], Baykal-Gursoy and Xiao [2] and D’Auria [5] in the special case when the random environment is Markovian and has only two states.

In [9] the authors looked at the case when the environment is given by a finite state Markov process and for this case they showed how to compute the factorial moments for the number of customers in the system in steady state. Here we extend their analysis to the case of a semi-Markovian random environment.

This extension is interesting as it makes the model more attractive for application purposes. Indeed, despite its simplicity, the M/M/∞ system is often used to model pure delay systems, such as highways, satellite links or long communication cables, or to approximate the behavior of multi server systems. When these kinds of systems are subject to external influences, such as day time changing rates, it is then helpful to look at extended models, such as the one proposed in this work, in order to analyze or predict their behaviors.

The methodology we use follows the technique developed in [5]. It consists on representing the stationary and isolate M/G/∞ system as a Poisson process on \mathbb{R}^2 and by computing the number of customers in the system by measuring a deterministic set according to the point process measure (see also [11, Sect. 3.3] and [6]). In this context the random environment can be expressed as a random modulation of the set and in the special case of exponential distributed service times its measure can be derived by solving a system of stochastic equations, see relation (4.5) below. We use this relation to compute the factorial moments for the number of customers in the system at steady state.

2 Model description

To start, we define the random environment $\{\Gamma(u), u \in \mathbb{R}\}$ as a semi-Markov chain with values in the finite state space $E = \{1, \dots, K\}$. We assume that the sojourn time in the state $k \in E$, denoted as T_k , is an independent positive random variable whose distribution function has Laplace transform denoted by $\tau_k(s) := \mathbb{E}[e^{-sT_k}]$. In the following we show that the Laplace transform is the only information we need to compute the moments. When the sojourn time in state $k \in E$ expires, the environment jumps to state $j \in E$ with probability p_{kj} . Denoting by $\mathbf{P} := \{p_{kj}\}_{k,j \in E}$ the routing matrix that we assume irreducible and with no loss of generality with $p_{kk} = 0$, we can define the reverse routing matrix

$$\mathbf{Q} := \mathbf{\Pi}^{-1} \mathbf{P}^{\top} \mathbf{\Pi}, \quad (2.1)$$

where \mathbf{P}^{\top} denotes the transpose of the matrix \mathbf{P} , $\mathbf{\Pi} := \mathbf{diag}(\vec{\pi})$ and $\vec{\pi}$ is the stationary distribution of the Markov chain generated by \mathbf{P} (see [3, Sect. 6.1]).

We assume that when the environment is in state $k \in E$ customers arrive according to a Poisson rate $\lambda_k \geq 0$. Each of them brings an independent request of service, σ , that is exponentially distributed with rate μ . All servers work at constant speed $\beta_k = \mu_k/\mu \leq 1$. To avoid trivial cases we assume that $\mu, \beta, \lambda > 0$ where $\lambda := \max_{k \in E} \lambda_k$ and $\beta := \max_{k \in E} \beta_k$.

By the results in [5] the system is stable and we are allowed to study its stationary regime.

We then look at the system at time 0 and we count the number of customers still in the system. We order them according to their arrival times $\{u_h\}_{h \in \mathbb{Z}}$ with $u_h < u_{h+1}$ and $u_{-1} < 0 \leq u_0$, and we denote by $G(\sigma) := 1 - e^{-\mu\sigma}$, $\sigma > 0$, the common exponential distribution function of the $\{\sigma_h\}_{h \in \mathbb{Z}}$.

The h -th customer, $h < 0$, will be in the system at time 0 iff its service time, σ_h , is greater than the work done by the server it has occupied during the time interval $[u_h, 0)$. We denote this quantity by $F_\Gamma(u_h)$ and, as the subscript shows, it is a random quantity that depends on the random environment Γ . Its value can be computed in the following way,

$$F_\Gamma(u) := \int_u^0 \beta_{\Gamma(t)} dt, \quad u \leq 0. \tag{2.2}$$

Denoting by N the number of customers in stationary regime we have that it is given by

$$N = \sum_{h < 0} 1\{\sigma_h > F_\Gamma(u_h)\}, \tag{2.3}$$

where $1\{\cdot\}$ is the indicator function of the set $\{\cdot\}$. It is helpful to rewrite the numerable collection of indicator functions appearing in expression (2.3) in the following equivalent way

$$1\{\sigma_h > F_\Gamma(u_h)\} = \delta_{(u_h, \sigma_h)}(A_\Gamma)$$

where $\delta_{(u, \sigma)}$ denotes a Dirac delta measure with center $(u, \sigma) \in \mathbb{R} \times \mathbb{R}^+$ and the set $A_\Gamma \subset \mathbb{R}^- \times \mathbb{R}^+$ is given by

$$A_\Gamma := \{(u, \sigma) \in \mathbb{R}^- \times \mathbb{R}^+ : \sigma > F_\Gamma(u)\}. \tag{2.4}$$

This alternative formulation allows the decoupling of the sequence $\{(u_h, \sigma_h)\}_{h \in \mathbb{Z}}$ and the function $F_\Gamma(u)$ both depending on the realization of the environment Γ in the computation of the quantity N . Indeed we can express the stationary number of customers in the system in the following way

$$N = \sum_{h < 0} \delta_{(u_h, \sigma_h)}(A_\Gamma) = \mathcal{N}_\Gamma(A_\Gamma), \tag{2.5}$$

where

$$\mathcal{N}_\Gamma := \sum_{h \in \mathbb{Z}} \delta_{(u_h, \sigma_h)} \tag{2.6}$$

is a point process which locates one Dirac delta measure at each arrival point $\{(u_h, \sigma_h)\}_{h \in \mathbb{Z}}$. For the theoretical background and definition of point processes see

