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On the first passage times of reflected O-U processes with two-sided barriers

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Abstract In this article, we give the Laplace transform of the first passage times of reflected Ornstein-Uhlenbeck process with two-sided barriers.

Keywords Ornstein-Uhlenbeck processes · Two-sided reflected processes · The first passage times

AMS Subject Classifications 60H10 · 60G40 · 90B05

1 Introduction

In [4], Ward and Glynn proposed a queueing system with reneging and then constructed a reflected Ornstein-Uhlenbeck (O-U) process via an appropriate Markovian approximation procedure. In their successive article [5], Ward and Glynn further derived the stationary distribution of the reflected process. Now, from the point of view of queueing system, it is natural to suggest a fluid model with finite buffer capacity. This leads to consider a refined queueing process with two-sided barriers. Hence motivated by Ward and Glynn's one-sided reflected O-U processes, we are concerned with the following reflected O-U process with two-sided barriers. Let $Z = \{Z_t, t \geq 0\}$ be an one-dimensional reflected Ornstein-Uhlenbeck process with barriers 0 and b ($b > 0$ is a constant), which is defined by

$$\begin{cases} dZ_t = (\mu - \alpha Z_t)dt + \sigma dB_t + dL_t - dU_t, \\ Z_0 = x \in [0, b], \end{cases} \quad (1.1)$$

where $B = \{B_t, t \geq 0\}$ is an one-dimensional standard Brownian motion, and $\mu \in \mathbb{R}$, α, $\sigma \in \mathbb{R}^+$. Here $L = \{L_t, t \geq 0\}$ and $U = \{U_t, t \geq 0\}$ are the regulators of point 0 and b, respectively. Further, the processes L and U are uniquely determined by the following properties (see e.g. [2]),

- Both $t \to L_t$ and $t \to U_t$ are continuous nondecreasing processes with $L_0 = U_0 = 0$ and $t \in \mathbb{R}^+$,
- L and U increase only when $Z = 0$ and $Z = b$, respectively, i.e., $\int_0^t \mathbb{I}_{\{Z_s=0\}} dL_s = L_t$ and $\int_0^t \mathbb{I}_{\{Z_s=b\}} dU_s = U_t$, for $t \geq 0$.

Recall (1.1), it is actually equivalent to

$$Z_t = x - \alpha \int_0^t Z_s ds + \sigma B_t + \mu t + L_t - U_t \in [0, b], \quad \text{for } t \geq 0. \quad (1.2)$$

As for the existence and uniqueness of the strong solutions of (1.1), readers might refer to Lions and Sznitman [3] for more details. In this note, our objective is to derive the Laplace

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transform on the first passage times of the reflected process Z with two-sided barriers.

2 Main result and its proof

In this section we consider the Eq. (1.1) with assumptions that $\mu = 0, \sigma = 1$ and $Z_0 = x \in [0, b]$. Let $y \in [0, b]$, define the first passage time by

$$T(y) := \inf\{t \geq 0 : \ Z_t = y\}, \tag{2.1}$$

with the usual convention $\inf \emptyset = \infty$. On the other hand, suppose $\lambda > 0$. For $f \in C^2([0, b])$, define a linear operator

$$A^{(\lambda)} f(x) := \frac{1}{2} f''(x) - \alpha x f'(x) - \lambda f(x), \quad \text{for } x \in [0, b].$$

With this we are going to give the expression of the Laplace transform of $T(y)$.

Theorem 2.1. *Let $x \in [0, b]$ and $\lambda > 0$. Suppose that $f_1^{(\lambda)}$ and $f_2^{(\lambda)}$ are the respective solutions of the following equations*

$$A^{(\lambda)} f_1^{(\lambda)}(y) = 0, \ \ y \in [0, b], \ \ \textit{and} \ \ f_1^{(\lambda)\prime}(0) = 0, \tag{2.2}$$

and

$$A^{(\lambda)} f_2^{(\lambda)}(y) = 0, \ \ y \in [0, b], \ \ \textit{and} \ \ f_2^{(\lambda)\prime}(b) = 0. \tag{2.3}$$

