Quantum Information Processing (2024) 23:264
https://doi.org/10.1007/s11128-024-04454-2

Lorentz invariants of pure three-qubit states

A. R. Usha Devi[1,2] · Sudha[2,3] · H. Akshata Shenoy[4] · H. S. Karthik[4] · B. N. Karthik[1]

Received: 11 October 2023 / Accepted: 5 June 2024 / Published online: 6 July 2024

Abstract
Extending the mathematical framework of Sudha et al. (Phys Rev A 102:052419, 2020), we construct Lorentz invariant quantities of pure three-qubit states. This method serves as a bridge between the well-known local unitary (LU) invariants of an arbitrary three-qubit pure state and the Lorentz invariants of its reduced two-qubit systems.

Keywords Three-qubit pure states · SL(2,C) canonical form · Lorentz invariants · Geometric picture

1 Introduction

The use of entanglement as a resource in quantum information processing tasks has accelerated research efforts on its quantification, characterization, and control over the past two decades [1–5]. While multipartite entanglement poses higher level of complexity than the bipartite case, it enriches our theoretical understanding and paves way for innovative applications in distributed quantum networks [6–12]. It has been recognized that geometry associated with particular symmetry transformations plays a vital

✉ A. R. Usha Devi
ushadevi@bub.ernet.in

Sudha
tthdrs@gmail.com

H. Akshata Shenoy
akshata.shenoy@ug.edu.pl

H. S. Karthik
karthik.hs@ug.edu.pl

B. N. Karthik
karthikbnj@gmail.com

1 Department of Physics, Bangalore University, Bangalore 560 056, India

2 Inspire Institute Inc., Alexandria, VA 22303, USA

3 Department of Physics, Kuvempu University, Shankaraghatta, Karnataka 577 451, India

4 International Centre for Theory of Quantum Technologies, University of Gdánsk, Gdánsk, Poland

role in exploring multipartite entanglement—especially in the distribution of entanglement among the constituent subsystems [13–19]. Study of geometric invariants and canonical forms of composite quantum states under *local* symmetry operations on subsystems serves as a powerful tool to probe different manifestations of entanglement. To this end, considerable progress has been evinced in exploring local invariant quantities, canonical forms of equivalence classes of states under local unitary (LU) transformations, stochastic local operations and classical communication (SLOCC) associated with local SL(2,C) transformations [20–40].

An essential feature of entanglement is that it remains invariant under LU operations. Any two arbitrary pure states $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent (written symbolically as $|\psi\rangle \sim |\phi\rangle$) if and only if they can be transformed into each other by local unitary operations. A complete set of polynomial quantities that remain unaltered under LU operations on subsystems are used to certify LU equivalence of multipartite states. Recognizing normal/canonical form of a composite system by using LU transformations on individual subsystems is advantageous in evaluating these polynomial invariants.

Acín et al. showed that a three-qubit pure state under LU transformations can be reduced to a canonical form given by [16]:

$$|\psi_{ABC}\rangle = \lambda_0|0,0,0\rangle + \lambda_1 e^{i\phi}|1,0,0\rangle + \lambda_2|1,0,1\rangle + \lambda_3|1,1,0\rangle + \lambda_4|1,1,1\rangle \tag{1.1}$$

in terms of *five* real entanglement parameters $\lambda_i \geq 0,\ i = 0,1,2,3,4$ satisfying $\sum_{i=0}^{4} \lambda_i^2 = 1$, and a phase ϕ ranging between 0 and π. This gives a minimal form of pure three-qubit states containing only five terms and is helpful for evaluating LU invariants.

We consider a set of *five* LU invariants [16] characterizing pure three-qubit states (apart from normalization):

$$\begin{aligned}
I_1 &= \mathrm{Tr}\,[\rho_{BC}^2] = \mathrm{Tr}\,[\rho_A^2] = 1 - 2\lambda_0^2(1-\lambda_0^2-\lambda_1^2),\\
I_2 &= \mathrm{Tr}\,[\rho_{AC}^2] = \mathrm{Tr}[\rho_B^2] = 1 - 2\lambda_0^2(1-\lambda_0^2-\lambda_1^2-\lambda_2^2) - 2\triangle,\\
I_3 &= \mathrm{Tr}\,[\rho_{AB}^2] = \mathrm{Tr}\,[\rho_C^2] = 1 - 2\lambda_0^2(1-\lambda_0^2-\lambda_1^2-\lambda_3^2) - 2\triangle,\\
I_4 &= \mathrm{Tr}\,[(\rho_A\otimes\rho_B)\,\rho_{AB}] = 1 + \lambda_0^2\left(\lambda_2^2\lambda_3^2 - \lambda_1^2\lambda_4^2 - 2\,\lambda_2^2 - 3\,\lambda_3^2 - 3\,\lambda_4^2\right)\\
&\qquad -(2-\lambda_0^2)\ \triangle\\
I_5 &= \lambda_0^4\lambda_4^4 = \frac{\tau^2}{16},
\end{aligned} \tag{1.2}$$

where $\rho_A = \mathrm{Tr}_{\mathrm{B,C}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}|$, $\rho_B = \mathrm{Tr}_{\mathrm{A,C}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}|$, $\rho_C = \mathrm{Tr}_{\mathrm{A,B}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}|$, $\rho_{AB} = \mathrm{Tr}_{\mathrm{C}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}|$, $\rho_{BC} = \mathrm{Tr}_{\mathrm{A}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}|$, $\rho_{AC} = \mathrm{Tr}_{\mathrm{B}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}|$ and

$$\triangle \equiv |\lambda_1\lambda_4 e^{i\phi} - \lambda_2\lambda_3|^2. \tag{1.3}$$

The first three invariants $I_1,\ I_2,\ I_3$ are related to the squares of the three one-to-other bipartite concurrences $C^2_{A(BC)},\ C^2_{B(AC)}$, and $C^2_{C(AB)}$, respectively [15, 41]. The fourth one, I_4, is related to the *Kempe invariant* [6, 17]

$$\begin{aligned}\mathcal{I}_4 &= 3\mathrm{Tr}\,[(\rho_A\otimes\rho_B)\,\rho_{AB}] - \mathrm{Tr}\,[\rho_A^3] - \mathrm{Tr}\,[\rho_B^3],\\ &= 3\mathrm{Tr}\,[(\rho_B\otimes\rho_C)\,\rho_{BC}] - \mathrm{Tr}\,[\rho_B^3] - \mathrm{Tr}\,[\rho_C^3],\\ &= 3\mathrm{Tr}\,[(\rho_A\otimes\rho_C)\,\rho_{AC}] - \mathrm{Tr}\,[\rho_A^3] - \mathrm{Tr}\,[\rho_C^3],\end{aligned} \tag{1.4}$$

which is symmetric under the permutation of qubits. This quantity, while algebraically independent of the other LU invariants, has no known implication toward the classification of three-qubit entanglement [28].

Writing $|\psi_{\mathrm{ABC}}\rangle = \sum_{i,j,k=0,1} c_{ijk}\,|i,j,k\rangle$ in the computational basis, the invariant I_5 (Cayley's hyperdeterminant [42, 43]) is expressed as

$$I_5 = \frac{1}{4}\,\left|\epsilon_{i_1\,i_2}\,\epsilon_{i_3\,i_4}\,\epsilon_{j_1\,j_2}\,\epsilon_{j_3\,j_4}\,\epsilon_{k_1\,k_3}\,\epsilon_{k_2\,k_4}\,c_{i_1j_1k_1}c_{i_2j_2k_2}c_{i_3j_3k_3}c_{i_4j_4k_4}\right|^2, \tag{1.5}$$

where ϵ_{ij} denote antisymmetric tensor of rank-2; repeated indices are to be summed over in (1.5). In Acín's canonical form (1.1) of the three-qubit state, one obtains a simple form $I_5 = \lambda_0^4\lambda_4^4$, which is related to the three-tangle $\tau = 4\,\lambda_0^2\lambda_4^2$, a measure of three-way entanglement of three qubits in a pure state [15].