[4, Sect. 7.1] or [11, Sect. 3.1]. The subscript Γ stays to denote that \mathcal{N}_Γ depends on the random environment by the sequence of arrival times $\{u_h\}_{h \in \mathbb{Z}}$. Indeed given a realization γ of the process Γ , the sequence $(\{u_h\}_{h \in \mathbb{Z}} | \Gamma = \gamma)$ belongs to an inhomogeneous Poisson process with intensity rate $\lambda_{\gamma(u)}$, $u \in \mathbb{R}$. By Proposition 3.8 in [11] it follows that $\mathcal{N}_\Gamma | \Gamma = \gamma$ is still a Poisson process, now on $\mathbb{R} \times \mathbb{R}^+$, with intensity measure

$$\lambda_\gamma(A) := E[\mathcal{N}_\Gamma(A) | \Gamma = \gamma] = \int_A \lambda_{\gamma(u)} du G(d\sigma), \quad A \subset \mathbb{R} \times \mathbb{R}^+.$$

Finally \mathcal{N}_Γ is a doubly stochastic Poisson process or, more briefly, a Cox process (see [4, Sect. 8.5]), i.e. a Poisson process whose intensity measure is itself random and given by

$$\lambda_\Gamma(A) := E[\mathcal{N}_\Gamma(A) | \Gamma] = \int_A \lambda_{\Gamma(u)} du G(d\sigma), \quad A \subset \mathbb{R} \times \mathbb{R}^+. \tag{2.7}$$

It is well known that the finite dimensional distributions of a Cox process are of mixed Poisson type (see [4, Corollary 8.5.II]), or equivalently that for any set $A \subset \mathbb{R} \times \mathbb{R}^+$

$$\mathcal{N}_\Gamma(A) \sim \text{Po}(|A|_\Gamma) \tag{2.8}$$

is a Poisson random variable whose parameter is itself random with value $|A|_\Gamma = \mathbb{E}[\mathcal{N}_\Gamma(A) | \Gamma]$. $|A|_\Gamma$ can be geometrically interpreted as the measure of the set A according to the measure $\lambda_\Gamma(\cdot)$, i.e. $|A|_\Gamma = \lambda_\Gamma(A)$.

From relations (2.5) and (2.8) we finally get that

$$N \sim \text{Po}(|A_\Gamma|_\Gamma), \tag{2.9}$$

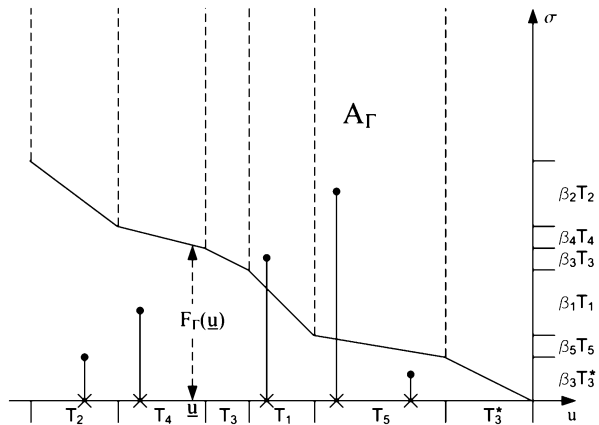
a mixed Poisson random variable with random parameter $|A_\Gamma|_\Gamma = \lambda_\Gamma(A_\Gamma)$. Figure 1 shows an example of realization where the random environment has $K = 5$ states: the dots are the centers of the Dirac deltas of the point process \mathcal{N}_Γ , while the piecewise linear function $F_\Gamma(u)$ denotes the lower bound of the set of integration A_Γ . The customers present in the system at time 0 are then the ones whose dots fall in the set A_Γ ; in the shown example $N = 2$.

Example 2.1 The easiest case is when the environment process is constant, $K = 1$, so that the system reduces to a classical M/M/∞ queue. In this case the set A_Γ is deterministic, given by $\{(u, \sigma) : u < 0, \sigma > \beta|u|\}$. From (2.7) we get

$$|A_\Gamma|_\Gamma = \int_{-\infty}^0 \int_{\sigma > \beta|u|} \lambda G(d\sigma) du = \frac{\lambda}{\beta} \int_0^\infty (1 - G(u)) du = \frac{\lambda}{\beta} \mathbb{E}[\sigma] = \frac{\lambda}{\beta\mu}$$

and we obtain the known information that $N \sim \text{Po}(\frac{\lambda}{\beta\mu})$, i.e. the stationary number of customers in the system is Poisson distributed.

Fig. 1 Example of realization



3 Computing the factorial moments

Before beginning to compute the factorial moments of the random variable N , it is worthwhile to review some basic results about the different kinds of moments and their relations with the various generating functions. A good reference about the following relations especially in connection with point processes is [4, Chap. 5].

Given a random variable X , we denote by $\psi_X(s) := \mathbb{E}[e^{sX}]$ its moment generating function and by $\phi_X(z) := \mathbb{E}[z^X]$ its probability generating function.

The factorial moment of order i of X , $f_X^{(i)}$ is defined as

$$f_X^{(i)} := \mathbb{E}[X^{\underline{i}}] = \sum_{n=0}^{\infty} n^{\underline{i}} p_n,$$

where $p_n = \Pr\{X = n\}$ and $n^{\underline{i}} := n(n - 1) \cdots (n - i + 1)$ is the falling factorial. It can be directly computed by the i -th derivative of the probability generating function, i.e. $f_X^{(i)} = \lim_{z \rightarrow 1} \phi_X^{(i)}(z)$. Knowing the factorial moments of X it is then easy to compute its moments, in the sequel called *raw moments* to distinguish them from the factorial ones. Indeed, by taking the expectations on both sides of the following known equivalence [1]

$$X^n = \sum_{i=0}^n \mathfrak{S}_n^{(i)} X^{\underline{i}},$$

where $\mathfrak{S}_n^{(i)}$ is a Stirling Number of the Second Kind, we obtain the following relation between the n -th moment of X , $m_X^{(n)} := \mathbb{E}[X^n]$ with $m_X^{(0)} := 1$, and the factorial moments of order $i \leq n$,

$$m_X^{(n)} = \sum_{i=0}^n \mathfrak{S}_n^{(i)} f_X^{(i)}. \tag{3.1}$$

The reverse relation is obtained by using the Stirling Numbers of the First Kind, $s_i^{(n)}$ (see [1]), that satisfy the following known relation

