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An inequality for entangled qutrits in SU(3) basis

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Abstract
It is well known from the representation theory of particle physics that the tensor product of two fundamental representation of SU(2) and SU(3) group can be decomposed to obtain the desired spectrum of the physical states. In this paper, we apply this tenet in case of two *non-local* qubits and qutrits, which leads the complete spectrum of their entangled states in their respective basis. For qutrit system, the study of their properties reveals the existence of a new $\sqrt{2}$ inequality, in addition to usual Bell-CHSH type $2\sqrt{2}$ inequality, which is significant from the experimental point of view.

Keywords Entanglement · SU(3) group · Qutrit states · Inequality for qutrits

1 Introduction

Quantum entanglement is considered as one of the most exquisite traits of the quantum world [1]. During the early years of its development, Einstein–Podolsky–Rosen (EPR) raised their doubt about the ability of quantum theory to explain the nature in a way that is consistent with both the special theory of relativity and the objective realism [2]. Their stance necessitates the postulation of the concept of hidden variables in support of their perspective and also to address some compatibility issues related to the foundational aspects of quantum mechanics. However, soon after that, Bohr refuted their viewpoint by emphasizing on the crucial role of the observer in the measurement process in the quantum realm, which invalidates the proposition of such variables [3]. Later, the issue became more popular when Bohm reformulated it in a more pragmatic footing [4], enabling Bell [5, 6], and thereafter Clauser–Horne–Shimony–Holt (CHSH) [7] to point out the existence of an experimentally testable identity.

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Finally, it was verified experimentally by Aspect and his coworkers [9], which was further refined by fixing several loopholes [10–12]. Although these studies have laid the foundation of the modern paradigm of quantum information science, the study of entanglement beyond qubit particularly in high dimension always remains at the center stage for harnessing faster information processing.

The simplest representative of an entangled system is the two-qubit system, formally known as the Bell states. This set of entangled states is constructed by the successive operations of the Hadamard and CNOT gates, some variant of the Pauli gates, on the two-qubit state which is intricately linked with the Bell-CHSH inequality [7, 8]. However, in a three-dimensional Hilbert space, the straightforward extension to an entangled qutrit system is not available for various reasons. Recently significant efforts have been made to understand various aspect of qutrits from different perspective. These endeavors include the exploration of the entanglement scenario of the qutrit system [13–19], the Bloch space structure of qutrits [20–26], etc. In addition, since the qutrit system is inherently a three-level system, several quantum-optical phenomena have been investigated within the framework of the SU(3) group [27–31]. Although these studies revealed the crucial role of the SU(3) group to understand various properties of the qutrit system, the information about all conceivable SU(3)-based entangled qutrit states is still unknown. The prime hindrance of it is due to the absence of experimentally verifiable inequality for the qutrits, non-availability of suitable representation of the qutrit gates, and added to them, the complexity arises due to our incomplete understanding of the entanglement scenario of high-dimensional Hilbert space.

There is another reason to study the qutrit system. Soon after the experimental breakthrough of the Bell-CHSH inequality [7, 9], it was well understood that an inequality could serve as a powerful tool to verify the non-local character of the quantum correlation. In that context, the qutrit-based inequality emerges as a natural choice to address the entanglement scenario of the high dimensional system. In the recent past, the Collins–Gisin–Linden–Massar–Popescu (CGLMP) inequality has garnered significant attention as a multi-partite entangled system, including the qutrit system [32–35]. The evaluation of the magnitude of violation from this inequality necessitates the calculation of the expectation value of the CGLMP joint probability operator using an appropriate qutrit state. In this context, the maximally entangled state (MES), namely $\psi_{MES} = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j\rangle_A \otimes |j\rangle_B$, which is a singlet state of SU(3) group, has been studied in the recent past [13, 14, 32, 36, 37]. Therefore, it is intriguing to explore other conceivable representations of the two-qutrit states using SU(3) group and then to look for the associated inequality. The measurement setting of the CGLMP operator is determined by various factors such as, the number of parties involved, the number of observable per party, and the number of outcomes per observable [32]. Such requirement imposes a stringent constraint on the geometry of the experimental setup, making it quite difficult for functional implementation. Therefore, it is worthwhile to search for an inequality for the qutrit system in a simpler framework which is closely parallel to the conventional Bell-CHSH setup.

