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Asymptotic analysis for loss probability of queues with finite $GI/M/1$ type structure

Jeongsim Kim · Bara Kim

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Abstract This paper discusses the asymptotic behavior of the loss probability for general queues with finite $GI/M/1$ type structure such as $GI/M/c/K$, $SM/M/1/K$ and $GI/MSP/1/K$ queues. We find an explicit expression for the asymptotic behavior of the loss probability as K tends to infinity. With the result, it is shown that the loss probability tends to 0 at a geometric rate.

Keywords $GI/M/1$ type Markov chain · Stationary distribution · Loss probability · Maximal eigenvalue · Matrix-valued function

Mathematics Subject Classification (2000) 60K25 · 68M20

1 Introduction

A Markov chain with state space $\{(0,j): 1 \le j \le m_0\} \cup \{(i,j): 1 \le i \le K, 1 \le j \le m\}$ is said to be finite $GI/M/1$ type if it has a transition probability matrix (TPM) of the form

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J. Kim
Department of Mathematics Education, Chungbuk National University, 12, Gaeshin-dong, Heungduk-ku, Cheongju, Chungbuk, 361-763, Republic of Korea
e-mail: jeongsimkim@chungbuk.ac.kr

B. Kim (✉)
Department of Mathematics and Telecommunication Mathematics Research Center, Korea University, 1, Anam-dong, Sungbuk-ku, Seoul, 136-701, Republic of Korea
e-mail: bara@korea.ac.kr

$$P^{(K)} = \begin{bmatrix} B_0 & C & O & O & \cdots & O \\ B_1 & A_1 & A_0 & O & \cdots & O \\ B_2 & A_2 & A_1 & A_0 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ B_{K-1} & A_{K-1} & A_{K-2} & \cdots & A_1 & A_0 \\ B_{K-1} & A_{K-1} & A_{K-2} & \cdots & A_1 & A_0 \end{bmatrix}, \qquad (1)$$

where $A_i, B_i, 0 \le i \le K-1$ and C are matrices. The matrices A_i, $0 \le i \le K-1$, are $m \times m$, the matrices B_i, $1 \le i \le K-1$, are $m \times m_0$, the matrix B_0 is $m_0 \times m_0$, and the matrix C is $m_0 \times m$. For each state (i,j), the first component i is called level and the second component j is called phase. Embedded Markov chains in several queueing models such as $GI/M/c/K$, $GI/MSP/1/K$, $SM/M/1/K$ queues, have the above finite $GI/M/1$ type structure. We abbreviate MSP and SM with Markovian service process and semi-Markov.

Let $\boldsymbol{\pi}^{(K)} = (\boldsymbol{\pi}_0^{(K)}, \dots, \boldsymbol{\pi}_K^{(K)})$, with $\boldsymbol{\pi}_0^{(K)} = (\pi_{01}^{(K)}, \pi_{02}^{(K)}, \dots, \pi_{0m_0}^{(K)})$ and $\boldsymbol{\pi}_i^{(K)} = (\pi_{i1}^{(K)}, \pi_{i2}^{(K)}, \dots, \pi_{im}^{(K)})$, $1 \le i \le K$, be a stationary distribution of $P^{(K)}$. That is, $\boldsymbol{\pi}^{(K)}$ is a solution of the linear system

$$\boldsymbol{\pi}^{(K)} P^{(K)} = \boldsymbol{\pi}^{(K)}, \qquad \sum_{i=0}^{K} \boldsymbol{\pi}_i^{(K)} \mathbf{1} = 1.$$

Here and subsequently, $\mathbf{1}$ denotes a column vector of appropriate size all of whose components are equal to 1. As we see in several examples in Sect. 5, the loss probability of specific queues with finite $GI/M/1$ type structure is expressed in terms of $\boldsymbol{\pi}_K^{(K)}$. Thus the asymptotic behavior of the loss probability is uncovered if we investigate the asymptotic behavior of $\boldsymbol{\pi}_K^{(K)}$. The main objective in this paper is to do this.

It is not a difficult task to show that $\boldsymbol{\pi}_K^{(K)}$ satisfies

$$\boldsymbol{\pi}_K^{(K)} = c^{(K)} \sigma^K (\boldsymbol{g} + o(1)), \quad \text{as } K \to \infty$$

with $c^{(K)}$ such that

$$\lim_{K\to\infty}\frac{1}{K}\log c^{(K)}=0.$$

However, it turns out that obtaining the exact value of $\lim_{K\to\infty} c^{(K)}$ is challenging, meaningful and interesting in our judgment.

Throughout the paper, $o(\cdot)$ may denote a scalar-, a vector-, or a matrix-valued function. When $o(\cdot)$ indicates a vector- or a matrix-valued function, it should be interpreted as componentwise.

In the main theorem of this paper, we find the following asymptotic behavior of $\boldsymbol{\pi}_K^{(K)}$ (see Theorem 1):

$$\boldsymbol{\pi}_K^{(K)}=c\sigma^K\boldsymbol{g}+o(\sigma^K),\quad \text{as } K\to\infty, \tag{2}$$

where $c>0$, $0<\sigma<1$, and $\boldsymbol{g}$ is an m-dimensional probability vector.

From the result (2), we can use the value $c\sigma^K\boldsymbol{g}$ as an approximation of $\boldsymbol{\pi}_K^{(K)}$. Our approximation may reduce the computational complexity drastically for the following reason: once we compute c, σ and $\boldsymbol{g}$, we can obtain $c\sigma^K\boldsymbol{g}$ immediately for various values of K, while the exact value $\boldsymbol{\pi}_K^{(K)}$ must be computed again whenever K is changed.

For finite buffer queues, the loss behavior has been one of the important topics of research in telecommunication systems and should be estimated accurately for buffer dimensioning. Many researchers have analyzed asymptotic loss probability for various queueing systems with finite buffer. Baiocchi found the asymptotic loss probability of the $M/G/1/K$ queue [1] and the $MAP/G/1/K$ queue [2]. Baiocchi and Blefari-Melazzi [3] studied approximations of the loss probability of the $MAP/G/1/K$ queue by the technique in [2]. Approximations for the loss probability in the $M/G/c/K$ queue are proposed in Miyazawa [13]. Baiocchi [1] and Choi et al. [5] gave the asymptotic loss probability of the $GI/M/1/K$ queue. Recently, Miyazawa et al. [14] obtained the asymptotic behavior of the loss probability for a finite queue with QBD structure.

In addition, there are several works on analysis of asymptotic loss probability of queues with batch arrivals. A simple approximation for the loss probability was proposed by Tijms [19] for the $GI^X/G/c/K+c$ queue under the partial rejection strategy. Gouweleeuw [8] extended the approximation in [19] to the $GI^X/G/c/K+c$ queue under the complete rejection strategy. Kim and Choi [11] studied the asymptotic loss probability for the $GI^X/M/c/K$ queue under the partial and complete rejection strategies. In another paper by Gouweleeuw [9] an approximation was suggested to evaluate the loss probability in a single server queue where the arrival process is a batch Markovian arrival process.