$$X^i = \sum_{n=0}^i s_i^{(n)} X^n,$$

so that, taking the expectations of both sides, finally we get

$$f_X^{(i)} = \sum_{n=0}^i s_i^{(n)} m_X^{(n)}. \tag{3.2}$$

It is interesting to notice that relation (3.1) comes directly from using the fact that $\psi_X(s) = \phi_X(e^s)$ and that $m_X^{(n)} = \lim_{s \rightarrow 0} \psi_X^{(n)}(s)$. Indeed,

$$\lim_{s \rightarrow 0} \psi_X^{(n)}(s) = \lim_{s \rightarrow 0} \frac{d^n}{ds^n} \phi_X(e^s) = \sum_{i=0}^n \mathfrak{S}_n^{(i)} \phi_X^{(i)}(1),$$

where in the last equation we used Faá di Bruno’s formula for the expansion of derivatives of order n for composition of functions (see [1]) and the fact that $\lim_{s \rightarrow 0} \frac{d^n}{ds^n} e^s = 1$.

A random variable X is called mixed Poisson if there exists a non-negative random variable Y such that $X \stackrel{d}{=} \text{Po}(Y)$, or equivalently $(X|Y = y) \stackrel{d}{=} \text{Po}(y)$, where the operator $\stackrel{d}{=}$ denotes equality in distribution. In the case X were a mixed Poisson random variable we would have that

$$\phi_X(z) = \psi_Y(z - 1),$$

so that taking the derivatives of order n , we get

$$\lim_{z \rightarrow 1} \phi_X^{(n)}(z) = \lim_{z \rightarrow 1} \psi_Y^{(n)}(z - 1) = \lim_{s \rightarrow 0} \psi_Y^{(n)}(s),$$

or, in other words, that the factorial moments of X are equal to the raw moments of Y ,

$$f_X^{(n)} = m_Y^{(n)},$$

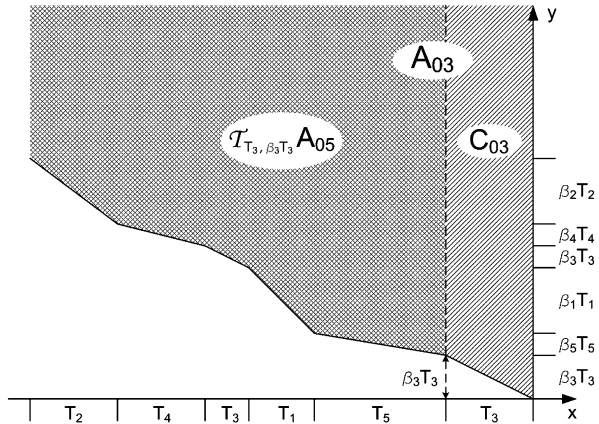
and the latter often are easier to compute.

This is exactly what happens in our case where, as shown by relation (2.9), N is a mixed Poisson and that is why we are interested into its factorial moments rather than directly its raw moments. Indeed we have that the following relation holds

$$f_N^{(n)} = m_{|A_\Gamma|_\Gamma}^{(n)}, \tag{3.3}$$

and our task reduces to the computation of the raw moments of the measure of the random set A_Γ .

Fig. 2 Decomposition of A_{03} as $C_{03} \cup \mathcal{T}_{T_3, \beta_3 T_3} A_{05}$



4 Computing the raw moments of $|A_\Gamma|_\Gamma$

In this section we compute the raw moments of the measure of the set A_Γ , defined in (2.4), when measured by the random intensity measure λ_Γ , defined in (2.7). We use a fixed point technique and to this aim we look at a modified environment process, Γ_0 , that is the Palm version of the process Γ , i.e. we assume that at time 0 it has a transition. We denote by $k \in E$ the last state it has assumed before 0, i.e. $k := \Gamma_0(0^-)$, and by T_k its corresponding sojourn time. While, as depicted in Figure 1, for the process Γ the sojourn time in the last state before 0 would be given by a residual sojourn time, for the process Γ_0 it is distributed as any other sojourn time corresponding to the same state. We define by $A_{0k} := (A_{\Gamma_0} | \Gamma_0(0^-) = k)$, $k \in E$, the set A_{Γ_0} conditioned to the event that the last state occupied by the environment before 0 is the state k , and we call $|A_{0k}|$ its measure, i.e. $|A_{0k}| := (\lambda_{\Gamma_0}(A_{\Gamma_0}) | \Gamma_0(0^-) = k)$.

Figure 2 shows an example of the set A_{0k} when $k = 3$, together with its decomposition in the set C_{0k} and the set $\mathcal{T}_{T_3, \beta_3 T_3} A_{05}$. To this we have defined by C_{0k} the restriction of the set A_{0k} up the last transition of the process Γ_0 before time 0, i.e.

$$C_{0k} := A_{0k} \cap \{(x, y) \in \mathbb{R}^- \times \mathbb{R}^+ | |x| < T_k\}, \tag{4.1}$$

$$|C_{0k}| := (\lambda_{\Gamma_0}(C_{0k}) | \Gamma_0(0^-) = k)$$

and by $\mathcal{T}_{s,t} A$ the $(-s, t)$ -translated version of the set A , i.e.

$$\mathcal{T}_{s,t} A := \{(x, y) \in \mathbb{R}^2 | (x + s, y - t) \in A\}. \tag{4.2}$$

We denote by $j \in E$ the state of the environment before the last transition before time 0, i.e. $j := \Gamma_0(-T_k^-)$, so that, $-T_k$ being a regeneration point for the process Γ_0 , we have the independence of the sets C_{0k} and $\mathcal{T}_{T_k, \beta_k T_k} A_{0j}$ conditioned to the values of the states before and after the transitions, i.e. j and k . $\beta_k T_k = F_{\Gamma_0}(-T_k)$ is the exact amount of work the non-empty servers have done during the time interval $[-T_k, 0)$ being in state k .