The aim of this paper is to generate the complete spectrum of entangled qutrit states within framework of SU(3) group and to find the associated inequality. To accomplish

this objective, the paper is structured as follows: In Section II, we develop a heuristic scheme to derive the Bell states from the SU(2) representation theory and review its connection with the Bell-CHSH-Tsirelson bound [7, 8]. In Section III, we extend this methodology to construct all qutrit states using the SU(3) as the basis group and then, discuss their properties in IV. In Section V, we demonstrate how these states can be utilized to develop a pair of inequalities, which we refer as qutrit inequality. Finally, we conclude by summarizing the key findings of the paper and discuss the future prospects.

2 SU(2) group and bell states

To construct the Bell states, let q_i^α ($q_i^1 = u_i$, $q_i^2 = d_i$) be the fundamental representation of $SU(2)$ group in two-dimensional Hilbert space $\mathcal{H}_i^2$ with site index i ($i = A, B$). Following the decomposition of the SU(2) representation of particle physics [38, 39], the tensor product of two non-local doublets at site A and B can be expressed as their direct sum,

$$\{\bar{\mathbf{2}}\}_A \otimes \{\mathbf{2}\}_B = \mathbf{3}_{AB} \oplus \mathbf{1}_{AB}, \tag{1}$$

where $\mathbf{3}_{AB}$ and $\mathbf{1}_{AB}$ represent three triplet and one singlet states forming a composite entangled system AB in Hilbert space $\mathcal{H}_{AB}^{2\otimes2}$.

To construct Bell states, we adopt a heuristic approach and define the following *star product* ($\star$) between the non-local doublets,

$$q_A^{\alpha T} \star \sigma_0 \star q_B^\alpha := u_A \star u_B + d_A \star d_B, \tag{2a}$$

$$q_A^{\alpha T} \star \sigma_1 \star q_B^\alpha := u_A \star d_B + d_A \star u_B, \tag{2b}$$

$$q_A^{\alpha T} \star i\sigma_2 \star q_B^\alpha := u_A \star d_B - d_A \star u_B, \tag{2c}$$

$$q_A^{\alpha T} \star \sigma_3 \star q_B^\alpha := u_A \star u_B - d_A \star d_B, \tag{2d}$$

where σ_i ($i = 0, 1, 2, 3$) is the Pauli matrix. Now our method involves two steps: a) promote the star product to tensor product of the basis states ($\star \to \otimes$), i.e.,

$$u_A \star u_B + d_A \star d_B \to |\sigma_0\rangle_{AB} := u_A \otimes u_B + d_A \otimes d_B, \tag{3a}$$

$$u_A \star d_B + d_A \star u_B \to |\sigma_1\rangle_{AB} := u_A \otimes d_B + d_A \otimes u_B, \tag{3b}$$

$$u_A \star d_B - d_A \star u_B \to |\sigma_2\rangle_{AB} := u_A \otimes d_B - d_A \otimes u_B, \tag{3c}$$

$$u_A \star u_B - d_A \star d_B \to |\sigma_3\rangle_{AB} := u_A \otimes u_B - d_A \otimes d_B, \tag{3d}$$

and then, b) identify the components of the doublets with the standard qubit basis, namely, $u_i := |0\rangle_i = (1, 0)_i^T$, $d_i := |1\rangle_i = (0, 1)_i^T$. Thus the normalized Bell states obtained from Eq.(3) are given by [1, 6] (For notational convenience, we write

$|0\rangle_A \otimes |0\rangle_B = |00\rangle$, $|1\rangle_A \otimes |0\rangle_B = |10\rangle$, etc., and drop local indices $i = A, B$),

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \tag{4a}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \tag{4b}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big), \tag{4c}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big), \tag{4d}$$

where $|\sigma_0\rangle_{AB} := |\Phi^+\rangle$, $|\sigma_1\rangle_{AB} := |\Psi^+\rangle$, $|\sigma_2\rangle_{AB} := |\Psi^-\rangle$ and $|\sigma_3\rangle_{AB} := |\Phi^-\rangle$.