Any two pure three-qubit states $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent if and only if they are mutually interconvertible by means of local invertible transformations:

$$|\psi\rangle \sim A\otimes B\otimes C\,|\phi\rangle \tag{1.6}$$

where $A,\ B,\ C\in$ SL(2,C) denote 2×2 complex matrices with determinant unity. Because local protocols are unable to generate entanglement, invariant quantities under local SL(2,C) transformations are used for classification and also quantification of entanglement. Equivalence classes of pure three-qubit states under local invertible operations were explored in the celebrated work by Dür et al. [14], where it was shown that there exist *two* inequivalent tripartite entanglement classes—represented by the Greenberger–Horne–Zeilinger (GHZ) state [44]

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\,(|0,0,0\rangle + |1,1,1\rangle) \tag{1.7}$$

and the W state [14]

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\,(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)\,. \tag{1.8}$$

There has been a large effort toward gaining deeper insight into the structure of local SL(2,C) invariant quantities, where three-qubit pure state is considered as a test bed [15, 25, 27–31, 33, 36].

In this paper, we extend the mathematical framework of Ref. [38] to construct Lorentz invariants of pure three-qubit states. In the following section, we describe the basic formalism of Ref. [38]. Mainly we highlight here that the transformation property of the real 4×4 matrix parametrization Λ_{AB} of a two-qubit density matrix ρ_{AB} paves the way to identify *Lorentz invariance* of the eigenvalues μ_α^{AB}, $\alpha = 0, 1, 2, 3$ of the matrix $\Gamma_{AB} = G\,\Lambda_{AB}^T\,G\,\Lambda_{AB}$. Section 3 is devoted to explore the properties of the Lorentz invariant eigenvalues of $\Gamma_{ij},\ ij = AB,\ BC,\ AC$ associated with the reduced two-qubit density matrices ρ_{ij} of a pure three-qubit state. We recognize that (i) the matrices $\Gamma_{ij},\ ij = AB,\ BC,\ AC$ associated with a pure three-qubit state have at most *two* distinct Lorentz invariant eigenvalues; (ii) difference between the two eigenvalues is symmetric under the interchange of qubits and is equal to the three-tangle τ of the three-qubit state; (iii) the smallest Lorentz invariant eigenvalue of Γ_{ij} is equal to the squared concurrence $C^2_{ij},\ ij = AB,\ BC,\ AC$ of the two-qubit subsystems. We explicitly illustrate these features in pure permutation symmetric three-qubit states in Subsect. 3.1. Construction of a set of *five* local SL(2,C) invariants, which turn out to be the algebraic analogues of corresponding set of LU invariants of the three-qubit pure state, is outlined in Subsect. 3.2. A summary of our results is given in Sect. 4.

2 Transformation of two-qubit state under local SL(2,C) operations

Let us consider an arbitrary two-qubit density matrix ρ_{AB}, expanded in the Hilbert–Schmidt basis $\{\sigma_\alpha \otimes \sigma_\beta, \alpha, \beta = 0, 1, 2, 3\}$:

$$\rho_{AB} = \frac{1}{4}\sum_{\alpha,\,\beta=0}^{3} (\Lambda_{AB})_{\alpha\,\beta}\,\left(\sigma_\alpha \otimes \sigma_\beta\right), \qquad (\Lambda_{AB})_{\alpha\,\beta} = \mathrm{Tr}\left[\rho_{AB}\,(\sigma_\alpha \otimes \sigma_\beta)\right] \tag{2.1}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

It is convenient to express the expansion coefficients $(\Lambda_{AB})_{\mu\,\nu}$ in (2.1) as a 4×4 real matrix:

$$\Lambda_{AB} = \begin{pmatrix} 1 & s_{B1} & s_{B2} & s_{B3} \\ s_{A1} & t_{11}^{AB} & t_{12}^{AB} & t_{13}^{AB} \\ s_{A2} & t_{21}^{AB} & t_{22}^{AB} & t_{23}^{AB} \\ s_{A3} & t_{31}^{AB} & t_{32}^{AB} & t_{33}^{AB} \end{pmatrix}. \tag{2.3}$$

Here $\mathbf{s}_A$, $\mathbf{s}_B$ are Minkowski four vectors with components $s_{A\alpha}$, $s_{B\alpha}$, $\alpha = 0, 1, 2, 3$, respectively, and $T^{AB} = (t_{ij}^{AB}),\ i,\ i = 1, 2, 3$ denotes the two-qubit correlation matrix:

$$s_{A\alpha} = (\Lambda_{AB})_{\alpha\,0} = \mathrm{Tr}\,[\rho_{AB}\,(\sigma_\alpha \otimes \sigma_0)\,] = \mathrm{Tr}\,[\rho_A\,\sigma_\alpha]\,, \tag{2.4}$$

$$s_{B\alpha} = (\Lambda_{AB})_{0\,\alpha} = \mathrm{Tr}\,[\rho_{AB}\,(\sigma_0 \otimes \sigma_\alpha)\,] = \mathrm{Tr}\,[\rho_B\,\sigma_\alpha]\,,\ \alpha = 0, 1, 2, 3 \tag{2.5}$$

$$t_{ij}^{AB} = (\Lambda_{AB})_{i\,j} = \mathrm{Tr}\left[\rho_{AB}\,(\sigma_i \otimes \sigma_j)\right], \quad i,\ j = 1,\ 2,\ 3. \tag{2.6}$$

Under local SL(2,C) operations, the two-qubit state ρ_{AB} transforms as

$$\rho_{AB} \longrightarrow \bar{\rho}_{AB} = \frac{(A \otimes B)\, \rho_{AB}\, (A^{\dagger} \otimes B^{\dagger})}{\mathrm{Tr}\left[\rho_{AB}\, (A^{\dagger}\, A \otimes B^{\dagger}\, B)\right]} \tag{2.7}$$

where $A, B \in \mathrm{SL}(2, \mathrm{C})$ denote 2×2 complex matrices with unit determinant. As a result, one finds that

$$\Lambda_{AB} \longrightarrow \bar{\Lambda}_{AB} = \frac{L_A\, \Lambda_{AB}\, L_B^T}{\left(L_A\, \Lambda_{AB}\, L_B^T\right)_{00}} \tag{2.8}$$

where $L_A, L_B \in SO(3,1)$ are 4×4 proper orthochronous Lorentz transformation matrices [45] corresponding to $A, B \in \mathrm{SL}(2, \mathrm{C})$, respectively, and the superscript 'T' denotes transpose operation.

We construct a 4×4 real matrix

$$\Gamma_{AB} = G\, \Lambda_{AB}^T\, G\, \Lambda_{AB}, \tag{2.9}$$

where $G = \mathrm{diag}\,(1, -1, -1, -1)$ denotes the Minkowski metric [45]. It is readily identified that the matrix Γ_{AB} undergoes a similarity transformation (up to an overall factor) [38]:

$$\begin{aligned}
\Gamma_{AB} \to \bar{\Gamma}_{AB} &= G\, \bar{\Lambda}_{AB}^T\, G\, \bar{\Lambda}_{AB} \\
&= G\, \left(L_A\, \Lambda_{AB}\, L_B^T\right)^T\, G\; L_A\, \Lambda_{AB} L_B^T \\
&= G\, L_B\, \Lambda_{AB}^T\, L_A^T\, G\, L_A\, \Lambda_{AB} L_B^T \\
&= (G\, L_B\, G)\; G\, \Lambda_{AB}^T\, \left(L_A^T\, G\, L_A\right)\; \Lambda_{AB} L_B^T \\
&= \left(L_B^T\right)^{-1}\; \Gamma_{AB}\, L_B^T
\end{aligned} \tag{2.10}$$

where the defining property [45] $L^T\, G\, L = G$ of Lorentz transformation is used.

The matrix Γ_{AB}, constructed using the real matrix parametrization Λ_{AB} of the two-qubit density matrix ρ_{AB} (see (2.1), (2.3)), exhibits the following important properties (see Theorem of Ref. [38] on the nature of eigenvalues and eigenvectors of the matrix Γ_{AB}):

(i) It possesses *non-negative* eigenvalues $\mu_0^{AB} \geq \mu_1^{AB} \geq \mu_2^{AB} \geq \mu_3^{AB} \geq 0$.
(ii) Four-eigenvector X associated with the highest eigenvalue μ_0^{AB} of the matrix Γ_{AB} satisfies one of the following Lorentz invariant properties:

$$X^T\, G\, X > 0 \tag{2.11}$$

or

$$X^T\, G\, X = 0. \tag{2.12}$$

The condition (2.12) is accompanied by the observation that the matrix Γ_{AB} has only *two* eigenvalues μ_0^{AB}, μ_2^{AB} with $\mu_0^{AB} \geq \mu_2^{AB}$, both of which are doubly degenerate.