This paper is organized as follows. In Sect. 2, we present without proof the asymptotic behavior of $\boldsymbol{\pi}_K^{(K)}$ as K tends to infinity. Section 3 is devoted to the derivation of useful analytical properties which will be exploited in Sect. 4, where we give the proof of the asymptotic behavior of $\boldsymbol{\pi}_K^{(K)}$, stated without proof in Sect. 2. In Sect. 5, we apply the asymptotic behavior of $\boldsymbol{\pi}_K^{(K)}$ to several queueing models with finite $GI/M/1$ type structure to find the asymptotic behavior of the loss probability.

2 Asymptotic behavior of $\boldsymbol{\pi}_K^{(K)}$ as $K\to\infty$

Consider a Markov chain of finite $GI/M/1$ type with TPM (1). We assume that the Markov chain is irreducible for sufficiently large K. We also assume that the matrix $A\equiv\sum_{i=0}^{\infty}A_i$ is irreducible. Here A is substochastic but not necessarily stochastic. When A is stochastic, $\boldsymbol{\alpha}$ denotes the stationary probability vector of A, that is, $\boldsymbol{\alpha}$ is the unique vector satisfying $\boldsymbol{\alpha}A=\boldsymbol{\alpha}$, $\boldsymbol{\alpha}\mathbf{1}=1$. When A is stochastic, we assume that

$$\rho^{-1}\equiv\boldsymbol{\alpha}\sum_{i=1}^{\infty}iA_i\mathbf{1}>1.$$

Under the conditions that $P^{(K)}$ is irreducible for sufficiently large K and that A is irreducible, it is not difficult to see that the following are equivalent (see Neuts [16, 17]):

(i) A is not stochastic, or A is stochastic and $\boldsymbol{\alpha}\sum_{i=1}^{\infty}iA_i\mathbf{1}>1$.

(ii) $0<\mathrm{sp}(G)<1$, where G is the minimal nonnegative solution of the non-linear matrix equation

$$G=\sum_{i=0}^{\infty}A_iG^i. \tag{3}$$

Here and subsequently, $\mathrm{sp}(\cdot)$ denotes the spectral radius.

(iii) $0<\mathrm{sp}(R)<1$, where R is the minimal nonnegative solution of the non-linear matrix equation

$$R=\sum_{i=0}^{\infty}R^iA_i. \tag{4}$$

(iv) The equation

$$\chi(z)=z,\quad 0<z<1 \tag{5}$$

has a unique solution, where $\chi(z)$ is the maximal eigenvalue of the matrix-valued function $A(z)\equiv\sum_{n=0}^{\infty}A_nz^n$, i.e., $\chi(z)\equiv\mathrm{sp}(A(z))$.

Let σ denote the unique solution in $(0,1)$ of (5). It is well known that both $\mathrm{sp}(G)$ and $\mathrm{sp}(R)$ are equal to σ, that is, σ is the maximal eigenvalue of G and R. Let $\boldsymbol{g}$ and $\boldsymbol{\xi}$ be respectively left and right maximal eigenvectors of G, scaled by

$$\boldsymbol{g}\mathbf{1}=\boldsymbol{g}\boldsymbol{\xi}=1.$$

Let $\boldsymbol{\eta}$ and $\boldsymbol{r}$ be respectively left and right maximal eigenvectors of R, scaled by

$$\boldsymbol{\eta}\boldsymbol{\xi} = \boldsymbol{\eta}\boldsymbol{r} = 1.$$

To describe our main result, let us introduce the corresponding infinite $GI/M/1$ type Markov chain whose TPM $P^{(\infty)}$ is given by

$$P^{(\infty)} = \begin{bmatrix} B_0 & C & O & O & O & \cdots \\ B_1 & A_1 & A_0 & O & O & \cdots \\ B_2 & A_2 & A_1 & A_0 & O & \cdots \\ B_3 & A_3 & A_2 & A_1 & A_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6}$$

From the assumption that A is not stochastic, or A is stochastic and $\boldsymbol{\alpha}\sum_{i=1}^{\infty} iA_i\mathbf{1} > 1$, it is well known that the above infinite $GI/M/1$ type Markov chain has the unique stationary distribution $\boldsymbol{\pi}^{(\infty)} = (\boldsymbol{\pi}_0^{(\infty)}, \boldsymbol{\pi}_1^{(\infty)}, \ldots)$, with $\boldsymbol{\pi}_0^{(\infty)} = (\pi_{01}^{(\infty)}, \ldots, \pi_{0m_0}^{(\infty)})$ and $\boldsymbol{\pi}_i^{(\infty)} = (\pi_{i1}^{(\infty)}, \ldots, \pi_{im}^{(\infty)})$, $i \geq 1$.

Now we are ready to present our main result. We defer the proof to Sect. 4.

Theorem 1 *The probability vector $\boldsymbol{\pi}_K^{(K)}$ satisfies*

$$\boldsymbol{\pi}_K^{(K)} = c\sigma^K \boldsymbol{g} + o(\sigma^K), \quad \text{as } K \to \infty,$$

with

$$c = \frac{\boldsymbol{\pi}_1^{(\infty)} \boldsymbol{r}(1 - \chi'(\sigma))}{\sigma(1-\sigma)},$$

where $\chi'(\sigma) = \frac{d}{dz}\chi(z)|_{z=\sigma}$.

Remark Theorem 1 may be generalized to the Markov chain with a TPM that has different transition probabilities for the boundary level K. However, the generalization seems to require much more effort than the case of the Markov chain with TPM (1). Further Theorem 1 is sufficient for the study of the asymptotic behavior of queueing models represented as Markov chains of finite $GI/M/1$ type.

When $m_0 = m$, and $C = A_0$, it is known that $\boldsymbol{\pi}_1^{(\infty)} = \boldsymbol{\pi}_0^{(\infty)} R$, see Chap. 1 of Neuts [16]. Since $R\boldsymbol{r} = \sigma\boldsymbol{r}$, the next corollary is immediate from Theorem 1.

