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Tail asymptotics for the fundamental period in the MAP/G/1 queue

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Abstract This paper studies the tail behavior of the fundamental period in the MAP/G/1 queue. We prove that if the service time distribution has a regularly varying tail, then the fundamental period distribution in the MAP/G/1 queue has also regularly varying tail, and vice versa, by finding an explicit expression for the asymptotics of the tail of the fundamental period in terms of the tail of the service time distribution. Our main result with the matrix analytic proof is a natural extension of the result in (de Meyer and Teugels, J. Appl. Probab. 17: 802–813, 1980) on the M/G/1 queue where techniques rely heavily on analytic expressions of relevant functions.

Keywords MAP/G/1 queue · Fundamental period · Regular variation · Abelian-Tauberian theorem

Mathematics Subject Classification (2000) 60K25

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1 Introduction

We consider a MAP/G/1 queueing system, where customers arrive according to a Markovian arrival process (MAP) with a representation (C, D) and service times are independent and identically distributed with a distribution function $\tilde{B}$. The underlying Markov process of the MAP is denoted by $\{J(t) : t \geq 0\}$, which has the Q-matrix $C + D$ and state space $\{1, \ldots, m\}$. The service policy is assumed to be work conserving. Though the order of service is not essential in this work, the 'first-come-first-served' policy is assumed for simplicity of description.

Let $N(t)$, $t \geq 0$, be the number of customers in the system at time t. Then $\{(N(t), J(t)) : t \geq 0\}$ becomes a Markov regenerative process with service completion points as Markov renewal epochs. Let T_n be the nth service completion epoch. Then the semi-Markov kernel $P = (P_{(k,i)(l,j)}(x))$, $k, l = 0, 1, 2, \ldots, i, j = 1, \ldots, m$, $x \geq 0$, defined as

$$P_{(k,i)(l,j)}(x) = \mathbb{P}\big((N(T_{n+1}), J(T_{n+1})) = (l, j),\\ T_{n+1} - T_n \leq x \mid (N(T_n), J(T_n)) = (k, i)\big)$$

is given by

$$P(x) = \begin{bmatrix} \tilde{A}_0(x) & \tilde{A}_1(x) & \tilde{A}_2(x) & \tilde{A}_3(x) & \ldots \\ A_0(x) & A_1(x) & A_2(x) & A_3(x) & \ldots \\ 0 & A_0(x) & A_1(x) & A_2(x) & \ldots \\ 0 & 0 & A_0(x) & A_1(x) & \ldots \\ & \ddots & \ddots & \ddots & \end{bmatrix},$$

where $A_k(x)$, $k = 0, 1, 2, \ldots$, $x \geq 0$, are $m \times m$ matrices determined by

$$\sum_{k=0}^{\infty} A_k(x) z^k = \int_0^x e^{(C+Dz)y}\, d\tilde{B}(y), \quad |z| \leq 1, \tag{1}$$

and

$$\tilde{A}_k(x) = \int_0^x e^{Cy} D A_k(x-y)\, dy.$$

When $N(T_n) = k \geq 1$, the duration from T_n till the first hitting time of $k-1$ for $N(t)$, $t \geq T_n$, is called the fundamental period whose distribution is independent of k and n.

The object of this paper is to investigate the regularly varying tail behavior of the fundamental period distribution in the MAP/G/1 queue. Our topic is motivated by de Meyer and Teugels [7] and Zwart [13], where regularly varying tail behavior of busy period distribution was studied for M/G/1 queue and GI/G/1 queue, respectively. We prove that if the service time distribution has a regularly varying tail then the fundamental period distribution in the MAP/G/1 queue has also regularly varying tail of the same index, and vice versa. Our matrix analytic proof is substantially inspired by de Meyer and Teugels [7] in the context of Abelian and Tauberian arguments. It turns out that the matrix computations involved are challenging, meaningful, and interesting in our judgment. Another possible way for the proof of our result is to employ a sample path approach that was used in Zwart [13]. The sample path approach is an attractive and modern technique that may provide an elegant proof for some part of our result. Actually the authors are sure that the sample path approach is suitable for the proof of one direction of the result: if the service time distribution has a regularly varying tail, then so does the fundamental period distribution in the MAP/G/1 queue. However, it seems to be very hard to employ the sample path approach to prove that the regularly varying tail of the fundamental period distribution in the MAP/G/1 queue implies the regularly varying tail of the service time distribution.

The tail behavior of the busy period in single server queues has been studied by many authors. Abate and Whitt [1] studied the tail behavior of the busy period in an M/G/1 queue under Cramér-type assumption. De Meyer and Teugels [7] proved that the busy period distribution in an M/G/1 queue has a regularly varying tail if and only if the service time distribution has a regularly varying tail. Our main result extends de Meyer and Teugels [7] to MAP/G/1 queue. Zwart [13] generalized de Meyer and Teugels [7] to GI/G/1 queue in different context. He showed that if the service time B in the GI/G/1 queue has a regularly varying tail distribution, then so is the busy period P with

$$\mathbb{P}(P > x) \sim \mathbb{E}Z\, \mathbb{P}(B > (1-\rho)x), \quad \text{as } x \to \infty, \tag{2}$$

where Z is the number of number of customers served during a busy period and ρ is the offered load. It is worthwhile to compare the insightful and probabilistic techniques employed in Zwart [13], and the analytic proofs in [7] and in our work. Asmussen [2] showed that (2) does not hold if the service time B has a tail lighter than $e^{-\sqrt{x}}$. Jelenković and Momčilović [8] proved that (2) holds in a GI/G/1 queue if the service time B has a subexponential distribution with tail heavier than $e^{-\sqrt{x}}$ but lighter than any polynomial. Baltrūnas et al. [3] also exploited the tail behavior of the busy period of the GI/G/1 queue with subexponential service time distribution under some assumptions.

2 Main result

Recall a MAP/G/1 queueing system, where customers arrive according to a MAP with a representation (C, D). We assume that the underlying Markov process $\{J(t) : t \geq 0\}$ of the MAP is irreducible with state space $\{1, \ldots, m\}$. Let π be the stationary probability vector of $\{J(t) : t \geq 0\}$, which is given by a unique solution of $\pi(C + D) = 0$ and $\pi \mathbf{e} = 1$, where $\mathbf{e}$ is the m-dimensional column vector with all ones. Denote by λ the fundamental arrival rate of the MAP, i.e, $\lambda = \pi D \mathbf{e}$. Service times are assumed to be independent and identically distributed. Let B be the generic random variable representing a service time and $\tilde{B}$ the distribution function of B. We assume that B has a finite mean β and that the stability condition $\rho \equiv \lambda\beta < 1$ holds.

Let $V(t)$, $t \geq 0$, be the unfinished work in the system at time t. Then $\{(V(t), J(t)) : t \geq 0\}$ is a Markov process. Let us denote by $\mathbb{P}_{(v,i)}$ the probability measure given $(V(0), J(0)) = (v, i)$, and by $\mathbb{E}_{(v,i)}$ the expectation with respect to $\mathbb{P}_{(v,i)}$. Let τ be the first hitting time of 0 for $\{V(t) : t \geq 0\}$, i.e.,

$$\tau \equiv \inf\{t \geq 0 : V(t) = 0\}.$$

Define $m \times m$ matrix-valued functions $\tilde{G}^v$ and $\tilde{G}$ as, for $1 \leq i, j \leq m$ and $t \geq 0$,

$$\tilde{G}^v_{ij}(t) \equiv \mathbb{P}_{(v,i)}\{\tau \leq t,\ J(\tau) = j\} \quad \text{and}$$

$$\tilde{G}_{ij}(t) \equiv \mathbb{P}_{(\tilde{B},i)}\{\tau \leq t,\ J(\tau) = j\},$$

where $\mathbb{P}_{(\tilde{B},i)}$ is the probability measure defined as $\mathbb{P}_{(\tilde{B},i)}(\cdot) \equiv \int_{[0,\infty)} \mathbb{P}_{(v,i)}(\cdot)\, d\tilde{B}(v)$.

When $V(0) = B$ in distribution and $J(0) = i$, it can be shown that the distribution of τ is the same as that of the fundamental period with the underlying Markov process of the MAP starting at i. By a slight abuse of terminology, we call τ the fundamental period when $V(0) = B$ in distribution. Hence, for each $i \in \{1, \ldots, m\}$, the distribution on $\mathbb{R} \times \{1, \ldots, m\}$ induced by $\tilde{G}_{ij}(t)$, $t \in \mathbb{R}$, $j \in \{1, \ldots, m\}$, in the usual way is the conditional joint distribution of a fundamental period τ and $J(\tau)$ given $J(0) = i$. Moreover, $\sum_{j=1}^{m} \tilde{G}_{ij}(t)$, $t \in \mathbb{R}$, is the distribution function of the busy period that starts with the underlying Markov process of the MAP at i, for each $i \in \{1, \dots, m\}$. Also, the distribution function of an arbitrary busy period is obtained as follows: Let κ_i be the probability that the underlying Markov process of the MAP is at i immediately after the beginning of a busy period. Then the distribution function $\tilde{T}$ of an arbitrary busy period is given by

$$\tilde{T}(t) = \sum_{i=1}^{m}\sum_{j=1}^{m} \kappa_i \tilde{G}_{ij}(t) = \kappa \tilde{G}(t)\mathbf{e}, \tag{3}$$

where $\kappa \equiv (\kappa_1, \dots, \kappa_m)$.

For the sake of simplicity of notation, let $G \equiv \tilde{G}(\infty)$. It is well known that G satisfies $G = \sum_{k=0}^{\infty} A_k(\infty)G^k$, where $A_k(\infty)$, $k = 0, 1, 2, \dots$, are determined by (1). Since $A_0(\infty) = \int_0^\infty e^{Ct}\, d\tilde{B}(t)$, $(A_0(\infty))_{jj} > 0$, $j = 1, \dots, m$. Hence $G_{jj} \geq (A_0(\infty))_{jj} > 0$, $j = 1, \dots, m$. Since $C + D$ is irreducible, $\sum_{k=0}^{\infty} A_k(\infty) = \int_0^\infty e^{(C+D)t}\, d\tilde{B}(t)$ is a positive matrix. Therefore, for every i and j, there is l such that $(A_l(\infty))_{ij} > 0$. Hence $G_{ij} = \sum_{k=0}^{\infty}(A_k(\infty)G^k)_{ij} \geq (A_l(\infty))_{ij}(G_{jj})^l > 0$. Therefore G is a positive matrix. Let $\mathbf{g} \equiv (g_1, \dots, g_m)$ denote the stationary probability vector of G.

Now we state the main theorem which classifies the heavy tail behaviors of the service time distribution and that of the fundamental period distribution. The proof is given in Sect. 4. In what follows, $o(f(t))$, as $t \to \infty$ ($t \to 0$, respectively), denotes a function whose ratio to $f(t)$ vanishes as $t \to \infty$ ($t \to 0$, respectively). Denote by $O(f(t))$, as $t \to \infty$ ($t \to 0$, respectively), a function whose ratio to $f(t)$ has finite lim sup and lim inf as $t \to \infty$ ($t \to 0$, respectively). By a little abuse of notation, a matrix-valued or vector-valued function is also denoted by $o(f(t))$ ($O(f(t))$, respectively) if all of its components are $o(f(t))$ ($O(f(t))$, respectively).

Theorem 1 *Let L be a slowly varying function and $\alpha > 1$. Then the following are equivalent*:

(i) $1 - \tilde{B}(t) = ct^{-\alpha}L(t) + o(t^{-\alpha}L(t))$, *as* $t \to \infty$, *for some* $c \geq 0$.
(ii) $G - \tilde{G}(t) = Ht^{-\alpha}L(t) + o(t^{-\alpha}L(t))$, *as* $t \to \infty$, *for some* $m \times m$ *nonnegative matrix* H.

In either case, c *and* H *are related as follows*:

$$H = c(1-\rho)^{-\alpha}\mu\mathbf{g}, \tag{4}$$

where

$$\mu \equiv (\mathbf{eg} - C - D - DG + GD - \beta\mathbf{eg}D)^{-1}\mathbf{e}. \tag{5}$$

Remark The intuitive meaning of μ is given in Lemma 3.

It is interesting to establish the analogous results for an arbitrary busy period and a busy period starting at a fixed unfinished work when the service time is regularly varying. We present two corollaries which are more intrinsic and subtle. The proofs are deferred to Appendices 4 and 5, respectively.