(iii) Suppose the eigenvector X satisfies the Lorentz invariant condition (2.11). Then there exist suitable SL(2,C) transformations $A_1, B_1 \in$ SL(2,C) (with corresponding Lorentz transformations $L_{A_1}, L_{B_1} \in$ SO(3,1), respectively) such that the matrix Γ_{AB} assumes a diagonal canonical form:

$$\bar{\Gamma}^{(I_c)}_{AB} = \left(L^T_{B_{I_c}}\right)^{-1} \Gamma_{AB}\, L^T_{B_{I_c}} = \text{diag}\left(\mu_0^{AB},\, \mu_1^{AB},\, \mu_2^{AB},\, \mu_3^{AB}\right). \quad (2.13)$$

(iv) Associated with the standard form $\bar{\Gamma}^{(I_c)}_{AB}$, it is seen that [38]

$$\rho_{AB} \longrightarrow \bar{\rho}^{\,I_c}_{AB} = \frac{(A_{I_c} \otimes B_{I_c})\, \rho_{AB}\, (A^\dagger_{I_c} \otimes B^\dagger_{I_c})}{\text{Tr}\left[\rho_{AB}\, (A^\dagger_{I_c} A_{I_c} \otimes B^\dagger_{I_c} B_{I_c})\right]}$$

reduces to the Bell diagonal form

$$\bar{\rho}^{\,I_c}_{AB} = \frac{1}{4}\left(\sigma_0 \otimes \sigma_0 + \sum_{i=1,2} \sqrt{\frac{\mu_i^{AB}}{\mu_0^{AB}}}\, \sigma_i \otimes \sigma_i \pm \sqrt{\frac{\mu_3^{AB}}{\mu_0^{AB}}}\, \sigma_3 \otimes \sigma_3\right) \quad (2.14)$$

under local SL(2,C) operations. Here the sign $\pm$ is chosen based on $\text{sgn}[\det(\Lambda_{AB})] = \pm$.

(v) Whenever the eigenvector X obeys the condition (2.12), suitable local SL(2,C) transformations $A_{II_c}, B_{II_c} \in$ SL(2,C) (associated Lorentz transformations denoted, respectively, by $L_{A_{II_c}}, L_{B_{II_c}} \in$ SO(3,1)) exist such that the real symmetric matrix Γ_{AB} takes the following canonical form:

$$\bar{\Gamma}^{(II_c)}_{AB} = \left(L^T_{B_{II_c}}\right)^{-1} \Gamma_{AB}\, L^T_{B_{II_c}} = \begin{pmatrix} \phi_0^{AB} & 0 & 0 & \phi_0^{AB} - \mu_0^{AB} \\ 0 & \mu_2^{AB} & 0 & 0 \\ 0 & 0 & \mu_2^{AB} & 0 \\ \mu_0^{AB} - \phi_0^{AB} & 0 & 0 & 2\,\mu_0^{AB} - \phi_0^{AB}, \end{pmatrix} \quad (2.15)$$

where

$$\phi_0^{AB} = \left(L_{B_{II_c}}\, \Gamma_{AB}\, L^T_{B_{II_c}}\right)_{00}. \quad (2.16)$$

(vi) Consequently, the canonical form of the two-qubit density matrix is given by

$$\rho_{AB} \longrightarrow \bar{\rho}^{\,II_c}_{AB} = \frac{(A_{II_c} \otimes B_{II_c})\, \rho_{AB}\, (A^\dagger_{II_c} \otimes B^\dagger_{II_c})}{\text{Tr}\left[\rho_{AB}\, (A^\dagger_{II_c} A_{II_c} \otimes B^\dagger_{II_c} B_{II_c})\right]},$$

$$= \frac{1}{4}\left[\sigma_0\otimes\sigma_0 + (1-\gamma_0^{AB})\,\sigma_0\otimes\sigma_3 + \gamma_2^{AB}\,(\sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2) + \gamma_0^{AB}\,\sigma_3\otimes\sigma_3\right] \quad (2.17)$$

where

$$\gamma_0^{AB} = \frac{\mu_0^{AB}}{\phi_0^{AB}}, \quad \gamma_2^{AB} = \sqrt{\frac{\mu_2^{AB}}{\phi_0^{AB}}}, \quad 0 \le \left(\gamma_2^{AB}\right)^2 \le \gamma_0^{AB} \le 1. \quad (2.18)$$

(vii) Corresponding to the canonical form $\bar{\Gamma}^{(I_c)}_{AB}$ (see (2.13)), an elegant geometric visualization in terms of *an ellipsoid inscribed inside the Bloch sphere* [38], with semiaxes lengths (i) $\left(\sqrt{\frac{\mu_1^{AB}}{\mu_0^{AB}}}, \sqrt{\frac{\mu_2^{AB}}{\mu_0^{AB}}}, \sqrt{\frac{\mu_3^{AB}}{\mu_0^{AB}}}\right)$ and with center coinciding with that of the Bloch sphere, obeying the equation

$$\left(\frac{\mu_0^{AB}}{\mu_1^{AB}}\right) x^2 + \left(\frac{\mu_0^{AB}}{\mu_2^{AB}}\right) y^2 + \left(\frac{\mu_0^{AB}}{\mu_3^{AB}}\right) z^2 = 1. \quad (2.19)$$

Associated with the Lorentz canonical structure $\bar{\Gamma}^{(II_c)}_{AB}$ (see (2.15)), a shifted spheroid [38], with semiaxes lengths $(\sqrt{\gamma_1^{AB}}, \sqrt{\gamma_1^{AB}}, \sqrt{\gamma_0^{AB}})$ (see (2.18)) and center $(0,\ 0,\ (1-\gamma_0^{AB}))$ inside the Bloch sphere, satisfying the equation

$$\frac{(x^2+y^2)}{\gamma_1^{AB}} + \frac{\left(z-(1-\gamma_0^{AB})\right)^2}{\gamma_0^{AB}} = 1. \quad (2.20)$$

represents the set of all two-qubit states on the SL(2,C) orbit of $\bar{\rho}^{\,II_c}_{AB}$ (see (2.17)).

(viii) The eigenvalues μ_α^{AB}, $\alpha = 0, 1, 2, 3$ of the 4×4 matrix Γ_{AB} are Lorentz invariant, i.e., they are unchanged under local SL(2,C) operations.

In the following section, we construct local SL(2,C) invariants, by exploiting the Lorentz transformation properties of the real matrices Λ_{AB}, Λ_{BC}, and Λ_{AC} characterizing the two-qubit subsystems of a pure three-qubit state.

3 Lorentz invariants of pure three-qubit state

Let us write the two-qubit reduced density matrices ρ_{AB}, ρ_{BC} and ρ_{AC} of a pure three-qubit state as

$$\rho_{AB} = \mathrm{Tr}_{\mathrm{C}}\,|\psi_{\mathrm{ABC}}\rangle\langle\psi_{\mathrm{ABC}}| = \frac{1}{4}\sum_{\alpha,\,\beta=0}^{3} (\Lambda_{AB})_{\alpha\beta}\,\left(\sigma_\alpha\otimes\sigma_\beta\right), \quad (3.1)$$

$$\rho_{BC} = \mathrm{Tr_A}\, |\psi_{\rm ABC}\rangle\langle\psi_{\rm ABC}| = \frac{1}{4}\sum_{\alpha,\,\beta=0}^{3} (\Lambda_{BC})_{\alpha\,\beta}\,\left(\sigma_\alpha\otimes\sigma_\beta\right), \tag{3.2}$$

$$\rho_{AC} = \mathrm{Tr_B}\, |\psi_{\rm ABC}\rangle\langle\psi_{\rm ABC}| = \frac{1}{4}\sum_{\alpha,\,\beta=0}^{3} (\Lambda_{AC})_{\alpha\,\beta}\,\left(\sigma_\alpha\otimes\sigma_\beta\right). \tag{3.3}$$