Corollary 1 *If $m_0 = m$, and $C = A_0$, then*

$$\boldsymbol{\pi}_K^{(K)} = c_1\sigma^K \boldsymbol{g} + o(\sigma^K), \quad \text{as } K \to \infty,$$

with

$$c_1 = \frac{\pi_0^{(\infty)} \boldsymbol{r}(1 - \chi'(\sigma))}{(1-\sigma)}.$$

3 A transient Markov chain of $M/G/1$ type

We consider an $M/G/1$ type Markov chain with TPM $\widetilde{P}$ given by

$$\widetilde{P} = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 & \cdots \\ A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{7}$$

The reason for considering the Markov chain with TPM $\widetilde{P}$ is as follows. Relabeling the level of the Markov chain with TPM (1), we obtain a Markov chain with TPM $\widetilde{P}^{(K)}$

$$\widetilde{P}^{(K)} = \begin{bmatrix} A_0 & A_1 & \cdots & A_{K-2} & A_{K-1} & B_{K-1} \\ A_0 & A_1 & \cdots & A_{K-2} & A_{K-1} & B_{K-1} \\ & \ddots & \ddots & \vdots & \vdots & \vdots \\ O & \cdots & A_0 & A_1 & A_2 & B_2 \\ O & \cdots & O & A_0 & A_1 & B_1 \\ O & \cdots & O & O & C & B_0 \end{bmatrix}. \tag{8}$$

Letting $K \to \infty$ in (8) yields, formally, $\widetilde{P}$ in (7). Therefore, in order to obtain the asymptotic behavior of the stationary distribution for the Markov chain with TPM $P^{(K)}$ it seems natural to consider the Markov chain with TPM $\widetilde{P}$ first. Actually, the results on the Markov chain with TPM $\widetilde{P}$ in this section are essential for the proof of Theorem 1 in Sect. 4.

Every state of the Markov chain with TPM (7) is transient because A is not stochastic or A is stochastic and $\boldsymbol{\alpha}\sum_{i=1}^{\infty} iA_i\mathbf{1} > 1$. Further, if the Markov chain has a stationary measure $\widetilde{\boldsymbol{\pi}} = (\widetilde{\boldsymbol{\pi}}_0, \widetilde{\boldsymbol{\pi}}_1, \ldots)$, where $\widetilde{\boldsymbol{\pi}}_n$ is an m-dimensional vector for $n = 0, 1, 2, \ldots$, then we have

$$\widetilde{\boldsymbol{\pi}}_n = \widetilde{\boldsymbol{\pi}}_0 A_n + \sum_{i=1}^{n+1} \widetilde{\boldsymbol{\pi}}_i A_{n-i+1}, \quad n = 0, 1, 2 \ldots. \tag{9}$$

By (9), the 'formal' generating function $\widetilde{\Pi}(z) \equiv \sum_{n=0}^{\infty} \widetilde{\boldsymbol{\pi}}_n z^n$ satisfies

$$z\widetilde{\Pi}(z) = \widetilde{\Pi}(z)A(z) + \widetilde{\boldsymbol{\pi}}_0(z-1)A(z).$$

From this we obtain the following 'formal' equation:

$$\widetilde{\Pi}(z) = \widetilde{\boldsymbol{\pi}}_0 \Phi(z),$$

where

$$\Phi(z) \equiv (z-1)z(zI - A(z))^{-1} - (z-1)I$$

with $m \times m$ identity matrix I.

Now we investigate the matrix-valued function $\Phi(z)$ on $\{z \in \mathbb{C} : |z| < 1, \det(zI - A(z)) \neq 0\}$. First we study the factor $(zI - A(z))^{-1}$ in $\Phi(z)$.

Proposition 1 *For* $|z| < 1$, *we have*

$$zI - A(z) = H_G(z)(zI - G), \tag{10}$$

where $H_G(z) \equiv I - \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} A_k G^{k-1-i} z^i$.

Proof Observe that

$$H_G(z)(zI - G) = zI - G - \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} A_k G^{k-i} z^i + \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} A_k G^{k-i} z^i.$$

Hence we may immediately write

$$\begin{aligned} H_G(z)(zI - G) &= zI - G - \sum_{i=1}^{\infty} \left(\sum_{k=i}^{\infty} A_k G^{k-i} - \sum_{k=i+1}^{\infty} A_k G^{k-i} \right) z^i + \sum_{k=1}^{\infty} A_k G^k \\ &= zI - G - \sum_{i=1}^{\infty} A_i z^i + \sum_{k=1}^{\infty} A_k G^k, \end{aligned}$$

so (10) follows from (3). □

Proposition 2 *The matrix-valued function* $H_G(z)$ *is nonsingular on* $\{z \in \mathbb{C} : |z| < 1\}$.

Proof By Theorems 3 and 4 in Gail et al. [7], $\det(zI - A(z))$ has m zeros (counting multiplicities) in the open unit disk. Since $\mathrm{sp}(G) < 1$, $\det(zI - G)$ has m zeros (counting multiplicities) in the open unit disk. Therefore $\det(H_G(z))$ has no zeros in the open unit disk. □

Let $z_1, z_2, \ldots, z_m$ (not necessarily distinct) be the zeros of $\det(zI - A(z))$ on $\{z \in \mathbb{C} : |z| < 1\}$, with $z_1 = \sigma$. Then σ is a simple zero of $\det(zI - A(z))$ and $|z_i| < \sigma$ for $i = 2, 3, \ldots, m$.

Consider the following Laurent series expansion of $\Phi(z)$ at $z = 0$:

$$\Phi(z) = \sum_{i=-v}^{\infty} \Phi_i z^i, \quad 0 < |z| < \min\{|z_i| : z_i \neq 0, i = 1, \ldots, m\},$$

where $v \equiv \max_{1 \le i,j \le m}\{$the order of pole of the (i, j)-component of $\Phi(z)$ at $z = 0\}$. In the case where a function has a removable singularity at a point, we say that the function is analytic at the point, by abuse of terminology. When a function is analytic at a point, we define the order of pole at the point as 0.

Proposition 3 *Suppose that* $\boldsymbol{x}$ *is an* m*-dimensional row vector such that* $\boldsymbol{x}(zI - A(z))^{-1}$ *is analytic at* $z = 0$. *Let* $\boldsymbol{y}_i \equiv \boldsymbol{x}\Phi_i$, $i = 0, 1, 2, \ldots$, *be an* m*-dimensional row vector. Then*

(i) $\boldsymbol{y}_0 = \boldsymbol{x}$,
(ii) $\boldsymbol{y}_0 A_i + \sum_{k=1}^{i+1} \boldsymbol{y}_k A_{i+1-k} = \boldsymbol{y}_i$, $i = 0, 1, 2, \ldots$.

Proof (i) We have

$$\boldsymbol{x}\Phi(z) = (z-1)z\boldsymbol{x}(zI - A(z))^{-1} - (z-1)\boldsymbol{x}, \quad \text{for } |z| < 1,\ z \neq z_1, \ldots, z_m.$$

Since $\boldsymbol{x}(zI - A(z))^{-1}$ is analytic in $z = 0$, so is $\boldsymbol{x}\Phi(z)$. Letting $z \to 0$ leads to $\boldsymbol{x}\Phi_0 = \boldsymbol{x}$.