Corollary 1 *Let L be a slowly varying function. Suppose that $\alpha > 1$ and $c \geq 0$. If*

$$1 - \tilde{B}(t) = ct^{-\alpha}L(t) + o(t^{-\alpha}L(t)), \quad \text{as } t \to \infty,$$

then the distribution function $\tilde{T}$ of an arbitrary busy period satisfies

$$1 - \tilde{T}(t) = dt^{-\alpha}L(t) + o(t^{-\alpha}L(t)), \quad \text{as } t \to \infty,$$

where

$$d = c(\mathbf{g}D\mathbf{e})^{-1}\lambda(1-\rho)^{-\alpha-1}.$$

Corollary 2 *Let L be a slowly varying function. Suppose that $\alpha > 1$, $c \geq 0$ and $v > 0$. If*

$$1 - \tilde{B}(t) = ct^{-\alpha}L(t) + o(t^{-\alpha}L(t)), \quad \text{as } t \to \infty,$$

then

$$e^{(C+DG)v} - \tilde{G}^v(t) = H_v t^{-\alpha}L(t) + o(t^{-\alpha}L(t)), \quad \text{as } t \to \infty,$$

where

$$H_v = c(1-\rho)^{-\alpha}\int_0^v e^{(C+DG)x}\, dx\, D\mu\mathbf{g}. \tag{6}$$

Remark (i) The matrix $\int_0^v e^{(C+DG)x}\, dx$ on the right hand side of (6) is written as

$$\int_0^v e^{(C+DG)x}\, dx = \left(v\mathbf{eg} + I - e^{(C+DG)v}\right)(\mathbf{eg} - C - DG)^{-1}.$$

(ii) The ith component of the column vector $\int_0^v e^{(C+DG)x}\, dx\, D\mu$ is the mean number of arrivals before the system becomes empty given $V(0) = v$ and $J(0) = i$.

3 Preliminaries

First we introduce notation related to the Laplace Stieltjes Transform (LST) of the fundamental period distribution. Let $\mathcal{G}$ be the LST of $\tilde{G}$, i.e.,

$$\mathcal{G}(s) \equiv \int_{[0,\infty)} e^{-st}\, d\tilde{G}(t), \quad s \geq 0. \tag{7}$$

Since G is a positive matrix, so is $\mathcal{G}(s)$ for all $s \geq 0$. Moreover, it can be rewritten (see, for example, (50) in [10]) as

$$\mathcal{G}(s) = \int_{[0,\infty)} e^{[-sI+C+D\mathcal{G}(s)]t}\, d\tilde{B}(t), \quad s \geq 0. \tag{8}$$

This exponential representation leads to an explicit and convenient formula which relates $\tilde{B}$ and $\tilde{G}$. Since the Q-matrix $C+D$ is irreducible and $\mathcal{G}(s)$ is positive for $s\ge 0$, $C+D\mathcal{G}(s)$ is an irreducible ML matrix for $s\ge 0$. (See p. 46 in [12] for the definition of an irreducible ML matrix.) Hence $-sI+C+D\mathcal{G}(s)$, $s\ge 0$, is also an irreducible ML matrix. For $s\ge 0$, let $-\chi(s)$ be the eigenvalue of $-sI+C+D\mathcal{G}(s)$ with the largest real part. It is well known that the eigenvalue with the largest real part has the algebraic multiplicity 1 for an irreducible ML matrix. Therefore the right eigenvector of $-sI+C+D\mathcal{G}(s)$ corresponding to the eigenvalue $-\chi(s)$ is unique up to scalar multiplication. Denote by $\xi(s)$, $s\ge 0$, the right eigenvector of $-sI+C+D\mathcal{G}(s)$ normalized by

$$\mathbf{g}\xi(s)=1. \tag{9}$$

Then we have

$$(-sI+C+D\mathcal{G}(s))\xi(s)=-\chi(s)\xi(s),\quad s\ge 0. \tag{10}$$

Equations (9) and (10) are equivalent to

$$\left[\chi(s)I-sI+C+D\mathcal{G}(s)+\mathbf{e}g\right]\xi(s)=\mathbf{e},\quad s\ge 0. \tag{11}$$

Since $\xi(s)$, $s\ge 0$, is uniquely determined by (9) and (10), the square matrix $\chi(s)I-sI+C+D\mathcal{G}(s)+\mathbf{e}g$ is invertible for all $s\ge 0$ and

$$\xi(s)=\left[\chi(s)I-sI+C+D\mathcal{G}(s)+\mathbf{e}g\right]^{-1}\mathbf{e},\quad s\ge 0. \tag{12}$$

By (8),

$$\mathcal{G}(s)\xi(s)=\mathcal{B}(\chi(s))\xi(s),\quad s\ge 0, \tag{13}$$

where $\mathcal{B}$ is the LST of $\tilde{B}$, i.e., $\mathcal{B}(s)\equiv\int_{[0,\infty)}e^{-st}\,d\tilde{B}(t)$, $s\ge 0$. Premultiplying $\mathbf{g}$ to both sides of (13) leads to

$$\mathbf{g}\mathcal{G}(s)\xi(s)=\mathcal{B}(\chi(s)),\quad s\ge 0. \tag{14}$$

Now we present two well-known lemmas modified into a suitable form for our purpose. We expect that only slight modification is required for the proofs. (See Theorem 8.1.6 in [4] and Theorem 2 in [6].) Let $\mathfrak{M}$ be the collection of all nondecreasing and right continuous functions $F:\mathbb{R}\to\mathbb{R}$ such that $F(x)=0$, $x<0$, and $F(\infty)\equiv\lim_{x\to\infty}F(x)<\infty$.

Lemma 1 *Let $F\in\mathfrak{M}$, and denote by $\mathcal{F}$ the LST of F. Suppose that $\alpha>1$ is not an integer and $n\equiv\lfloor\alpha\rfloor$, the integer part of α. Let L be a slowly varying function and $a\ge 0$. Then the following are equivalent*:

(i) $F(\infty)-F(x)=ax^{-\alpha}L(x)+o(x^{-\alpha}L(x))$, *as* $x\to\infty$.
(ii) $\mathcal{F}(s)=\sum_{i=0}^{n}\mathcal{F}_is^i-a\Gamma(1-\alpha)s^\alpha L(\frac{1}{s})+o(s^\alpha L(\frac{1}{s}))$, *as* $s\to 0+$, *for some* $\mathcal{F}_i\in\mathbb{R}$, $0\le i\le n$.

Proof A slight modification of Theorem 8.1.6 in [4] proves the lemma. □

Lemma 2 *Let $F\in\mathfrak{M}$, and denote by $\mathcal{F}$ the LST of F. For n a positive integer, b a nonnegative real number and L a slowly varying function, suppose that*

$$F(\infty)-F(t)=bt^{-n}L(t)+o(t^{-n}L(t)),\quad \text{as } t\to\infty. \tag{15}$$

Then the following hold:

(i) $\mathcal{F}(s)$ *is written as*

$$\mathcal{F}(s)=\sum_{i=0}^{n-1}\mathcal{F}_is^i+\frac{(-1)^n}{(n-1)!}s^n\tilde{L}\left(\frac{1}{s}\right),\quad s>0,$$

for some $\mathcal{F}_i\in\mathbb{R}$, $0\le i\le n-1$, *and a function* $\tilde{L}$ *such that*

$$\lim_{x\to\infty}\frac{\tilde{L}(tx)-\tilde{L}(x)}{L(x)}=b\log t \tag{16}$$

uniformly in t on every compact subset of $(0,\infty)$.

(ii) *Suppose $K\in\mathfrak{M}$, and denote by $\mathcal{K}$ the LST of K. If*

$$\lim_{s\to 0+}\frac{\mathcal{K}(s)-a\mathcal{F}(s)+\sum_{i=0}^{n}\theta_is^i}{s^nL(\frac{1}{s})}=0, \tag{17}$$

for some $a\ge 0$ and $\theta_i\in\mathbb{R}$, $i=0,\dots,n$, then

$$K(\infty)-K(t)=abt^{-n}L(t)+o(t^{-n}L(t)),\quad \text{as } t\to\infty.$$

Proof We defer the proof until Appendix 1. □

The following lemma is on the first moments of some quantities related to the fundamental period.

Lemma 3

(i) *The matrix* $\mathbf{eg}-C-D-DG+GD-\beta\mathbf{eg}D$ *is invertible.*
(ii) *The ith component μ_i of* $\mu\equiv(\mathbf{eg}-C-D-DG+GD-\beta\mathbf{eg}D)^{-1}\mathbf{e}$ *is 1 plus the mean number of arrivals during a fundamental period starting with* $J(0)=i$, $1\le i\le m$.
(iii) $\mathbf{g}\mu=\frac{1}{1-\rho}$.
(iv) $\mathbf{g}\int_0^\infty t\,d\tilde{G}(t)\mathbf{e}=\frac{\beta}{1-\rho}$.
(v) $\mathbf{g}D\mu=\frac{\lambda}{1-\rho}$.

Proof Proofs of (i) and (ii) are given in Appendices 2 and 3, respectively. Equations (iii) and (iv) are given in (3.1.14) in [11].

Now we prove (v). By (5),

$$(\mathbf{eg} - C - D - DG + GD - \beta\mathbf{eg}D)\mu = \mathbf{e}.$$

Premultiplying $\mathbf{g}$ to the above equation leads to $\mathbf{g}\mu - \mathbf{g}(C + DG)\mu + \mathbf{g}(G - I)D\mu - \beta\mathbf{g}D\mu = 1$. The identities $\mathbf{g}(C + DG) = 0$, $\mathbf{g}(G - I) = 0$ and (iii) complete the proof. □

Remark The column vector μ is also written as (see Theorem 3.1.1 in [11])

$$\mu = (I - G + \mathbf{eg})(I - A + (\mathbf{e} - \tilde{\beta})\mathbf{g})^{-1}\mathbf{e},$$

where $A \equiv \int_0^\infty e^{(C+D)t}\,d\tilde{B}(t)$ and $\tilde{\beta} \equiv \int_0^\infty e^{(C+D)t}D\mathbf{e}(1 - \tilde{B}(t))\,dt = (\beta\mathbf{e}\pi + I - A)(\mathbf{e}\pi - C - D)^{-1}D\mathbf{e}$.

Before we prove our main result, we need to establish a preliminary weak result. It is well known that the kth moment of the service time in a stable M/G/1 queue is finite if and only if the kth moment of the busy period is finite for $k = 1, 2, \ldots$. See, for example, Lemma 3 in [7]. It is natural to expect that the analogous result holds for the MAP/G/1 queues. Beforehand we need a technical lemma with its own interests regarding the relevant eigenvalue and the corresponding eigenvector.

Lemma 4 *Let k be a positive integer. Suppose that $\int_0^\infty x^k\,d\tilde{G}_{ij}(x) < \infty$, for all $1 \le i, j \le m$.*

(i) *The eigenvalue $-\chi(s)$ of $-sI + C + D\mathcal{G}(s)$, $s \ge 0$, with the largest real part satisfies*

$$\chi(s) = \sum_{l=1}^{k} \chi_l s^l + o(s^k), \quad \text{as } s \to 0+, \tag{18}$$

for some $\chi_l \in \mathbb{R}$, $1 \le l \le k$. Further, $\chi_1 = (1 - \rho)^{-1}$.

(ii) *The right eigenvector $\xi(s)$ of $-sI + C + D\mathcal{G}(s)$, $s \ge 0$, corresponding to the eigenvalue $-\chi(s)$ and normalized by (9) satisfies*

$$\xi(s) = \mathbf{e} + \sum_{l=1}^{k} \xi_l s^l + o(s^k), \quad \text{as } s \to 0+, \tag{19}$$

for some m-dimensional column vectors ξ_l, $1 \le l \le k$.

Proof By (7), $\mathcal{G}$ is C^∞ on $(0, \infty)$. Further, it is C^k on $[0, \infty)$. (See Remark below.) Since $-\chi(s)$ is an eigenvalue of $-sI + C + D\mathcal{G}(s)$, $s \ge 0$,

$$\det\big(\chi(s)I - sI + C + D\mathcal{G}(s)\big) = 0, \quad s \ge 0.$$

The eigenvalue $-\chi(s)$ of $-sI + C + D\mathcal{G}(s)$ has the algebraic multiplicity 1, and $-sI + C + D\mathcal{G}(s)$ is C^k on $[0, \infty)$. Therefore, by the implicit function theorem, $-\chi(s)$ is also C^k on $[0, \infty)$. Thus, (18) is derived.

Since $(C + D\mathcal{G}(0))\mathbf{e} = (C + DG)\mathbf{e} = 0$, the column vector $\mathbf{e}$ is a right eigenvector of $(C + D\mathcal{G}(0))$. Hence $\xi(0) = \mathbf{e}$. By (12), $\xi(s)$ is C^k on $[0, \infty)$. Thus (19) is derived.