We employ Acín's canonical form (1.1) of the pure three-qubit state, for evaluating the 4×4 real matrices $\Lambda_{AB},\ \Lambda_{BC}$ and Λ_{AC} explicitly:

$$\Lambda_{AB} = \begin{pmatrix} 1 & 2\,(\lambda_2\,\lambda_4+\lambda_1\lambda_3\,\cos\phi) & -2\,\lambda_1\,\lambda_3\,\sin\phi & 1-2\,(\lambda_3^2+\lambda_4^2) \\ 2\,\lambda_0\,\lambda_1\,\cos\phi & 2\,\lambda_0\,\lambda_3 & 0 & 2\,\lambda_0\,\lambda_1\,\cos\phi \\ 2\,\lambda_0\,\lambda_1\,\sin\phi & 0 & -2\,\lambda_0\,\lambda_3 & 2\,\lambda_0\,\lambda_1\,\sin\phi \\ 2\,\lambda_0^2-1 & -2\,(\lambda_2\,\lambda_4+\lambda_1\lambda_3\,\cos\phi) & 2\,\lambda_1\,\lambda_3\,\sin\phi & 1-2\,(\lambda_1^2+\lambda_2^2) \end{pmatrix}, \tag{3.4}$$

$$\Lambda_{BC} = \begin{pmatrix} 1 & 2\,(\lambda_3\,\lambda_4+\lambda_1\lambda_2\,\cos\phi) & -2\,\lambda_1\,\lambda_2\,\sin\phi & 1-2\,(\lambda_2^2+\lambda_4^2) \\ 2\,(\lambda_2\,\lambda_4+\lambda_1\lambda_3\,\cos\phi) & 2\,(\lambda_2\,\lambda_3+\lambda_1\lambda_4\,\cos\phi) & -2\,\lambda_1\,\lambda_4\,\sin\phi & -2\,(\lambda_2\,\lambda_4-\lambda_1\lambda_3\,\cos\phi) \\ -2\,\lambda_1\,\lambda_3\,\sin\phi & -2\,\lambda_1\,\lambda_4\,\sin\phi & 2\,(\lambda_2\,\lambda_3-\lambda_1\lambda_4\,\cos\phi) & -2\,\lambda_1\,\lambda_3\,\sin\phi \\ 1-2\,(\lambda_3^2+\lambda_4^2) & -2\,(\lambda_3\,\lambda_4-\lambda_1\lambda_2\,\cos\phi) & -2\,\lambda_1\,\lambda_2\,\sin\phi & 1-2\,(\lambda_2^2+\lambda_3^2) \end{pmatrix}, \tag{3.5}$$

$$\Lambda_{AC} = \begin{pmatrix} 1 & 2\,(\lambda_3\,\lambda_4+\lambda_1\lambda_2\,\cos\phi) & -2\,\lambda_1\,\lambda_2\,\sin\phi & 1-2\,(\lambda_2^2+\lambda_4^2) \\ 2\,\lambda_0\,\lambda_1\,\cos\phi & 2\,\lambda_0\,\lambda_2 & 0 & 2\,\lambda_0\,\lambda_1\,\cos\phi \\ 2\,\lambda_0\,\lambda_1\,\sin\phi & 0 & -2\,\lambda_0\,\lambda_2 & 2\,\lambda_0\,\lambda_1\,\sin\phi \\ 2\,\lambda_0^2-1 & -2\,(\lambda_3\,\lambda_4+\lambda_1\lambda_2\,\cos\phi) & 2\,\lambda_1\,\lambda_2\,\sin\phi & 1-2\,(\lambda_1^2+\lambda_3^2) \end{pmatrix}. \tag{3.6}$$

Let us recall the formula for the concurrence C_{AB} of an arbitrary two-qubit state ρ_{AB} introduced by Wootters [41]:

$$C_{AB} = \max\,\{0,\, \nu_1^{AB}-\nu_2^{AB}-\nu_3^{AB}-\nu_4^{AB}\} \tag{3.7}$$

where $\nu_i^{AB},\ i=1,2,3,4$ are the square roots of the eigenvalues of

$$\rho_{AB}\,\widetilde{\rho}_{AB} = \rho_{AB}\,(\sigma_2\otimes\sigma_2)\,\rho_{AB}^T\,(\sigma_2\otimes\sigma_2) \tag{3.8}$$

in decreasing order. While the matrix $\rho_{AB}\,\widetilde{\rho}_{AB}$ is non-hermitian, it has only real and positive eigenvalues [41].

To gain further insight into the structure of the non-hermitian matrix $\rho_{AB}\,\widetilde{\rho}_{AB}$, we express the spin flipped two-qubit density matrix $\widetilde{\rho}_{AB} = (\sigma_2\otimes\sigma_2)\,\rho_{AB}^T\,(\sigma_2\otimes\sigma_2)$ in the basis $\{\sigma_\alpha\otimes\sigma_\beta, \alpha,\beta=0,1,2,3\}$ to obtain

$$\widetilde{\rho}_{AB} = \frac{1}{4}\sum_{\alpha,\beta=0}^{3}\left(\Lambda^{'}_{AB}\right)_{\alpha\,\beta}\,\sigma_\alpha\otimes\sigma_\beta \tag{3.9}$$

where the 4×4 real matrix $\Lambda^{'}_{AB}$ characterizing the spin flipped two-qubit density matrix $\widetilde{\rho}_{AB}$ is found to be [46]

$$\Lambda^{'}_{AB} = G\,\Lambda_{AB}\,G. \tag{3.10}$$

We thus recognize (see (2.1),(3.9), (3.10) and (2.9)) that

$$\mathrm{Tr}[\rho_{AB}\,\widetilde{\rho}_{AB}] = \frac{1}{4}\,\mathrm{Tr}\,[G\,\Lambda_{AB}^T\,G\,\Lambda_{AB}] = \frac{1}{4}\,\mathrm{Tr}\,[\Gamma_{AB}]. \tag{3.11}$$

Evidently, $\mathrm{Tr}\,[\Gamma_{AB}] \geq 0$ as the 4×4 real matrix $\Gamma_{AB} = G\,\Lambda_{AB}^T\,G\,\Lambda_{AB}$ is non-negative [38] and this, in turn, justifies that trace of the non-hermitian matrix $\rho_{AB}\,\widetilde{\rho}_{AB}$ (LHS of (3.11)) is also positive.

In a pure entangled three-qubit state, every pair of qubits are entangled with the remaining qubit. Thus, the two-qubit subsystem density matrix of a pure three-qubit state has at most *two* nonzero eigenvalues. As a result, the matrix $\rho_{ij}\,\widetilde{\rho}_{ij}$ has only *two* nonzero eigenvalues $\left(\nu_1^{ij}\right)^2, \left(\nu_2^{ij}\right)^2$, $ij = AB, BC, AC$. Thus, the squared concurrence C_{ij}^2 of two-qubit subsystem state ρ_{ij} of a three-qubit pure state simplifies to

$$C_{ij}^2 = (\nu_1^{ij} - \nu_2^{ij})^2, \;\; ij = AB, BC, AC. \tag{3.12}$$

We are interested in recognizing local SL(2,C) invariants of three-qubit pure state, which are useful in determining the Lorentz invariant eigenvalues of the matrices Γ_{AB}, Γ_{BC}, and Γ_{AC}. In Acín's canonical form (1.1), the matrices Γ_{AB}, Γ_{BC} and Γ_{AC} have the following explicit structure:

$$\Gamma_{AB} = \begin{pmatrix} 4\,\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2) & 4\,\lambda_0^2\,\lambda_2\,\lambda_4 & 0 & 4\,\lambda_0^2\,\lambda_2^2 \\ -4\,\lambda_0^2\,\lambda_2\,\lambda_4 & 4\,\lambda_0^2\,\lambda_3^2 & 0 & -4\,\lambda_0^2\,\lambda_2\,\lambda_4 \\ 0 & 0 & 4\,\lambda_0^2\,\lambda_3^2 & 0 \\ -4\,\lambda_0^2\,\lambda_2^2 & -4\,\lambda_0^2\,\lambda_2\,\lambda_4 & 0 & 4\,\lambda_0^2(\lambda_3^2-\lambda_2^2+\lambda_4^2) \end{pmatrix} \tag{3.13}$$