(ii) We have

$$\boldsymbol{x}\Phi(z)(zI - A(z)) = (z-1)z\boldsymbol{x} - (z-1)\boldsymbol{x}(zI - A(z)), \quad |z| < 1,$$

so

$$z\boldsymbol{x}\Phi(z) = \boldsymbol{x}\Phi(z)A(z) + (z-1)\boldsymbol{x}A(z), \quad |z| < 1.$$

Hence

$$\begin{aligned} z\sum_{i=0}^{\infty} \boldsymbol{y}_i z^i &= \left(\sum_{i=0}^{\infty} \boldsymbol{y}_i z^i\right)\left(\sum_{i=0}^{\infty} A_i z^i\right) + z\boldsymbol{y}_0 \sum_{i=0}^{\infty} A_i z^i - \boldsymbol{y}_0 \sum_{i=0}^{\infty} A_i z^i \\ &= z\left(\sum_{i=0}^{\infty} \boldsymbol{y}_{i+1} z^i\right)\left(\sum_{i=0}^{\infty} A_i z^i\right) + z\boldsymbol{y}_0 \sum_{i=0}^{\infty} A_i z^i, \quad |z| < 1. \end{aligned}$$

Dividing both sides by z yields

$$\sum_{i=0}^{\infty} \boldsymbol{y}_i z^i = \left(\sum_{i=0}^{\infty} \boldsymbol{y}_{i+1} z^i\right)\left(\sum_{i=0}^{\infty} A_i z^i\right) + \boldsymbol{y}_0 \sum_{i=0}^{\infty} A_i z^i, \quad |z| < 1. \tag{11}$$

Comparing the coefficients of z^i in (11), we obtain the assertion (ii). □

Recall that the solution σ of (5) is the maximal eigenvalue of G and that $\boldsymbol{g}$ and $\boldsymbol{\xi}$ are respectively left and right maximal eigenvectors of G scaled by $\boldsymbol{g}\mathbf{1} = \boldsymbol{g}\boldsymbol{\xi} = 1$. The solution σ of (5) is also the maximal eigenvalue of R, and $\boldsymbol{\eta}$ and $\boldsymbol{r}$ are respectively left and right maximal eigenvectors of R scaled by $\boldsymbol{\eta}\boldsymbol{\xi} = \boldsymbol{\eta}\boldsymbol{r} = 1$.

Proposition 4 *We have that*

(i) $\boldsymbol{g}(zI-A(z))^{-1}$ *is analytic at* $z=0$,
(ii) $\boldsymbol{g}\Phi_n=\frac{1-\sigma}{(1-\chi'(\sigma))}\sigma^{-n}\boldsymbol{\eta}+o(\sigma^{-n})$, *as* $n\to\infty$.

Proof (i) Observe that

$$\boldsymbol{g}(zI-G)=(z-\sigma)\boldsymbol{g},$$

so

$$\boldsymbol{g}(zI-G)^{-1}=\frac{1}{z-\sigma}\boldsymbol{g},\quad \text{for } z\neq z_1,\dots,z_m.$$

Since $\boldsymbol{g}(zI-A(z))^{-1}=\boldsymbol{g}(zI-G)^{-1}H_G(z)^{-1}=\frac{1}{z-\sigma}\boldsymbol{g}H_G(z)^{-1}$, for $|z|<1, z\neq z_1,\dots,z_m$, and $H_G(z)$ is analytic in the unit disk, $\boldsymbol{g}(zI-A(z))^{-1}$ is analytic at $z=0$.

(ii) Observe that

$$\begin{aligned}\boldsymbol{g}\Phi(z)&=(z-1)z\boldsymbol{g}(zI-A(z))^{-1}-(z-1)\boldsymbol{g}\\&=\frac{(z-1)z}{z-\sigma}\boldsymbol{g}H_G(z)^{-1}-(z-1)\boldsymbol{g}\\&=\frac{(\sigma-1)\sigma}{z-\sigma}\boldsymbol{g}H_G(\sigma)^{-1}+\Psi(z),\end{aligned}$$

$$\text{for } |z|<1, z\neq z_1,\dots,z_m, \tag{12}$$

where $\Psi(z)=\frac{(z-1)z}{z-\sigma}\boldsymbol{g}H_G(z)^{-1}-\frac{(\sigma-1)\sigma}{z-\sigma}\boldsymbol{g}H_G(\sigma)^{-1}-(z-1)\boldsymbol{g}$. Note that $\Psi(z)$ is analytic in the unit disk (except the removable singularity at $z=\sigma$). Let $\Psi(z)=\sum_{i=0}^{\infty}\Psi_i z^i$ be the power series expansion at $z=0$. Then $\Psi_i=o((1+\epsilon)^i)$, as $i\to\infty$, for any $\epsilon>0$.

From (12) it follows that

$$\boldsymbol{g}\sum_{i=0}^{\infty}\Phi_i z^i=(1-\sigma)\sum_{i=0}^{\infty}\sigma^{-i}\boldsymbol{g}H_G(\sigma)^{-1}z^i+\sum_{i=0}^{\infty}\Psi_i z^i,\quad |z|<\sigma.$$

Comparing the coefficients of z^i on the both sides of the above equation, we have

$$\boldsymbol{g}\Phi_i=(1-\sigma)\sigma^{-i}\boldsymbol{g}H_G(\sigma)^{-1}+\Psi_i,\quad i=0,1,\dots,$$

so

$$\boldsymbol{g}\Phi_i=(1-\sigma)\sigma^{-i}\boldsymbol{g}H_G(\sigma)^{-1}+o(\sigma^{-i}),\quad \text{as } i\to\infty. \tag{13}$$

By (3) and (4), it can be shown that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are respectively right and left maximal eigenvectors of $A(\sigma)=\sum_{i=0}^{\infty}A_i\sigma^i$. By Lemma 5.8 in Falkenberg [6], we can see that $\frac{1}{z-\sigma}\frac{1}{1-\chi'(\sigma)}\boldsymbol{\xi\eta}$ is the principal part of $(zI-A(z))^{-1}$. Hence

$$\begin{aligned}\boldsymbol{g}H_G(\sigma)^{-1}&=\lim_{z\to\sigma}\boldsymbol{g}(zI-G)(zI-A(z))^{-1}\\&=\lim_{z\to\sigma}(z-\sigma)\boldsymbol{g}\left[\frac{1}{z-\sigma}\frac{1}{1-\chi'(\sigma)}\boldsymbol{\xi\eta}\right.\\&\quad\left.+\left((zI-A(z))^{-1}-\frac{1}{z-\sigma}\frac{1}{1-\chi'(\sigma)}\boldsymbol{\xi\eta}\right)\right]\\&=\frac{1}{1-\chi'(\sigma)}\boldsymbol{\eta}.\end{aligned} \tag{14}$$

Substituting (14) into (13) completes the proof of (ii). □

4 Proof of Theorem 1

Before proving Theorem 1, we introduce several notations. For the notion of matrices introduced in this section, we refer the reader to Chap. 5 of Latouche and Ramaswami [12].