If $f_1^{(\lambda)}(y) \neq 0$ for $x \leq y \leq b$, and $f_2^{(\lambda)}(y) \neq 0$ for $0 \leq y \leq x$, then

$$\mathbb{E}_x\big[e^{-\lambda T(y)}\big] = \frac{f_1^{(\lambda)}(x)}{f_1^{(\lambda)}(y)}, \quad \textit{for } x \leq y \leq b, \tag{2.4}$$

and

$$\mathbb{E}_x[e^{-\lambda T(y)}] = \frac{f_2^{(\lambda)}(x)}{f_2^{(\lambda)}(y)}, \quad \textit{for } 0 \ \leq y \leq x. \tag{2.5}$$

Proof: Applying the Itô formula for $h(t, x) := \exp(-\lambda t) f(x)$ with $f \in C_b^2([0, b])$, we have

$$\begin{aligned} h(t, Z_t) &= h(0, Z_0) + \int_0^t \frac{\partial h}{\partial s}(s, Z_s)ds + \int_0^t \frac{\partial h}{\partial x}(s, Z_s)dZ_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 h}{\partial x^2}(s, Z_s)d\langle Z, Z\rangle_s \\ &= f(Z_0) + \int_0^t e^{-\lambda s}\Big[-\lambda f(Z_s) - \alpha Z_s f'(Z_s) \\ &\quad + \frac{1}{2} f''(Z_s)\Big]ds + \int_0^t e^{-\lambda s} f'(Z_s)dL_s \\ &\quad - \int_0^t e^{-\lambda s} f'(Z_s)dU_s + \int_0^t e^{-\lambda s} f'(Z_s)dB_s \\ &= f(Z_0) + \int_0^t e^{-\lambda s} A^{(\lambda)} f(Z_s)ds + f'(0)\int_0^t e^{-\lambda s} dL_s \\ &\quad - f'(b)\int_0^t e^{-\lambda s} dU_s + \int_0^t e^{-\lambda s} f'(Z_s)dB_s, \end{aligned} \tag{2.6}$$

since L and U are finite variation (FV) processes. Let $T < \infty$ be a stopping time and $x \in [0, b]$. It follows from martingale optional theorem, that

$$\begin{aligned} &\mathbb{E}_x[e^{-\lambda T} f(Z_T)] \\ &\quad = f(x) + \mathbb{E}_x\left[\int_0^T e^{-\lambda s} A^{(\lambda)} f(Z_s)ds\right] \\ &\quad + f'(0)\mathbb{E}_x\left[\int_0^T e^{-\lambda s} dL_s\right] - f'(b)\mathbb{E}_x\left[\int_0^T e^{-\lambda s} dU_s\right]. \end{aligned} \tag{2.7}$$

In particular, take $T = T(y)$ for $y \in [0, b]$, and note that

$$\mathbb{E}_x\left[\int_0^{T(y)} e^{-\lambda s} dU_s\right] = 0, \quad \text{for } x \leq y \leq b,$$

and

$$\mathbb{E}_x\left[\int_0^{T(y)} e^{-\lambda s} dL_s\right] = 0, \quad \text{for } 0 \leq y \leq x.$$

Then,

$$\mathbb{E}_x\big[e^{-\lambda T(y)}f(Z_{T(y)})\big] = f(x) + \mathbb{E}_x\left[\int_0^{T(y)} e^{-\lambda s}A^{(\lambda)}f(Z_s)ds\right] + f'(0)\mathbb{E}_x\left[\int_0^{T(y)} e^{-\lambda s}dL_s\right], \quad \text{for } x \le y \le b, \quad (2.8)$$

and

$$\mathbb{E}_x\big[e^{-\lambda T(y)}f(Z_{T(y)})\big] = f(x) + \mathbb{E}_x\left[\int_0^{T(y)} e^{-\lambda s}A^{(\lambda)}f(Z_s)ds\right] - f'(b)\mathbb{E}_x\left[\int_0^{T(y)} e^{-\lambda s}dU_s\right], \quad \text{for } 0 \le y \le x. \quad (2.9)$$

Replace f by $f_1^{(\lambda)}$ in (2.8) and by $f_2^{(\lambda)}$ in (2.9), we immediately get (2.4) and (2.5) by $Z_{T(y)} = y$, $f_1^{(\lambda)\prime}(0) = 0$ and $f_2^{(\lambda)\prime}(b) = 0$. Thus the proof of the theorem is completed. □

Remark 2.1. Although neither $f_1^{(\lambda)}$ nor $f_2^{(\lambda)}$ in the theorem is unique, each of their ratios by (2.4) and (2.5) is unique, which determine the first passage time $T(y)$.