By noticing that the set $\mathcal{T}_{T_k, \beta_k T_k} A_{0j}$ has measure equal in distribution to $|\mathcal{T}_{0, \beta_k T_k} A_{0j}| := (\lambda_{\Gamma_0}(\mathcal{T}_{0, \beta_k T_k} A_{0j}) | \Gamma_0(0^-) = j)$, we can write down the following set

of stochastic equations

$$|A_{0k}| \stackrel{d}{=} |C_{0k}| + \sum_{j=1}^K 1\{k \leftarrow j\} |\mathcal{T}_{0,\beta_k T_k} A_{0j}|, \tag{4.3}$$

where the indicator function $1\{k \leftarrow j\}$ selects the backward state transition of the environment from the state k to the state j ; this would happen, according to definition (2.1), with probability q_{kj} .

Thanks to the fact that along the vertical axis the measure function is given by G that is exponential we have that the following result holds:

Lemma 4.1 *Given the transformation $\mathcal{T}_{s,t}$, defined in (4.2), we have that*

$$|\mathcal{T}_{0,t} A_\Gamma|_\Gamma = e^{-\mu t} |A_\Gamma|_\Gamma, \tag{4.4}$$

for any random set $A_\Gamma : \Gamma \rightarrow \mathcal{B}(\mathbb{R}^- \times \mathbb{R}^+)$.

Proof By using definition (2.7) we have

$$\begin{aligned} \lambda_\Gamma(\mathcal{T}_{0,t} A_\Gamma) &= \int_{\mathcal{T}_{0,t} A_\Gamma} \lambda_{\Gamma(u)} du e^{-\mu\sigma} d\sigma \\ &= e^{-\mu t} \int_{\mathcal{T}_{0,t} A_\Gamma} \lambda_{\Gamma(u)} du e^{-\mu(\sigma-t)} d\sigma = e^{-\mu t} \lambda_\Gamma(A_\Gamma). \end{aligned} \quad \square$$

By using Lemma 4.1, (4.3) simplifies in the following

$$|A_{0k}| \stackrel{d}{=} |C_{0k}| + e^{-\mu\beta_k T_k} \sum_{j=1}^K 1\{k \leftarrow j\} |A_{0j}|, \tag{4.5}$$

that is the starting point to prove the following main result:

Theorem 4.2 *Let us define $\vec{m}_0^{(n)} \in \mathbb{R}^K$ as the column vector whose k -th coordinate is the n -th moment of the random variable $|A_{0k}|$, i.e. $m_{0k}^{(n)} := m_{|A_{0k}|}^{(n)}$ then the following relation holds*

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \mathbf{R}^{n-i} \mathbf{B}_n \vec{m}_0^{(i)} = 0, \tag{4.6}$$

where $\mathbf{R} := \text{diag}(\rho_k)$, $\rho_k := \lambda_k / \mu_k$ and the matrix $\mathbf{B}_n := \text{diag}(\tau_k^{-1}(n\mu_k)) - \mathbf{Q}$. The matrix \mathbf{B}_n , $n > 0$, is invertible and therefore it is possible to express the n -th moment vector $\vec{m}_0^{(n)}$ in terms of the previous ones, $\vec{m}_0^{(i)}$, $i = 0, \dots, n - 1$, in the following way

$$\vec{m}_0^{(n)} = \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n}{i} \mathbf{B}_n^{-1} \mathbf{R}^{n-i} \mathbf{B}_n \vec{m}_0^{(i)}. \tag{4.7}$$

Proof We first compute the values of the variable $|C_{0k}|$ in the following way

$$|C_{0k}| = \lambda_k \int_0^{T_k} e^{-\mu_k x} dx = \rho_k (1 - e^{-\mu_k T_k}).$$

Then substituting its value in (4.5), it gives

$$|A_{0k}| \stackrel{d}{=} \rho_k (1 - e^{-\mu_k T_k}) + e^{-\mu_k T_k} \sum_{j=1}^K 1\{k \leftarrow j\} |A_{0j}|, \tag{4.8}$$

that can be rewritten as

$$|A_{0k}| - \rho_k \stackrel{d}{=} \sum_{j=1}^K 1\{k \leftarrow j\} (|A_{0j}| - \rho_k) e^{-\mu_k T_k}. \tag{4.9}$$

We denote by $\psi_{0k}(s) := \mathbb{E}[e^{s|A_{0k}|}]$ the moment generating function of $|A_{0k}|$ so that applying the exponential function to both members of (4.9) previously multiplied by s and then taking the expectation, we obtain

$$\begin{aligned} \psi_{0k}(s) e^{-s\rho_k} &= \mathbb{E} \left[\sum_{j=1}^K q_{kj} e^{s(|A_{0j}| - \rho_k) e^{-\mu_k T_k}} \right] \\ &= \mathbb{E} \left[\sum_{j=1}^K q_{kj} \psi_{0j}(s e^{-\mu_k T_k}) e^{-s\rho_k e^{-\mu_k T_k}} \right]. \end{aligned}$$

The last expression can be written in matrix form in the following way

$$e^{-s\mathbf{R}} \vec{\psi}_0(s) = \mathbb{E} \left[e^{-s\mathbf{R}\mathbf{T}} (\mathbf{Q} \vec{\psi}_0)(s\mathbf{T}) \right], \tag{4.10}$$

where $\mathbf{T} := \mathbf{diag}(e^{-\mu_k T_k})$ and where with notation $\vec{v}(\mathbf{W})$, with \mathbf{W} a diagonal matrix, we denote a vector whose k -th component is $v_k(w_{kk})$. We use then the following matrix formulas for derivatives

$$D^{(n)}[e^{-s\mathbf{W}} \vec{v}(s)] = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} e^{-s\mathbf{W}} \mathbf{W}^{n-i} D^{(i)}[\vec{v}(s)], \tag{4.11}$$

and

$$D^{(n)}[\vec{v}(s\mathbf{W})] = \mathbf{W}^n \vec{v}^{(n)}(s\mathbf{W}), \tag{4.12}$$

to compute the n -th derivative of both sides of (4.10) so that

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} e^{-s\mathbf{R}} \mathbf{R}^{n-i} \vec{\psi}_0^{(i)}(s)$$

$$\begin{aligned}
 &= \mathbb{E} \left[\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} e^{-s\mathbf{R}\mathbf{T}} \mathbf{R}^{n-i} \mathbf{T}^{n-i} D^{(i)}[\mathbf{Q}\vec{\psi}_0(s\mathbf{T})] \right] \\
 &= \mathbb{E} \left[\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} e^{-s\mathbf{R}\mathbf{T}} \mathbf{R}^{n-i} \mathbf{T}^n (\mathbf{Q}\vec{\psi}_0^{(i)})(s\mathbf{T}) \right].
 \end{aligned}$$