Let $\{(X_t^{(K)},J_t^{(K)}):t=0,1,2,\dots\}$ be the Markov chain with TPM $P^{(K)}$ in (1). For $n,p,q=0,1,\dots,K$, let ${}_nR_{pq}^{(K)}$ be the $m\times m$ matrix whose (i,j)-component is

$$({}_nR_{pq}^{(K)})_{ij}\equiv\mathbb{E}\left(\sum_{t=0}^{\tau_n^{(K)}-1}1_{\{X_t^{(K)}=q,\,J_t^{(K)}=j\}}\,\middle|\,X_0^{(K)}=p,J_0^{(K)}=i\right),$$

where

$$\tau_n^{(K)}\equiv\inf\{t\ge 1:X_t^{(K)}=n\}.$$

Let $\{(X_t,J_t):t=0,1,2,\dots\}$ be the Markov chain with TPM $P^{(\infty)}$ in (6). For $n,p,q=0,1,2,\dots$, let ${}_nF_{pq}$ and ${}_nR_{pq}$ be the $m\times m$ matrices whose (i,j)-components are, respectively, given by

$$({}_nF_{pq})_{ij}\equiv\mathbb{P}(\tau_q\le\tau_n,J_{\tau_q}=j\mid X_0=p,J_0=i),$$

$$({}_nR_{pq})_{ij}\equiv\mathbb{E}\left(\sum_{t=0}^{\tau_n-1}1_{\{X_t=q,J_t=j\}}\,\middle|\,X_0=p,J_0=i\right),$$

where

$$\tau_n\equiv\inf\{t\ge 1:X_t=n\}.$$

Let $\{(\widetilde{X}_t,\widetilde{J}_t):t=0,1,2,\dots\}$ be the Markov chain with TPM $\widetilde{P}$ in (7). For $n,p,q=0,1,2,\dots$, let ${}_n\widetilde{R}_{pq}$ be the $m\times m$ matrix whose (i,j)-component is

$$({}_n\widetilde{R}_{pq})_{ij}\equiv\mathbb{E}\left(\sum_{t=0}^{\widetilde{\tau}_n-1}1_{\{\widetilde{X}_t=q,\,\widetilde{J}_t=j\}}\,\middle|\,\widetilde{X}_0=p,\widetilde{J}_0=i\right),$$

where

$$\tilde{\tau}_n \equiv \inf\{t \geq 1 : \tilde{X}_t = n\}.$$

To prove Theorem 1, we need the following lemma.

Lemma 1 *There is an m-dimensional row vector $\boldsymbol{\beta}$ such that*

$${}_1F_{1K} = \sigma^K \boldsymbol{r}\boldsymbol{\beta} + o(\sigma^K), \quad \textit{as } K \to \infty.$$

Proof Let $\{(\hat{X}_t, \hat{J}_t) : t = 0, 1, 2, \ldots\}$ be a Markov chain with TPM $\hat{P}$ of the following form:

$$\hat{P} = \begin{array}{c} \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & \cdots \end{array} \\ \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & & & \\ \cdots & A_3 & A_2 & A_1 & A_0 & & & \\ & \cdots & A_3 & A_2 & A_1 & A_0 & & \\ & & \cdots & A_3 & A_2 & A_1 & A_0 & \\ & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \end{array}.$$

Let $\hat{R}_{00}$ be the $m \times m$ matrix whose (i, j)-component is

$$(\hat{R}_{00})_{ij} \equiv \mathbb{E}\left(\sum_{t=0}^{\infty} 1_{\{\hat{X}_t=0, \hat{J}_t=j\}} \,\middle|\, \hat{X}_0 = 0, \hat{J}_0 = i \right).$$

By the monotone convergence theorem, we have

$$\lim_{K\to\infty} {}_1R_{KK} = \hat{R}_{00}. \tag{15}$$

On the other hand, from the relation $R^{K-1} = {}_1R_{1K} = {}_1F_{1K}\, {}_1R_{KK}$ (see Theorem 5.2.3 in Latouche and Ramaswami [12]), we have

$${}_1F_{1K} = R^{K-1}({}_1R_{KK})^{-1}. \tag{16}$$

By (15) and (16), we have

$$\begin{aligned} {}_1F_{1K} &= R^{K-1}((\hat{R}_{00})^{-1} + o(1)) \\ &= \sigma^{K-1}\boldsymbol{r}\boldsymbol{\eta}(\hat{R}_{00})^{-1} + o(\sigma^K), \end{aligned}$$

which completes the proof. □

Now we prove Theorem 1.

Proof of Theorem 1 Let $\boldsymbol{h}_i \equiv \boldsymbol{g}\Phi_i$, $i = 0, 1, \ldots$, be an m-dimensional row vector. Then, by Propositions 3 and 4(i), we have

$$\boldsymbol{h}_0 = \boldsymbol{g}, \quad \text{and}$$

$$\boldsymbol{h}_i = \boldsymbol{h}_0 A_i + \sum_{k=1}^{i+1} \boldsymbol{h}_k A_{i+1-k}, \quad i = 0, 1, 2, \ldots.$$

From this it follows that

$$\begin{aligned} &(\boldsymbol{h}_{K-2}, \ldots, \boldsymbol{h}_0) \\ &\quad = (\boldsymbol{h}_{K-1}, \ldots, \boldsymbol{h}_0) \begin{bmatrix} A_0 & O_{m\times(K-2)m} \\ Q^{(K)} & \end{bmatrix}, \end{aligned} \tag{17}$$

where $O_{m\times(K-2)m}$ denotes the $m \times (K-2)m$ zero matrix and $Q^{(K)}$ is given by

$$Q^{(K)} \equiv \begin{bmatrix} A_1 & A_0 & O & O & \cdots & O \\ A_2 & A_1 & A_0 & O & \cdots & O \\ A_3 & A_2 & A_1 & A_0 & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ A_{K-2} & A_{K-1} & A_{K-2} & \cdots & A_1 & A_0 \\ A_{K-2} & A_{K-1} & A_{K-2} & \cdots & A_1 & A_0 \end{bmatrix}.$$

Equation (17) is rewritten as

$$\begin{aligned} &(\boldsymbol{h}_{K-2}, \ldots, \boldsymbol{h}_0) \\ &\quad = \boldsymbol{h}_{K-1}[A_0 O_{m\times(K-2)m}](I_{(K-1)m} - Q^{(K)})^{-1}, \end{aligned}$$

where $I_{(K-1)m}$ denotes the $(K-1)m \times (K-1)m$ identity matrix. The above equation implies that

$$\begin{aligned} \boldsymbol{h}_0 = \boldsymbol{h}_{K-1}&[A_0\ O_{m\times(K-2)m}](I_{(K-1)m} - Q^{(K)})^{-1} \\ &\times \begin{bmatrix} O_{(K-2)m\times m} \\ I_m \end{bmatrix}. \end{aligned} \tag{18}$$