After developing the Bell states, we shall very briefly recall its connection with the Bell-CHSH inequality to understand the subsequent development of the paper. The correlation tensor in $\mathcal{H}_{AB}^{2\otimes2}$ read-off from this inequality is given by [1, 7, 8],

$$\hat{\mathcal{C}}^{2\otimes2} = \hat{Q} \otimes \hat{S} + \hat{R} \otimes \hat{S} + \hat{R} \otimes \hat{T} - \hat{Q} \otimes \hat{T}, \tag{5}$$

where $\hat{Q} = \hat{\sigma}_3$, $\hat{S} = -\frac{1}{\sqrt{2}}(\hat{\sigma}_3 + \hat{\sigma}_1)$, $\hat{R} = \hat{\sigma}_1$, and $\hat{T} = \frac{1}{\sqrt{2}}(\hat{\sigma}_3 - \hat{\sigma}_1)$. This setting can be equivalently expressed as,

$$\hat{\mathcal{C}}^{2\otimes2} = \begin{pmatrix} -\sqrt{2} & 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & 0 & -\sqrt{2} \end{pmatrix}. \tag{6}$$

It is easy to show that the X-structured correlation operator is equal to the outer product of two of the Bell states or, equivalently, the tensor product of the Pauli matrices (For detail see, Note added in proof),

$$\begin{aligned}\hat{\mathcal{C}}^{2\otimes2} &= 2\sqrt{2}\big(|\Psi^-\rangle\langle\Psi^-| - |\Phi^+\rangle\langle\Phi^+|\big) \\ &= -\sqrt{2}(\hat{\sigma}_3 \otimes \hat{\sigma}_3 + \hat{\sigma}_1 \otimes \hat{\sigma}_1).\end{aligned} \tag{7}$$

This ensures the nonzero expectation value giving Bell-CHSH-Tsirelson bound [8, 40],

$$\langle\Phi^+ | \hat{\mathcal{C}}^{2\otimes2} | \Phi^+\rangle = -2\sqrt{2}, \tag{8a}$$

$$\langle\Psi^- | \hat{\mathcal{C}}^{2\otimes2} | \Psi^-\rangle = 2\sqrt{2}, \tag{8b}$$

while for remaining two Bell states it vanishes. Equation(8) is the celebrated bound for the states $|\psi^i\rangle = \{|\Phi^+\rangle, |\Psi^-\rangle\}$ known as the Bell-CHSH inequality [1]

$$| \langle\psi^i | \hat{\mathcal{C}}^{2\otimes2} | \psi^i\rangle | \leq 2\sqrt{2}. \tag{9}$$

Thus, it is worth noting that even in the absence of specific information about quantum gates, e.g., Hadamard and CNOT gates, it is still possible to derive the complete spectrum of the entangled states and the inequality associated with it. We now proceed to extend above procedure to derive an inequality for the entangled qutrit system.

3 SU(3) group and entangled qutrit states

In three-dimensional Hilbert space $\mathcal{H}_i^3$ ($i = A$ or B), a qutrit system consists of three orthogonal basis states,

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle , \tag{10}$$

where the standard basis are given by,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{11}$$

with the amplitude normalized as $\sum_{i=0}^{2} |c_i|^2 = 1$. To obtain all possible entangled states in composite Hilbert space $\mathcal{H}_{AB}^{3\otimes3}$, we first consider the Gell-Mann matrices which in terms of the qutrit basis are given by,

$$\begin{aligned}
\lambda_0 &= |0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2| , \quad & \lambda_1 &= |0\rangle \langle 1| + |1\rangle \langle 0| \\
\lambda_2 &= i(|1\rangle \langle 0| - |0\rangle \langle 1|), \quad & \lambda_3 &= |0\rangle \langle 0| - |1\rangle \langle 1| , \\
\lambda_4 &= |0\rangle \langle 2| + |2\rangle \langle 0| , \quad & \lambda_5 &= i(|2\rangle \langle 0| - |0\rangle \langle 2|), \\
\lambda_6 &= |1\rangle \langle 2| + |2\rangle \langle 1| , \quad & \lambda_7 &= i(|2\rangle \langle 1| - |1\rangle \langle 2|), \\
\lambda_8 &= \frac{1}{\sqrt{3}}(|0\rangle \langle 0| + |1\rangle \langle 1| - 2 |2\rangle \langle 2|).
\end{aligned} \tag{12}$$

Here, the λ_i matrices are normalized as $\lambda_l \lambda_m = \delta_{lm} + d_{lmn}\lambda_n + f_{lmp}\lambda_p$ with d_{lmn} and f_{lmp} ($l, m, n, p = 1, 2, \ldots, 8$) as the completely symmetric and completely antisymmetric structure constants [39]. Similar to the previous section, we now consider two non-local triplet $q_i^\alpha = (u_i, d_i, s_i)^T \in \mathcal{H}_i^3$ and promote the *star product* into tensor product,

$$q_A^{\alpha T} \star \lambda_0 \star q_B^\alpha \rightarrow | \lambda_0\rangle_{AB} := u_A \otimes u_B + d_A \otimes d_B + s_A \otimes s_B, \tag{13a}$$