$$\Gamma_{BC} = \begin{pmatrix} 4\,\lambda_0^2\,(\lambda_3^2+\lambda_4^2)+4\,\triangle & 4\,\lambda_0^2\,\lambda_3\,\lambda_4 & 0 & 4\,\lambda_0^2\,\lambda_3^2 \\ -4\,\lambda_0^2\,\lambda_3\,\lambda_4 & 4\,\triangle & 0 & -4\,\lambda_0^2\,\lambda_3\,\lambda_4 \\ 0 & 0 & 4\,\triangle & 0 \\ -4\,\lambda_0^2\,\lambda_3^2 & -4\,\lambda_0^2\,\lambda_3\,\lambda_4 & 0 & 4\,\lambda_0^2\,(\lambda_4^2-\lambda_3^2)+4\,\triangle \end{pmatrix} \tag{3.14}$$

$$\Gamma_{AC} = \begin{pmatrix} 4\,\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2) & 4\,\lambda_0^2\,\lambda_3\,\lambda_4 & 0 & 4\,\lambda_0^2\,\lambda_3^2 \\ -4\,\lambda_0^2\,\lambda_3\,\lambda_4 & 4\,\lambda_0^2\,\lambda_2^2 & 0 & -4\,\lambda_0^2\,\lambda_3\,\lambda_4 \\ 0 & 0 & 4\,\lambda_0^2\,\lambda_2^2 & 0 \\ -4\,\lambda_0^2\,\lambda_3^2 & -4\,\lambda_0^2\,\lambda_3\,\lambda_4 & 0 & 4\,\lambda_0^2(\lambda_2^2-\lambda_3^2+\lambda_4^2) \end{pmatrix}. \tag{3.15}$$

- We find that these matrices Γ_{AB}, Γ_{BC}, and Γ_{AC} have *at most* two distinct eigenvalues:

$$\mu_0^{AB} = \mu_1^{AB} = 4\,\lambda_0^2\,(\lambda_3^2+\lambda_4^2),\;\; \mu_2^{AB} = \mu_3^{AB} = 4\,\lambda_0^2\,\lambda_3^2, \tag{3.16}$$

$$\mu_0^{BC} = \mu_1^{BC} = 4\,(\triangle + \lambda_0^2\,\lambda_4^2),\;\; \mu_2^{BC} = \mu_3^{BC} = 4\,\triangle, \tag{3.17}$$

$$\mu_0^{AC} = \mu_1^{AC} = 4\,\lambda_0^2\,(\lambda_2^2 + \lambda_4^2),\;\; \mu_2^{AC} = \mu_3^{AC} = 4\,\lambda_0^2\,\lambda_2^2. \tag{3.18}$$

- We notice an interesting feature that the differences between the largest and the smallest eigenvalues of $\Gamma_{AB}, \Gamma_{BC}, \Gamma_{AC}$ are identically equal to the three-tangle τ of the three-qubit state:

$$\mu_0^{ij} - \mu_2^{ij} = 4\,\lambda_0^2\,\lambda_4^2 = \tau, \quad ij = AB, BC, AC. \tag{3.19}$$

This reveals the fact that the LU invariant $I_5 = \frac{\tau^2}{16}$ of the three-qubit state (see last line of (1.2)) is a permutation symmetric local SL(2,C) invariant. It is worth noting that $\sqrt{I_5} = \lambda_0^2\,\lambda_4^2$ is equal to the product [15] $\nu_1^{AB}\,\nu_2^{AB} = \nu_1^{BC}\,\nu_2^{BC} = \nu_1^{AC}\,\nu_2^{AC}$ of the square root of the eigenvalues of $\rho_{ij}\,\widetilde{\rho}_{ij},\; ij = AB, BC, AC$. Thus,

$$\mu_0^{ij} - \mu_2^{ij} = 4\,\nu_1^{ij}\,\nu_2^{ij}. \tag{3.20}$$

- The Lorentz invariant eigenvalues of Γ_{AB}, Γ_{BC} and Γ_{AC} can be determined using I_5 along with *three* more invariants given by

$$\begin{aligned}
\mathcal{K}_1 &= \frac{1}{4}\,\mathrm{Tr}[\,\Gamma_{AB}] = \frac{1}{2}\,(\mu_0^{AB} + \mu_2^{AB}) \\
\mathcal{K}_2 &= \frac{1}{4}\,\mathrm{Tr}[\,\Gamma_{BC}] = \frac{1}{2}\,(\mu_0^{BC} + \mu_2^{BC}) \\
\mathcal{K}_3 &= \frac{1}{4}\,\mathrm{Tr}[\,\Gamma_{AC}] = \frac{1}{2}\,(\mu_0^{AC} + \mu_2^{AC}).
\end{aligned} \tag{3.21}$$

Substituting (3.11) in (3.21), we obtain

$$\begin{aligned}
\mathcal{K}_1 &= \mathrm{Tr}[\rho_{AB}\,\widetilde{\rho}_{AB}] = (\nu_1^{AB})^2 + (\nu_2^{AB})^2, \\
\mathcal{K}_2 &= \mathrm{Tr}[\rho_{BC}\,\widetilde{\rho}_{BC}] = (\nu_1^{BC})^2 + (\nu_2^{BC})^2, \\
\mathcal{K}_3 &= \mathrm{Tr}[\rho_{AC}\,\widetilde{\rho}_{AC}] = (\nu_1^{AC})^2 + (\nu_2^{AC})^2.
\end{aligned} \tag{3.22}$$

We proceed to prove the following theorem:

Theorem 1 *The squared concurrence C_{ij}^2 of the two-qubit subsystem ρ_{ij} of a pure three-qubit state is equal to the smallest Lorentz invariant eigenvalue μ_2^{ij} of the* 4×4 *matrix $\Gamma_{ij} = G\,\Lambda_{ij}^T\,G\,\Lambda_{ij},\quad ij = AB, BC, AC$.*

Proof Using (3.20), (3.21) and (3.22), we connect the eigenvalues of $\Gamma_{AB},\; \Gamma_{BC},\; \Gamma_{AC}$ with those of $\rho_{AB}\,\widetilde{\rho}_{AB},\; \rho_{BC}\,\widetilde{\rho}_{BC},\; \rho_{AC}\,\widetilde{\rho}_{AC}$, respectively:

$$\begin{aligned}
4\,\nu_1^{ij}\,\nu_2^{ij} &= \mu_0^{ij} - \mu_2^{ij}, \\
\left[\,(\nu_1^{ij})^2 + (\nu_2^{ij})^2\right] &= \frac{1}{2}\,(\mu_0^{ij} + \mu_2^{ij}), \quad ij = AB, BC, AC.
\end{aligned} \tag{3.23}$$

We thus obtain (see (3.12))

$$\mu_2^{ij} = \left(\nu_1^{ij} - \nu_2^{ij}\right)^2 = C_{ij}^2, \quad ij = AB, BC, AC. \tag{3.24}$$

The above theorem offers an interesting alternate method to evaluate concurrences of two-qubit subsystems $\rho_{ij},\ ij = AB, BC, AC$ of a pure three-qubit state, in terms of the smallest Lorentz invariant eigenvalues of Γ_{ij}. □