Observe that

$$\begin{aligned} &[A_0 O_{m\times(K-2)m}](I_{(K-1)m} - Q^{(K)})^{-1} \begin{bmatrix} O_{(K-2)m\times m} \\ I_m \end{bmatrix} \\ &\quad = {}_1R_{1K}^{(K)} = {}_1F_{1K}\, {}_1R_{KK}^{(K)} = {}_1F_{1K}\, {}_{K-1}\tilde{R}_{00}. \end{aligned}$$

Substituting the above equation and $\boldsymbol{h}_0 = \boldsymbol{g}$ into (18) yields

$$\boldsymbol{g} = \boldsymbol{h}_{K-1}\, {}_1F_{1K}\, {}_{K-1}\tilde{R}_{00}. \tag{19}$$

By the monotone convergence theorem, we have

$$\lim_{K\to\infty} {}_{K-1}\tilde{R}_{00} = \tilde{R}_{00}, \tag{20}$$

where $\tilde{R}_{00}$ is the $m \times m$ matrix with (i, j)-component

$$(\tilde{R}_{00})_{ij} \equiv \mathbb{E}\left(\sum_{t=0}^{\infty} 1_{\{\tilde{X}_t=0, \tilde{J}_t=j\}} \,\middle|\, \tilde{X}_0 = 0, \tilde{J}_0 = i \right).$$

Substituting Proposition 4(ii), Lemma 1 and (20) into (19) yields

$$\begin{aligned}\boldsymbol{g} &= \left(\frac{(1-\sigma)\sigma}{(1-\chi'(\sigma))\sigma^K}\boldsymbol{\eta} + o(\sigma^{-K})\right) \\ &\quad \times (\sigma^K \boldsymbol{r}\boldsymbol{\beta} + o(\sigma^K))(\widetilde{R}_{00} + o(1)) \\ &= \frac{(1-\sigma)\sigma}{1-\chi'(\sigma)}\boldsymbol{\beta}\widetilde{R}_{00} + o(1), \quad \text{as } K \to \infty.\end{aligned}$$

Therefore

$$\boldsymbol{\beta}\widetilde{R}_{00} = \frac{1-\chi'(\sigma)}{\sigma(1-\sigma)}\boldsymbol{g}. \tag{21}$$

On the other hand, by Theorem 5.2.1 in Latouche and Ramaswami [12], we have

$$\begin{aligned}\boldsymbol{\pi}_K^{(K)} &= \boldsymbol{\pi}_1^{(K)}\, {}_1R_{1K}^{(K)} = \boldsymbol{\pi}_1^{(K)}\, {}_1F_{1K}\, {}_1R_{KK}^{(K)} \\ &= \boldsymbol{\pi}_1^{(K)}\, {}_1F_{1K}\, {}_{K-1}\widetilde{R}_{00}.\end{aligned} \tag{22}$$

By (22), Lemma 1, (20) and the fact that $\boldsymbol{\pi}_1^{(K)} = \boldsymbol{\pi}_1^{(\infty)} + o(1)$ as $K \to \infty$, we have

$$\begin{aligned}\boldsymbol{\pi}_K^{(K)} &= (\boldsymbol{\pi}_1^{(\infty)} + o(1))(\sigma^K \boldsymbol{r}\boldsymbol{\beta} + o(\sigma^K))(\widetilde{R}_{00} + o(1)) \\ &= \boldsymbol{\pi}_1^{(\infty)}\sigma^K \boldsymbol{r}\boldsymbol{\beta}\widetilde{R}_{00} + o(\sigma^K), \quad \text{as } K \to \infty.\end{aligned} \tag{23}$$

Finally, substituting (21) into (23) yields

$$\boldsymbol{\pi}_K^{(K)} = \frac{\pi_1^{(\infty)}\boldsymbol{r}(1-\chi'(\sigma))}{\sigma(1-\sigma)}\sigma^K \boldsymbol{g} + o(\sigma^K), \quad \text{as } K \to \infty,$$

which completes the proof of Theorem 1. □

5 Examples

In this section, we apply Theorem 1 to investigate the asymptotic behavior of the loss probability for some specific queueing models.

Example 1 ($GI/M/c/K$ queue) We consider the $GI/M/c/K$ queue with general interarrival time distribution $H(\cdot)$. The service times of customers are exponentially distributed with mean μ^{-1}. The system has a finite buffer of size K to store incoming customers, including any customer in service. The offered load ρ is $\rho = (c\mu \int_0^\infty t\, dH(t))^{-1}$ and is assumed to be less than 1.

The system size embedded just before the arrival epoch constitutes a Markov chain with TPM $P^{(K-c+1)}$ given by

$$P^{(K-c+1)} = \begin{array}{c} 0 \\ 1 \\ \vdots \\ c-1 \\ c \\ c+1 \\ \vdots \\ K-1 \\ K \end{array} \left[\begin{array}{ccccc|ccccc} b_0^{(0)} & b_1^{(0)} & 0 & \cdots & 0 & & & & & \\ b_0^{(1)} & b_1^{(1)} & b_2^{(1)} & \cdots & 0 & & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & & & \\ b_0^{(c-1)} & b_1^{(c-1)} & b_2^{(c-1)} & \cdots & b_{c-1}^{(c-1)} & a_0 & & & & \\ \hline b_0^{(c)} & b_1^{(c)} & b_2^{(c)} & \cdots & b_{c-1}^{(c)} & a_1 & a_0 & & & \\ b_0^{(c+1)} & b_1^{(c+1)} & b_2^{(c+1)} & \cdots & b_{c-1}^{(c+1)} & a_2 & a_1 & a_0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ b_0^{(K-1)} & b_1^{(K-1)} & b_2^{(K-1)} & \cdots & b_{c-1}^{(K-1)} & a_{K-c} & a_{K-c-1} & a_{K-c-2} & \cdots & a_0 \\ b_0^{(K-1)} & b_1^{(K-1)} & b_2^{(K-1)} & \cdots & b_{c-1}^{(K-1)} & a_{K-c} & a_{K-c-1} & a_{K-c-2} & \cdots & a_0 \end{array}\right],$$

where $a_i = \int_0^\infty e^{-c\mu t}\frac{(c\mu t)^i}{i!}\, dH(t), i \ge 0$. For explicit expressions of $b_j^{(i)}$, $0 \le i \le K-1, 0 \le j \le \min\{i+1, c\}$, see Simonot [18].