$$q_A^{\alpha T} \star \lambda_1 \star q_B^\alpha \rightarrow | \lambda_1\rangle_{AB} := s_A \otimes v_B + v_B \otimes s_A, \tag{13b}$$

$$q_A^{\alpha T} \star i\lambda_2 \star q_B^\alpha \rightarrow | \lambda_2\rangle_{AB} := -s_A \otimes v_B + v_B \otimes s_A, \tag{13c}$$

$$q_A^{\alpha T} \star \lambda_3 \star q_B^\alpha \rightarrow | \lambda_3\rangle_{AB} := -v_A \otimes v_B + s_A \otimes s_B, \tag{13d}$$

$$q_A^{\alpha T} \star \lambda_4 \star q_B^\alpha \rightarrow | \lambda_4\rangle_{AB} := u_A \otimes s_B + s_B \otimes u_A, \tag{13e}$$

$$q_A^{\alpha T} \star i\lambda_5 \star q_B^\alpha \rightarrow | \lambda_5\rangle_{AB} := -u_A \otimes s_B + s_B \otimes u_A, \tag{13f}$$

$$q_A^{\alpha T} \star \lambda_6 \star q_B^\alpha \rightarrow |\lambda_6\rangle_{AB} := u_A \otimes v_B + v_B \otimes u_A, \tag{13g}$$

$$q_A^{\alpha T} \star i\lambda_7 \star q_B^\alpha \rightarrow |\lambda_7\rangle_{AB} := -u_A \otimes v_B + v_B \otimes u_A, \tag{13h}$$

$$q_A^{\alpha T} \star \lambda_8 \star q_B^\alpha \rightarrow |\lambda_8\rangle_{AB} := -2u_A \otimes u_B + d_A \otimes d_B + s_A \otimes s_B. \tag{13i}$$

If we identify the states $\{u_i, d_i, s_i\}$ with the standard basis in Eq.(12), i.e., $u_i = |0\rangle_i$, $d_i = |1\rangle_i$, $s_i = |2\rangle_i$, Eq.(13) leads to the normalized entangled qutrit states,

$$|\psi_{00}\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right), \tag{14a}$$

$$\left|\psi_{21}^+\right\rangle = \frac{1}{\sqrt{2}}\left(|21\rangle + |12\rangle\right), \tag{14b}$$

$$\left|\psi_{21}^-\right\rangle = \frac{1}{\sqrt{2}}\left(|21\rangle - |12\rangle\right), \tag{14c}$$

$$|\psi_{11}\rangle = \frac{1}{\sqrt{2}}\left(-|11\rangle + |22\rangle\right), \tag{14d}$$

$$\left|\psi_{20}^+\right\rangle = \frac{1}{\sqrt{2}}\left(|20\rangle + |02\rangle\right) \tag{14e}$$

$$\left|\psi_{20}^-\right\rangle = \frac{1}{\sqrt{2}}\left(|20\rangle - |02\rangle\right), \tag{14f}$$

$$\left|\psi_{10}^+\right\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right), \tag{14g}$$

$$\left|\psi_{10}^-\right\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right), \tag{14h}$$

$$|\psi_{22}\rangle = \frac{1}{\sqrt{6}}\left(-2|00\rangle + |11\rangle + |22\rangle\right). \tag{14i}$$

where $|\lambda_0\rangle_{AB} := |\psi_{00}\rangle$, $|\lambda_1\rangle_{AB} := \left|\psi_{21}^+\right\rangle$, etc. Thus, we note that, analogous to $SU(2)$ theory, two qutrits follow SU(3) decomposition, namely, $\{\bar{3}\}_A \otimes \{3\}_B = \mathbf{8}_{AB} \oplus \mathbf{1}_{AB}$ in Hilbert space $\mathcal{H}_{AB}^{3\otimes3}$ which leads to eight octet-like and one singlet-like entangled states. Among them, only the singlet state $|\psi_{00}\rangle$ has been recently investigated both theoretically [14, 32, 34] and experimentally [36, 37], while other states remain unexplored.

4 Properties of entangled qutrit states

Before delivering the inequalities associated with these states, we shall touch upon their properties. We note that, on swapping positions, out of the nine states, $|\psi_{00}\rangle$, $\left|\psi_{10}^+\right\rangle$, $\left|\psi_{20}^+\right\rangle$, $\left|\psi_{21}^+\right\rangle$ are symmetric states, $|\psi_{11}\rangle$, $\left|\psi_{10}^-\right\rangle$, $\left|\psi_{20}^-\right\rangle$, $\left|\psi_{21}^-\right\rangle$ are antisymmetric states, while $|\psi_{22}\rangle$ does not conform to this pattern.