Remembering that $\vec{m}_0^{(n)} = \lim_{s \rightarrow 0} \vec{\psi}_0^{(n)}(s)$ and taking the limit of the last expression as $s \rightarrow 0$, we get

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \mathbf{R}^{n-i} \vec{m}_0^{(i)} = \mathbb{E} \left[\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \mathbf{R}^{n-i} \mathbf{T}^n \mathbf{Q} \vec{m}_0^{(i)} \right]. \tag{4.13}$$

Multiplying on the left side by $(-1)^{-n} \mathbb{E}[\mathbf{T}^n]^{-1}$, the last expression can be easily rearranged in

$$\sum_{i=0}^n (-1)^{-i} \binom{n}{i} \mathbf{R}^{n-i} [\mathbb{E}[\mathbf{T}^n]^{-1} - \mathbf{Q}] \vec{m}_0^{(i)} = 0, \tag{4.14}$$

that gives the result. The invertibility of the matrix \mathbf{B}_n for $n > 0$ comes from Lemma A.2. □

It is remarkable that it is possible to express (4.6) in terms of the forward transition chain \mathbf{P} . The result is contained in the following corollary whose proof comes from simple matrix computations.

Corollary 4.3 *A result similar to (4.6) is valid for the row vector $\vec{m}_0^{\top(n)} := (\vec{m}_0^{(n)})^{\top}$, that involves the matrix \mathbf{P} instead of the matrix \mathbf{Q} , i.e.*

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \vec{m}_0^{\top(i)} \mathbf{P} \mathbf{B}'_n \mathbf{R}^{n-i} = 0, \tag{4.15}$$

where $\mathbf{B}'_n := \text{diag}(\tau_k^{-1}(n\mu_k)) - \mathbf{P}$. The matrix \mathbf{B}'_n is non-singular when $n > 0$.

Given the raw moments of $|A_{\Gamma_0}|$, we can successively compute the moments of the measures of the sets $A_k := (A_{\Gamma} | \Gamma(0) = k)$, $k \in E$. Following previous definitions we define $m_k^{(n)} := m_{|A_k|}^{(n)}$. Similarly to (4.3) we have the following equation

$$|A_k| \stackrel{d}{=} |C_k^*| + \sum_{j=1}^K 1\{k \leftarrow j\} |T_{0,\beta_k T_k^*} A_{0j}|, \tag{4.16}$$

with $|C_k^*| = \rho_k(1 - e^{-\mu_k T_k^*})$. T_k^* refers to a residual sojourn time of the environment in state $k \in E$. We define by $\tau_k^*(s)$ the Laplace transform of the distribution function of T_k^* and it is related to the one of T_k , $\tau_k(s)$, by the relation $\tau_k^*(s) = \bar{\tau}_k(1 - \tau_k(s))/s$, with $\bar{\tau}_k := \mathbb{E}[T_k]^{-1}$. For the vector of raw moments $\vec{m}^{(n)}$ the following theorem holds.

Theorem 4.4 *The vector $\vec{m}^{(n)}$ satisfies the following relation with the vector $\vec{m}_0^{(n)}$*

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \mathbf{R}^{n-i} [\vec{m}^{(i)} - \mathbf{E}_n \vec{m}_0^{(i)}] = 0, \tag{4.17}$$

with $\mathbf{E}_n := \text{diag}(\tau_k^*(n\mu_k)/\tau_k(n\mu_k))$. Therefore the vector $\vec{m}^{(n)}$ can be computed from the previous moments $\{\vec{m}^{(i)}\}_{i < n}$ and the corresponding vectors $\{\vec{m}_0^{(i)}\}_{i \leq n}$ in the following way

$$\vec{m}^{(n)} = \mathbf{E}_n \vec{m}_0^{(n)} + \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n}{i} \mathbf{R}^{n-i} [\vec{m}^{(i)} - \mathbf{E}_n \vec{m}_0^{(i)}], \tag{4.18}$$

finally

$$m^{(n)} := m_{A\Gamma}^{(n)} = \vec{m}^{(n)} \vec{\pi}. \tag{4.19}$$

Proof Starting by (4.16) and following the same calculations that brought us from (4.3) to (4.14), we get

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \mathbf{R}^{n-i} [\mathbb{E}[\mathbf{T}^{*n}]^{-1} \vec{m}^{(i)} - \mathbf{Q} \vec{m}_0^{(i)}] = 0, \tag{4.20}$$

that after subtracting equation (4.14) gives

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \mathbf{R}^{n-i} [\mathbb{E}[\mathbf{T}^{*n}]^{-1} \vec{m}^{(i)} - \mathbb{E}[\mathbf{T}^n]^{-1} \vec{m}_0^{(i)}] = 0, \tag{4.21}$$

and by multiplying on the left by $\mathbb{E}[\mathbf{T}^{*n}]$ we finally get the result. □

In order to check our results we compare (4.15) for the exponential case with results in [9] here repeated in formula (4.22). For this case since $T_k^* \stackrel{d}{=} T_k$, we have that $\vec{m}^{(n)} = \vec{m}_0^{(n)}$.

Remark 4.5 It is worth to notice that in [9], they actually computed the factorial moments of the random row vector $(N \mathbb{1}\{\Gamma_0 = k\})_{k \in E}$ while here we compute the factorial moments of the row vector $(N|\Gamma_0 = k)_{k \in E}$. This explains the presence, in formula (4.22), of the additional factor given by the diagonal matrix $\tilde{\mathbf{\Pi}}$ that has the diagonal equal to the stationary distribution of the Markovian Environment.