In the following, we are going to give an explicit expression on the Laplace transform of $T(y)$. Actually we can get a solution $g^{(\lambda)}$ of (2.2) and a solution $l^{(\lambda)}$ of (2.3) as well:

$$g^{(\lambda)}(x) := \sum_{k=0}^{\infty}\frac{(2x^2)^k}{(2k)!}\prod_{j=0}^{k-1}(\lambda + 2j\alpha), \quad \text{for } x \in [0, b],$$

and

$$l^{(\lambda)}(x) := g^{(\lambda)}(x) - C^{(\lambda)}h^{(\lambda)}(x), \quad \text{for } x \in [0, b],$$

where the constant $C^{(\lambda)}$ and the function $h^{(\lambda)}(\cdot)$ are determined respectively by

$$C^{(\lambda)} := \frac{\sum_{k=1}^{\infty}\frac{2^k b^{2k-1}}{(2k-1)!}\prod_{j=0}^{k-1}(\lambda + 2j\alpha)}{\sum_{k=0}^{\infty}\frac{(2b^2)^k}{(2k)!}\prod_{j=0}^{k-1}[\lambda + (2j+1)\alpha]} < \infty,$$

and

$$h^{(\lambda)}(x) := \sum_{k=0}^{\infty}\frac{2^k x^{2k+1}}{(2k+1)!}\prod_{j=0}^{k-1}[\lambda + (2j+1)\alpha] < \infty, \quad \text{for } x \in [0, b].$$

Hence by (2.4) and (2.5) in the theorem,

$$\mathbb{E}_x[e^{-\lambda T(y)}] = \frac{g^{(\lambda)}(x)}{g^{(\lambda)}(y)}, \quad \text{for } x \le y \le b, \quad (2.10)$$

and

$$\mathbb{E}_x[e^{-\lambda T(y)}] = \frac{l^{(\lambda)}(x)}{l^{(\lambda)}(y)}, \quad \text{for } 0 \le y \le x. \quad (2.11)$$

By the way, if let $y = 0$ or b, then we get the first passage times at 0 and b respectively:

$$\mathbb{E}_x[e^{-\lambda T(0)}] = \frac{l^{(\lambda)}(x)}{l^{(\lambda)}(0)} = l^{(\lambda)}(x),$$

and

$$\mathbb{E}_x[e^{-\lambda T(b)}] = \frac{g^{(\lambda)}(x)}{g^{(\lambda)}(b)}.$$

Where we adopt the convention $\prod_{j=0}^{-1}(\lambda + 2j\alpha) = \prod_{j=0}^{-1}(\lambda + (2j+1)\alpha) = 1$.

Remark 2.2. As for the case $\mu \neq 0$ and $\sigma \neq 1$, in order to derive Laplace transform of $T(y)$, one needs to solve the following equations,

$$\tilde{A}^{(\lambda)}\tilde{f}_1^{(\lambda)}(y) = 0, \ y \in [0, b], \quad \text{and} \quad \tilde{f}_1^{(\lambda)\prime}(0) = 0, \quad (2.12)$$

and

$$\tilde{A}^{(\lambda)}\tilde{f}_2^{(\lambda)}(y) = 0, \ y \in [0, b], \quad \text{and} \quad \tilde{f}_2^{(\lambda)\prime}(b) = 0, \quad (2.13)$$

where

$$\tilde{A}^{\lambda}f(x) := \frac{\sigma^2}{2}f''(x) + (\mu - \alpha x)f'(x) - \lambda f(x), \quad \text{for } f \in C^2([0, b]).$$

For the Eqs. (2.12) and (2.13), it is not hard to get their power series solutions. However it seems to be difficult to give the explicit forms of the solutions as above $g^{(\lambda)}$ and $l^{(\lambda)}$ for $\mu = 0$ and $\sigma = 1$.

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