To see whether their subsystems are pure or mixed state, it is customary to consider the reduced density matrix of the system. Taking the partial trace over one subsystem

(say, B-subsystem) we obtain,

$$\rho_{00}^A = \frac{1}{3}\big(|0\rangle_A \langle 0| + |1\rangle_A \langle 1| + |2\rangle_A \langle 2|\big),$$

$$\rho_{10}^{+A} = \rho_{10}^{-A} = \frac{1}{2}(|0\rangle_A \langle 1| + |1\rangle_A \langle 0|),$$

$$\rho_{20}^{+A} = \rho_{20}^{-A} = \frac{1}{2}\big(|0\rangle_A \langle 2| + |2\rangle_A \langle 0|\big),$$

$$\rho_{21}^{+A} = \rho_{21}^{-A} = \rho_{11}^A = \frac{1}{2}\big(|1\rangle_A \langle 2| + |2\rangle_A \langle 1|\big),$$

$$\rho_{22}^A = \frac{1}{6}\big(4|0\rangle_A \langle 0| + |1\rangle_A \langle 1| + |2\rangle_A \langle 2|\big),$$

(15)

We note that the reduced density matrices satisfy the properties $Tr[\rho^A] = 1$ and $Tr[\rho^{A^2}] < 1$, indicating that they are essentially a mixed state. The maximally mixed state can be corroborated by noting the entropy, $S_{00}^A = \log_2 3 := 1.585$ for the singlet state in comparison with the other states, $S_{ij}^A = 1$ and $S_{22}^A = 1.2516$.

For completeness, we finally discuss the change of basis of the bipartite qutrit system. In the computational basis, most general entangled two-qutrit system is given by,

$$
\begin{aligned}
|\Psi_T\rangle = &\, c_{00}|00\rangle + c_{01}|01\rangle + c_{02}|02\rangle \\
&+ c_{10}|10\rangle + c_{11}|11\rangle + c_{12}|12\rangle \\
&+ c_{20}|20\rangle + c_{21}|21\rangle + c_{22}|22\rangle.
\end{aligned}
$$

(16)

where the amplitudes are normalized as $\sum_{i,j}|c_{ij}|^2 = 1$. Plucking back $|00\rangle$, $|01\rangle$, etc., from Eq.(14) into Eq.(16), the wave function can be transformed into $SU(3)$ basis,

$$
\begin{aligned}
\big|\Psi_{SU(3)}\big\rangle = &\, b_{00}|\psi_{00}\rangle + b_{21}^+ \big|\psi_{21}^+\big\rangle + b_{21}^- \big|\psi_{21}^-\big\rangle, \\
&+ b_{11}|\psi_{11}\rangle + b_{20}^+ \big|\psi_{20}^+\big\rangle + b_{20}^- \big|\psi_{02}^-\big\rangle, \\
&+ b_{10}^+ \big|\psi_{10}^+\big\rangle + b_{10}^- \big|\psi_{10}^-\big\rangle + b_{22}|\psi_{22}\rangle,
\end{aligned}
$$

(17)

where the amplitudes are given by,

$$b_{00} = \frac{1}{\sqrt{3}}(c_{00} - \sqrt{2}c_{22}), \quad b_{21}^+ = \frac{1}{\sqrt{2}}(c_{01} + c_{02}), \quad b_{21}^- = \frac{1}{\sqrt{2}}(c_{01} - c_{02}),$$

$$b_{11} = \frac{1}{\sqrt{6}}(\sqrt{2}c_{20} - \sqrt{3}c_{02} + c_{22}), \quad b_{20}^+ = \frac{1}{\sqrt{2}}(c_{20} + c_{21}), \quad b_{20}^- = \frac{1}{\sqrt{2}}(c_{20} - c_{21}),$$

$$b_{10}^+ = \frac{1}{\sqrt{2}}(c_{11} + c_{12}), \quad b_{10}^- = \frac{1}{\sqrt{2}}(c_{11} - c_{12}), \quad b_{22} = \frac{1}{\sqrt{6}}(\sqrt{2}c_{00} + \sqrt{3}c_{10} + c_{22}).$$

(18)

In this new basis, the amplitudes once again are normalized, i.e., $\sum_{i,j}|b_{ij}|^2 = 1$, indicating the consistency of our construction of all entangled qutrit states. In both

basis, the density matrix satisfies the pure state condition, $\rho_T^2 = \rho_T$ and $\rho_{SU(3)}^2 = \rho_{SU(3)}$.