Corollary 4.6 *In case the sojourn times T_k are exponentially distributed with parameters $\bar{\tau}_k$ we have that with $n > 0$*

$$(\vec{m}^\top(n) \tilde{\mathbf{\Pi}})(n\mathbf{M} - \mathbf{G}) = n(\vec{m}^\top(n-1) \tilde{\mathbf{\Pi}}) \mathbf{\Lambda} \tag{4.22}$$

where $\mathbf{M} := \text{diag}(\mu_k)$, $\mathbf{\Lambda} := \text{diag}(\lambda_k)$, $\tilde{\mathbf{\Pi}} := \bar{\tau} \mathbf{\Pi} \bar{\mathbf{T}}^{-1}$ is the stationary distribution of the Markovian Random Environment, $1/\bar{\tau} := \sum_k \pi_k / \bar{\tau}_k$, while $\mathbf{G} := \bar{\mathbf{T}}(\mathbf{P} - \mathbf{I})$ is its generator, $\bar{\mathbf{T}} := \text{diag}(\bar{\tau}_k)$.

Proof When the sojourn times T_k are exponentially distributed we have that

$$\mathbb{E}[\mathbf{T}^n]^{-1} = n\mathbf{M}\bar{\mathbf{T}}^{-1} + \mathbf{I},$$

that implies

$$\bar{\mathbf{T}}\mathbf{B}_n = n\mathbf{M} - \mathbf{H},$$

with $\mathbf{H} := \bar{\mathbf{T}}(\mathbf{Q} - \mathbf{I})$ being the generator of the reverse-time Markov process. By multiplying on the left both sides of (4.6) by $\bar{\mathbf{T}}$ and noticing that it can commute with the powers of the matrix \mathbf{R} , we can rewrite (4.6) as

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \mathbf{R}^{n-i} [n\mathbf{M} - \mathbf{H}]\vec{m}^{(i)} = 0. \tag{4.23}$$

By Lemma A.1 with $\mathbf{U} = \mathbf{M}$, $\mathbf{D} = \mathbf{R}$, $\mathbf{V}_n = \mathbf{H}$ and $\vec{v}^{(n)} = \vec{m}^{(n)}$, and imposing $\vec{m}^{(0)} = \vec{1}$, we get that the unique solution of (4.23) is given by

$$(n\mathbf{M} - \mathbf{H})\vec{m}^{(n)} = n\mathbf{R}\mathbf{M}\vec{m}^{(n-1)},$$

that transposed reduces to

$$\vec{m}^{\top(n)} (\tilde{\mathbf{\Pi}} \tilde{\mathbf{\Pi}}^{-1})(n\mathbf{M} - \mathbf{H}^{\top}) = n\vec{m}^{\top(n-1)} \mathbf{\Lambda}.$$

Multiplying on the right side by $\tilde{\mathbf{\Pi}}$ and simplifying we get

$$\vec{m}^{\top(n)} \tilde{\mathbf{\Pi}}(n\mathbf{M} - \tilde{\mathbf{\Pi}}^{-1}\mathbf{H}^{\top}\tilde{\mathbf{\Pi}}) = n\vec{m}^{\top(n-1)} \tilde{\mathbf{\Pi}}\mathbf{\Lambda},$$

that gives the result after noticing that $\mathbf{G} = \tilde{\mathbf{\Pi}}^{-1}\mathbf{H}^{\top}\tilde{\mathbf{\Pi}}$. □

5 Some explicit formulas—Case $K = 2$

Formulas (4.7) and (4.18) show that generally to find the n -th moment of the random number of users in the system involves in a complex way the knowledge of all previous moments. Reversely the exponential case, that was already solved in [9], is easier as the n -th vector of moments is related only by a factor to the $(n - 1)$ -th one. That was anyway hidden in a non-trivial way in formula (4.7) so that there could be some other special cases where an easier expression holds.

In this section we have a look to the case when the environment has only two stages, i.e. $K = 2$.

This is a very special case and when the sojourn times are all assumed exponentially distributed, it is known how to compute the complete distribution of the number of customers in the system at steady state (see [2, 7] and [5]).

We give for this case an explicit formula to calculate the factorial moments in terms of the Laplace transform of the sojourn time in state 1, when the sojourn time in state 2 is exponentially distributed.

By rewriting in more explicit form (4.9) for the case $K = 2$ we get

$$|A_{01}| - \rho_1 \stackrel{d}{=} (|A_{02}| - \rho_1)e^{-\mu_1 T_1} \tag{5.1}$$

$$|A_{02}| - \rho_2 \stackrel{d}{=} (|A_{01}| - \rho_2)e^{-\mu_2 T_2}. \tag{5.2}$$

We define $\tilde{m}_{0k}^{(n)} := \mathbb{E}[(|A_{0k}| - \rho_1)^n]$ and take the mean of the n -powers of expression (5.1) so getting

$$\tilde{m}_{01}^{(n)} = \tilde{m}_{02}^{(n)} \tau_1(n\mu_1). \tag{5.3}$$

By adding and subtracting ρ_1 to both sides of (5.2) we get

$$(|A_{02}| - \rho_1) - \rho_* \stackrel{d}{=} (|A_{01}| - \rho_1)e^{-\mu_2 T_2} - \rho_* e^{-\mu_2 T_2},$$

with $\rho_* = \rho_2 - \rho_1$. Then using (5.1) we obtain a recursive equation involving only $|A_{02}| - \rho_1$,

$$(|A_{02}| - \rho_1) - \rho_* \stackrel{d}{=} (|A_{02}| - \rho_1)e^{-\mu_1 T_1} e^{-\mu_2 T_2} - \rho_* e^{-\mu_2 T_2}.$$

Taking the n -th power and then the expectation of both sides we get

$$\begin{aligned} \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \rho_*^{n-i} \tilde{m}_{02}^{(i)} &= \tau_2(n\mu_2) \mathbb{E}[((|A_{02}| - \rho_1)e^{-\mu_1 T_1} - \rho_*)^n] \\ &= \tau_2(n\mu_2) \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \rho_*^{n-i} \tau_1(i\mu_1) \tilde{m}_{02}^{(i)}, \end{aligned}$$

that, taking into account (5.3), can be rearranged to get the following

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \rho_*^{n-i} (\tau_2^{-1}(n\mu_2) \tau_1^{-1}(i\mu_1) - 1) \tilde{m}_{01}^{(i)} = 0. \tag{5.4}$$