By differentiating (14) at $s = 0+$, we obtain $\mathbf{g}\mathcal{G}_1\mathbf{e} + \mathbf{g}\xi_1 = -\beta\chi_1$, where $\mathcal{G}_1 \equiv \frac{d}{ds}\mathcal{G}(s)|_{s=0+} = -\int_0^\infty x\,d\tilde{G}(x)$. By taking derivative of (9) at $s = 0+$, $\mathbf{g}\xi_1 = 0$. Therefore $\chi_1 = \frac{1}{\beta}\mathbf{g}\int_0^\infty x\,d\tilde{G}(x)\mathbf{e}$, which is $(1 - \rho)^{-1}$ by Lemma 3(iv). □

Remark The statement "the matrix function $\mathcal{G}$ is C^k on $[0, \infty)$" means that there exists $\epsilon > 0$ such that $\mathcal{G}$ can be extended to a C^k matrix function on $(-\epsilon, \infty)$. This does not imply that the right hand side of (7) is finite for some $s < 0$.

Theorem 2 *For $k = 1, 2, 3, \ldots$, the following are equivalent:*

(i) $\int_0^\infty x^k\,d\tilde{B}(x) < \infty$.
(ii) $\int_0^\infty x^k\,d\tilde{G}_{ij}(x) < \infty$, *for all $1 \le i, j \le m$.*

Proof (i) ⇒ (ii): It is known that (ii) holds for $k = 1$ (see, for example, Chap. 3 in [11]). Now we assume that (ii) holds for $k = l \ge 1$ and that (i) holds for $k = l + 1$. The proof is completed by induction on k if we derive (ii) for $k = l + 1$. Since (i) holds for $k = l + 1$, $\mathcal{B}(s)$ is expressed as

$$\mathcal{B}(s) = 1 - \beta s + \sum_{i=2}^{l+1} \mathcal{B}_i s^i + o(s^{l+1}), \quad \text{as } s \to 0+, \tag{20}$$

for some $\mathcal{B}_i$, $2 \le i \le l + 1$. By substituting (20) into (13),

$$\mathcal{G}(s)\xi(s) = \left(1 - \beta\chi(s) + \sum_{i=2}^{l+1} \mathcal{B}_i(\chi(s))^i + o\big((\chi(s))^{l+1}\big)\right) \times \xi(s),$$

$$\text{as } s \to 0+. \tag{21}$$

Since (ii) holds for $k = l$, $\mathcal{G}(s)$ is expressed as

$$\mathcal{G}(s) = \sum_{i=0}^{l} \mathcal{G}_i s^i + \mathcal{G}_l^*(s), \quad s \ge 0, \tag{22}$$

where $\mathcal{G}_i$, $0 \le i \le l$, and $\mathcal{G}_l^*(s)$, $s \ge 0$, are $m \times m$ matrices, and $\mathcal{G}_l^*(s) = o(s^l)$, as $s \to 0+$. Note that $\mathcal{G}_0 = G$. By Lemma 4,

$$\chi(s) = \sum_{i=1}^{l} \chi_i s^i + \chi_l^*(s), \quad s \ge 0, \tag{23}$$

$$\xi(s) = \sum_{i=0}^{l} \xi_i s^i + \xi_l^*(s), \quad s \ge 0, \tag{24}$$

where $\chi_i \in \mathbb{R}$, $1 \le i \le l$, $\chi_l^*(s) \in \mathbb{R}$, $s \ge 0$, and $\chi_l^*(s) = o(s^l)$, as $s \to 0+$, and ξ_i, $0 \le i \le l$, and $\xi_l^*(s)$, $s \ge 0$, are m-dimensional column vectors, and $\xi_l^*(s) = o(s^l)$, as $s \to 0+$. Note that $\chi_1 = (1-\rho)^{-1}$ and $\xi_0 = \mathbf{e}$ by Lemma 4. By substituting (22–24) into (21), and comparing $o(s^l)$ terms,

$$G\xi_l^*(s) + \mathcal{G}_l^*(s)\mathbf{e} = \xi_l^*(s) - \beta\chi_l^*(s)\mathbf{e} + O(s^{l+1}), \quad \text{as } s \to 0+. \tag{25}$$

By substituting (22–24) into (10) and comparing $o(s^l)$ terms,

$$(C + DG)\xi_l^*(s) + D\mathcal{G}_l^*(s)\mathbf{e} = -\chi_l^*(s)\mathbf{e} + O(s^{l+1}), \quad \text{as } s \to 0+. \tag{26}$$

By the identity $(G - I) = -(\mathbf{e}g - C - DG)^{-1}(G - I)(C + DG)$ and (26),

$$\begin{aligned}(G - I)\xi_l^*(s) &= -(\mathbf{e}g - C - DG)^{-1}(G - I)(C + DG)\xi_l^*(s) \\ &= (\mathbf{e}g - C - DG)^{-1}(G - I)D\mathcal{G}_l^*(s)\mathbf{e} + O(s^{l+1}), \\ &\quad \text{as } s \to 0+.\end{aligned} \tag{27}$$

By (25) and (27),

$$\big(I + (\mathbf{e}g - C - DG)^{-1}(G - I)D\big)\mathcal{G}_l^*(s)\mathbf{e} = -\beta\chi_l^*(s)\mathbf{e} + O(s^{l+1}), \quad \text{as } s \to 0+.$$

Premultiplying $\mathbf{e}g - C - DG$ to the above equation leads to

$$(\mathbf{e}g - C - DG + GD - D)\mathcal{G}_l^*(s)\mathbf{e} = -\beta\chi_l^*(s)\mathbf{e} + O(s^{l+1}), \quad \text{as } s \to 0+. \tag{28}$$

By premultiplying $\mathbf{g}$ to (26),

$$\mathbf{g}D\mathcal{G}_l^*(s)\mathbf{e} = -\chi_l^*(s) + O(s^{l+1}), \quad \text{as } s \to 0+. \tag{29}$$

Substituting (29) into (28) leads to

$$(\mathbf{e}g - C - DG + GD - D - \beta\mathbf{e}\mathbf{g}D)\mathcal{G}_l^*(s)\mathbf{e} = O(s^{l+1}), \quad \text{as } s \to 0+.$$

Therefore, by Lemma 3(i), $\mathcal{G}_l^*(s)\mathbf{e} = O(s^{l+1})$, as $s \to 0+$, which implies (ii) for $k = l + 1$.

(ii) ⇒ (i): This direction is immediate from

$$\tilde{B}(x) \ge \sum_{j=1}^{m} \tilde{G}_{ij}(x), \quad x \ge 0, \ i = 1, \ldots, m.$$

□

The following lemma provides a crucial role in establishing the proof of Theorem 1. They are resemblant to Lemma 4 which is employed in the process of proving Theorem 2.

Lemma 5 *Suppose that* (ii) *in Theorem* 1 *holds and that* $\alpha > 1$ *is not an integer. Let* $n = \lfloor\alpha\rfloor$, *the integer part of* α.

(i) *The eigenvalue* $-\chi(s)$ *of* $-sI + C + D\mathcal{G}(s)$, $s \ge 0$, *with the largest real part satisfies*

$$\chi(s) = \sum_{i=1}^{n} \chi_i s^i + \chi_\alpha s^\alpha L\left(\frac{1}{s}\right) + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+, \tag{30}$$

for some $\chi_i \in \mathbb{R}$, $1 \le i \le n$ *and* $\chi_\alpha \in \mathbb{R}$. *Further*, $\chi_1 = (1-\rho)^{-1}$.

The inverse function of χ, *denoted by* $\chi^{-1} : [0,\infty) \to [0,\infty)$, *satisfies*

$$\chi^{-1}(y) = \sum_{i=1}^{n} \tilde{\chi}_i y^i + \tilde{\chi}_\alpha y^\alpha L\left(\frac{1}{y}\right) + o\left(y^\alpha L\left(\frac{1}{y}\right)\right), \quad \text{as } y \to 0+, \tag{31}$$

for some $\tilde{\chi}_i \in \mathbb{R}$, $1 \le i \le n$ *and* $\tilde{\chi}_\alpha \in \mathbb{R}$. *Further*, $\tilde{\chi}_1 = (1-\rho)$.

(ii) *The right eigenvector* $\xi(s)$ *of* $-sI + C + D\mathcal{G}(s)$, $s \ge 0$, *corresponding to the eigenvalue* $-\chi(s)$ *and normalized by* (9) *satisfies*

$$\xi(s) = \mathbf{e} + \sum_{i=1}^{n} \xi_i s^i + \xi_\alpha s^\alpha L\left(\frac{1}{s}\right) + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+, \tag{32}$$

for some m*-dimensional column vectors* ξ_i, $1 \le i \le n$, *and* ξ_α.

Proof (i) The eigenvalue $-\chi(s)$ satisfies

$$\det[\chi(s)I - sI + C + D\mathcal{G}(s)] = 0, \quad s \ge 0. \tag{33}$$

By Lemma 4, $\chi(s)$ can be written as

$$\chi(s) = \sum_{i=1}^{n} \chi_i s^i + \chi_n^*(s), \quad s \ge 0, \tag{34}$$

where $\chi_1 = (1-\rho)^{-1}$, $\chi_i \in \mathbb{R}$, $2 \le i \le n$, and $\chi_n^*(s) = o(s^n)$, as $s \to 0+$. By the condition (ii) in Theorem 1 and Lemma 1,

$$\mathcal{G}(s) = \sum_{i=0}^{n} \mathcal{G}_i s^i - \Gamma(1-\alpha)Hs^\alpha L\left(\frac{1}{s}\right) + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+, \tag{35}$$

for some $m \times m$ matrices $\mathcal{G}_i$, $0 \le i \le n$. By substituting (34) and (35) into (33),

$$\det\Bigg(\chi_n^*(s)I - sI + (C+DG) + \sum_{i=1}^n (\chi_i I + D\mathcal{G}_i)s^i - \Gamma(1-\alpha)DHs^\alpha L\left(\frac{1}{s}\right)\Bigg) = o\left(s^\alpha L\left(\frac{1}{s}\right)\right),$$

as $s \to 0+$, which leads to

$$\det\big(\chi_n^*(s)I + (C+DG)\big) = \sum_{i=1}^n \gamma_i s^i + \gamma_\alpha s^\alpha L\left(\frac{1}{s}\right) + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+, \tag{36}$$

for some $\gamma_i \in \mathbb{R}$, $1 \le i \le n$, and $\gamma_\alpha \in \mathbb{R}$. Let $0, a_2, \ldots, a_m$ be the eigenvalues of $C + DG$, written with repetition by multiplicities. Then, the left hand side of (36) becomes $\chi_n^*(s)\prod_{i=2}^m(\chi_n^*(s) + a_i)$, which is $\chi_n^*(s)(\prod_{i=2}^m a_i + o(s^n))$, as $s \to 0+$, because $\chi_n^*(s) = o(s^n)$, as $s \to 0+$. Therefore (36) leads to

$$\chi_n^*(s) = \left(\prod_{i=2}^m a_i\right)^{-1} \gamma_\alpha s^\alpha L\left(\frac{1}{s}\right) + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+,$$

and (30) is obtained.

By the inverse value theorem, χ^{-1} is C^n on $[0,\infty)$. Express $\chi^{-1}(y)$ as

$$\chi^{-1}(y) = \sum_{i=1}^n \tilde{\chi}_i y^i + \tilde{\chi}_n^*(y), \quad y \ge 0,$$

where $\tilde{\chi}_i \in \mathbb{R}$, $1 \le i \le n$, and $\tilde{\chi}_n^*$ is a function such that $\tilde{\chi}_n^*(y) = o(y^n)$, as $y \to 0+$. Then,

$$\begin{aligned} y &= \chi(\chi^{-1}(y)) \\ &= \sum_{i=1}^n \chi_i \left(\sum_{j=1}^n \tilde{\chi}_j y^j + \tilde{\chi}_n^*(y)\right)^i + \chi_\alpha \left(\sum_{j=1}^n \tilde{\chi}_j y^j + \tilde{\chi}_n^*(y)\right)^\alpha L\left(\frac{1}{y}\right) + o\left(y^\alpha L\left(\frac{1}{y}\right)\right), \quad \text{as } y \to 0+, \\ &= \chi_1 \left(\sum_{j=1}^n \tilde{\chi}_j y^j + \tilde{\chi}_n^*(y)\right) + \sum_{i=2}^n \chi_i \left(\left(\sum_{j=1}^n \tilde{\chi}_j y^j\right)^i + o(y^{n+1})\right) \\ &\quad + \chi_\alpha(\tilde{\chi}_1 y(1+o(1)))^\alpha L\left(\frac{1}{y}\right) + o\left(y^\alpha L\left(\frac{1}{y}\right)\right), \quad \text{as } y \to 0+. \end{aligned}$$

Hence, for some $r_i \in \mathbb{R}$, $2 \le i \le n$,

$$y = \chi_1 \tilde{\chi}_1 y + \sum_{i=2}^n r_i y^i + \chi_1 \tilde{\chi}_n^*(y) + \chi_\alpha \tilde{\chi}_1^\alpha y^\alpha L\left(\frac{1}{y}\right) + o\left(y^\alpha L\left(\frac{1}{y}\right)\right), \quad \text{as } y \to 0+.$$

Therefore $\chi_1\tilde{\chi}_1 = 1$, $r_i = 0$, $2 \le i \le n$, and $\tilde{\chi}_n^*(y) = -\frac{\chi_\alpha \tilde{\chi}_1^\alpha}{\chi_1} y^\alpha L(\frac{1}{y}) + o(y^\alpha L(\frac{1}{y}))$, as $y \to 0+$. Thus, the proof is completed.