5 Qutrit inequality

To this end, we are finally in position to discuss the inequality associated with the states obtained above. To construct the correlation tensor using SU(3) as the basis group, we adopt the methodology developed above. Similar to Eq.(6), the correlation tensor for the qutrit system should satisfy the following properties: i) it must be a symmetric matrix, ii) it should be diagonal in the qutrit basis states $|\psi_{ij}\rangle$ and, iii) it should be expressed as the tensor product of the $SU(3)$ matrices [40]. After a lengthy but straight forward calculation yields,

$$
\hat{\mathcal{C}}^{3\otimes3} = \begin{pmatrix}
0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} \\
0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\
0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}
\end{pmatrix}. \tag{19}
$$

It is worth mentioning here that, similar to Bell system, the diagonalization of above correlation matrix precisely gives the Bell-like qutrit states Eq.(14) as its eigenvectors [40]. Finally it is easy to see that the matrix can be decomposed in terms of their outer product (see Note added in proof),

$$
\begin{aligned}
\hat{\mathcal{C}}^{3\otimes3} &= \sqrt{2}\big(|\psi_{11}\rangle\langle\psi_{11}| - |\psi_{22}\rangle\langle\psi_{22}|\big) \\
&\quad + 2\sqrt{2}\big(|\psi_{00}\rangle\langle\psi_{00}| + |\psi_{10}^-\rangle\langle\psi_{10}^-| - |\psi_{12}^-\rangle\langle\psi_{12}^-| - |\psi_{20}^-\rangle\langle\psi_{20}^-|\big)
\end{aligned} \tag{20}
$$

which can be further expressed as the tensor product of Gell-Mann matrices,

$$
\begin{aligned}
\hat{\mathcal{C}}^{3\otimes3} &= \sqrt{2}(\lambda_4 \otimes \lambda_4 - \lambda_2 \otimes \lambda_2) + \frac{1}{\sqrt{2}}(\lambda_6 \otimes \lambda_6 + \lambda_7 \otimes \lambda_7) \\
&\quad - \frac{1}{2\sqrt{2}}(\lambda_3 \otimes \lambda_3 - 5\lambda_8 \otimes \lambda_8) + \frac{1}{\sqrt{6}}(\lambda_8 \otimes \lambda_0 + \lambda_0 \otimes \lambda_8) \\
&\quad - \frac{1}{2\sqrt{6}}(\lambda_3 \otimes \lambda_8 + \lambda_8 \otimes \lambda_3) - \frac{1}{3\sqrt{2}}(\lambda_0 \otimes \lambda_3 + \lambda_3 \otimes \lambda_0).
\end{aligned} \tag{21}
$$

The basis in which the correlation matrix is diagonal leads to the following bounds,

$$
\begin{aligned}
\langle \psi_{ij} \mid \hat{\mathcal{C}}^{3\otimes3} \mid \psi_{ij} \rangle &= \sqrt{2} \quad \text{for} \quad \{|\psi_{11}\rangle\}, \\
\langle \psi_{ij} \mid \hat{\mathcal{C}}^{3\otimes3} \mid \psi_{ij} \rangle &= 2\sqrt{2} \quad \text{for} \quad \{|\psi_{00}\rangle, |\psi_{10}^-\rangle\}, \\
\langle \psi_{ij} \mid \hat{\mathcal{C}}^{3\otimes3} \mid \psi_{ij} \rangle &= -\sqrt{2} \quad \text{for} \quad |\{\psi_{22}\rangle\}, \\
\langle \psi_{ij} \mid \hat{\mathcal{C}}^{3\otimes3} \mid \psi_{ij} \rangle &= -2\sqrt{2} \quad \text{for} \quad \{|\psi_{12}^-\rangle, |\psi_{20}^-\rangle\}.
\end{aligned}
\tag{22}
$$

while for the remaining states, namely, $\{|\psi_{12}^+\rangle, |\psi_{20}^+\rangle, |\psi_{10}^+\rangle\}$, it vanishes. Thus, we have two distinct set of inequalities for the entangled qutrit system,

$$
\begin{aligned}
|\langle \psi_{ij} \mid \hat{\mathcal{C}}^{3\otimes3} \mid \psi_{ij} \rangle| &\leq 2\sqrt{2} \quad \text{for} \quad \{|\psi_{00}\rangle, |\psi_{12}^-\rangle, |\psi_{20}^-\rangle, |\psi_{20}^-\rangle\}, \\
|\langle \psi_{ij} \mid \hat{\mathcal{C}}^{3\otimes3} \mid \psi_{ij} \rangle| &\leq \sqrt{2}, \quad \text{for} \quad \{|\psi_{11}\rangle, |\psi_{22}\rangle\}.
\end{aligned}
\tag{23}
$$