Theorem 5.1 *Assuming that the sojourn times of state 2 are exponentially distributed, i.e. $T_2 \sim \text{Exp}(\bar{\tau}_2)$, the solution of formula (5.4) is given by*

$$\tilde{m}_{01}^{(n)} = \left(\frac{\mu_2 \rho_*}{\bar{\tau}_2} \right)^n \prod_{i=1}^n \frac{i \tau_1^{-1}((i-1)\mu_1)}{\tau_1^{-1}(i\mu_1) \tau_2^{-1}(i\mu_2) - 1}, \tag{5.5}$$

and therefore

$$\tilde{m}_{02}^{(n)} = \left(\frac{\mu_2 \rho_*}{\bar{\tau}_2} \right)^n \tau_1^{-1}(n\mu_1) \prod_{i=1}^n \frac{i \tau_1^{-1}((i-1)\mu_1)}{\tau_1^{-1}(i\mu_1) \tau_2^{-1}(i\mu_2) - 1}. \tag{5.6}$$

Finally

$$m_{0k}^{(n)} = \sum_{i=0}^n \binom{n}{i} \rho_1^{n-i} \tilde{m}_{0k}^{(i)}, \quad k = 1, 2. \tag{5.7}$$

Proof Substituting $\tau_2^{-1}(s) = 1 + s/\bar{\tau}_2$ in (5.4) and rearranging it, we get

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \rho_*^{n-i} \left(n \frac{\mu_2}{\bar{\tau}_2} - (\tau_1(i\mu_1) - 1) \right) \frac{\tilde{m}_{01}^{(i)}}{\tau_1(i\mu_1)} = 0. \tag{5.8}$$

By applying Lemma A.1 in the scalar case, with $\mathbf{U} = (\mu_2/\bar{\tau}_2)$, $\mathbf{D} = (\rho_*)$, $\mathbf{V}_n = (\tau_1(n\mu_1) - 1)$ and $\tilde{v}^{(n)} = (\tilde{m}_{01}^{(n)}/\tau_1(n\mu_1))$, we notice that a set of solutions is given by

$$\left(\tau_2^{-1}(n\mu_2) - \tau_1(n\mu_1) \right) \frac{\tilde{m}_{01}^{(n)}}{\tau_1(n\mu_1)} = n \rho_* \frac{\mu_2}{\bar{\tau}_2} \frac{\tilde{m}_{01}^{(n-1)}}{\tau_1((n-1)\mu_1)}, \tag{5.9}$$

that is then uniquely defined given that $\tilde{m}_{01}^{(0)} = 1$. Therefore (5.5) holds. Equation (5.6) results by applying (5.3) to (5.5) and finally (5.7) comes from the fact that $m_{0k}^{(n)} = \mathbb{E}[(|A_{0k}| - \rho_1) + \rho_1]^n$. \square

Example 5.2 (Case $T_1 \sim \text{Exp}(\bar{\tau}_1)$) In this special case (5.5) simplifies in

$$\tilde{m}_{01}^{(n)} = \rho_*^n \frac{(\bar{\tau}_1/\mu_1)^{\bar{n}}}{(\bar{\tau}_1/\mu_1 + \bar{\tau}_2/\mu_2 + 1)^{\bar{n}}},$$

with $i^{\bar{n}} := i(i+1) \cdots (i+n-1)$ being the rising factorial (in [1] it is denoted by $(i)_n$). Therefore the moment generating function of $|A_{01}| - \rho_1$ is given by the Kummer function $M(\bar{\tau}_1/\mu_1, \bar{\tau}_1/\mu_1 + \bar{\tau}_2/\mu_2 + 1, \rho_*s)$ (see [1]), in accordance to what is shown in [2, 5] (in there, it is denoted by $\phi_{\text{ON}}(-s)$).

The following example is a new result that generalizes the one of Example 5.2.

Example 5.3 (Case $T_1 \sim \text{Erlang}(\kappa, \bar{\tau}_1^{-1})$) For this case we have that $\tau_1^{-1}(k\mu_1) = (1 + k\mu_1/\bar{\tau}_1)^\kappa$. Therefore (5.5) simplifies in

$$\tilde{m}_{01}^{(n)} = \frac{n! \rho_*^n [(\bar{\tau}_1/\mu_1)^{\bar{n}}]^\kappa}{\prod_{i=1}^n [(\bar{\tau}_1/\mu_1 + i)^\kappa (\bar{\tau}_2/\mu_2 + i) - (\bar{\tau}_1/\mu_1)^\kappa (\bar{\tau}_2/\mu_2)]}.$$

6 Conclusions

In this paper we showed that using a matrix-geometric approach it is possible to solve the problem to find the factorial moments of the random number of customers in an M/M/∞ system when its parameters are modulated by a semi-Markovian random environment. We showed that this is possible by looking at this random variable as the random measure of a bidimensional random set by a mixed Poisson process. Finally the case when the environment has only 2 states is more deeply investigated and it is shown that explicit formulas are obtainable given that one state has exponential sojourn times. It is then plausible to believe that for this last case it would be possible to get an explicit expression for the complete characteristic function.

Appendix: Technical lemmas

Lemma A.1 *Given the matrices $\mathbf{U}, \mathbf{D}, \{\mathbf{V}_n\}_{n \geq 0}$ such that for any $0 < n \leq l$ the matrix $(n\mathbf{U} - \mathbf{V}_n)$ is invertible then the system of equations*

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \mathbf{D}^{n-i} [n\mathbf{U} - \mathbf{V}_i] \vec{v}^{(i)} = 0, \quad 0 < n \leq l \tag{A.1}$$

has a family of solutions $\vec{v}^{(n)}, 0 < n \leq l$, given by

$$(n\mathbf{U} - \mathbf{V}_n) \vec{v}^{(n)} = n \mathbf{D} \mathbf{U} \vec{v}^{(n-1)}, \tag{A.2}$$

that reduces to a unique solution once given the vector $\vec{v}^{(0)}$ that has to satisfy the relation $\mathbf{V}_0 \vec{v}^{(0)} = 0$.