(ii) Substituting (30) and (35) into (12) leads to (32). □

4 Proof of Theorem 1

The proof of Theorem 1 proceeds based on two separate arguments:

- (i) implies (ii) and (4).
- (ii) implies (i).

To make clear, we explain two procedures separately in the following subsections.

4.1 (i) implies (ii) and (4).

We prove the assertion by showing that for all $1 \le i, j \le m$,

$$\lim_{t\to\infty} \sum_{j=1}^m \frac{G_{ij} - \tilde{G}_{ij}(t)}{t^{-\alpha}L(t)} = c(1-\rho)^{-\alpha}\mu_i,$$

$$\liminf_{t\to\infty} \frac{G_{ij} - \tilde{G}_{ij}(t)}{t^{-\alpha}L(t)} \ge c(1-\rho)^{-\alpha}\mu_i g_j.$$

The former is proved in Lemma 6 and the latter is proved in Lemma 7.

Lemma 6 *If* (i) *in Theorem* 1 *holds*, *then*

$$\mathbf{e} - \tilde{G}(t)\mathbf{e} = c(1-\rho)^{-\alpha}\mu t^{-\alpha}L(t) + o(t^{-\alpha}L(t)), \quad \textit{as } t \to \infty. \tag{37}$$

Proof We deal with two cases separately: 'α is not an integer' and 'α is an integer'.

Case 1 $\alpha > 1$ *is not an integer.*

Let $n = \lfloor \alpha \rfloor$, the integer part of α. By Lemma 1, $\mathcal{B}(s)$ is expressed as

$$\mathcal{B}(s) = 1 - \beta s + \sum_{i=2}^{n} \mathcal{B}_i s^i - c\Gamma(1-\alpha) s^\alpha L\left(\frac{1}{s}\right) + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+, \tag{38}$$

for some $\mathcal{B}_i$, $2 \le i \le n$. By substituting (38) into (13),

$$\mathcal{G}(s)\xi(s) = \left(1 - \beta\chi(s) + \sum_{i=2}^{n} \mathcal{B}_i (\chi(s))^i - c\Gamma(1-\alpha)(\chi(s))^\alpha L\left(\frac{1}{\chi(s)}\right) + o\left(\chi(s)^\alpha L\left(\frac{1}{\chi(s)}\right)\right)\right)\xi(s), \quad \text{as } s \to 0+. \tag{39}$$

By Theorem 2, $\int_0^\infty x^n \, d\tilde{G}_{ij}(x) < \infty$ for all $1 \le i, j \le m$. Hence $\mathcal{G}(s)$ is expressed as

$$\mathcal{G}(s) = G + \sum_{i=1}^{n} \mathcal{G}_i s^i + \mathcal{G}_n^*(s), \quad s \ge 0, \tag{40}$$

where $\mathcal{G}_i$, $1 \le i \le n$, and $\mathcal{G}_n^*(s)$, $s \ge 0$, are $m \times m$ matrices, and $\mathcal{G}_n^*(s) = o(s^n)$, as $s \to 0+$. By Lemma 4,

$$\chi(s) = \sum_{i=1}^{n} \chi_i s^i + \chi_n^*(s), \quad s \ge 0, \tag{41}$$

$$\xi(s) = \mathbf{e} + \sum_{i=1}^{n} \xi_i s^i + \xi_n^*(s), \quad s \ge 0, \tag{42}$$

where $\chi_i \in \mathbb{R}$, $1 \le i \le n$, $\chi_n^*(s) \in \mathbb{R}$, $s \ge 0$, and $\chi_n^*(s) = o(s^n)$, as $s \to 0+$, and ξ_i, $1 \le i \le n$, and $\xi_n^*(s)$, $s \ge 0$, are m-dimensional column vectors, and $\xi_n^*(s) = o(s^n)$, as $s \to 0+$. By substituting (40–42) into (39), and comparing $o(s^n)$ terms,

$$G\xi_n^*(s) + \mathcal{G}_n^*(s)\mathbf{e} = \xi_n^*(s) - \beta\chi_n^*(s)\mathbf{e} - c\Gamma(1-\alpha)\chi_1^\alpha s^\alpha L\left(\frac{1}{s}\right)\mathbf{e} + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+. \tag{43}$$

By substituting (40–42) into (10) and comparing $o(s^n)$ terms,

$$(C + DG)\xi_n^*(s) + D\mathcal{G}_n^*(s)\mathbf{e} = -\chi_n^*(s)\mathbf{e} + O(s^{n+1}), \quad \text{as } s \to 0+. \tag{44}$$

Hence

$$\begin{aligned}(G - I)\xi_n^*(s) &= -(\mathbf{eg} - C - DG)^{-1}(G - I)(C + DG)\xi_n^*(s) \\ &= (\mathbf{eg} - C - DG)^{-1}(G - I)D\mathcal{G}_n^*(s)\mathbf{e} + O(s^{n+1}), \quad \text{as } s \to 0+.\end{aligned} \tag{45}$$

By (43) and (45),

$$\left(I + (\mathbf{eg} - C - DG)^{-1}(G - I)D\right)\mathcal{G}_n^*(s)\mathbf{e} = -\beta\chi_n^*(s)\mathbf{e} - c\Gamma(1-\alpha)\chi_1^\alpha s^\alpha L\left(\frac{1}{s}\right)\mathbf{e} + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+.$$

Premultiplying $\mathbf{eg} - C - DG$ to the above equation leads to

$$(\mathbf{eg} - C - DG + GD - D)\mathcal{G}_n^*(s)\mathbf{e} = -\beta\chi_n^*(s)\mathbf{e} - c\Gamma(1-\alpha)\chi_1^\alpha s^\alpha L\left(\frac{1}{s}\right)\mathbf{e} + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+. \tag{46}$$

By premultiplying $\mathbf{g}$ to (44),

$$\mathbf{g}D\mathcal{G}_n^*(s)\mathbf{e} = -\chi_n^*(s) + O(s^{n+1}), \quad \text{as } s \to 0+. \tag{47}$$

Substituting (47) into (46) leads to

$$(\mathbf{eg} - C - DG + GD - D - \beta\mathbf{eg}D)\mathcal{G}_n^*(s)\mathbf{e} = -c\Gamma(1-\alpha)\chi_1^\alpha s^\alpha L\left(\frac{1}{s}\right)\mathbf{e} + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+.$$

Therefore, by Lemma 3(i) and (ii), $\mathcal{G}_n^*(s)\mathbf{e} = -c\Gamma(1-\alpha) \times \chi_1^\alpha s^\alpha L(\frac{1}{s})\mu + o(s^\alpha L(\frac{1}{s}))$, as $s \to 0+$, which completes the proof by Lemma 1.

Case 2 $\alpha > 1$ *is an integer.*

By Lemma 2(i), $\mathcal{B}(s)$ is expressed as

$$\mathcal{B}(s) = 1 - \beta s + \sum_{i=2}^{\alpha-1} \mathcal{B}_i s^i + \frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \tilde{L}\left(\frac{1}{s}\right), \quad s > 0, \tag{48}$$

where $\mathcal{B}_i \in \mathbb{R}$ and $\tilde{L}$ is a function that satisfies

$$\lim_{x\to\infty} \frac{\tilde{L}(tx) - \tilde{L}(x)}{L(x)} = c \log t \tag{49}$$

uniformly in t on every compact subset of $(0, \infty)$. By substituting (48) into (13),

$$\mathcal{G}(s)\xi(s) = \Bigg(1 - \beta\chi(s) + \sum_{i=2}^{\alpha-1} \mathcal{B}_i(\chi(s))^i + \frac{(-1)^\alpha}{(\alpha-1)!}(\chi(s))^\alpha \tilde{L}\left(\frac{1}{\chi(s)}\right)\Bigg)\xi(s). \tag{50}$$

Let $0 < \epsilon < 1$. Since

$$1 - \tilde{B}(t) = o(t^{-\alpha+1-\epsilon}), \quad \text{as } t \to \infty,$$

by the non-integer case of this Lemma, $\mathcal{G}(s)$ is expressed as

$$\mathcal{G}(s) = G + \sum_{i=1}^{\alpha-1} \mathcal{G}_i s^i + \mathcal{G}^*_{\alpha-1}(s), \quad s \ge 0, \tag{51}$$

where $\mathcal{G}_i$, $0 \le i \le \alpha - 1$, and $\mathcal{G}^*_{\alpha-1}(s)$, $s \ge 0$, are $m \times m$ matrices, and $\mathcal{G}^*_{\alpha-1}(s) = o(s^{\alpha-1+\epsilon})$, as $s \to 0+$. By Lemma 5,

$$\chi(s) = \sum_{i=1}^{\alpha-1} \chi_i s^i + \chi^*_{\alpha-1}(s), \quad s \ge 0, \tag{52}$$

$$\xi(s) = \mathbf{e} + \sum_{i=1}^{\alpha-1} \xi_i s^i + \xi^*_{\alpha-1}(s), \quad s \ge 0, \tag{53}$$

where $\chi_i \in \mathbb{R}$, $1 \le i \le \alpha - 1$, $\chi^*_{\alpha-1}(s) \in \mathbb{R}$, $s \ge 0$, and $\chi^*_{\alpha-1}(s) = o(s^{\alpha-1+\epsilon})$, as $s \to 0+$, and ξ_i, $0 \le i \le n$, and $\xi^*_{\alpha-1}(s)$, $s \ge 0$, are m-dimensional column vectors, and $\xi^*_{\alpha-1}(s) = o(s^{\alpha-1+\epsilon})$, as $s \to 0+$. By substituting (51–53) into (50), and comparing $o(s^{\alpha-1+\epsilon})$ terms,

$$\begin{aligned} &G\xi^*_{\alpha-1}(s) + \mathcal{G}^*_{\alpha-1}(s)\mathbf{e} \\ &\quad = \xi^*_{\alpha-1}(s) - \beta\chi^*_{\alpha-1}(s)\mathbf{e} + \frac{(-1)^\alpha}{(\alpha-1)!}(\chi(s))^\alpha \tilde{L}\left(\frac{1}{\chi(s)}\right)\mathbf{e} \\ &\qquad + \phi_1 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+, \end{aligned} \tag{54}$$

for some m-dimensional column vector ϕ_1. By substituting (51–53) into (10) and comparing $o(s^{\alpha-1+\epsilon})$ terms,

$$\begin{aligned} &(C + DG)\xi^*_{\alpha-1}(s) + D\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} \\ &\quad = -\chi^*_{\alpha-1}(s)\mathbf{e} + \phi_2 s^\alpha + o(s^{\alpha+\epsilon}), \quad \text{as } s \to 0+, \end{aligned} \tag{55}$$

for some m-dimensional column vector ϕ_2. Hence

$$\begin{aligned} &(G - I)\xi^*_{\alpha-1}(s) \\ &\quad = -(\mathbf{eg} - C - DG)^{-1}(G - I)(C + DG)\xi^*_{\alpha-1}(s) \\ &\quad = (\mathbf{eg} - C - DG)^{-1}(G - I)D\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} \\ &\qquad + \phi_3 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+, \end{aligned} \tag{56}$$

for some m-dimensional column vector ϕ_3. By (54) and (56),

$$\begin{aligned} &\big(I + (\mathbf{eg} - C - DG)^{-1}(G - I)D\big)\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} \\ &\quad = -\beta\chi^*_{\alpha-1}(s)\mathbf{e} + \frac{(-1)^\alpha}{(\alpha-1)!}(\chi(s))^\alpha \tilde{L}\left(\frac{1}{\chi(s)}\right)\mathbf{e} \\ &\qquad + \phi_4 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+, \end{aligned}$$

for some m-dimensional column vector ϕ_4. Premultiplying $\mathbf{eg} - C - DG$ to the above equation leads to

$$\begin{aligned} &\big(\mathbf{eg} - C - DG + GD - D\big)\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} \\ &\quad = -\beta\chi^*_{\alpha-1}(s)\mathbf{e} + \frac{(-1)^\alpha}{(\alpha-1)!}(\chi(s))^\alpha \tilde{L}\left(\frac{1}{\chi(s)}\right)\mathbf{e} \\ &\qquad + \phi_5 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+, \end{aligned} \tag{57}$$

for some m-dimensional column vector ϕ_5. By premultiplying $\mathbf{g}$ to (55),

$$\begin{aligned} &\mathbf{g}D\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} = -\chi^*_{\alpha-1}(s) + \mathbf{g}\phi_2 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+. \end{aligned} \tag{58}$$