These are desired qutrit inequality for the qutrit system. The emergence of the Bell-CHSH-like $2\sqrt{2}$ inequality in the sideline of our treatment shows the consistency of the SU(3) based approach of which SU(2) is an subgroup.

6 Conclusion

In this paper, using the representation theory of the SU(2) and SU(3) group, we present a method to derive the Bell states for qubit and Bell-like entangled states for qutrit systems. Our approach, which is inspired by particle physics, uses a heuristic *star product* method to generate the entire spectrum of entangled states without prior information about the quantum gates. After discussing the key properties of these states and their special attributes, we proceed to find the correlation function of such system which can be expressed in terms of the tensor product of the Gell-Mann matrices. It is noteworthy that the appearance of a new $\sqrt{2}$ inequality, apart from the well-known Bell-CHSH $2\sqrt{2}$ inequality, without prior knowledge of the detector setting, is a natural outcome of our analysis, which may have experimental importance. In the ever-expanding landscape of quantum technology, the qutrits may play a significant role in shaping the future quantum computers with improved algorithms, augmented communication ability and multifaceted security.

Note added in proof:

Here, we give an outline of the derivation of Eq.(7) and Eq.(21) from the entangled states of the Bell and Bell-like qutrit systems, which is crucial to understand the derivation of the corresponding inequalities.

To derive the correlation matrix for the Bell states, we start with the linear combination of all possible density matrix operators obtained by taking the outer product of the Bell states,

$$
a_1 \rho_{\Phi^+} + a_2 \rho_{\Phi^-} + a_3 \rho_{\Psi^+} + a_4 \rho_{\Psi^-}
$$

$$= a_1 \left|\Phi^+\right\rangle\!\left\langle\Phi^+\right| + a_2 \left|\Phi^-\right\rangle\!\left\langle\Phi^-\right| + a_3 \left|\Psi^+\right\rangle\!\left\langle\Psi^+\right| + a_4 \left|\Psi^-\right\rangle\!\left\langle\Psi^-\right|$$

$$= \begin{pmatrix} \frac{1}{2}(a_1 + a_2) & 0 & 0 & \frac{1}{2}(a_1 - a_2) \\ 0 & \frac{1}{2}(a_3 + a_4) & \frac{1}{2}(a_3 - a_4) & 0 \\ 0 & \frac{1}{2}(a_3 - a_4) & \frac{1}{2}(a_3 + a_4) & 0 \\ \frac{1}{2}(a_1 - a_2) & 0 & 0 & \frac{1}{2}(a_1 + a_2) \end{pmatrix}, \tag{24}$$

where a_i ($i = 1, 2, 3, 4$) be constants to be determined. Comparing Eq.(24) with Eq.(6), the constants are found to be $a_1 = -2\sqrt{2}$, $a_2 = a_3 = 0$, $a_4 = 2\sqrt{2}$ and we have,

$$a_1 \rho_{\Phi^+} + a_2 \rho_{\Phi^-} + a_3 \rho_{\Psi^+} + a_4 \rho_{\Psi^-}$$
$$= 2\sqrt{2}\left(\left|\Psi^-\right\rangle\!\left\langle\Psi^-\right| - \left|\Phi^+\right\rangle\!\left\langle\Phi^+\right|\right)$$
$$= -\sqrt{2}(\hat{\sigma}_3 \otimes \hat{\sigma}_3 + \hat{\sigma}_1 \otimes \hat{\sigma}_1) = \hat{\mathcal{C}}^{2\otimes 2} \quad Q.E.D. \tag{25}$$

In the last step, we used the fact that the outer product of the Bells states is numerically equal to the tensor product of the Pauli matrices.