- With the help of the following Lorentz transformations

$$L_B = \begin{pmatrix} \frac{\tau+2\,C_{AC}^2}{2\,C_{AC}\,\sqrt{\tau}} & \frac{C_{AC}}{\sqrt{\tau}} & 0 & \frac{\sqrt{\tau}}{2\,C_{AC}} \\ -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \frac{\tau-2\,C_{AC}^2}{2\,C_{AC}\,\sqrt{\tau}} & -\frac{C_{AC}}{\sqrt{\tau}} & 0 & \frac{\sqrt{\tau}}{2\,C_{AC}} \end{pmatrix} \tag{3.25}$$

and

$$L_C = \begin{pmatrix} \frac{\tau+2\,C_{AB}^2}{2\,C_{AB}\,\sqrt{\tau}} & \frac{C_{AB}}{\sqrt{\tau}} & 0 & \frac{\sqrt{\tau}}{2\,C_{AB}} \\ -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \frac{\tau-2\,C_{AB}^2}{2\,C_{AB}\,\sqrt{\tau}} & -\frac{C_{AB}}{\sqrt{\tau}} & 0 & \frac{\sqrt{\tau}}{2\,C_{AB}} \end{pmatrix} \tag{3.26}$$

it is seen that

$$\left(L_B^T\right)^{-1}\Gamma_{AB}L_B^T = \bar{\Gamma}_{AB}^{(II_c)} = \begin{pmatrix} C_{AB}^2+\tau & 0 & 0 & 0 \\ 0 & C_{AB}^2 & 0 & 0 \\ 0 & 0 & C_{AB}^2 & 0 \\ 0 & 0 & 0 & C_{AB}^2+\tau \end{pmatrix} \tag{3.27}$$

$$\left(L_C^T\right)^{-1}\Gamma_{BC}L_C^T = \bar{\Gamma}_{BC}^{(II_c)} = \begin{pmatrix} C_{BC}^2+\tau & 0 & 0 & 0 \\ 0 & C_{BC}^2 & 0 & 0 \\ 0 & 0 & C_{BC}^2 & 0 \\ 0 & 0 & 0 & C_{BC}^2+\tau \end{pmatrix} \tag{3.28}$$

$$\left(L_C^T\right)^{-1}\Gamma_{AC}L_C^T = \bar{\Gamma}_{AC}^{(II_c)} = \begin{pmatrix} C_{AC}^2+\tau & 0 & 0 & 0 \\ 0 & C_{AC}^2 & 0 & 0 \\ 0 & 0 & C_{AC}^2 & 0 \\ 0 & 0 & 0 & C_{AC}^2+\tau \end{pmatrix}. \tag{3.29}$$

It follows that (see (2.16),(2.18))

$$\gamma_0^{ij} = 1, \ \ \gamma_2^{ij} = \frac{C_{ij}}{\sqrt{C_{ij}^2+\tau}}, \ ij = AB,\ BC,\ AC. \tag{3.30}$$

A spheroid inside the Bloch sphere (see Fig. 1) with semiaxes lengths

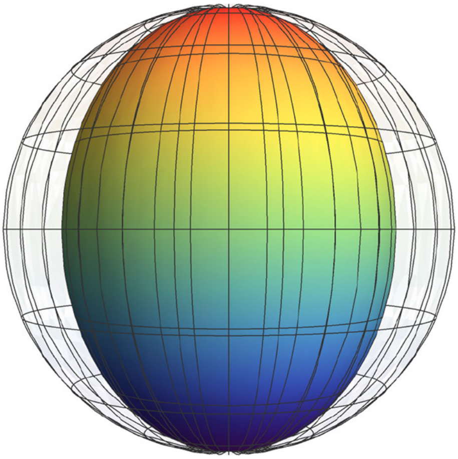

Fig. 1 Spheroid centered at the origin of the Bloch sphere with semiaxes lengths $\left(\frac{C}{\sqrt{C^2+\tau}}, \frac{C}{\sqrt{C^2+\tau}}, 1\right)$, for $C = 0.8$ and $\tau = 0.5$ (Color figure online)

$\left(\frac{C_{ij}}{\sqrt{C_{ij}^2+\tau}}, \frac{C_{ij}}{\sqrt{C_{ij}^2+\tau}}, 1\right)$ and origin (0,0,0) (see (2.20), (3.30) offers geometrical visualization of the reduced two-qubit systems of the three-qubit pure state $|\psi_{ABC}\rangle$ given in (1.1).

3.1 Permutation symmetric three-qubit pure states

It is well known that permutation symmetric states offer conceptual clarity and computational simplicity in the analysis of local invariants [34, 39, 47–49]. In this subsection, we illustrate the effectiveness of our framework to evaluate concurrence and tangle in pure three-qubit permutation symmetric states, where we make use of the explicit parametrization given by Meill and Meyer [34] recently.

Consider a one-parameter family of three-qubit permutation symmetric state [34, 39]:

$$|\psi_{\rm sym}(\beta)\rangle = \frac{1}{\sqrt{2+\cos\beta}}\left(\sqrt{3}\,\cos\frac{\beta}{2}\,|0_A\,0_B\,0_C\rangle + \sin\frac{\beta}{2}\,|{\rm W}\rangle\right) \qquad (3.31)$$

where $0 < \beta \leq \pi$. We find that [39]

$$\begin{aligned}\Gamma_{\rm sym}(\beta) &= \Gamma_{AB}(\beta) = \Gamma(\beta)_{BC} = \Gamma_{AC}(\beta)\\ &= \begin{pmatrix} 2\,u(\beta) & 0 & 0 & u(\beta)\\ 0 & u(\beta) & 0 & 0\\ 0 & 0 & u(\beta) & 0\\ -u(\beta) & 0 & 0 & 0\end{pmatrix}, \quad u(\beta) = \left[\frac{1-\cos\beta}{3(2+\cos\beta)}\right]^2. \end{aligned} \qquad (3.32)$$

The Lorentz invariant eigenvalues of $\Gamma_{\rm sym}(\beta)$ are equal, i.e.,

$$\mu_0(\beta) = \mu_2(\beta) = \left[\frac{1-\cos\beta}{3(2+\cos\beta)}\right]^2, \tag{3.33}$$

and hence, the concurrence of $\rho_{\rm sym}(\beta) = \rho_{AB}(\beta) = \rho_{BC}(\beta) = \rho_{AC}(\beta)$ is given by

$$C(\beta) = \frac{1-\cos\beta}{3(2+\cos\beta)}, \tag{3.34}$$

in perfect agreement with the result given by Meill and Meyer [34]. Substitution of (3.33) in the LHS of (3.19) confirms that the three-tangle $\tau(\beta)$ for the state (3.31) is zero.

We proceed further with a three-parameter family of pure three-qubit permutation symmetric state [34]

$$|\psi_{\rm sym}(y,\beta,\phi)\rangle = N\left(|0\rangle^{\otimes 3} + y\,e^{i\,\phi}\,|\beta\rangle^{\otimes 3}\right), \tag{3.35}$$

where $|\beta\rangle = \cos\frac{\beta}{2}\,|0\rangle + \sin\frac{\beta}{2}\,|1\rangle,\ 0 < y \leq 1,\ 0 \leq \phi \leq 2\pi,\ 0 < \beta \leq \pi$. We evaluate the matrix $\Gamma_{\rm sym}(y,\beta,\phi) \equiv \Gamma_{AB}(y,\beta,\phi) = \Gamma_{BC}(y,\beta,\phi) = \Gamma_{AC}(y,\beta,\phi)$ in the state (3.35) (see Ref. [39] for details):

$$\Gamma_{\rm sym}(y,\beta,\phi) = \mathcal{B}(y,\beta,\phi)\begin{pmatrix} 3+\cos\beta & \sin\beta & 0 & 1+\cos\beta \\ -\sin\beta & (1+\cos\beta) & 0 & -\sin\beta \\ 0 & 0 & (1+\cos\beta) & 0 \\ -(1+\cos\beta) & -\sin\beta & 0 & (1-\cos\beta) \end{pmatrix} \tag{3.36}$$

where

$$\mathcal{B}(y,\beta,\phi) = \frac{y^2(1-\cos\beta)^2}{2\left(1+y^2+2\,y\cos\phi\cos^3\frac{\beta}{2}\right)^2}. \tag{3.37}$$

The Lorentz invariant eigenvalues of $\Gamma_{\rm sym}(y,\beta,\phi)$ are found to be

$$\mu_0(y,\beta,\phi) = 2\,\mathcal{B}(y,\beta,\phi),\ \ \mu_2(y,\beta,\phi) = \mathcal{B}(y,\beta,\phi)\ (1+\cos\beta). \tag{3.38}$$

The concurrence for the two-qubit subsystem density matrices $\rho_{\rm sym}(y,\beta,\phi) = \rho_{AB}(y,\beta,\phi) = \rho_{BC}(y,\beta,\phi) = \rho_{AC}(y,\beta,\phi)$ drawn from the three-qubit pure symmetric state (3.35) is thus given by

$$\begin{aligned} C(y,\beta,\phi) &= \sqrt{\mathcal{B}(y,\beta,\phi)\ (1+\cos\beta)} \\ &= \frac{2\,y\ \sin\beta\ \sin\frac{\beta}{2}}{\left(1+y^2+2\,y\cos\phi\cos^3\frac{\beta}{2}\right)} \end{aligned} \tag{3.39}$$

which matches exactly with the formula derived in Ref. [34].