Substituting (58) into (57) leads to

$$\begin{aligned} &\big(\mathbf{eg} - C - DG + GD - D - \beta\mathbf{eg}D\big)\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} \\ &\quad = \frac{(-1)^\alpha}{(\alpha-1)!}(\chi(s))^\alpha \tilde{L}\left(\frac{1}{\chi(s)}\right)\mathbf{e} + \phi_6 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+, \end{aligned}$$

for some m-dimensional column vector ϕ_6. Therefore, by Lemma 3(i) and (ii),

$$\begin{aligned} &\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} = \frac{(-1)^\alpha}{(\alpha-1)!}(\chi(s))^\alpha \tilde{L}\left(\frac{1}{\chi(s)}\right)\mu + \phi_7 s^\alpha + o(s^{\alpha+\epsilon}), \\ &\quad \text{as } s \to 0+, \end{aligned}$$

for some m-dimensional column vector ϕ_7. Hence

$$\begin{aligned} &\lim_{s\to 0+} \frac{\mathcal{G}^*_{\alpha-1}(s)\mathbf{e} - \frac{(-1)^\alpha}{(\alpha-1)!}(\chi_1 s)^\alpha \tilde{L}\big(\frac{1}{\chi_1 s}\big)\mu - \phi_7 s^\alpha}{s^\alpha L\big(\frac{1}{s}\big)} \\ &\quad = \frac{(-1)^\alpha}{(\alpha-1)!} \lim_{s\to 0+} \frac{(\chi(s))^\alpha \tilde{L}\big(\frac{1}{\chi(s)}\big) - (\chi_1 s)^\alpha \tilde{L}\big(\frac{1}{\chi_1 s}\big)}{s^\alpha L\big(\frac{1}{s}\big)}\mu \end{aligned}$$

$$= \frac{(-1)^\alpha}{(\alpha-1)!}\lim_{s\to 0+}\left(\frac{\chi(s)}{s}\right)^\alpha \frac{\tilde{L}(\frac{1}{\chi(s)}) - \tilde{L}(\frac{1}{\chi_1 s})}{L(\frac{1}{s})}\mu$$

$$+ \frac{(-1)^\alpha}{(\alpha-1)!}\lim_{s\to 0+}\left(\left(\frac{\chi(s)}{s}\right)^\alpha - \chi_1^\alpha\right)\frac{\tilde{L}(\frac{1}{\chi_1 s})}{L(\frac{1}{s})}\mu. \quad (59)$$

Since the convergence in (49) is uniform in t on every compact subset of $(0,\infty)$, the first term on the right hand side of (59) vanishes. By (52), $\frac{\chi(s)}{s} = \chi_1(1+o(s^\epsilon))$, as $s\to 0+$. Hence $(\frac{\chi(s)}{s})^\alpha = \chi_1^\alpha(1+o(s^\epsilon))$, as $s\to 0+$. Therefore the last term on the right hand side of (59) vanishes. Thus

$$\lim_{s\to 0+}\frac{\mathcal{G}(s)\mathbf{e} - \mathcal{B}(\chi_1 s)\mu + \sum_{i=0}^{\alpha}\theta_i s^i}{s^\alpha L(\frac{1}{s})} = 0,$$

for some m-dimensional column vectors θ_i, $0\le i\le\alpha$. Since $\mathcal{B}(\chi_1 s)$, $s\ge 0$, is the LST of $\tilde{B}(\chi_1^{-1}t)$, $t\in\mathbb{R}$, and

$$\tilde{B}(\infty) - \tilde{B}(\chi_1^{-1}t) = \chi_1^\alpha t^{-\alpha}L(t) + o(t^{-\alpha}L(t)),$$

as $t\to\infty$,

Lemma 2(ii) completes the proof of (37). □

Lemma 7 *Let L be a slowly varying function and $\alpha > 1$. If* (i) *in Theorem 1 holds, then*

$$\liminf_{t\to\infty}\frac{G_{ij} - \tilde{G}_{ij}(t)}{t^{-\alpha}L(t)} \ge c(1-\rho)^{-\alpha}\mu_i g_j,$$

for $1\le i,j\le m$. (60)

We proceed the proof through 5 steps.

Step 1 *Let $\epsilon > 0$ and $\delta > 0$ be given. Then, there exists $x_0 > 0$ such that for all $1\le i,j\le m$,*

$$\mathbb{P}_{((1-\rho+\delta)x,i)}(\tau > x,\ J(\tau) = j) > g_j - \epsilon \quad \text{if } x\ge x_0. \quad (61)$$

Proof Let $U(t)$, $t\ge 0$, be the amount of work arrived during $(0,t]$. For each $v\ge 0$ and $1\le i\le m$,

$$\lim_{t\to\infty}\frac{U(t)}{t} = \lambda\beta = \rho, \quad \mathbb{P}_{(v,i)}\text{-a.s.}$$

Hence, for each $v\ge 0$ and $1\le i\le m$, there exists $K>0$ such that

$$\mathbb{P}_{(v,i)}(U(t) > (\rho-\delta)t \text{ for all } t\ge K) > 1 - \frac{\epsilon}{2}. \quad (62)$$

We can choose $K>0$ uniformly for $v\ge 0$ and $1\le i\le m$, because the left hand side of the above is independent of $v\ge 0$ and the range of i is finite. If $V(0) > K$, then $\tau \ge V(0) > K$. Hence, if $V(0) > K$,

$U(t) > (\rho-\delta)t$ for all $t\ge K$ implies

$U(\tau) > (\rho-\delta)\tau$.

Since $\tau = V(0) + U(\tau)$,

$U(t) > (\rho-\delta)t$ for all $t\ge K$ implies

$\tau > V(0) + (\rho-\delta)\tau$,

if $V(0) > K$. Therefore, for all $v > K$ and $1\le i\le m$,

$$\mathbb{P}_{(v,i)}\left(\tau > \frac{v}{1-\rho+\delta}\right) \ge \mathbb{P}_{(v,i)}(U(t) > (\rho-\delta)t \text{ for all } t\ge K). \quad (63)$$

By (62) and (63), for all $v > K$ and $1\le i\le m$,

$$\mathbb{P}_{(v,i)}\left(\tau > \frac{v}{1-\rho+\delta}\right) > 1 - \frac{\epsilon}{2}. \quad (64)$$

On the other hand,

$$\lim_{v\to\infty}\mathbb{P}_{(v,i)}(J(\tau) = j) = \lim_{v\to\infty}\left(e^{(C+DG)v}\right)_{ij} = g_j. \quad (65)$$

By (64) and (65), we obtain (61). □

Step 2 *There exists $x_0 > 0$ such that for all $v\ge 0$ and $1\le i,j\le m$,*

$$\mathbb{P}_{(v,i)}(\tau > x,\ J(\tau) = j) \ge (g_j - \epsilon)\mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau)$$

if $x\ge x_0$,

where $\sigma_{(1-\rho+\delta)x} \equiv \inf\{t\ge 0: V(t) > (1-\rho+\delta)x\}$.

Proof Since $\{(V(t), J(t)): t\ge 0\}$ is a piecewise deterministic Markov process [5, 9], it is a strong Markov process. Since $\sigma_{(1-\rho+\delta)x}$ is a stopping time, by the strong Markov property,

$$\begin{aligned}
&\mathbb{P}_{(v,i)}(\tau > x,\ J(\tau) = j)\\
&\quad\ge \mathbb{P}_{(v,i)}(\tau > \sigma_{(1-\rho+\delta)x} + x,\ J(\tau) = j)\\
&\quad= \mathbb{E}_{(v,i)}\big(\mathbf{1}_{\{\sigma_{(1-\rho+\delta)x} < \tau\}}\\
&\qquad\times \mathbb{P}_{(V(\sigma_{(1-\rho+\delta)x}), J(\sigma_{(1-\rho+\delta)x}))}(\tau > x,\ J(\tau) = j)\big).
\end{aligned}$$

Hence, by Step 1, there exists x_0 such that for all $v\ge 0$ and $1\le i\le m$,

$$\mathbb{P}_{(v,i)}(\tau > x,\ J(\tau) = j) \ge \mathbb{E}_{(v,i)}(\mathbf{1}_{\{\sigma_{(1-\rho+\delta)x} < \tau\}}(g_j - \epsilon))$$

if $x\ge x_0$,

which completes the proof. □

Step 3 *For $v\ge 0$ and $1\le i\le m$,*

$$\begin{aligned}
&\mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau)\\
&\quad\ge \sum_{k=1}^{m}\sum_{l=1}^{m}\int_0^\infty \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} > t,\ \tau > t,\ J(t) = k)\\
&\qquad\times D_{kl}(1 - \tilde{B}((1-\rho+\delta)x))\,dt. \quad (66)
\end{aligned}$$

Proof Observe that

$$\begin{aligned}
&\mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau) \\
&\quad = \sum_{k=1}^{m} \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau,\ J(\sigma_{(1-\rho+\delta)x}-) = k) \\
&\quad \ge \sum_{k=1}^{m} \mathbb{P}_{(v,i)}\big(\sigma_{(1-\rho+\delta)x} < \tau,\ J(\sigma_{(1-\rho+\delta)x}-) = k, \\
&\qquad\qquad V(\sigma_{(1-\rho+\delta)x}) - V(\sigma_{(1-\rho+\delta)x}-) > (1-\rho+\delta)x\big).
\end{aligned} \tag{67}$$

Since

$$\begin{aligned}
&\mathbb{P}_{(v,i)}\big(\sigma_{(1-\rho+\delta)x} < \tau,\ t < \sigma_{(1-\rho+\delta)x} \le t + \Delta t, \\
&\qquad J(\sigma_{(1-\rho+\delta)x}-) = k, \\
&\qquad V(\sigma_{(1-\rho+\delta)x}) - V(\sigma_{(1-\rho+\delta)x}-) > (1-\rho+\delta)x\big) \\
&\quad = \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} > t,\ \tau > t,\ J(t) = k) \\
&\qquad \times \sum_{l=1}^{m} D_{kl}(1 - \tilde{B}((1-\rho+\delta)x))\Delta t + o(\Delta t), \\
&\quad \text{as } \Delta \to 0+,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{d}{dt}\mathbb{P}_{(v,i)}\big(\sigma_{(1-\rho+\delta)x} < \tau,\ \sigma_{(1-\rho+\delta)x} \le t, \\
&\qquad J(\sigma_{(1-\rho+\delta)x}-) = k, \\
&\qquad V(\sigma_{(1-\rho+\delta)x}) - V(\sigma_{(1-\rho+\delta)x}-) > (1-\rho+\delta)x\big) \\
&\quad = \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} > t,\ \tau > t,\ J(t) = k) \\
&\qquad \times \sum_{l=1}^{m} D_{kl}(1 - \tilde{B}((1-\rho+\delta)x)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{P}_{(v,i)}\big(\sigma_{(1-\rho+\delta)x} < \tau,\ J(\sigma_{(1-\rho+\delta)x}-) = k, \\
&\qquad V(\sigma_{(1-\rho+\delta)x}) - V(\sigma_{(1-\rho+\delta)x}-) > (1-\rho+\delta)x\big) \\
&\quad = \int_0^\infty \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} > t,\ \tau > t,\ J(t) = k) \\
&\qquad \times \sum_{l=1}^{m} D_{kl}(1 - \tilde{B}((1-\rho+\delta)x))\, dt.
\end{aligned} \tag{68}$$

Substituting (68) into (67) completes the proof. □

Step 4 *For* $1 \le i \le m$,

$$\liminf_{x\to\infty} \frac{\int_{[0,(1-\rho+\delta)x]} \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau)\, d\tilde{B}(v)}{1 - \tilde{B}((1-\rho+\delta)x)} \ge \mu_i - 1.$$

Proof We have

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\int_{[0,(1-\rho+\delta)x]} \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau)\, d\tilde{B}(v)}{1 - \tilde{B}((1-\rho+\delta)x)} \\
&\quad \ge \int_{[0,\infty)} \liminf_{x\to\infty} \frac{\mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} < \tau)}{1 - \tilde{B}((1-\rho+\delta)x)}\, d\tilde{B}(v) \\
&\quad \ge \int_{[0,\infty)} \liminf_{x\to\infty} \sum_{k=1}^{m}\sum_{l=1}^{m} \\
&\qquad \int_0^\infty \mathbb{P}_{(v,i)}(\sigma_{(1-\rho+\delta)x} > t,\ \tau > t,\ J(t) = k) \\
&\qquad \times D_{kl}\, dt\, d\tilde{B}(v) \\
&\quad \ge \int_{[0,\infty)} \sum_{k=1}^{m}\sum_{l=1}^{m} \int_0^\infty \mathbb{P}_{(v,i)}(\tau > t,\ J(t) = k) \\
&\qquad \times D_{kl}\, dt\, d\tilde{B}(v),
\end{aligned} \tag{69}$$

where we have used Fatou's Lemma in the first and the last inequalities and (66) in the second inequality.