To derive the correlation tensor for the two-qutrit system, similar to the previous case, we consider the linear combination of the density matrices from all nine two-qutrit states derived in Eq.(14),

$$a_{00}\rho_{\psi_{00}} + a_{12}^+\rho_{\psi_{12}^+} + a_{12}^-\rho_{\psi_{12}^-} + a_{11}\rho_{\psi_{11}}$$
$$+ a_{20}^+\rho_{\psi_{20}^+} + a_{20}^-\rho_{\psi_{20}^-} + a_{10}^+\rho_{\psi_{10}^+} + a_{10}^-\rho_{\psi_{10}^-} + a_{22}\rho_{\psi_{22}}$$
$$= a_{00}\left|\psi_{00}\right\rangle\!\left\langle\psi_{00}\right| + a_{12}^+\left|\psi_{12}^+\right\rangle\!\left\langle\psi_{12}^+\right| + a_{12}^-\left|\psi_{12}^-\right\rangle\!\left\langle\psi_{12}^-\right|$$
$$+ a_{11}\left|\psi_{11}\right\rangle\!\left\langle\psi_{11}\right| + a_{20}^+\left|\psi_{20}^+\right\rangle\!\left\langle\psi_{20}^+\right| + a_{20}^-\left|\psi_{20}^-\right\rangle\!\left\langle\psi_{20}^-\right|$$
$$+ a_{10}^+\left|\psi_{10}^+\right\rangle\!\left\langle\psi_{10}^+\right| + a_{10}^-\left|\psi_{10}^-\right\rangle\!\left\langle\psi_{10}^-\right| + a_{22}\left|\psi_{22}\right\rangle\!\left\langle\psi_{22}\right| \tag{26}$$

In the absence of the correlation matrix for the qutrit system, aforesaid procedure of evaluating constants by simple comparison does not work. We therefore adopt a procedure where the diagonalization of that correlation matrix must give all two-qutrit states as its eigenstates [40]. This requirement, supplemented by rigorous algebraic manipulation of a set of coupled linear equations gives, $a_{00} = 2\sqrt{2}$, $a_{12}^+ = 0$, $a_{12}^- = -2\sqrt{2}$, $a_{11} = \sqrt{2}$, $a_{20}^+ = 0$, $a_{20}^- = -2\sqrt{2}$, $a_{10}^+ = 0$, $a_{10}^- = 2\sqrt{2}$, $a_{22} = -\sqrt{2}$. Plugging them back in Eq.(26), we obtain the qutrit correlation matrix given by Eq.(20), which can be also written as the tensor product of the Gell-Mann matrices,

$$a_{00}\rho_{\psi_{00}} + a_{12}^+\rho_{\psi_{12}^+} + a_{12}^-\rho_{\psi_{12}^-} + a_{11}\rho_{\psi_{11}}$$
$$+ a_{20}^+\rho_{\psi_{20}^+} + a_{20}^-\rho_{\psi_{20}^-} + a_{10}^+\rho_{\psi_{10}^+} + a_{10}^-\rho_{\psi_{10}^-} + a_{22}\rho_{\psi_{22}}$$
$$= \sqrt{2}\left(\left|\psi_{11}\right\rangle\!\left\langle\psi_{11}\right| - \left|\psi_{22}\right\rangle\!\left\langle\psi_{22}\right|\right)$$
$$+ 2\sqrt{2}\left(\left|\psi_{00}\right\rangle\!\left\langle\psi_{00}\right| + \left|\psi_{10}^-\right\rangle\!\left\langle\psi_{10}^-\right| - \left|\psi_{12}^-\right\rangle\!\left\langle\psi_{12}^-\right| - \left|\psi_{20}^-\right\rangle\!\left\langle\psi_{20}^-\right|\right)$$
$$= \sqrt{2}(\lambda_4 \otimes \lambda_4 - \lambda_2 \otimes \lambda_2) + \frac{1}{\sqrt{2}}(\lambda_6 \otimes \lambda_6 + \lambda_7 \otimes \lambda_7)$$

$$- \frac{1}{2\sqrt{2}}(\lambda_3 \otimes \lambda_3 - 5\lambda_8 \otimes \lambda_8) + \frac{1}{\sqrt{6}}(\lambda_8 \otimes \lambda_0 + \lambda_0 \otimes \lambda_8)$$

$$- \frac{1}{2\sqrt{6}}(\lambda_3 \otimes \lambda_8 + \lambda_8 \otimes \lambda_3) - \frac{1}{2\sqrt{3}}(\lambda_0 \otimes \lambda_3 + \lambda_3 \otimes \lambda_0),$$

$$= \hat{\mathcal{C}}^{3\otimes 3} \qquad Q.E.D. \tag{27}$$

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