Substituting (3.38) in (3.19), we evaluate the three-tangle $\tau(y,\beta,\phi)$ in (3.35) to obtain

$$\begin{aligned}\tau(y,\beta,\phi) &= \mu_0(y,\beta,\phi) - \mu_2(y,\beta,\phi)\\ &= \mathcal{B}(y,\beta,\phi)(1-\cos\beta)\\ &= \left(\frac{2\,y\,\sin^3\frac{\beta}{2}}{1+y^2+2\,y\cos\phi\cos^3\frac{\beta}{2}}\right)^2,\end{aligned} \tag{3.40}$$

in agreement with the expression for τ^2 given in Ref. [34] for the state (3.35).

3.2 LU versus local SL(2,C) invariants of pure three-qubit state

Taking a closer look at the set of *five* LU invariants (1.2) of pure three-qubit states, it is seen that $I_1 = \mathrm{Tr}[\rho_{AB}^2]$, $I_2 = \mathrm{Tr}[\rho_{AC}^2]$, $I_3 = \mathrm{Tr}[\rho_{AB}^2]$ get replaced by their local SL(2,C) counterparts (see (3.22)) $\mathcal{K}_1 = \mathrm{Tr}[\rho_{AB}\,\widetilde{\rho}_{AB}]$, $\mathcal{K}_2 = \mathrm{Tr}[\rho_{BC}\,\widetilde{\rho}_{BC}]$, $\mathcal{K}_3 = \mathrm{Tr}[\rho_{AC}\,\widetilde{\rho}_{AC}]$. It is seen that the LU invariant $I_5 = \tau^2/16$ enjoys a higher level of invariance, by remaining unchanged when a three-qubit pure state undergoes local SL(2,C) transformation. Thus, we have *four* local SL(2,C) invariants, which encode information about the entanglement content in the three-qubit pure state since it is possible to reconstruct concurrences C_{AB}, C_{BC}, C_{AC} and three-tangle τ using them. On the other hand, the Kempe invariant $\mathcal{I}_4$ given by (1.4) (which is a permutation symmetric extension of the LU invariant I_4 listed in (1.2)) is known to be algebraically independent of concurrences and three tangles [28]. In order to complete the set of SLOCC invariants, we study the structure of I_4 with an intention to find its Lorentz invariant analogue. Using (2.1), (2.4), (2.5), we obtain

$$I_4 = \mathrm{Tr}\left[(\rho_A\otimes\rho_B)\,\rho_{AB}\right] = \frac{1}{4}\,\left(\mathbf{s}_A^T\,\Lambda_{AB}\,\mathbf{s}_B\right). \tag{3.41}$$

We consider [50]

$$\mathcal{K}_4 = \mathrm{Tr}\left[(\rho_A\otimes\rho_B)\,\widetilde{\rho}_{AB}\right] = \frac{1}{4}\,\left(\mathbf{s}_A^T\,G\,\Lambda_{AB}\,G\,\mathbf{s}_B\right). \tag{3.42}$$

to be the Lorentz invariant analogue replacing I_4. Thus, we have the following set of *five* local SL(2,C) invariants

$$\begin{aligned}\mathcal{K}_1 &= 2\,\mathrm{Tr}[\rho_{AB}\,\widetilde{\rho}_{AB}] = 4\,\lambda_0^2\,\lambda_3^2 + 2\,\lambda_0^2\lambda_4^2 = C_{AB}^2 + \frac{\tau}{2},\\ \mathcal{K}_2 &= 2\,\mathrm{Tr}[\rho_{BC}\,\widetilde{\rho}_{BC}] = 4\,\triangle + 2\,\lambda_0^2\lambda_4^2 = C_{BC}^2 + \frac{\tau}{2},\\ \mathcal{K}_3 &= 2\,\mathrm{Tr}[\rho_{AC}\,\widetilde{\rho}_{AC}] = 4\,\lambda_0^2\,\lambda_2^2 + 2\,\lambda_0^2\,\lambda_4^2 = C_{AC}^2 + \frac{\tau}{2},\\ \mathcal{K}_4 &= \mathrm{Tr}\left[(\rho_A\otimes\rho_B)\,\widetilde{\rho}_{AB}\right] = \lambda_0^2\,\left(\triangle + \lambda_2^2\lambda_3^2 - \lambda_1^2\lambda_4^2 + \lambda_3^2 + \lambda_4^2\right),\end{aligned}$$

$$\mathcal{K}_5 \equiv I_5 = \lambda_0^4\,\lambda_4^4 = \frac{\tau^2}{16}, \tag{3.43}$$

which are the algebraic counterparts of the LU invariants of the three-qubit pure state.

4 Summary

In this paper, we have extended the mathematical framework of Ref. [38] to explore local SL(2,C) invariants of pure three-qubit states. This method enables one to evaluate concurrences and tangle in terms of the *Lorentz invariant eigenvalues* of the 4×4 real positive matrices $\Gamma_{ij} = G\,\Lambda_{ij}^T\,G\,\Lambda_{ij}$, constructed from the real parametrizations Λ_{ij} of the two-qubit subsystem density matrices $\rho_{ij},\ ij = AB, BC, CA$ of a pure state of three qubits. In particular, we have shown that (i) the matrices Γ_{ij} evaluated in a pure three-qubit state have *at most* two distinct eigenvalues, (ii) the squared concurrence C_{ij}^2 is equal to the least eigenvalue of Γ_{ij}, and (iii) the three-tangle τ is equal to the difference between the highest and the smallest eigenvalues of Γ_{ij}. This is illustrated in the example of permutation symmetric three-qubit pure states. Finally, we have given a set of *five* local SL(2,C) invariants $\{\mathcal{K}_1,\ \mathcal{K}_2,\ \mathcal{K}_3,\ \mathcal{K}_4,\ \mathcal{K}_5\}$, which are the natural algebraic generalizations of Acín's LU invariants $\{I_1,\ I_2,\ I_3,\ I_4,\ I_5\}$.

Acknowledgements ARU, Sudha, and BNK are supported by the Department of Science and Technology (DST), India, through Project No. DST/ICPS/QUST/2018/107. ASH is supported by the Foundation for Polish Science (IRAP Project, ICTQT, contract no. MAB/2018/5, co-financed by EU within Smart Growth Operational Programme). HSK is supported by the Institute of Information & Communications Technology Planning & 14 Evaluation (IITP) Grant funded by the Korean government (MSIT) (No.2022-0-00463, Development of a quantum repeater in optical fiber networks for quantum internet). We thank Markus Grassl for insightful suggestions.

Author Contributions A.R.U proposed the idea and wrote the manuscript. Sudha, H.A.S, H.S.K, and B.N.K worked out the required mathematical details. All authors reviewed the manuscript.

Funding Department of Science and Technology, Ministry of Science and Technology, India (DST/ICPS/QuST/Theme-2/2019/Q107), Foundation for Polish Science, IRAP Project, ICTQT (Contract no. MAB/2018/5), Institute of Information & Communications Technology Planning & 14 Evaluation (IITP) (No. 2022-0-00463), Development of a quantum repeater in optical fiber networks for quantum internet.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Amico, L., Fazio, R., Osterloh, R., Vedral, V.: Entanglement in many-body systems. Rev. Mod. Phys. **80**, 517 (2008)
2. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865 (2009)
3. Eltschka, C., Siewert, J.: Quantifying entanglement resources. J. Phys. A Math. Theor. **47**, 424005 (2014)
4. Chitambar, E., Gaur, G.: Quantum resource theories. Rev. Mod. Phys. **91**, 025001 (2019)