Let $M(t)$, $t \ge 0$, be the number of arrivals during $(0, t]$. Since

$$\begin{aligned}
&\mathbb{E}_{(v,i)}[(M(\tau \wedge (t + \Delta t)) - M(\tau \wedge t))] \\
&\quad = \sum_{k=1}^{m} \mathbb{E}_{(v,i)}[(M(\tau \wedge (t + \Delta t)) - M(\tau \wedge t))\mathbf{1}_{\{J(t)=k\}}] \\
&\quad = \sum_{k=1}^{m} \mathbb{P}_{(v,i)}(\tau > t,\ J(t) = k) \sum_{l=1}^{m} D_{kl}\Delta t + o(\Delta t), \\
&\quad \text{as } \Delta t \to 0+,
\end{aligned}$$

we have

$$\frac{d}{dt}\mathbb{E}_{(v,i)}[M(\tau \wedge t)] = \sum_{k=1}^{m} \mathbb{P}_{(v,i)}(\tau > t,\ J(t) = k) \sum_{l=1}^{m} D_{kl},$$

where $a \wedge b$ denotes $\min\{a, b\}$. Thus

$$\mathbb{E}_{(v,i)}[M(\tau)] = \sum_{k=1}^{m}\sum_{l=1}^{m} \int_0^\infty \mathbb{P}_{(v,i)}(\tau > t,\ J(t) = k) D_{kl}\, dt. \tag{70}$$

On the other hand, by Lemma 3(ii) (see also (94)),

$$\mu_i = 1 + \int_{[0,\infty)} \mathbb{E}_{(v,i)}[M(\tau)]\, d\tilde{B}(v).$$

Therefore, by (70),

$$\mu_i = 1 + \int_{[0,\infty)} \sum_{k=1}^{m}\sum_{l=1}^{m} \int_0^\infty \mathbb{P}_{(v,i)}(\tau > t,\ J(t) = k) D_{kl}\, dt. \tag{71}$$

By (69) and (71), the assertion follows.

Step 5 *Equation* (60) *holds.*

Proof Write

$$\liminf_{x\to\infty}\frac{\int_{[0,\infty)}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}$$
$$\geq\liminf_{x\to\infty}\frac{\int_{[0,(1-\rho+\delta)x]}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}$$
$$+\liminf_{x\to\infty}\frac{\int_{((1-\rho+\delta)x,\infty)}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}. \tag{72}$$

By Step 1,

$$\liminf_{x\to\infty}\frac{\int_{((1-\rho+\delta)x,\infty)}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}\geq g_j-\epsilon. \tag{73}$$

By Steps 2 and 4,

$$\liminf_{x\to\infty}\frac{\int_{[0,(1-\rho+\delta)x]}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}\geq(g_j-\epsilon)(\mu_i-1). \tag{74}$$

By substituting (73) and (74) into (72), and letting $\epsilon\to 0+$,

$$\liminf_{x\to\infty}\frac{\int_{[0,\infty)}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}\geq\mu_i g_j.$$

Hence

$$\liminf_{x\to\infty}\frac{G_{ij}-\tilde{G}_{ij}(x)}{x^{-\alpha}L(x)}=\liminf_{x\to\infty}\frac{\int_{[0,\infty)}\mathbb{P}_{(v,i)}(\tau>x,\ J(\tau)=j)\,d\tilde{B}(v)}{1-\tilde{B}((1-\rho+\delta)x)}\times\frac{1-\tilde{B}((1-\rho+\delta)x)}{x^{-\alpha}L(x)}\geq\mu_i g_j c(1-\rho+\delta)^{-\alpha}.$$

Letting $\delta\to 0+$ completes the proof. □

4.2 (ii) implies (i)

We deal with two cases separately: 'α is not an integer' and 'α is an integer'.

Case 1 $\alpha>1$ *is not an integer.*

Suppose that (ii) in Theorem 1 holds. Let $n=\lfloor\alpha\rfloor$. By Theorem 2, $\int_0^\infty x^n\,d\tilde{B}(x)<\infty$. Hence $\mathcal{B}(s)$ is written as

$$\mathcal{B}(s)=1-\beta s+\sum_{i=2}^{n}\mathcal{B}_i s^i+\mathcal{B}_n^*(s),\quad s\geq 0,$$

for some $\mathcal{B}_i\in\mathbb{R}$, $2\leq i\leq n$, and $\mathcal{B}_n^*(s)$, $s\geq 0$, such that $\mathcal{B}_n^*(s)=o(s^n)$ as $s\to 0+$. Thus

$$\mathcal{B}(\chi(s))=1-\beta\chi(s)+\sum_{i=2}^{n}\mathcal{B}_i(\chi(s))^i+\mathcal{B}_n^*(\chi(s)),\quad s\geq 0. \tag{75}$$

On the other hand, by (14), (32) and (35),

$$\begin{aligned}\mathcal{B}(\chi(s))&=\mathbf{g}\mathcal{G}(s)\xi(s)\\&=\mathbf{g}\mathcal{G}(s)\mathbf{e}+\mathbf{g}(\mathcal{G}(s)-G)(\xi(s)-\mathbf{e})\\&=1+\mathbf{g}\mathcal{G}_1\mathbf{e}s+\sum_{k=2}^{n}\gamma_k s^k-\Gamma(1-\alpha)\mathbf{g}H\mathbf{e}s^\alpha L\left(\frac{1}{s}\right)\\&\quad+o\left(s^\alpha L\left(\frac{1}{s}\right)\right),\quad\text{as } s\to 0+,\end{aligned} \tag{76}$$

for some γ_k, $2\leq k\leq n$. By (30), (75) and (76),

$$\mathcal{B}_n^*(\chi(s))=(\beta\chi_\alpha-\Gamma(1-\alpha)\mathbf{g}H\mathbf{e})s^\alpha L\left(\frac{1}{s}\right)+o\left(s^\alpha L\left(\frac{1}{s}\right)\right),\quad\text{as } s\to 0+,$$

where χ_α is given in Lemma 5. Hence, by (30),

$$\mathcal{B}_n^*(s)=(1-\rho)^\alpha(\beta\chi_\alpha-\Gamma(1-\alpha)\mathbf{g}H\mathbf{e})s^\alpha L\left(\frac{1}{s}\right)+o\left(s^\alpha L\left(\frac{1}{s}\right)\right),\quad\text{as } s\to 0+,$$

Thus (i) in Theorem 1 is derived by Lemma 1.

Case 2 $\alpha>1$ *is an integer.*

Suppose that (ii) in Theorem 1 holds. Let $\epsilon\in(0,1)$. Since $G-\tilde{G}(t)=o(t^{-(\alpha-1+\epsilon)})$ as $t\to\infty$, the non-integer case of Theorem 1 and Lemma 1 give

$$\mathcal{B}(s)=1-\beta s+\sum_{i=2}^{\alpha-1}\mathcal{B}_i s^i+\mathcal{B}_{\alpha-1}^*(s),\quad s\geq 0,$$

where $\mathcal{B}_i \in \mathbb{R}$, $2 \le i \le \alpha - 1$, $\mathcal{B}^*_{\alpha-1}(s) \in \mathbb{R}$, $s \ge 0$, and $\mathcal{B}^*_{\alpha-1}(s) = o(s^{\alpha-1+\epsilon})$ as $s \to 0+$. Hence

$$\mathcal{B}(\chi(s)) = 1 - \beta\chi(s) + \sum_{i=2}^{\alpha-1} \mathcal{B}_i(\chi(s))^i + \mathcal{B}^*_{\alpha-1}(\chi(s)), \quad s \ge 0. \tag{77}$$

By Lemma 2, $\mathcal{G}(s)$ is written as

$$\mathcal{G}(s) = G + \sum_{i=1}^{\alpha-1} \mathcal{G}_i s^i + \frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \tilde{L}\left(\frac{1}{s}\right), \quad s > 0, \tag{78}$$

where $\mathcal{G}_i$, $1 \le i \le \alpha$, are $m \times m$ matrices and $\tilde{L}$ is an $m \times m$ matrix-valued function such that

$$\lim_{x\to\infty} \frac{\tilde{L}(tx) - \tilde{L}(x)}{L(x)} = H \log t, \tag{79}$$

uniformly in t on every compact subset of $(0, \infty)$. Since $G - \tilde{G}(t) = o(t^{-(\alpha-1+\epsilon)})$ as $t \to \infty$, Lemma 5 gives expressions

$$\chi(s) = \sum_{i=1}^{\alpha-1} \chi_i s^i + \chi^*_{\alpha-1}(s), \quad s \ge 0, \tag{80}$$

$$\xi(s) = \mathbf{e} + \sum_{i=1}^{\alpha-1} \xi_i s^i + \xi^*_{\alpha-1}(s), \quad s \ge 0, \tag{81}$$

where χ_i and $\chi^*_{\alpha-1}(s)$ are real numbers, $\chi^*_{\alpha-1}(s) = o(s^{\alpha-1+\epsilon})$ as $s \to 0+$, ξ_i and $\xi^*_{\alpha-1}$ are m-dimensional column vectors, and $\xi^*_{\alpha-1}(s) = o(s^{\alpha-1+\epsilon})$ as $s \to 0+$. By (14), (78) and (81),

$$\begin{aligned}\mathcal{B}(\chi(s)) &= \mathbf{g}\mathcal{G}(s)\xi(s) \\ &= \mathbf{g}\mathcal{G}(s)\mathbf{e} + \mathbf{g}(\mathcal{G}(s) - G)(\xi(s) - \mathbf{e}) \\ &= 1 + \mathbf{g}\mathcal{G}_1\mathbf{e}s + \sum_{k=2}^{\alpha} \gamma_k s^k + \frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \mathbf{g}\tilde{L}\left(\frac{1}{s}\right)\mathbf{e} \\ &\quad + o(s^{\alpha+\epsilon}),\end{aligned}$$
$$\text{as } s \to 0+, \tag{82}$$

for some γ_k, $2 \le k \le \alpha$. By (77), (80) and (82),

$$\begin{aligned}&-\beta\chi^*_{\alpha-1}(s) + \mathcal{B}^*_{\alpha-1}(\chi(s)) \\ &\quad = \psi_1 s^\alpha + \frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \mathbf{g}\tilde{L}\left(\frac{1}{s}\right)\mathbf{e} + o(s^{\alpha+\epsilon}),\end{aligned}$$
$$\text{as } s \to 0+, \tag{83}$$

for some $\psi_1 \in \mathbb{R}$. Since

$$\begin{aligned}&\mathbf{g}(-sI + C + D\mathcal{G}(s))\mathbf{e} \\ &\quad = \mathbf{g}(-sI + C + D\mathcal{G}(s))\xi(s) \\ &\qquad + \mathbf{g}(-sI + C + D\mathcal{G}(s))(\mathbf{e} - \xi(s)) \\ &\quad = -\chi(s) + \mathbf{g}(-sI + C + D\mathcal{G}(s))(\mathbf{e} - \xi(s)),\end{aligned}$$

we have

$$\frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \mathbf{g}D\tilde{L}\left(\frac{1}{s}\right)\mathbf{e} = -\chi^*_{\alpha-1}(s) + \psi_2 s^\alpha + o(s^{\alpha+\epsilon}),$$
$$\text{as } s \to 0+, \tag{84}$$

for some $\psi_2 \in \mathbb{R}$. By (83) and (84),

$$\begin{aligned}\mathcal{B}^*_{\alpha-1}(\chi(s)) &= \frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \mathbf{g}(I - \beta D)\tilde{L}\left(\frac{1}{s}\right)\mathbf{e} + \psi_3 s^\alpha \\ &\quad + o(s^{\alpha+\epsilon}),\end{aligned}$$
$$\text{as } s \to 0+, \tag{85}$$

for some $\psi_3 \in \mathbb{R}$. By (31) and (79),

$$\begin{aligned}\mathcal{B}^*_{\alpha-1}(y) &= \frac{(-1)^\alpha}{(\alpha-1)!}((1-\rho)y)^\alpha \mathbf{g}(I - \beta D)\tilde{L}\left(\frac{1}{(1-\rho)y}\right)\mathbf{e} \\ &\quad + (\gamma_\alpha + \psi)(1-\rho)^\alpha y^\alpha + o(y^{\alpha+\epsilon}),\end{aligned}$$
$$\text{as } y \to 0+.$$