Proof The proof of the lemma is immediate once we prove that a set of vectors obeying to relations (A.2) for $n \leq l$ with $\mathbf{V}_0 \vec{v}^{(0)} = 0$ satisfies as well the following equation for $0 < n \leq l$

$$\sum_{i=0}^{n-1} (-1)^{l-1-i} \binom{l}{i} \mathbf{D}^{l-i} [l\mathbf{U} - \mathbf{V}_i] \vec{v}^{(i)} = (-1)^{l-n} \binom{l}{n} \mathbf{D}^{l-n} [n\mathbf{U} - \mathbf{V}_n] \vec{v}^{(n)}. \tag{A.3}$$

We prove it by induction. Assuming $n = 1$, we have that

$$\begin{aligned} (-1)^{l-1} \mathbf{D}^l [l\mathbf{U} - \mathbf{V}_0] \vec{v}^{(0)} &= (-1)^{l-1} \mathbf{D}^l [l\mathbf{U}] \vec{v}^{(0)} \\ &= (-1)^{l-1} l \mathbf{D}^{l-1} \mathbf{D} \mathbf{U} \vec{v}^{(0)} \\ &= (-1)^{l-1} \binom{l}{1} \mathbf{D}^{l-1} (\mathbf{U} - \mathbf{V}_1) \vec{v}^{(1)}, \end{aligned}$$

so that the base of the induction holds. Now assuming (A.3) valid for $n < l$ we have for $n + 1$ that

$$\begin{aligned} &\sum_{i=0}^n (-1)^{l-1-i} \binom{l}{i} \mathbf{D}^{l-i} [l\mathbf{U} - \mathbf{V}_i] \vec{v}^{(i)} \\ &= (-1)^{l-n} \binom{l}{n} \mathbf{D}^{l-n} [n\mathbf{U} - \mathbf{V}_n] \vec{v}^{(n)} + (-1)^{l-1-n} \binom{l}{n} \mathbf{D}^{l-n} [l\mathbf{U} - \mathbf{V}_n] \vec{v}^{(n)} \\ &= (-1)^{l-1-n} \binom{l}{n} \mathbf{D}^{l-n} (l - n) \mathbf{U} \vec{v}^{(n)} \\ &= (-1)^{l-1-n} \binom{l}{n} \mathbf{D}^{l-n-1} (l - n) \mathbf{D} \mathbf{U} \vec{v}^{(n)} \end{aligned}$$

and by (A.2),

$$= (-1)^{l-1-n} \binom{l}{n} \frac{l-n}{n+1} \mathbf{D}^{l-n-1} ((n+1)\mathbf{U} - \mathbf{V}_{n+1}) \vec{v}^{(n+1)}$$

$$= (-1)^{l-(n+1)} \binom{l}{n+1} \mathbf{D}^{l-(n+1)} ((n+1)\mathbf{U} - \mathbf{V}_{n+1}) \vec{v}^{(n+1)}. \quad \square$$

The proof of the next result follows closely the one of Lemma B.1 in [12].

Lemma A.2 *The matrix \mathbf{B}_n , $n > 0$ in (4.6) is invertible.*

Proof By noticing that all the diagonal entries of the matrix $\mathbf{diag}(\tau_k^{-1}(n\mu_k))$ are strictly positive, in order to prove the invertibility of the matrix \mathbf{B}_n we are left with proving the non-singularity of the matrix $(\mathbf{I} - \mathbf{D}_n)$, with

$$\mathbf{D}_n := \mathbf{diag}(\tau_k(n\mu_k))\mathbf{Q}.$$

We assume with no loss of generality that the states of the random environment are ordered according to the increasing values of β_k , so that the states with null betas have the lowest indexes. Being $\beta > 0$ we know that they are in number $K_0 < K$. If we compute the l th power of the matrix \mathbf{D}_n we get that in the limit it converges elementwise to

$$\mathbf{D}_n^l \rightarrow \begin{pmatrix} \vec{1} \vec{\pi}_0^\top & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{as } l \rightarrow \infty, \tag{A.4}$$

where $\vec{\pi}_0$ is the vector containing only the first K_0 coordinates of the vector $\vec{\pi}$ and $\vec{1}$ is defined as a vector with all coordinates equal to 1 and whose dimension depends on the context.

The above result comes from the fact that $\mathbf{Q}^l \rightarrow \vec{1} \vec{\pi}^\top$ as $l \rightarrow \infty$ and from knowing that $\tau_k(n\mu_k) = 1$ when $k \leq K_0$ and $\tau_k^l(n\mu_k) \rightarrow 0$ when $K_0 < k \leq K$ as $l \rightarrow \infty$. For any $l > 0$ the following relation is valid

$$(\mathbf{I} - \mathbf{D}_n)(\mathbf{I} + \mathbf{D}_n + \dots + \mathbf{D}_n^{l-1}) = \mathbf{I} - \mathbf{D}_n^l,$$

and the matrix on the right side converges to

$$\mathbf{I} - \mathbf{D}_n^l \rightarrow \begin{pmatrix} \mathbf{I} - \vec{1} \vec{\pi}_0^\top & 0 \\ 0 & \mathbf{I} \end{pmatrix} \quad \text{as } l \rightarrow \infty.$$

The determinant of the limit matrix in the right side of the last relation is equal to the determinant of the matrix $(\mathbf{I} - \vec{1} \vec{\pi}_0^\top)$ that is positive by applying Lemma B.1 in [12] to the strictly substochastic matrix $\vec{1} \vec{\pi}_0^\top$.

Following the reasoning in Lemma B.1 in [12], as the determinant is a continuous function in the space of matrices with elementwise convergence we get that there exists some $l > 0$ such that the determinant of the matrix $(\mathbf{I} - \mathbf{D}_n^l)$ is positive. Therefore it follows that the product of the determinants of the two matrix factors in the left side of (A.4) has to be positive, which concludes the proof. \square

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