5. Xie, S., Eberly, J.H.: Managing the three-party entanglement challenge. Contemp. Phys. **62**, 189 (2021)
6. Kempe, J.: Multiparticle entanglement and its applications to cryptography. Phys. Rev. **60**, 910 (1999)
7. Epping, M., Kampermann, H., Macchiavello, C., Bruß, D.: Multi-partite entanglement can speed up quantum key distribution in networks. New J. Phys. **19**, 093012 (2017)
8. Pivoluska, M., Huber, M., Malik, M.: Layered quantum key distribution. Phys. Rev. A **97**, 032312 (2018)
9. Yamasaki, H., Pirker, A., Murao, M., Dür, W., Kraus, B.: Multipartite entanglement outperforming bipartite entanglement under limited quantum system sizes. Phys. Rev. A **98**, 052313 (2018)
10. Cunha, M.M., Fonseca, A., Silva, E.O.: Tripartite entanglement: foundations and applications. Universe **5**, 209 (2019)
11. Navascués, M., Wolfe, E., Rosset, D., Pozas-Kerstjens, A.: Genuine network multipartite entanglement. Phys. Rev. Lett. **125**, 240505 (2020)
12. Grasselli, F., Murta, G., de Jong, J., Hahn, F., Bruß, D., Kampermann, H., Pappa, A.: Secure anonymous conferencing in quantum networks. PRX Quantum **3**, 040306 (2022)
13. Schlienz, J., Mahler, G.: The maximal entangled three-particle state is unique. Phys. Lett. A **224**, 39 (1996)
14. Dür, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A **62**, 062314 (2000)
15. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A **61**, 052306 (2000)
16. Acín, A., Andrianov, A., Costa, L., Jané, E., Latorre, J.I., Tarrach, R.: Generalized schmidt decomposition and classification of three-quantum-bit states. Phys. Rev. Lett. **85**, 1560 (2000)
17. Sudbery, A.: On local invariants of pure three-qubit states. J. Phys. A Math. Gen. **34**, 643 (2001)
18. Brun, T., Cohen, O.: Parametrization and distillability of three-qubit entanglement. Phys. Lett. A **281**, 88 (2001)
19. Rau, A.R.P.: Symmetries and geometries of qubits, and their uses. Symmetry **13**, 1732 (2021)
20. Grassl, M., Rotteler, M., Beth, T.: Computing local invariants of quantum-bit systems. Phys. Rev. A **58**, 1833 (1998)
21. Linden, N., Popescu, S.: On multi-particle entanglement. Fortschr. Phys. **46**, 567 (1998)
22. Albeverio, S., Fei, S.-M.: A note on invariants and entanglements. J. Opt. B Quantum Semiclass. Opt. **3**, 223 (2001)
23. Teodorescu-Frumosu, M., Jaeger, G.: Quantum Lorentz-group invariants of n-qubit systems. Phys. Rev. A **67**, 052305 (2003)
24. Verstraete, F., Dehaene, J., De Moor, B.: Local filtering operations on two qubits. Phys. Rev. A **64**, 010101(R) (2001)
25. Verstraete, F., Dehaene, J., De Moor, B.: Lorentz singular-value decomposition and its applications to pure states of three qubits. Phys. Rev. A **65**, 032308 (2002)
26. Albeverio, S., Fei, S.-M., Parashar, P., Yang, W.-L.: Nonlocal properties and local invariants for bipartite systems. Phys. Rev. A **68**, 010303(R) (2003)
27. Arrighi, P., Patricot, C.: A note on the correspondence between qubit quantum operations and special relativity. J. Phys. A Math. Gen. **36**, L287 (2003)
28. Osterloh, A.: Classification of qubit entanglement: SL(2, C) versus SU(2) invariance. Appl. Phys. B **98**, 609 (2010)
29. Williamson, M.S., Ericsson, M., Johansson, M., Sjöqvist, E., Sudbery, A., Vedral, V., Wootters, W.K.: Geometric local invariants and pure three-qubit states. Phys. Rev. A **83**, 062308 (2011)
30. de Vicente, J.I., Carle, T., Streitberger, C., Kraus, B.: Complete set of operational measures for the characterization of three-qubit entanglement. Phys. Rev. Lett. **108**, 060501 (2012)
31. Tajima, H.: Deterministic LOCC transformation of three-qubit pure states and entanglement transfer. Ann. Phys. **329**, 1 (2013)
32. Torun, G., Yildiz, A.: Canonical operators and the optimal concentration of three-qubit Greenberger-Horne-Zeilinger states. Phys. Rev. A **89**, 032320 (2014)
33. Eltschka, C., Siewert, J.: Monogamy equalities for qubit entanglement from Lorentz invariance. Phys. Rev. Lett. **114**, 140402 (2015)
34. Meill, A., Meyer, D.A.: Symmetric three-qubit-state invariants. Phys. Rev. A **96**, 062310 (2017)
35. Sun, B.-Z., Fei, S.-M., Wang, Z.-X.: On local unitary equivalence of two and three-qubit states. Sci. Rep. **7**, 4869 (2017)

36. Li, D.: Stochastic local operations and classical communication (SLOCC) and local unitary operations (LU) classifications of n qubits via ranks and singular values of the spin-flipping matrices. Quantum Inf. Proc. **17**, 132 (2018)
37. Torun, G., Yildiz, A.: Deterministic transformations of three-qubit entangled pure states. Phys. Rev. A **100**, 022320 (2019)
38. Sudha, Karthik, H.S., Pal, R., Akhilesh, K.S., Ghosh, S., Mallesh, K.S., Usha Devi, A.R.: Canonical forms of two-qubit states under local operations. Phys. Rev. A **102**, 052419 (2020)
39. Anjali, K., Reena, I., Sudha, Divyamani, B.G., Karthik, H.S., Mallesh, K.S., Usha Devi, A.R.: Geometric picture for SLOCC classification of pure permutation symmetric Threequbit states. Quantum Inf. Process. **21**, 326 (2022)
40. Divyamani, B.G., Reena, I., Panigrahi, P.K., Usha Devi, A.R., Sudha: Canonical steering ellipsoids of pure symmetric multiqubit states with two distinct spinors and volume monogamy of steering. Phys. Rev. A **107**, 042207 (2023)
41. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. **80**, 2245 (1998)
42. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants. Resultants and Multidimensional Determinants. Birkhäuser, Boston (1994)
43. Miyake, A.: Classification of multipartite entangled states by multidimensional determinants. Phys. Rev. A **67**, 012108 (2003)
44. Greenberger, D.M., Horne, M., Zeilinger, A.: Going Beyond Bell's Theorem. In: Kafatos, M. (ed.) Bell's Theorem, Quantum Theory, and Conceptions of the Universe, p. 69. Kluwer, Dordrecht (1989)
45. Srinivasa Rao, K.N.: The rotation and Lorentz groups and their representations for physicists. Wiley Eastern, New Delhi (1988)
46. It may be noted that $\sigma_2\,\sigma_\alpha\,\sigma_2 = \sum_{\alpha'=0,1,2,3} g_{\alpha\alpha'}\,\sigma'_\alpha$, where $g_{\alpha\alpha'}$ are the elements of the Minkowski metric $G = \mathrm{diag}(1,-1,-1,-1) = G^T$. Thus, $\sum_{\alpha,\beta} \Lambda_{\alpha\,\beta}\,(\sigma_2\,\sigma_\alpha\,\sigma_2)\otimes(\sigma_2\,\sigma_\beta\,\sigma_2) = \sum_{\alpha,\beta,\alpha',\beta'} \Lambda_{\alpha\,\beta}\,g_{\alpha\alpha'}\,g_{\beta\beta'} = \sum_{\alpha',\beta'} (G\,\Lambda\,G)_{\alpha'\beta'}\,\sigma'_\alpha \otimes \sigma'_\beta$
47. Usha Devi, A.R., Uma, M.S., Prabhu, R., Sudha: Non-local properties of a symmetric two-qubit system. J. Opt. B Quantum Semiclass. Opt. **7**, S740 (2005)
48. Usha Devi, A.R., Uma, M.S., Prabhu, R.: Sudha: local invariants and pairwise entanglement in symmetric multiqubit system. Int. J. Mod. Phys. B **20**, 1917 (2006)
49. Usha Devi, A.R., Uma, M.S., Prabhu, R., Rajagopal, A.K.: Constraints on the uncertainties of entangled symmetric qubits. Phys. Lett. A **364**, 203 (2007)
50. When $\mathbf{s}_A \rightarrow L_A\,\mathbf{s}_A$, $\mathbf{s}_B \rightarrow L_B\,\mathbf{s}_B$ and $\Lambda_{AB} \rightarrow L_A\,\Lambda_{AB}\,L_B^T$, L_A, L_B $\in$SO(3,1) the quantity $\mathcal{K}_{AB} = \mathbf{s}_A^T\,G\,\Lambda_{AB}\,G\,\mathbf{s}_B$ remains invariant