Therefore

$$\begin{aligned}&\lim_{y\to 0+}\left[y^\alpha L\left(\frac{1}{y}\right)\right]^{-1} \\ &\quad \times \Bigg[(\mathcal{B}(y) + \beta\mathbf{g}D\mathcal{G}((1-\rho)y)\mathbf{e}) \\ &\quad - \mathbf{g}\mathcal{G}((1-\rho)y)\mathbf{e} + \sum_{i=0}^{\alpha}\theta_i y^i\Bigg] = 0\end{aligned}$$

for some $\theta_i \in \mathbb{R}$, $0 \le i \le \alpha$. Lemma 2 asserts that

$$\begin{aligned}&(1 - \tilde{B}(t)) + \beta\mathbf{g}D(G - \tilde{G}((1-\rho)^{-1}t))\mathbf{e} \\ &\quad = \mathbf{g}(G - \tilde{G}((1-\rho)^{-1}t))\mathbf{e} + o(t^{-\alpha}L(t)),\end{aligned}$$
$$\text{as } t \to \infty.$$

Hence

$$1 - \tilde{B}(t) = ct^{-\alpha}L(t) + o(t^{-\alpha}L(t)), \quad \text{as } t \to \infty,$$

where $c = (1-\rho)^\alpha \mathbf{g}(I - \beta D)H\mathbf{e}$. □

Appendix 1: Proof of Lemma 2

(i) By (15), $\mathcal{F}(s)$ is written as

$$\mathcal{F}(s) = \sum_{i=0}^{n-1} \mathcal{F}_i s^i + \frac{(-1)^n}{(n-1)!} s^n \tilde{L}\left(\frac{1}{s}\right), \quad s > 0, \tag{86}$$

for some $\mathcal{F}_i \in \mathbb{R}$, $0 \le i \le n-1$, and a function $\tilde{L}$ such that $\tilde{L}(\frac{1}{s}) = o(\frac{1}{s})$ as $s \to 0+$. For $x \ge 0$, let

$$R(x) = \begin{cases} F(x), & \text{if } n = 1, \\ \int_0^x \int_{x_{n-1}}^\infty \cdots \int_{x_2}^\infty (F(\infty) - F(x_1)) \\ \quad \times dx_1 \ldots dx_{n-2}\, dx_{n-1}, & \text{if } n \ge 2. \end{cases} \tag{87}$$

Denote by $\mathcal{R}$ the LST of R. Then (86) leads to

$$\tilde{L}\left(\frac{1}{s}\right) = (n-1)! \frac{R(\infty) - \mathcal{R}(s)}{s}, \quad s > 0.$$

By (15), we have

$$R(\infty) - R(x) = \frac{b}{(n-1)!} \frac{1}{x} L(x) + o\left(\frac{1}{x} L(x)\right), \quad \text{as } x \to \infty. \tag{88}$$

A slight modification of Theorem 2 in [6] asserts that

$$\lim_{x\to\infty} \frac{\int_0^x (R(\infty) - R(y))\, dy - \frac{1}{(n-1)!} \tilde{L}(x)}{L(x)} = \frac{b}{(n-1)!} \gamma, \tag{89}$$

where γ is the Euler constant. By (88) and (89),

$$\begin{aligned} &\frac{1}{(n-1)!} \lim_{x\to\infty} \frac{\tilde{L}(tx) - \tilde{L}(x)}{L(x)} \\ &= \lim_{x\to\infty} \frac{\int_x^{tx} (R(\infty) - R(y))\, dy}{L(x)} \\ &\quad + \lim_{x\to\infty} \frac{\int_0^x (R(\infty) - R(y))\, dy - \frac{1}{(n-1)!}\tilde{L}(x)}{L(x)} \\ &\quad - \lim_{x\to\infty} \frac{\int_0^{tx} (R(\infty) - R(y))\, dy - \frac{1}{(n-1)!}\tilde{L}(tx)}{L(x)} \\ &= \frac{b}{(n-1)!} \log t. \end{aligned} \tag{90}$$

Further, the convergence in (90) is uniform in t on every compact subset of $(0, \infty)$, because $\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1$ uniformly in t on every compact subset of $(0, \infty)$.

(ii) By (17), we have

$$\mathcal{K}(s) = \sum_{i=0}^{n-1} (a\mathcal{F}_i - \theta_i) s^i - \theta_n s^n + \frac{a(-1)^n}{(n-1)!} s^n \tilde{L}\left(\frac{1}{s}\right) + o\left(s^n L\left(\frac{1}{s}\right)\right), \quad \text{as } s \to 0+.$$

Hence

$$\mathcal{K}(s) = \sum_{i=0}^{n-1} (a\mathcal{F}_i - \theta_i) s^i + \frac{(-1)^n}{(n-1)!} s^n \tilde{L}_1\left(\frac{1}{s}\right), \quad s > 0, \tag{91}$$

for a function $\tilde{L}_1$ such that

$$\tilde{L}_1(x) = (-1)^{n-1}(n-1)!\theta_n + a\tilde{L}(x) + o(L(x)), \quad \text{as } x \to \infty. \tag{92}$$

For $x \ge 0$, let

$$R_1(x) = \begin{cases} K(x), & \text{if } n = 1, \\ \int_0^x \int_{x_{n-1}}^\infty \cdots \int_{x_2}^\infty (K(\infty) - K(x_1)) \\ \quad \times dx_1 \ldots dx_{n-2}\, dx_{n-1}, & \text{if } n \ge 2. \end{cases}$$

Denote by $\mathcal{R}_1$ the LST of R_1. Then (91) leads to

$$\tilde{L}_1\left(\frac{1}{s}\right) = (n-1)! \frac{R_1(\infty) - \mathcal{R}_1(s)}{s}, \quad s > 0.$$

By (16) and (92),

$$\lim_{s\to 0+} \frac{\tilde{L}_1(\frac{t}{s}) - \tilde{L}_1(\frac{1}{s})}{L(\frac{1}{s})} = ab \log t \quad \text{for all } t > 0.$$

A slight modification of Theorem 2 in [6] asserts that

$$R_1(\infty) - R_1(x) = \frac{ab}{(n-1)!} \frac{1}{x} L(x) + o\left(\frac{1}{x} L(x)\right), \quad \text{as } x \to \infty,$$

which implies that

$$K(\infty) - K(x) = ab\, x^{-n} L(x) + o(x^{-n} L(x)), \quad \text{as } x \to \infty.$$

Appendix 2: Proof of Lemma 3(i)

We use the following fact:

Fact 1 *Let M_1 and M_2 be square matrices of the same size. Then,*

$$\mathrm{sp}(M_1 M_2) < 1 \quad \text{if and only if} \quad \mathrm{sp}(M_2 M_1) < 1,$$

where $\mathrm{sp}(M)$ denotes the spectral radius of M.

The above fact is easily proved by noting that for any square matrix M,

$$\mathrm{sp}(M) < 1 \quad \text{if and only if} \quad \sum_{k=0}^\infty M^k \text{ converges}.$$

Now we prove that $\mathbf{eg} - C - D - DG + GD - \beta \mathbf{eg} D$ is invertible. Write

$$\begin{aligned} &\mathbf{eg} - C - D - DG + GD - \beta \mathbf{eg} D \\ &= (\mathbf{eg} - C - DG) \\ &\quad \times \left(I - (\mathbf{eg} - C - DG)^{-1}(I - G + \beta \mathbf{eg}) D\right). \end{aligned}$$

Since $\mathbf{eg} - C - DG$ is invertible, the proof is completed by Fact 1 if we show that

$$\mathrm{sp}\big(D(\mathbf{eg} - C - DG)^{-1}(I - G + \beta\mathbf{eg})\big) < 1. \tag{93}$$

Since

$$\begin{aligned}
&\int_0^\infty e^{(C+DG)x}(1-\tilde{B}(x))\,dx \\
&\quad = (\mathbf{eg} - C - DG)^{-1} \\
&\qquad \times \int_0^\infty \big(\mathbf{eg} - (C+DG)e^{(C+DG)x}\big)(1-\tilde{B}(x))\,dx \\
&\quad = (\mathbf{eg} - C - DG)^{-1} \\
&\qquad \times \left(\beta\mathbf{eg} + I - \int_0^\infty e^{(C+DG)x}\,d\tilde{B}(x)\right) \\
&\quad = (\mathbf{eg} - C - DG)^{-1}(\beta\mathbf{eg} + I - G),
\end{aligned}$$

$D(\mathbf{eg} - C - DG)^{-1}(I - G + \beta\mathbf{eg})$ is nonnegative. Since $\mathbf{eg} - C - DG$ and $I - G + \beta\mathbf{eg}$ are commute and $\mathbf{g}(\mathbf{eg} - C - DG)^{-1} = \mathbf{g}$,

$$\begin{aligned}
&\pi D(\mathbf{eg} - C - DG)^{-1}(I - G + \beta\mathbf{eg}) \\
&\quad = \pi D(I - G + \beta\mathbf{eg})(\mathbf{eg} - C - DG)^{-1} \\
&\quad = \pi(D - DG)(\mathbf{eg} - C - DG)^{-1} + \beta\pi D\mathbf{eg} \\
&\quad = -\pi(C + DG)(\mathbf{eg} - C - DG)^{-1} + \rho\mathbf{g} \\
&\quad = \pi(\mathbf{eg} - C - DG - \mathbf{eg})(\mathbf{eg} - C - DG)^{-1} + \rho\mathbf{g} \\
&\quad = \pi(I - \mathbf{eg}) + \rho\mathbf{g} \\
&\quad = \pi - (1-\rho)\mathbf{g}.
\end{aligned}$$

Thus

$$\pi D(\mathbf{eg} - C - DG)^{-1}(I - G + \beta\mathbf{eg}) < \pi,$$

which implies (93).

Appendix 3: Proof of Lemma 3(ii)

Let $M(t)$, $t \ge 0$, be the number of arrivals during $(0, t]$. We prove (ii) in Lemma 3 by showing that

$$\mu_i = \int_{[0,\infty)} \mathbb{E}_{(v,i)}(1 + M(\tau))\,d\tilde{B}(v), \quad 1 \le i \le m. \tag{94}$$

Let

$$\tilde{\mathcal{G}}_{ij}(z) \equiv \int_{[0,\infty)} \mathbb{E}_{(v,i)}\big(z^{1+M(\tau)}\mathbf{1}_{\{J(\tau)=j\}}\big)\,d\tilde{B}(v),$$
$$0 < z \le 1,\ 1 \le i, j \le m.$$

Note that $\tilde{\mathcal{G}}_{ij}(1) = G$. It is known (see, for example, (50) in [10]) that

$$\tilde{\mathcal{G}}(z) = z\int_{[0,\infty)} e^{(C+D\tilde{\mathcal{G}}(z))x}\,d\tilde{B}(x), \quad 0 < z \le 1. \tag{95}$$

Let $-\tilde{\chi}(z)$, $0 < z \le 1$, be the eigenvalue of $C + D\tilde{\mathcal{G}}(z)$ with largest real part, and $\tilde{\xi}(z)$, $0 < z \le 1$, the right eigenvector corresponding to the eigenvalue $-\tilde{\chi}(z)$ and normalized by $\mathbf{g}\tilde{\xi}(z) = 1$. Note that $\tilde{\chi}(1) = 0$ and $\tilde{\xi}(1) = \mathbf{e}$.

Observe that $\tilde{\mathcal{G}}(z)$ is C^1 with respect to z on (0, 1], since $\int_{[0,\infty)} \mathbb{E}_{(v,i)}(1 + M(\tau))\,d\tilde{B}(v) < \infty$ (see Theorem 3.1.1 in [11]). Since the eigenvalue $-\tilde{\chi}(z)$ of $C + D\tilde{\mathcal{G}}(z)$ has the algebraic multiplicity 1 and $C + D\tilde{\mathcal{G}}(z)$ is C^1 on (0, 1], $\tilde{\chi}(z)$ is also C^1 on (0, 1] by the implicit function theorem. The right eigenvector $\tilde{\xi}(z)$ is determined by $(C + D\tilde{\mathcal{G}}(z))\tilde{\xi}(z) = -\tilde{\chi}(z)\tilde{\xi}(z)$ and $\mathbf{g}\tilde{\xi}(z) = 1$. Hence $\tilde{\xi}(z)$ is a unique solution of $(\tilde{\chi}(z)I + C + D\tilde{\mathcal{G}}(z) + \mathbf{eg})\tilde{\xi}(z) = \mathbf{e}$. Therefore $(\tilde{\chi}(z)I + C + D\tilde{\mathcal{G}}(z) + \mathbf{eg})$ is invertible and $\tilde{\xi}(z) = (\tilde{\chi}(z)I + C + D\tilde{\mathcal{G}}(z) + \mathbf{eg})^{-1}\mathbf{e}$. Thus $\tilde{\xi}(z)$ is also C^1 on (0, 1].