In the notation of Sects. 1 and 2 we have

$$A_i = a_i, \quad i = 0, 1, 2, \ldots,$$

$$B_0 = \begin{bmatrix} b_0^{(0)} & b_1^{(0)} & 0 & \cdots & 0 \\ b_0^{(1)} & b_1^{(1)} & b_2^{(1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_0^{(c-1)} & b_1^{(c-1)} & b_2^{(c-1)} & \cdots & b_{c-1}^{(c-1)} \end{bmatrix},$$

$$B_i = (b_0^{(i+c-1)}, b_1^{(i+c-1)}, \ldots, b_{c-1}^{(i+c-1)}), \quad i = 1, 2, \ldots,$$

$$C = (0, \ldots, 0, a_0)^\top,$$

where the superscript $\top$ stands for vector transposition. Note that $A(z) = \sum_{i=0}^\infty a_i z^i = H^*(c\mu - c\mu z)$, where $H^*(s) = \int_0^\infty e^{-st}\, dH(t)$. Further σ is the unique solution in (0, 1) satisfying $H^*(c\mu - c\mu z) = z$.

Let $\boldsymbol{x}^{(K)} = (x_0^{(K)}, \ldots, x_K^{(K)})$ be the stationary probability vector of $P^{(K-c+1)}$. In the notation of Sects. 1 and 2,

$$\boldsymbol{\pi}_0^{(K-c+1)} = (x_0^{(K)}, \ldots, x_{c-1}^{(K)}), \quad \text{and}$$

$$\pi_i^{(K-c+1)} = x_{i+c-1}^{(K)}.$$

By Theorem 1, the loss probability $P_{\text{Loss}}^{(K)}$ of the $GI/M/c/K$ queue has the following asymptotic behavior:

$$\begin{aligned}P_{\text{Loss}}^{(K)} &= x_K^{(K)} = \pi_{K-c+1}^{(K-c+1)} \\ &= \frac{x_c^{(\infty)}(1+c\mu H^{*\prime}(c\mu - c\mu\sigma))}{\sigma(1-\sigma)}\sigma^{K-c+1} \\ &\quad + o(\sigma^K), \quad \text{as } K \to \infty.\end{aligned}$$

It is well known that $x_i^{(\infty)} = x_{c-1}^{(\infty)}\,\sigma^{i-c+1}$, $i \geq c-1$. Especially $x_c^{(\infty)} = x_{c-1}^{(\infty)}\sigma$. Hence

$$\begin{aligned}P_{\text{Loss}}^{(K)} &= \frac{x_{c-1}^{(\infty)}(1+c\mu H^{*\prime}(c\mu - c\mu\sigma))}{\sigma^{c-1}(1-\sigma)}\sigma^K \\ &\quad + o(\sigma^K), \quad \text{as } K \to \infty.\end{aligned}$$

Remark When $c = 1$, the above result is reduced to Theorem 3 in Choi et al. [5]. More precisely, for the $GI/M/1/K$ queue,

$$P_{\text{Loss}}^{(K)} = (1+\mu H^{*\prime}(\mu - \mu\sigma))\sigma^K + o(\sigma^K), \quad \text{as } K \to \infty.$$

Example 2 ($GI/PH/1/K$ queue) We consider the $GI/PH/1/K$ queue with general interarrival time distribution $H(\cdot)$ with mean λ^{-1}. The service time distribution is phase-type with irreducible representation $(\boldsymbol{\gamma}, T)$, and mean μ^{-1}, where $\boldsymbol{\gamma}$ is an m-dimensional row vector and T is an $m \times m$ matrix. The system has a finite buffer of size K. The offered load ρ is $\rho = \frac{\lambda}{\mu}$ and is assumed to be less than 1.

By $N_n^{(K)}$, we denote the number of customers in the system immediately before the nth customer's arrival. $Y_n^{(K)}$ denotes

$$Y_n^{(K)} = \begin{cases}\text{phase of the service at that time} & \text{if } N_n^{(K)} \geq 1, \\ 0, & \text{if } N_n^{(K)} = 0.\end{cases}$$

Then $\{(N_n^{(K)}, Y_n^{(K)})\}$ is a finite $GI/M/1$ type Markov chain whose TPM is given by (1), where A_n is the $m \times m$ matrix whose (i, j)-component is the probability that exactly n departures occur during an interarrival interval and the service phase is j at the end of that interval when there are sufficiently many customers to be served in the system and the service phase is i at the beginning of that interval. The matrices B_n, $n = 0, 1, \ldots, K-1$, and C are given by

$$\begin{aligned}B_0 &= 1 - \boldsymbol{\gamma} A_0 \mathbf{1}, \\ B_n &= \sum_{k=n+1}^{\infty} A_k \mathbf{1}, \quad n = 1, 2, \ldots, K-1, \\ C &= \boldsymbol{\gamma} A_0.\end{aligned}$$

The matrix-valued function $A(z) \equiv \sum_{n=0}^{\infty} A_i z^i$ is expressed as follows (Neuts [16], p. 65):

$$A(z) = \sum_{n=0}^{\infty} A_n z^n = \int_0^{\infty} e^{(T+T^0\boldsymbol{\gamma} z)t}\, dH(t),$$

with $T^0 = -T\mathbf{1}$.

By Theorem 1, the loss probability $P_{\text{Loss}}^{(K)}$ of the $GI/PH/1/K$ queue has the following asymptotic behavior:

$$\begin{aligned}P_{\text{Loss}}^{(K)} &= \boldsymbol{\pi}_K^{(K)}\mathbf{1} \\ &= \frac{\boldsymbol{\pi}_1^{(\infty)}\boldsymbol{r}(1-\chi'(\sigma))}{\sigma(1-\sigma)}\sigma^K + o(\sigma^K), \quad \text{as } K \to \infty.\end{aligned} \tag{24}$$

It is known that $\boldsymbol{\pi}_n^{(\infty)} = \pi_0^{(\infty)}\boldsymbol{\gamma} R^n$, $n = 1, 2, \ldots$. Thus $\pi_0^{(\infty)} = (\boldsymbol{\gamma}(I-R)^{-1}\mathbf{1})^{-1}$ and

$$\boldsymbol{\pi}_1^{(\infty)}\boldsymbol{r} = \pi_0^{(\infty)}\boldsymbol{\gamma} R\boldsymbol{r} = \sigma(\boldsymbol{\gamma}(I-R)^{-1}\mathbf{1})^{-1}(\boldsymbol{\gamma}\boldsymbol{r}). \tag{25}$$