By (95),

$$\tilde{\mathcal{G}}(z)\tilde{\xi}(z) = z\mathcal{B}(\tilde{\chi}(z))\tilde{\xi}(z), \quad 0 < z \le 1.$$

By differentiating the above equation with respect to z and evaluating at $z = 1-$,

$$G\tilde{\xi}_1 + \tilde{\mathcal{G}}_1\mathbf{e} = \mathbf{e} - \beta\tilde{\chi}_1\mathbf{e} + \tilde{\xi}_1, \tag{96}$$

where $\tilde{\mathcal{G}}_1 \equiv \frac{d}{dz}\tilde{\mathcal{G}}(z)|_{z=1-}$, $\tilde{\chi}_1 \equiv \frac{d}{dz}\tilde{\chi}(z)|_{z=1-}$ and $\tilde{\xi}_1 \equiv \frac{d}{dz}\tilde{\xi}(z)|_{z=1-}$. By differentiating $(C + D\tilde{\mathcal{G}}(z))\tilde{\xi}(z) = -\tilde{\chi}(z)\tilde{\xi}(z)$ with respect to z and evaluating at $z = 1-$,

$$(C + DG)\tilde{\xi}_1 + D\tilde{\mathcal{G}}_1\mathbf{e} = -\tilde{\chi}_1\mathbf{e}. \tag{97}$$

By the identity $(G - I) = -(\mathbf{e}g - C - DG)^{-1}(G - I)(C + DG)$ and (97),

$$\begin{aligned}
(G - I)\tilde{\xi}_1 &= -(\mathbf{e}g - C - DG)^{-1}(G - I)(C + DG)\tilde{\xi}_1 \\
&= (\mathbf{e}g - C - DG)^{-1}(G - I)D\tilde{\mathcal{G}}_1\mathbf{e}.
\end{aligned} \tag{98}$$

By (96) and (98),

$$\big(I + (\mathbf{e}g - C - DG)^{-1}(G - I)D\big)\tilde{\mathcal{G}}_1\mathbf{e} = \mathbf{e} - \beta\tilde{\chi}_1\mathbf{e}.$$

Premultiplying $\mathbf{e}g - C - DG$ to the above equation leads to

$$(\mathbf{eg} - C - DG + GD - D)\tilde{\mathcal{G}}_1\mathbf{e} = \mathbf{e} - \beta\tilde{\chi}_1\mathbf{e}. \tag{99}$$

By premultiplying $\mathbf{g}$ to (97),

$$\mathbf{g}D\tilde{\mathcal{G}}_1\mathbf{e} = -\tilde{\chi}_1. \tag{100}$$

Substituting (100) into (99) leads to

$$(\mathbf{eg} - C - DG + GD - D - \beta\mathbf{eg}D)\tilde{\mathcal{G}}_1\mathbf{e} = \mathbf{e}.$$

Therefore,

$$\mu = \tilde{\mathcal{G}}_1\mathbf{e} = (\mathbf{eg} - C - DG + GD - D - \beta\mathbf{eg}D)^{-1}\mathbf{e}.$$

Appendix 4: Proof of Corollary 1

The discrete time process embedded in the underlying Markov process of the MAP immediately after the beginnings of busy periods becomes a Markov process with the one step transition probability matrix $G(-C)^{-1}D$. Since κ is the stationary probability vector of the embedded Markov process, it is determined by a unique solution of

$$\kappa G(-C)^{-1}D = \kappa \quad \text{and} \quad \kappa\mathbf{e} = 1. \tag{101}$$

Using the identity $\mathbf{g}(C + DG) = 0$, it can be shown that $(\mathbf{g}D\mathbf{e})^{-1}\mathbf{g}D$ is the solution of (101). Hence κ is written as

$$\kappa = (\mathbf{g}D\mathbf{e})^{-1}gD. \tag{102}$$

The proof is completed by (3), (102), Theorem 1 and Lemma 3(v).

Appendix 5: Proof of Corollary 2

We deal with two cases separately: 'α is not an integer' and 'α is an integer'.

Case 1 *$\alpha > 1$ is not an integer.*

Let

$$\mathcal{G}^v(s) \equiv \int_0^\infty e^{-st}\,d\tilde{G}^v(t), \quad s \geq 0.$$

Then, it is well known that

$$\mathcal{G}^v(s) = e^{(-sI+C+D\mathcal{G}(s))v}, \quad s \geq 0. \tag{103}$$

For $m \times m$ matrices X and ΔX, by the multivariate Taylor's theorem,

$$\begin{aligned} e^{X+\Delta X} &= e^X + \sum_{i=1}^m\sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\Delta X_{ij} \\ &\quad + \sum_{k=2}^n \frac{1}{k!}\sum_{i_1=1}^m\sum_{j_1=1}^m \cdots \sum_{i_k=1}^m\sum_{j_k=1}^m \frac{\partial^k e^X}{\partial X_{i_1j_1}\ldots\partial X_{i_kj_k}} \\ &\quad \times \Delta X_{i_1j_1}\ldots\Delta X_{i_kj_k} + O(|\Delta X|^{n+1}), \end{aligned} \tag{104}$$

as $|\Delta X| \to 0$, where $|\Delta X| \equiv \max_{1\leq i,j\leq m}|\Delta X_{ij}|$.

By Theorem 1 and Lemma 1, $\mathcal{G}(s)$ is expressed as

$$\mathcal{G}(s) = G + \sum_{k=1}^n \mathcal{G}_k s^k + \mathcal{G}_n^*(s),$$

for some $m \times m$ matrices $\mathcal{G}_k$, $1 \leq k \leq n$, and $m \times m$ matrix-valued function $\mathcal{G}_n^*$ such that

$$\begin{aligned} \mathcal{G}_n^*(s) &= -\Gamma(1-\alpha)c(1-\rho)^{-\alpha}\mu\mathbf{g}s^\alpha L\left(\frac{1}{s}\right) \\ &\quad + o\left(s^\alpha L\left(\frac{1}{s}\right)\right), \end{aligned}$$

as $s \to 0+$.

By substituting

$$X + \Delta X = (-sI + C + D\mathcal{G}(s))v,$$

$$X = (C + DG)v, \quad \text{and}$$

$$\Delta X = -svI + vD\sum_{k=1}^n \mathcal{G}_k s^k + vD\mathcal{G}_n^*(s)$$

into (104),

$$\begin{aligned} \mathcal{G}^v(s) &= e^{(-sI+C+D\mathcal{G}(s))v} \\ &= e^{(C+DG)v} + \sum_{k=1}^n \mathcal{G}_k^v s^k \\ &\quad + \sum_{i=1}^m\sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\bigg|_{X=(C+DG)v}(vD\mathcal{G}_n^*(s))_{ij} \\ &\quad + O(s^{n+1}), \end{aligned}$$

as $s \to 0+$, for some $m \times m$ matrices $\mathcal{G}_k^v$, $1 \leq k \leq n$. Therefore Lemma 1 asserts that

$$\begin{aligned} &\lim_{t\to\infty}\frac{\tilde{G}^v(\infty) - \tilde{G}^v(t)}{t^{-\alpha}L(t)} \\ &\quad = \sum_{i=1}^m\sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\bigg|_{X=(C+DG)v}(vDc(1-\rho)^{-\alpha}\mu\mathbf{g})_{ij}. \end{aligned} \tag{105}$$

Let E_{ij} be the $m \times m$ 0–1 matrix whose elements are all zeros but the (i, j)th one which is 1. Using the identity

$$\begin{aligned} \frac{\partial e^X}{\partial X_{ij}} &= \frac{\partial}{\partial X_{ij}}\sum_{k=0}^\infty \frac{1}{k!}X^k \\ &= \sum_{k=1}^\infty \frac{1}{k!}\sum_{l=1}^k X^{l-1}E_{ij}X^{k-l}, \end{aligned}$$

we have

$$\sum_{i=1}^m \sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\bigg|_{X=v(C+DG)} (vDc(1-\rho)^{-\alpha}\mu\mathbf{g})_{ij}$$
$$= c(1-\rho)^{-\alpha} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^\infty \frac{1}{k!} \sum_{l=1}^k (v(C+DG))^{l-1} E_{ij} (v(C+DG))^{k-l} (vD\mu\mathbf{g})_{ij}$$
$$= c(1-\rho)^{-\alpha} \sum_{k=1}^\infty \frac{1}{k!} \sum_{l=1}^k (v(C+DG))^{l-1} (vD\mu\mathbf{g})(v(C+DG))^{k-l}$$
$$= c(1-\rho)^{-\alpha} \sum_{k=1}^\infty \frac{1}{k!} (C+DG)^{k-1} D\mu\mathbf{g} v^k$$
$$= c(1-\rho)^{-\alpha} \sum_{k=1}^\infty \frac{1}{k!} (C+DG)^{k-1} v^k \big(\mathbf{eg}(\mathbf{eg}-C-DG)^{-1} - (C+DG)(\mathbf{eg}-C-DG)^{-1}\big) D\mu\mathbf{g}$$
$$= c(1-\rho)^{-\alpha} \left(v\mathbf{eg} - \sum_{k=1}^\infty \frac{1}{k!}(C+DG)^k v^k\right)(\mathbf{eg}-C-DG)^{-1} D\mu\mathbf{g}$$
$$= c(1-\rho)^{-\alpha} \big(v\mathbf{eg} + I - e^{(C+DG)v}\big)(\mathbf{eg}-C-DG)^{-1} D\mu\mathbf{g}$$
$$= H_v, \tag{106}$$

where we have used $\mathbf{g}(C+DG)=0$ and $(C+DG)\mathbf{e}=0$ for the 3rd and 5th equalities, respectively, and H_v is given by (6). The proof is completed by (105) and (106).

Case 2 $\alpha > 1$ *is an integer.*

Let

$$\mathcal{G}^v(s) \equiv \int_0^\infty e^{-st}\, d\tilde{G}^v(t), \quad s \ge 0.$$

By Theorem 1 and Lemma 2, $\mathcal{G}(s)$ is expressed as

$$\mathcal{G}(s) = G + \sum_{k=1}^{\alpha-1} \mathcal{G}_k s^k + \mathcal{G}^*_\alpha(s),$$

where $\mathcal{G}_k$, $1 \le k \le \alpha-1$, are $m\times m$ matrices and $\mathcal{G}^*_\alpha(s) = \frac{(-1)^\alpha}{(\alpha-1)!} s^\alpha \tilde{L}(\frac{1}{s})$ for some $m\times m$ matrix-valued function $\tilde{L}$ such that

$$\lim_{x\to\infty} \frac{\tilde{L}(tx)-\tilde{L}(x)}{L(x)} = c(1-\rho)^{-\alpha}\mu\mathbf{g}\log t$$

uniformly in t on every compact subset of $(0,\infty)$. By a similar way as in the non-integer case, it can be shown that, for any $\epsilon \in (0,1)$,

$$\mathcal{G}^v(s) = e^{(C+DG)v} + \sum_{k=1}^\alpha \mathcal{G}^v_k s^k + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\bigg|_{X=(C+DG)v} (vD\mathcal{G}^*_\alpha(s))_{ij} + o(s^{\alpha+\epsilon}),$$

as $s\to 0+$, for some $m\times m$ matrices $\mathcal{G}^v_k$, $1\le k\le\alpha$. Hence

$$\lim_{s\to 0+}\left[s^\alpha L\left(\frac{1}{s}\right)\right]^{-1} \times \left[\mathcal{G}^v(s) - \sum_{i=1}^m \sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\bigg|_{X=(C+DG)v} (\mathcal{G}(s))_{ij} + \sum_{k=0}^\alpha \Theta_k s^k\right] = 0,$$

for some $m\times m$ matrices Θ_i, $0\le i\le\alpha$. Therefore Lemma 2 asserts

$$\lim_{t\to\infty} \frac{\tilde{G}^v(\infty)-\tilde{G}^v(t)}{t^{-\alpha}L(t)} = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial e^X}{\partial X_{ij}}\bigg|_{X=(C+DG)v} (vDc(1-\rho)^{-\alpha}\mu\mathbf{g})_{ij}.$$

The proof is completed by the identity (106).

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