Now we calculate σ and $\chi'(\sigma)$. For $z \in (0, 1)$, let $-\nu(z)$ be the eigenvalue of $T + \boldsymbol{T}^0\boldsymbol{\gamma} z$ with the largest real part. Then $\chi(z) = H^*(\nu(z))$, where $H^*(s) = \int_0^{\infty} e^{-st} dH(t)$, $s > 0$. Hence σ is the unique solution of

$$H^*(\nu(\sigma)) = \sigma, \quad 0 < \sigma < 1.$$

Further

$$\chi'(\sigma) = H^{*\prime}(\nu(\sigma))\nu'(\sigma). \tag{26}$$

Let $\boldsymbol{\zeta}(z)$, $0 < z < 1$, be the right maximal eigenvector of $A(z)$ with $\boldsymbol{\eta}\boldsymbol{\zeta}(z) = 1$. We note that $\boldsymbol{\zeta}(\sigma) = \boldsymbol{\xi}$ and that $\boldsymbol{\zeta}(z)$, $0 < z < 1$, is the right eigenvector of $T + \boldsymbol{T}^0\boldsymbol{\gamma} z$ corresponding to the eigenvalue $-\nu(z)$. Thus $(T + \boldsymbol{T}^0\boldsymbol{\gamma} z)\boldsymbol{\zeta}(z) = -\nu(z)\boldsymbol{\zeta}(z)$, from which we obtain

$$\boldsymbol{T}^0\boldsymbol{\gamma}\boldsymbol{\xi} + (T + \boldsymbol{T}^0\boldsymbol{\gamma}\sigma)\boldsymbol{\zeta}'(\sigma) = -\nu'(\sigma)\boldsymbol{\xi} - \nu(\sigma)\boldsymbol{\zeta}'(\sigma).$$

Noting that $\boldsymbol{\eta}(T + \boldsymbol{T}^0\boldsymbol{\gamma} z) = -\nu(z)\boldsymbol{\eta}$, we have

$$\nu'(\sigma) = -(\boldsymbol{\eta}\boldsymbol{T}^0)(\boldsymbol{\gamma}\boldsymbol{\xi}). \tag{27}$$

Substituting (25–27) into (24) leads to

$$\begin{aligned}P_{\text{Loss}}^{(K)} &= \frac{(\boldsymbol{\gamma}\boldsymbol{r})(1+H^{*\prime}(\nu(\sigma))(\boldsymbol{\eta}\boldsymbol{T}^0)(\boldsymbol{\gamma}\boldsymbol{\xi}))}{\boldsymbol{\gamma}(I-R)^{-1}\mathbf{1}(1-\sigma)}\sigma^K \\ &\quad + o(\sigma^K), \quad \text{as } K \to \infty.\end{aligned}$$

Example 3 ($SM/M/1/K$ queue) We consider the $SM/M/1/K$ queue in which the interarrival times form a semi-Markov process with semi-Markov kernel $H(\cdot) = (H_{ij}(\cdot))_{i,j=1,2,\ldots,m}$, with fundamental mean λ^{-1} defined as $\boldsymbol{\kappa}\int_0^{\infty} t\,dH(t)\mathbf{1}$, where $\boldsymbol{\kappa}$ is the stationary probability vector of $H(\infty)$. The service times are exponentially distributed with mean μ^{-1}. The system has a finite buffer of size K. The offered load ρ is $\rho = \frac{\lambda}{\mu}$ and is assumed to be less than 1.

By observing the system immediately before the arrival, we construct an embedded Markov chain with TPM given in (1), where

$$A_n = \int_0^\infty e^{-\mu t}\frac{(\mu t)^n}{n!}dH(t), \quad n = 0, 1, \ldots, K-1,$$

$$B_n = \sum_{k=n+1}^{\infty} A_k, \quad n = 0, 1, \ldots, K-1,$$

$$C = A_0.$$

Further, $A(z) = \sum_{n=0}^{\infty} A_n z^n = \int_0^\infty e^{\mu(z-1)t}dH(t) = H^*(\mu - \mu z)$, where $H^*(s) = \int_0^\infty e^{-st}dH(t)$. For a deeper discussion of this queueing system, we refer to Sect. 4 of Neuts [15].

By Corollary 1, the loss probability $P_{\text{Loss}}^{(K)}$ of the $SM/M/1/K$ queue has the following asymptotic behavior:

$$P_{\text{Loss}}^{(K)} = \boldsymbol{\pi}_K^{(K)}\mathbf{1}$$
$$= \frac{\boldsymbol{\pi}_0^{(\infty)}\boldsymbol{r}(1-\chi'(\sigma))}{(1-\sigma)}\sigma^K + o(\sigma^K), \quad \text{as } K \to \infty. \tag{28}$$

Since $\boldsymbol{\pi}_n^{(\infty)} = \boldsymbol{\pi}_0^{(\infty)}R^n$, $n = 0, 1, 2, \ldots$, and $\sum_{n=0}^{\infty}\boldsymbol{\pi}_n^{(\infty)} = \boldsymbol{\kappa}$, we have

$$\boldsymbol{\pi}_0^{(\infty)}\boldsymbol{r} = \boldsymbol{\kappa}(I-R)\boldsymbol{r} = (1-\sigma)\boldsymbol{\kappa}\boldsymbol{r}. \tag{29}$$

Let $\boldsymbol{\zeta}(z)$, $0 < z < 1$, be the right maximal eigenvalue of $A(z)$ with $\boldsymbol{\eta}\boldsymbol{\zeta}(z) = 1$. Then $A(z)\boldsymbol{\zeta}(z) = \chi(z)\boldsymbol{\zeta}(z)$, $0 < z < 1$, gives

$$A'(\sigma)\boldsymbol{\xi} + A(\sigma)\boldsymbol{\zeta}'(\sigma) = \chi'(\sigma)\boldsymbol{\xi} + \sigma\boldsymbol{\zeta}'(\sigma).$$

Since $\boldsymbol{\eta}A(\sigma) = \sigma\boldsymbol{\eta}$, we have

$$\chi'(\sigma) = \boldsymbol{\eta}A'(\sigma)\boldsymbol{\xi} = \mu\boldsymbol{\eta}\int_0^\infty te^{\mu(\sigma-1)t}dH(t)\boldsymbol{\xi}. \tag{30}$$

Substituting (29) and (30) into (28) leads to

$$P_{\text{Loss}}^{(K)} = \boldsymbol{\kappa}\boldsymbol{r}\left(1 - \mu\boldsymbol{\eta}\int_0^\infty te^{\mu(\sigma-1)t}dH(t)\,\boldsymbol{\xi}\right)\sigma^K$$
$$+ o(\sigma^K), \quad \text{as } K \to \infty.$$

Example 4 ($GI/MSP/1/K$ queue) We consider the $GI/MSP/1/K$ queue where the interarrival times are independent and identically distributed and the service process is Markovian service process (MSP). The system has a finite buffer of size K. For a deeper discussion of this queueing system, we refer to Bocharov et al. [4], Gupta and Banik [10]. The asymptotic of the loss probability for the $SM/M/1/K$ queue can also be obtained by Theorem 1 or Corollary 1.

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